Y. Yoshino
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MODULES WITH LINEAR RESOLUTION OVER A POLYNOMIAL RING IN TWO VARIABLES

YUJI YOSHINO

§1. Introduction and main theorem

Let k be a field and let S be a polynomial ring $k[x_1, x_2, \cdots, x_n]$ over k in n variables. An S-module M is called a module with linear resolution if M has a free resolution;

$$0 \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0 \longrightarrow M \longrightarrow 0$$

where, after taking suitable bases of free modules, all f_i's are matrices consisting of linear forms of S. The reader should be referred to Eisenbud-Goto [2, Sections 0 and 1] for elementary facts concerning modules with linear resolution.

The purpose of this note is to give a complete classification of modules with linear resolution over a polynomial ring in two variables. The main theorem is the following.

THEOREM (1.1). *Let k be an algebraically closed field of any characteristic and let S denote a polynomial ring $k[x, y]$ in two variables. Then any finitely generated indecomposable module with linear resolution over S is isomorphic to one of the following;*

(i) $S/(x, y)S \simeq k$

(ii) $(x, y)^n$ $(n \in \mathbb{N})$ *or*

(iii) *the module* $M(n, p)$ $(n \in \mathbb{N},\ p \in \mathbb{P}^1_k = k \cup \{\infty\})$, *where* $M(n, p)$ *is given as the cokernel of a linear mapping;*

$$S^n \xrightarrow{A(n, p)} S^n$$

defined by the $n \times n$ *matrix*

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$$A(n,p)=\begin{pmatrix} x+py & y & 0 & 0\cdots\cdots 0 \\ 0 & x+py & y & 0\cdots\cdots 0 \\ 0 & 0 & x+py & y\cdots\cdots 0 \\ 0 & 0 & 0 & x+py\cdots\cdot 0 \\ \vdots & \vdots & \vdots & \vdots \quad\quad \vdots \\ 0 & 0 & 0 & 0\cdots\cdots y \\ 0 & 0 & 0 & 0\cdots\ x+py \end{pmatrix} \quad \textit{if } p\in k\,,$$

and

$$\begin{pmatrix} y & x & 0\cdots 0 & 0 \\ 0 & y & x\cdots 0 & 0 \\ 0 & 0 & y\cdots 0 & 0 \\ \vdots & \vdots & \vdots \quad \vdots & \vdots \\ 0 & 0 & 0\cdots y & x \\ 0 & 0 & 0\cdots 0 & y \end{pmatrix} \quad \textit{if } p=\infty\,.$$

A proof of the theorem will be given in Section 3. To say roughly, it will be shown that the proof is almost equivalent to finding all the indecomposable representations of the euclidean graph of type $\tilde{A}_1$. And hence, even if k is not algebraically closed, the theorem will be valid after a slight modification. See Section 3 for more detail. We shall also establish in Section 2 an equivalence between the category of linear complexes over a polynomial ring in n variables and the category of representations of the Grassmann algebra over n-dimensional vector space. Using this equivalence, we will be able to classify the modules as in the theorem. Further remarks for the case of three variables will be made in Section 4.

§ 2. Linear complexes

In this section S always denotes a polynomial ring $k[x_1, x_2, \cdots, x_n]$ in n variables over an arbitrary field k. Let V be a k-vector space of dimension n with a basis $\{e_1, e_2, \cdots, e_n\}$ and let V^* denote the dual space of V with the dual basis $\{x_1, x_2, \cdots, x_n\}$, that is, $x_i(e_j) = \delta_{ij}$ for any i and j. We regard S as the symmetric algebra $S(V^*)$ over V^*.

