H. Kitahara, H. Kawakami and J. S. Pak
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ON A CONSTRUCTION OF COMPLETE SIMPLY-CONNECTED RIEMANNIAN MANIFOLDS WITH NEGATIVE CURVATURE

HARUO KITAHARA, HAJIME KAWAKAMI AND JIN SUK PAK

Let M be a complete simply-connected riemannian manifold of even dimension m. J. Dodziuk and I.M. Singer ([D1]) have conjectured that $H_2^p(M) = 0$ if $p \neq m/2$ and dim $H_2^{m/2}(M) = \infty$, where $H_2^*(M)$ is the space of L_2-harmonic forms on M.

Recently, M. T. Anderson ([An]) constructed manifolds which are counterexamples to the J. Dodziuk-I. M. Singer conjecture. In this paper, we will discuss how to construct complete simply-connected riemannian manifolds with negative sectional curvature, by the idea of M. T. Anderson and a private advice of J. Dodziuk ([D2]).

THEOREM. *Let B be a complele riemannian C^∞-manifold with C^∞-connected boundary ∂B and f a C^∞-function on B. Suppose that B and f satisfy the following conditions;*

(B.1) *B has the riemannian simple double $2B$, that is the canonically endowed continuous metric of $2B$ is smooth.*

(B.2) *The sectional curvature K_B of B is negative, or $B := [0, \infty)$,*

(B.3) *B is simply-connected,*

(F.1) *f is a function of the geodesic distance r from ∂B,*

(F.2) *f is an odd function of r on a neighborhood of $r = 0$ and satisfies that $f'(0) = 1$, $f''(r) > 0$ for $r > 0$, and $f'''(0) > 0$.*

Let $M := (B \setminus \partial B) \times_{f|_{B\setminus\partial B}} S^n(1)$. Then there is the unique complete simply-connected riemannian manifold $\mathscr{M}$ with negative curvature which is the completion of M.

Remark. Any function on $[0, \infty)$ can be considered as a function satisfying (F.1) under the assumptions (B.1)-(B.3).

Manifolds are supposed to be connected paracompact Hausdorff spaces.

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1.

J. Kazdan-F. Warner ([K-W]) proved that, for a C^∞-metric g on $R^2\setminus\{0\}$, there is a C^∞-metric $\tilde{g}$ on R^2 such that $\tilde{g}$ restricted to $R^2\setminus\{0\}$ is g. First, we will generalize their result.

LEMMA 1.1 (cf. [K-W], [O-N, p. 31]). *If $f(t)$ is a real valued C^∞-even function on R, then $f(r)$ is a C^∞-function on R^n, where $r := ((x^1)^2 + \cdots + (x^n)^2)^{1/2}$.*

LEMMA 1.2. *Let $f\colon R^m \times R^{1+n} \to R$ be a continuous function. If f satisfies the following conditions;*

(1.2.1) *f is of class C^∞ on $(R^m \times R^{1+n})\setminus(R^m \times \{0\})$,*

(1.2.2) *f is invariant under $\{I_m\} \times O(n+1)$, where $\{I_m\}$ is the unit group on R^m and $O(n+1)$ is the rotation group on R^{1+n},*

(1.2.3) *f is of class C^∞ on $R^m \times l$ for any straight line $l \subset R^{1+n}$ through the origin, then f is of class C^∞ on $R^m \times R^{1+n}$.*

Proof. We introduce two coordinates on $R^m \times R^{1+n}$, one is the usual Cartesian coordinates $(x^1, \cdots, x^m, z^0, z^1, \cdots, z^n)$ and one is $(x^1, \cdots, x^m, r, y^1, \cdots, y^n)$ where $(r, y^1, \cdots, y^n)$, $(r > 0)$, the polar coordinates on R^{1+n}. By (1.2.2), we can consider that f is a function with only $(x^1, \cdots, x^m, r)$ variables.

Step 1. We take a point $x_o := (x_o^1, \cdots, x_o^m)$ and fix it. (1.2.3), (1.2.2) and Lemma 1.1 imply that $f_o(r) := f(x_o, r)$, $r := ((z^1)^2 + \cdots + (z^n)^2)^{1/2}$, can be considered to be of class C^∞ on R^{1+n}. Since $(\partial/\partial x^i)f$ are invariant under $\{I_m\} \times O(n+1)$ and are of class C^∞ on $R^m \times l$ for a fixed l, if we choose any sequence $\{(x_n, z_n)\}$ in $R^m \times R^{1+n}$ converging to $(x_o, 0)$, then we have

$$\left|\left(\frac{\partial}{\partial x^j}\right)f(x_n, z_n) - \left(\frac{\partial}{\partial x^j}\right)f(x_o, 0)\right|$$
$$= \left|\left(\frac{\partial}{\partial x^j}\right)f(x_n, \pi(z_n)) - \left(\frac{\partial}{\partial x^j}\right)f(x_o, 0)\right| \longrightarrow 0$$
$$((x_n, z_n) \longrightarrow (x_o, 0)),$$

