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GENERAL SOLUTIONS DEPENDING ALGEBRAICALLY ON ARBITRARY CONSTANTS

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§ 1. Introduction

In his famous lectures [7] Painlevé investigates general solutions of algebraic differential equations which depend algebraically on some of arbitrary constants. Although his discussions are beyond our understanding, the rigorous and accurate interpretation to make his intuition true would be possible. Successful accomplishments have been done by some authors, for example, Kimura [1], Umemura [8, 9]. From differential algebraic viewpoint in [5] the author introduces the notion of rational dependence on arbitrary constants of general solutions of algebraic differential equations, and in [6] clarifies the relation between it and the notion of strong normality. Here we aim at generalizing to higher order case the result in [4] that in the first order case solutions of equations depend algebraically on those of equations free from moving singularities which are determined uniquely as the closest ones to the given. Part of our result can be seen in [7].

Let K be an algebraically closed ordinary differential field of characteristic zero. In what follows we tacitly assume every differential field extension of K is a finitely generated one and is contained in a fixed universal differential field extension of K. In order to reckon the degree of algebraic dependence of a differentially algebraic element, we begin with explaining an ordinary ordering among multi-indices. By a multi-index we mean a sequence $J = (j_n)$, where n runs through all nonnegative integers and the j's are nonnegative integers, being zero except for a finite number of the j's. Let $I = (i_n)$ and $J = (j_n)$ be two multi-indices. We say I is lower than J or J is higher than I if there is an integer m with $i_m < j_m$ and $i_n = j_n$ for all $n > m$. Let E be a differential field extension of K and $E\{Y\}$ denote the algebra of differential polynomials

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with coefficients in E. The element A of $E\{Y\}$ is expressed as $A = \sum a_J Y^J$, where J runs through all multi-indices, the a's are in E, being zero except for a finite number of the a's, and $Y^J = \prod_{0 \leqq n} Y_n^{j_n}$. The rank of nonzero differential polynomial A is defined as the highest multi-index J with $a_J \neq 0$. Let x be a differentially algebraic element over E. Then there exists a nonzero differential polynomial A over E such that $A(x) = 0$ and the rank of A is the lowest one among all the ranks of nonzero differential polynomials over E which vanish at x. Now we define the rank of x over E as the rank of the above A and denote it by $\text{rank}_E(x)$. Note that this definition does not depend on the choice of such an A. For a differentially algebraic element x over K by a symbol $r(x)$ we denote the lowest rank among all $\text{rank}_{EC_{\langle Ex\rangle}}(x)$, where E denotes any differential field extension of K from which $K\langle x\rangle$ is linearly disjoint over K and $C_{E\langle x\rangle}$ denotes the field of constants of $E\langle x\rangle$. Clearly $r(x)$ is lower than $\text{rank}_K(x)$. From the definitions it is seen that $r(x) = (j_n)$ with $j_0 = 1$ and $j_n = 0$ for other n means the differential field extension $K\langle x\rangle$ of K depends rationally on arbitrary constants.

THEOREM. *Let x be a differentially algebraic element over K. Then there exists a differential field extension S of K which is included in $K\langle x\rangle$ and depends rationally on arbitrary constants with $r(x) = \text{rank}_S(x)$. Moreover any differential field extension of K which is included in $K\langle x\rangle$ and depends rationally on arbitrary constants is included in S.*

In particular if $r(x) = (j_n)$ with $j_n = 0$ for positive n we say $K\langle x\rangle$ depends algebraically on arbitrary constants, which is in conformity with intuition. In this case and additionally if x satisfies a first order algebraic differential equation over K, the entry j_0 of $r(x)$ indicates in fact the degree of $K\langle x\rangle$ over S, and S is a differential function field of one variable without movable singularities (see Matsuda [3]). Thus our theorem is thought to be a generalization of the theorem in [4].

§ 2. Two lemmas

Let U denote a universal differential field extensipn of K. For any intermediate differential field L between K and U, we denote by C_L the field of constants of L.

LEMMA 1. *Let L and M be intermediate differential fields between K*

and U. Suppose M is a finitely generated differential field extension of L which is contained in LC_U. Then $M = LC_M$.