A linear complex over S;

$$F^{\cdot}:\ \cdots \longrightarrow F^p \xrightarrow{\varphi_p} F^{p+1} \xrightarrow{\varphi_{p+1}} \cdots \longrightarrow F^{q-1} \xrightarrow{\varphi_{q-1}} F^q \longrightarrow \cdots$$

is a complex consisting of finite free modules and matrices of linear forms of S. Note that a linear complex is completely determined by a set of matrices of linear forms $\{\varphi_i\}$ satisfying $\varphi_{i+1}\cdot\varphi_i = 0$ for all i. Let $\{\varphi_i\}$ and $\{\psi_i\}$ give linear complexes, and let $\{f_i\}$ be a morphism of complexes between them, i.e. $f_{i+1}\cdot\varphi_i = \psi_i\cdot f_i$ for all i. Then $\{f_i\}$ is said to be a linear morphism if each f_i is a matrix consisting of elements in $k = S_0(V^*)$. We denote by $\mathscr{L}(S)$ the category of all linear complexes over S and all linear morphisms. We also denote by $\mathscr{L}^b(S)$ the full subcategory of $\mathscr{L}(S)$ consisting of all bounded linear complexes. On the other hand, let G be a Grassmann algebra $\Lambda^{\cdot} V$ over V which we always regard as a $\mathbb{Z}$-graded algebra over k. The category of all graded left G-modules is denoted by $\mathscr{M}_{gr}(G)$ and we denote by $\mathscr{M}^f_{gr}(G)$ the full subcategory consisting of all finite modules. We want to establish the following equivalence.

THEOREM (2.1). *There is a category equivalence between $\mathscr{L}(S)$ and $\mathscr{M}_{gr}(G)$. And by this equivalence $\mathscr{L}^b(S)$ is equivalent to $\mathscr{M}^f_{gr}(G)$.*

Proof. We define a functor Φ: $\mathscr{M}_{gr}(G) \to \mathscr{L}(S)$ as follows. If $M = \sum M_i$ is a graded G-module, then each $v \in V$ gives a k-linear mapping $\varphi_i(v)$: $M_i \to M_{i+1}$ for any i, where each φ_i must be linear in v by definition, hence φ_i's are matrices consisting of elements in $S_1(V^*)$. They satisfy $\varphi_{i+1}\cdot\varphi_i = 0$ for all i, since the action of a square of any $v \in V$ on M is trivial and since each $\varphi_{i+1}\cdot\varphi_i$ is a matrix of quadratic forms. Thus the set of matrices $\{\varphi_i\}$ defines a linear complex which we denote by $\Phi(M)$. Note that if M is a finite G-module, then $\Phi(M)$ is a bounded complex. Next consider a morphism $f = \sum f_i$: $M = \sum M_i \to N = \sum N_i$ in $\mathscr{M}_{gr}(G)$. If $\{\varphi_i\}$ and $\{\psi_i\}$ define respectively $\Phi(M)$ and $\Phi(N)$, then regarding each f_i as a matrix consisting of elements in $S_0(V^*) = k$, it holds that $f_{i+1}\cdot\varphi_i = \psi_i\cdot f_i$ for all i as matrices over S, hence $\{f_i\}$ gives a linear morphism from $\Phi(M)$ to $\Phi(N)$, denoted by $\Phi(f)$.

A functor in the opposite direction will be defined by following the above in reverse. In fact, if $F^{\cdot}$: $\cdots \to F^i \xrightarrow{\varphi_i} F^{i+1} \xrightarrow{\varphi_{i+1}} F^{i+2} \to \cdots$ is a linear complex, then we consider k-spaces M_i of the same rank as free modules F^i, and denote $M = \sum M_i$ as a k-space. We make M a graded G-module by defining the action of $v \in V$ on M_i as $\varphi_i(v)$. We denote by $\Psi(F^{\cdot})$ the graded G-module M and may define a functor Ψ: $\mathscr{L}(S) \to \mathscr{M}_{gr}(G)$. Then it is easy to see that $\Phi\cdot\Psi = 1_{\mathscr{L}(S)}$ and $\Psi\cdot\Phi = 1_{\mathscr{M}_{gr}(G)}$, hence the categories are equivalent. It is also obvious that the functors Φ and Ψ give an equivalence between $\mathscr{L}^b(S)$ and $\mathscr{M}^f_{gr}(G)$. Q.E.D.

Remark (2.2). By the equivalence above the grade shifting on modules in $\mathscr{M}_{gr}(G)$ corresponds to the degree shifting on complexes. That is, if $M = \sum M_i$ is in $\mathscr{M}_{gr}(G)$ and if $F^{\cdot}$ is in $\mathscr{L}(S)$, then the following holds;

$$\Phi(F^{\cdot}[j]) = \Phi(F^{\cdot})[j] \quad \text{and} \quad \Psi(M[j]) = \Psi(M)[j]$$

where $(F^{\cdot}[j])^i = F^{j+i}$ and $(M[j])_i = M_{j+i}$.