where $\pi\colon R^{1+n} \to l_+ := \{r \in l \mid r \geq 0\}$ is the canonical projection. Thus, together with by (1.2.1), we have that $(\partial/\partial x^j)f$ are continuous on $R^m \times R^{1+n}$, and, inductively, $(\partial^{\alpha_1+\cdots+\alpha_k}/(\partial x^{i_1})^{\alpha_1}\cdots(\partial x^{i_k})^{\alpha_k})f$ are continuuos on $R^m \times R^{1+n}$.

Step 2. We set

$$F(x, r) := \left(\frac{\partial^{\alpha_1+\cdots+\alpha_k}}{(\partial x^{i_1})^{\alpha_1}\cdots(\partial x^{i_k})^{\alpha_k}}\right) f(x, r), \ \alpha_1 + \cdots + \alpha_k \geq 0 .$$

Note that $F_o(r)$ is of class C^∞ on R^{1+n}. For example, since

$$\frac{\partial^2}{\partial z^\alpha \partial z^\beta} F(x, r) = \begin{cases} \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 F(x, r) z^\alpha z^\beta, & \alpha \neq \beta , \\ \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 F(x, r)(z^\alpha)^2 + \frac{1}{r}\frac{\partial}{\partial r} F(x, r), & \alpha = \beta , \end{cases}$$

and $(1/r \cdot \partial/\partial r)^p F(x, r)$ $(p = 0, 1, 2, \cdots)$ are even functions in r, by the same way as Step 1, $(\partial^2/\partial z^\alpha \partial z^\beta)F(x, r)$ are continuous on $R^m \times R^{1+n}$. More generally, we have

$$\left(\frac{\partial^{\beta_1+\cdots+\beta_s+\alpha_1+\cdots+\alpha_k}}{(\partial z^{j_1})^{\beta_1}\cdots(\partial z^{j_s})^{\beta_s}(\partial x^{i_1})^{\alpha_1}\cdots(\partial x^{i_k})^{\alpha_k}}\right) f \quad (\beta_1 + \cdots + \beta_s > 0, \ \alpha_1 + \cdots + \alpha_k \geq 0)$$

are continuous on $R^m \times R^{1+n}$. Therefore, f is of class C^∞ on $R^m \times R^{1+n}$. □

PROPOSITION 1.3. *Let B be a complete riemannian manifold with C^∞-boundary ∂B and f a C^∞-function on B. Suppose that B and f satisfy the following conditions;*

(1.B.1) *B has the riemannian simple double $2B$,*

(1.F.1) *$f(x) > 0$ if $x \in B \setminus \partial B$, and f is an odd function on a neighbourhood of ∂B of the arc-length r in the inner normal direction to ∂B.*

(1.F.2) $\|\operatorname{grad} f\|(x) = 1$ *if* $x \in \partial B$.

Let M be $(B \setminus \partial B) \times_{f|_{B\setminus\partial B}} S^n(1)$. Then there is the unique complete riemannian manifold $\mathcal{M}$ without boundary such that $\mathcal{M}$ is the completion of M.

Proof. Let (U, φ) be a local path of ∂B and N the ε-collar neighborhood of U in B, We define a manifold $\mathcal{N}$ by

$$\mathcal{N} := (N \setminus U) \times_{f|_{N/U}} S^n(1) .$$

Imbedding of $S^n(1)$ into R^{1+n}, we define a diffeomorphism Ψ of $\mathcal{N}$ into $R^m \times R^{1+n}$ by

$$\Psi: ((x, \exp rX), y) \longrightarrow (\varphi(x), r(y)) ,$$

where $X \in T_x B$ is the unit inner normal vector to ∂B and $0 < r < \varepsilon$.

We take the riemannian metric g of $\Psi(\mathscr{N})$ so that Ψ may become an isometry. Note that g can be extended to the continuous metric $\bar{g}$ of $\overline{\Psi(\mathscr{N})}$ by the natural way. We have only to show that $\bar{g}$ is of class C^∞ at the origin. Let $(x^1, \cdots, x^m, x^{m+1}, \cdots, x^{m+1+n})$ be the Cartesian coordinates of $R^m \times R^{1+n}$. And we adopt the ranges of indices;

$$1 \leq i, j \leq m \quad \text{and} \quad m+1 \leq \alpha, \beta \leq m+1+n .$$