As a corollary of Theorem (2.1) we remark the following fact which, a priori, is non trivial.

COROLLARY (2.3). *The category $\mathscr{L}^b(S)$ of linear bounded complexes over S admits the Krull-Schmidt theorem.*

§ 3. Modules over a Grassmann algebra

By Theorem (2.1) in order to analyze the category $\mathscr{L}(S)$ it will be enough to classify the modules over a Grassmann algebra. If $\dim(V) = 1$, it will be easy. In fact the Grassmann algebra G over V is a commutative algebra $k[e_1]/(e_1^2)$ and hence G and k are all the indecomposable G-modules, correspondingly we may conclude that all the linear complexes over $S = k[x]$ are

$$0 \longrightarrow S \xrightarrow{x} S \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow S \longrightarrow 0\,.$$

However the circumstances are more difficult in the case $S = k[x, y]$. If this is the case, the Grassmann algebra G is a four dimensional k-algebra having a basis $\{1, e_1, e_2, e_1 \wedge e_2 = - e_2 \wedge e_1\}$, and it is easily seen that G is self-injective. Let R be a quotient algebra $G/(e_1 \wedge e_2) = k[e_1, e_2]/(e_1^2, e_1e_2, e_2^2)$, then the following holds true, and is easy to prove.

LEMMA (3.1). *If M is an indecomposable left G-module which is not free, then M is annihilated by $e_1 \wedge e_2$, i.e. M is a module over R.*

By this lemma it is sufficient to consider the modules over R. We note that the category of modules over R is stably equivalent to the category of representations of the euclidean graph $\tilde{A}_1$, hence the classification of those modules are known as a solution to Kronecker's problem. See [1] or [3] for the detail. We exhibit in the following the classification of R-modules in the convenient form for us.

LEMMA (3.2). *Let T be a commutative k-algebra $k[e_1, e_2]/(e_1^2, e_2^2)$ where k is an algebraically closed field. Then any finite indecomposable R-module is isomorphic to one of the following modules.*

(1) *the modules* $L_n (n \in \mathbb{Z})$ *defined as follows;* $L_0 = k$, *if* $n < 0$ *then* L_n *is the* $(-n)$*th syzygy module of* k *as a* T*-module (not as an* R*-module) and if* $n > 0$ *then* L_n *is the* T*-dual* $\mathrm{Hom}_T(L_{-n}, T)$ *of* L_{-n}.

(2) *the modules* $M(n, p)$ $(n \in \mathbb{N},\ p \in \mathbb{P}^1_k = k \cup \{\infty\})$ *defined as follows;* $M(n, p)$ *is a graded* R*-module* $W_1 \oplus W_2$ *having only two graded pieces, where* $W = W_1 = W_2$ *is an* n*-dimensional* k*-space, and* e_1 *and* e_2 *act as mappings of degree* 1. *If* $p \in k$, *then* e_1 *acts on* W_1 *as the identity (to* W_2*) and* e_2 *as a matrix of Jordan canonical form;* $\begin{bmatrix} p & 1 & 0 \cdots 0 & 0 \\ 0 & p & 1 \cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots p & 1 \\ 0 & 0 & 0 \cdots 0 & p \end{bmatrix}$.

If $k = \infty$, *then* e_1 *acts on* W_1 *as a matrix;* $\begin{bmatrix} 0 & 1 & 0 \cdots 0 & 0 \\ 0 & 0 & 1 \cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots 0 & 1 \\ 0 & 0 & 0 \cdots 0 & 0 \end{bmatrix}$ *and* e_2 *as the identity.*

Remark (3.3). Even if k is not algebraically closed, the lemma above holds true after a slight modification. In fact, it is enough to replace (2) in Lemma (3.2) by the below:

Let $\mathbb{P}^1_k$ be a set of all monic irreducible polynomials in $k[e_1]$ and a symbol ∞. If $p(e_1) = e_1^m + c_1 e_1^{m-1} + \cdots + c_{m-1} e_1 + c_m$ is an irreducible polynomial, then we denote by $J(p)$ the $m \times m$ matrix;

$$\begin{bmatrix} 0 & 1 & 0 \cdots\cdots 0 & 0 \\ 0 & 0 & 1 \cdots\cdots 0 & 0 \\ \vdots & \vdots & \vdots \quad\quad \vdots & \vdots \\ 0 & 0 & 0 \cdots\cdots 0 & 1 \\ -c_m & -c_{m-1} & -c_{m-2} \cdots -c_2 & -c_1 \end{bmatrix}$$

and denote by $I(p)$ the $m \times m$ matrix;

$$\begin{bmatrix} 0 & 0 \cdots 0 & 0 \\ 0 & 0 \cdots 0 & 0 \\ \vdots & \vdots \quad \vdots & \vdots \\ 0 & 0 \cdots 0 & 0 \\ 1 & 0 \cdots 0 & 0 \end{bmatrix}.$$

Then $M(n,p)$ is a graded module $W_1 \oplus W_2$ where $W = W_1 = W_2$ is an nm-dimensional space such that if $p \neq \infty$ then e_1 acts on W_1 as the identity and e_2 as a matrix;

$$\begin{bmatrix} J(p) & I(p) & 0 \cdots\cdots 0 & 0 \\ 0 & J(p) & I(p) \cdots 0 & 0 \\ \vdots & \vdots & \vdots \quad\quad \vdots & \vdots \\ 0 & 0 & 0 \cdots J(p) & I(p) \\ 0 & 0 & 0 \cdots\cdots 0 & J(p) \end{bmatrix},$$

and if $p = \infty$ then $M(n,p)$ is the same as the one in Lemma (3.2).

By Lemma (3.2) and Remark (3.3) above we see that every module over the Grassmann algebra G has a graded structure, and the modules in the list of Lemma (3.2) are thus giving all indecomposable objects in $\mathcal{M}^f_{gr}(G)$ up to grade shifting. Combining this with Theorem (2.1) we can classify all the indecomposable bounded linear complexes over $k[x, y]$.

THEOREM (3.4). *Let S be a polynomial ring $k[x,y]$ over an algebraically closed field k. Then any indecomposable bounded linear complex is one of the following*:

(1) *the Koszul complex*; $K^{\cdot}$: $0 \to S \xrightarrow[(x,\,-y)]{} S^2 \xrightarrow[\binom{y}{x}]{} S \to 0$,

(2) *the complexes* $L^{\cdot}_n$ $(n \in \mathbb{Z})$; $L^{\cdot}_0$: $0 \to S \to 0$, *if* $n > 0$ *then* $L^{\cdot}_n$: $0 \to S^n \xrightarrow[\begin{bmatrix} y & x & 0 & 0\cdots 0 & 0 \\ 0 & y & x & 0\cdots 0 & 0 \\ 0 & 0 & y & x\cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \quad \vdots & \vdots \\ 0 & 0 & 0 & 0\cdots x & 0 \\ 0 & 0 & 0 & 0\cdots y & x \end{bmatrix}]{} S^{n+1} \to 0$ *and if* $n < 0$ *then* $L^{\cdot}_n$ *is the dual complex* $\mathrm{Hom}_S(L^{\cdot}_{-n}, S)$ *of* $L^{\cdot}_{-n}$.

(3) *the complexes* $M^{\cdot}(n,p)$ $(n \in \mathbb{N},\ p \in \mathbb{P}^1_k)$; *if* $p \in k$ *then* $M^{\cdot}(n,p)$: $0 \to S^n \xrightarrow[\begin{bmatrix} x+py & y & 0\cdots\cdots 0 \\ 0 & x+py & y\cdots\cdots 0 \\ \vdots & \vdots & \vdots \quad\quad \vdots \\ 0 & 0 & 0 \quad\quad y \\ 0 & 0 & 0\cdots x+py \end{bmatrix}]{} S^n \to 0$ *and if* $p = \infty$ *then*

$$M^{\cdot}(n,p)\colon\ 0 \to S^n \xrightarrow[\begin{pmatrix} y & x & 0 \cdots 0 \\ 0 & y & x \cdots 0 \\ \vdots & \vdots & \vdots \quad \vdots \\ 0 & 0 & 0 \cdots x \\ 0 & 0 & 0 \cdots y \end{pmatrix}]{\qquad\qquad} S^n \to 0.$$

Remark (3.5.) If k is not algebraically closed, then (3) in Theorem (3.4) should read as follows:

Let $\mathbb{P}^1_k$ stand for the set of all monic irreducible polynomials in $k[e_1]$ and the symbol ∞. The complex $M^{\cdot}(n, \infty)$ is the same as in Theorem (3.4). For a $p \neq \infty$ in $\mathbb{P}^1_k$, let $I(p)$ and $J(p)$ be as in Remark (3.3). Further we denote by $E(p)$ the identity matrix of the size $\deg(p)$. Then the complex $M^{\cdot}(n,p)$ $(p \neq \infty)$ is the complex;

$$0 \longrightarrow S^{n\cdot\deg(p)} \xrightarrow[\begin{pmatrix} xE(p)+yJ(p) & yI(p) \cdots\cdots & 0 \\ 0 & xE(p)+yJ(p) \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \cdots\cdots & yI(p) \\ 0 & 0 \cdots & xE(p)+yJ(p) \end{pmatrix}]{\qquad\qquad} S^{n\cdot\deg(p)} \longrightarrow 0.$$

Using Theorem (3.4) we are now able to prove Theorem (1.1). In fact, if M is a finitely generated indecomposable module with linear resolution over $S = k[x,y]$ with k algebraically closed, then the resolution $F^{\cdot}$ of M is an indecomposable linear complex, hence it must be one of the complexes in Theorem (3.2). Since $F^{\cdot}$ is acyclic, it is either $K^{\cdot}$, $L^{\cdot}_n$ $(n \geq 0)$ or $M^{\cdot}(n,p)$ $(n \in \mathbb{N}, p \in \mathbb{P}^1_k)$, which gives that M is either k, $(x,y)^n$ $(n \geq 0)$ or $M(n,p)$ as in Theorem (1.1). Q.E.D.

Remark (3.6). Even if k is not algebraically closed, Theorem (1.1) will be valid after replacing $M(n,p)$ in the theorem by the homology module of the complex in Remark (3.5).

§ 4. Modules with linear resolution over $k[x, y, z]$

In this section we are concerned with the modules with linear resolution over a polynomial ring $S = k[x,y,z]$ in three variables. By the notation in the beginning of Section 2, S is the symmetric algebra over $V^* = \langle x, y, z\rangle$ where V denotes the three dimensional k space with a basis $\{e_1, e_2, e_3\}$ which is dual to $\{x, y, z\}$. We consider a category $\mathscr{T}(S)$ consisting of all finitely generated graded torsion S-modules M with linear resolution and with the condition that $\operatorname{depth}(M) = 2$ (or equivalently $\operatorname{pd}(M) = 1$) and that $M/(x, y)M$ is a direct sum of copies of k, and morphisms in $\mathscr{T}(S)$ are graded homomorphisms of degree 0. For a good evidence that the classification of modules with linear resolution over S would be hopeless, we show that the subcategory $\mathscr{T}(S)$ in the category of all modules with linear resolution is equivalent to the category of all finite modules over the free algebra $k\langle e_1, e_2\rangle$ in two variables.

THEOREM (4.1). *There is a category equivalence between* $\mathscr{T}(S)$ *and* ($k\langle e_1, e_2\rangle$-mod.).

Proof. Let M be a module in $\mathscr{T}(S)$. Since M is a torsion module and has depth 2, it has the free resolution;

$$0 \longrightarrow S^n \xrightarrow{\varphi} S^n \longrightarrow M \longrightarrow 0,$$

where φ is a matrix of linear forms in S. Consider the graded module N over the Grassmann algebra $G = \Lambda\langle e_1, e_2, e_3\rangle$ corresponding to this linear complex under the equivalence of Theorem (2.1). Note that N has only two graded pieces, and we may write $N = W_1 \oplus W_2$, where $W = W_1 = W_2$ is an n-dimensional k-space. The condition that $M/(x, y)M \simeq k^n$ says that, after taking a suitable basis, the matrix $\varphi \otimes S/(x, y)S$ is the identity matrix times z, and hence the action of e_3 on W_1 to W_2 is the identity mapping. The actions of e_1 and e_2 on $W_1 = W$ to $W_2 = W$ define a $k\langle e_1, e_2\rangle$-module structure on W, which we denote by $\Phi(M)$.