It is clear from Lemma 1.2 that $\bar{g}_{ij}: = \bar{g}(\partial/\partial x^i, \partial/\partial x^j)$ is of class C^∞. It follows from Lemma 1.2 again that $(1/r)\bar{g}(\partial/\partial x^i, \partial/\partial r)$ is of class C^∞. Therefore $\bar{g}_{i\alpha}: = \bar{g}(\partial/\partial x^i, \partial/\partial x^\alpha) = x^\alpha(1/r)\bar{g}(\partial/\partial x^i, \partial/\partial r)$ is of class C^∞. Finally, we have that

$$\begin{aligned} \bar{g}_{\alpha\beta}: &= \bar{g}(\partial/\partial x^\alpha, \partial/\partial x^\beta) \\ &= \tilde{g}_{\alpha\beta} + \frac{f^2(x, r) - r^2}{r^4} r^4 g_{S^n}(\partial/\partial x^\alpha, \partial/\partial x^\beta) \\ &= \tilde{g}_{\alpha\beta} + \frac{f^2(x, r) - r^2}{r^4} (r^2 \tilde{g}_{\alpha\beta} - x^\alpha x^\beta) \end{aligned}$$

where $\tilde{g}$ is the standard metric on $R^m \times R^{1+n}$. It follows from Lemma 1.2 that $(f^2 - r^2)/r^4$ is of class C^∞. Therefore, $\bar{g}_{\alpha\beta}$ is of class C^∞. □

Remark 1.4 ([B] p. 269). If $m = 0$ in Proposition 1.3, we can get a theorem of J. Kazdan-F. Warner; If we identify $\{x \in R^{1+n} \mid 0 < |x| < \varepsilon\}$ with $(0, \varepsilon) \times S^n$ in polar coordinates, the C^∞-riemannian metric $dt^2 + \varphi(t)^2 \hat{g}_0$ (where t is the parameter on $(0, \varepsilon)$ and $\hat{g}_0$ a metric on S^n) extends to a C^∞-riemannian metric on $\{x \in R^n \mid |x| < \varepsilon\}$ if and only if $\hat{g}_0$ is λg_{can} where g_{can} is the canonical metric on S^n and λ some positive constant, and $(1/\lambda)\varphi$ is the restriction on $(0, \varepsilon)$ of *a* C^∞ odd function on $(0, \varepsilon)$ with $(1/\lambda)\varphi'(0) = 1$.

OBSERVATION 1.5. Since $\mathscr{M}$ is a completion of M as a metric space, by means of theory of metric spaces, we can see that the condition (1.B.1) is necessary for the existence of $\mathscr{M}$. The condition (1.B.1) is strictly stronger than the condition that ∂B is totally geodesic. For example, consider the surface of revolution of the graph

$$x \in [0, \infty) \longrightarrow x^3 - 3x^2 + 6 \in R .$$

2.

LEMMA 2.1 ([B-O]). *Let $M: = B \times_f F$ be a warped product with a warping function f where B and F are any riemannian manifolds. Let* π_1

and π_2 *be the natural projections of* M *onto* B *and* F *respectively. Let* Π *be a 2-plane tangent to* M *at* x *and* $\{X+V, Y+W\}$ *an orthonormal basis for* Π, *where* $X, Y \in T_{\pi_1(x)}B$ *and* $V, W \in T_{\pi_2(x)}F$. *The sectional curvature* $K(\Pi)$ *of* Π *in* M *is given by*

$$K(\Pi) = K^1_{X,Y} + K^2_{X,Y,V,W} + K^3_{V,W}\,,$$

where

$$\begin{aligned} K^1_{X,Y} &:= K_B(X, Y)\|X \wedge Y\|^2_B\,, \\ K^2_{X,Y,V,W} &:= -f(\pi_1(x))\{\|W\|^2_F((\nabla_B)^2 f)(X, X) - 2\langle V, W\rangle_F((\nabla_B)^2 f)(X, Y) \\ &\qquad + \|V\|^2_F((\nabla_B)^2 f)(Y, Y)\}\,, \\ K^3_{V,W} &:= f^2(\pi_1(x))\{K_F(V, W) - \|\operatorname{grad} f\|^2_B\}\|V \wedge W\|^2_F\,, \end{aligned}$$

and $\nabla_{(.)}$ *and* $K_{(.)}$ *are the covariant derivative and the sectional curvature of* $(\,.\,)$ *respectively and* $(\nabla_B)^2 f$ *is the Hessian of* f.

We shall prove Theorem. By the conditions of B, there is a diffeomorphism Ψ: $\partial B \times [0, \infty) \to B$ such that, for any $x \in \partial B$, $\tau_x(r) := \Psi(x, r)$ is the geodesic parametrized by the arc-length r, starting at x and normal to ∂B. (Thus, Remark after Theorem holds.) Moreover, we have that $\pi_1(\mathcal{M}) = \pi_1(\partial B \times R^{1+n}) = \pi_1(\partial B) = 0$, because ∂B is simply-connected by the conditions. Since Lemma 2.1, (B.2) and (F.2) imply that K^1, K^2 and K^3 are non-positive on M and at least one of them is strictly negative on M, it is enough to show that at least one of K^1, K^2 and K^3 is strictly negative if $r \to 0$. Let x_0 be any point of ∂B and X_r, Y_r, V_r, W_r any vector fields along $\tau_{x_0}(r)$, where X_r, Y_r are horizontal and V_r, W_r are vertical if $r \neq 0$.

Case 1. The case that X_0 and Y_0 are linearly independent. We have

$$K^1_{X_0,Y_0} < 0\,.$$

Case 2. The case that V_0 and W_0 are linearly independent. (F.1) and (F.2) imply that

$$f^2(r) = r^2 + 2ar^4 + \cdots, \qquad a > 0$$

and

$$\begin{aligned} \|\operatorname{grad} f(r)\|^2_B &\geq \langle \operatorname{grad} f(r),\ \partial/\partial r\rangle^2_B \\ &= \left(\frac{\partial f}{\partial r}\right)^2 \\ &= 1 + 6ar^2 + \cdots\,. \end{aligned}$$

Then we have

$$\frac{1-\|\operatorname{grad} f(r)\|_B^2}{f^2(r)} \leq \frac{1-(1+6ar^2+\cdots)}{r^2+2ar^4+\cdots} = \frac{-6a+O(r)}{1+O(r)}\,.$$

Therefore we have

$$\lim_{r\to 0} K^3_{V_r,W_r} \leq -6a < 0\,.$$

Case 3. The case except Case 1 and Case 2. We can choose X_r, Y_r, V_r and W_r such that $Y_r = c_1X_r$ and $W_r = c_2V_r$, where c_1 and c_2 are constants with $c_1 \neq c_2$. Let Π_r be the 2-plane spanned by the orthonormal basis $\{X_r + V_r,\ Y_r + W_r\}$. Then we have

$$K(\Pi_r) = -\frac{((\nabla_B)^2 f)_{X_r,X_r}}{f(r)\langle X_r, X_r\rangle_B}\,.$$

To get $\lim_{r\to 0} K(\Pi_r) < 0$, it is enough to show that

$$\lim_{r\to 0}\frac{((\nabla_B)^2 f)_{X_r,X_r}}{f(r)} > 0$$

under the assumption $\|X_r\|_B = 1$.

$$\frac{((\nabla_B)^2 f)_{X_r,X_r}}{f(r)} = \frac{f''(r)(\nabla_{X_r} r)^2 + f'(r)(\nabla^2 r)_{X_r,X_r}}{f(r)}\,,$$

and (F.2) imply the claim. Therefore we have Theorem.

EXAMPLE 2.2 (cf. [M]). Let R^m be given a negatively curved metric, and $B := [0,\infty)\times_\varphi R^m$ the warped product with the warping function φ such that (1) φ is a C^∞-even function in a neighbourhood of 0, (2) $\varphi > 0$, and (3) $\varphi'' > 0$. Then B satisfies the conditions of Theorem.

Comment of counter example of M. T. Anderson. If, in Theorem, we set the following, we can get his example; $2B := H^{2p}(-a^2)$, $\partial B :=$ the totally geodesic hyperplane H^{2p-1} of $H^{2p}(-a^2)$ and $f(r) := \sinh r$.

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REFERENCES

[A] M. F. Atiyah, Elliptic operators, discrete groups and von Neumann algebras, Astérisque, **32–33** (1976), 43–72.

[An] M. T. Anderson, L^2-harmonic forms and a conjecture of Dodziuk-Singer, Bull. Amer. Math. Soc., **13** (1985), 163–165.

[B] A. L. Besse, Einstein manifolds, Ergeb. Math. Grenzgeb. 3. folge. Band **10**, 1987, Springer-Verlag.

[B-O] R. L. Bishop-B. O'Neill, Manifolds of negative curvature, Trans. Amer. Math. Soc., **145** (1969), 1–49.

[D1] J. Dodziuk, L^2-harmonic forms on rotationally symmetric Riemannian manifolds, Proc. of Amer. Math. Soc., **77** (1979), 395–400.

[D2] J. Dodzuik, A private letter.

[K-W] J. L. Kazdan-F. W. Warner, Curvature functions for open 2-manifolds, Ann. of Math., **99** (1974), 203–219.

[M] H. Mori, Remarks on the Riemannian metric on the double of a manifold with boundary, preprint.

[O-N] T. Ochiai-J. Noguchi, Geometric Function Theory (in Japanese), Iwanami, 1986.

H. Kitahara
H. Kawakami
Department of Mathematics
Kanazawa University
Kanazawa, 920 Japan

Jin Suk Pak
Department of Mathematics
Kyungpook National University
Taegu, 635 Korea