Let f be a graded homomorphism from M to M', where M and M' are in $\mathscr{T}(S)$. Then f induces a morphism (f_0, f_1) between the linear resolutions as follows;

$$\begin{tikzcd}
0 \arrow[r] & S^n \arrow[r, "\varphi"] \arrow[d, "f_1"'] & S^n \arrow[r] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\
0 \arrow[r] & S^{n'} \arrow[r, "\varphi'"'] & S^{n'} \arrow[r] & M' \arrow[r] & 0 .
\end{tikzcd}$$

Since f preserves the degree, (f_0, f_1) must be a linear morphism. If $N = W_1 \oplus W_2$ (resp. $N' = W_1' \oplus W_2'$) is the graded G-module corresponding to M (resp. M'), then (f_0, f_1) gives a graded G-homomorphism $g = g_1 \oplus g_2$ from N to N' by Theorem (2.1). Since the action of e_3 is the identity on

W_1 to W_2 (resp. on W_1' to W_2'), the diagram; $\begin{array}{ccc} W_1 & \xrightarrow{id} & W_2 \\ {\scriptstyle g_1}\big\downarrow & & \big\downarrow{\scriptstyle g_2} \\ W_1' & \xrightarrow[id]{} & W_2' \end{array}$ is commutative.

We denote by $\Phi(f)$ the linear mapping $g_1\colon W_1 \to W_1'$ (or equivalently g_2: $W_2 \to W_2'$). Since the following diagram is commutative, we see that $\Phi(f)$ is actually a $k\langle e_1, e_2\rangle$ homomorphism from $\Phi(M)$ to $\Phi(M)$.

$$\begin{array}{ccc} W_1 & \xrightarrow{e_i} & W_2 \\ {\scriptstyle g_1}\big\downarrow & & \big\downarrow{\scriptstyle g_2} \\ W_1' & \xrightarrow[e_i]{} & W_2' \end{array} \qquad (i = 1, 2)\,.$$

Thus we defined a functor Φ from $\mathscr{T}(S)$ to ($k\langle e_1, e_2\rangle$-mod.).

Next we want to define a reverse functor. For this let W be a finite module over $k\langle e_1, e_2\rangle$. Then we denote $N = W_1 \oplus W_2$ as a k-space where $W = W_1 = W_2$, and define a G-module structure by the following;

> $e_i W_2 = 0$ $(i = 1, 2, 3)$, $e_i\colon W_1 \to W_2$ $(i = 1$ or $2)$ is the same as the original action as a $k\langle e_1, e_2\rangle$-module, and e_3 acts on W_1 to W_2 as the identity mapping.

Thus we obtain a graded G-module N, and consider the linear complex corresponding to N under the equivalence in Theorem (2.1), say

$$0 \longrightarrow S^n \xrightarrow{\varphi} S^n \longrightarrow 0\,.$$

Here that the action of e_3 is the identity gives that $\varphi \otimes S/(x, y)S$ is a diagonal matrix with diagonal elements z. In particular this implies that φ is injective and the cokernel $\Psi(W)$ of φ is in $\mathscr{T}(S)$. For a $k\langle e_1, e_2\rangle$-homomorphism $h\colon W \to W'$, we may define a linear mapping g from $N = W_1 \oplus W_2$ $(W = W_1 = W_2)$ to $N' = W_1' \oplus W_2'$ $(W' = W_1' = W_2')$ by $g = h \oplus h$. Then it is easy to see that g is, in fact, a G-homomorphism, and hence it gives an S-homomorphism $\Psi(h)$ from $\Psi(W)$ to $\Psi(W')$ by Theorem (2.1). This defines the functor $\Psi\colon (k\langle e_1, e_2\rangle\text{-mod.}) \to \mathscr{T}(S)$.

From the definition of the functors one can easily show that $\Phi \cdot \Psi = 1$ and $\Psi \cdot \Phi = 1$, and this completes the proof. Q.E.D.

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Department of Mathematics
Nagoya University
Chikusa-ku, Nagoya 464
Japan