Proof. By our assumption there is a subfield D of C_U such that D is a finitely generated field extension of C_L and LD contains M. Since D and L are linearly disjoint over C_L, it follows that

$$\begin{aligned}\text{tr.deg. } LD/L &= \text{tr.deg. } LD/M + \text{tr.deg. } M/L \\ &\geqq \text{tr.deg. } D/C_M + \text{tr.deg. } C_M/C_L \\ &= \text{tr.deg. } D/C_L \\ &= \text{tr.deg. } LD/L\,,\end{aligned}$$

and hence

$$\text{tr.deg. } M/L = \text{tr.deg. } C_M/C_L\,.$$

Put $N = LD \cap \overline{LC_M}$. Then it is of finite degree over LC_M. The algebraic closedness of C_N in D implies that of LC_M in LD. Thus $N = LC_N$. We conclude $M = LC_M$ because

$$\begin{aligned}[N: M] &\geqq [C_N: C_M] = [LC_N: LC_M] = [N: LC_M] \\ &\geqq [N: M]\,.\end{aligned}$$

From this lemma we have immediately the fact: Let R be a differential field extension of K depending rationally on arbitrary constants and S be an intermediate differential field between K and R. Then S depends rationally on arbitrary constants. In fact let E be a differential field extension of K such that E and R are free from over K and $ER = EC_{ER}$. Then ES is contained in EC_{ER} and therefore by Lemma 1 $ES = EC_{ES}$, which justifies our assertion.

LEMMA 2. *Let R and S be two differential field extensions of K depending rationally on arbitrary constants. Then the comositum RS depends rationally on arbitrary constants.*

Proof. By assumption there are finitely generated differential field extensions E, F of K such that E and R are linearly disjoint over K, $ER = EC_{ER}$, F and S are linearly disjoint over K, $FS = FC_{FS}$. There is a differential field extension L of K being differentially isomorphic to E over K such that L and FRS are linearly disjoint over K. Since L and R are linearly disjoint over K and hence LR and ER are differentially isomorphic over R, we get $LR = LC_{LR}$. It is seen easily that LF and LS are linearly disjoint over L and

$$LFLS = LFS = LFC_{FS} = LFC_{LFLS}\,.$$

There is a differential field extension M of L such that M and LF are differentially isomorphic and linearly disjoint over L. We find M and RS are linearly disjoint over K. Since MS and LFS are differentially isomorphic over LS and $LFLS = LFC_{LFLS}$, it follows $MS = MC_{MS}$. This derives

$$MRS = MSLR = MC_{MS}LC_{LR} = MC_{MRS}$$

and completes the proof.

We remark that if R and S are strongly normal over K and $C_{RS} = C_K$ then the compositum RS is also strongly normal over K (cf. [2]).

§ 3. Proof of Theorem

By our assumption we have a differential field extension E of K which is linearly disjoint from $K\langle x\rangle$ over K with $r(x) = \mathrm{rank}_{EC_{E\langle x\rangle}}(x)$. Let $R = r(x)$. Let F be any differential field extension of E from which $K\langle x\rangle$ is linearly disjoint over K. The minimality of $r(x)$ implies $R = \mathrm{rank}_{FC_{F\langle x\rangle}}(x)$ since $\mathrm{rank}_{EC_{E\langle x\rangle}}(x)$ is not higher than $\mathrm{rank}_{FC_{F\langle x\rangle}}(x)$ from the definition of the rank. The element x annuls a nonzero differential polynomial A over $EC_{E\langle x\rangle}$: $A = \sum a_J Y^J$, where $a_J = 0$ or all J higher than R and $a_R = 1$. Since each a_J is contained in $F\langle x\rangle$ we have a representation:

$$a_J = (\textstyle\sum_j a_{J,j}x_j)/(1 + \sum_h b_{J,h}y_{J,h})\,,$$

where x_j, $y_{J,h}$ are in $K\langle x\rangle$ and $a_{J,j}$, $b_{J,h}$ are in F. Among such representations we select one which is subject to the following conditions: the quantities (x_j) are linearly independent over K, the number of nonzero terms $b_{J,h}y_{J,h}$ appearing in the denominator is taken as small as possible. In this case by $d_{J,F}$ we denote the number of nonzero terms $b_{J,h}y_{J,h}$.

In what follows we fix a multi-index I. Let E_I be a differential field extension of E such that E_I and $K\langle x\rangle$ are linearly disjoint over K and d_{I,E_I} is the minimum. There exists a differential field extension F of K which is differentially isomorphic to E_I over K and from which $E_I\langle x\rangle$ is linearly disjoint over K. This follows from the fact that the differential fields E_I and $E_I\langle x\rangle$ are spontaneously embedded into the quotient field of $E_I \otimes_K E_I\langle x\rangle$, which is a finitely generated differential field extension of K and therefore embedded into the universal differential field extension of K. Let f be the differential isomorphism of E_I onto F. It can be extended to a differential isomorphism, say the same f, of $E_I\langle x\rangle$ onto $F\langle x\rangle$ over $K\langle x\rangle$ since $K\langle x\rangle$ and $H = FE_I$ are linearly disjoint over K. Since each a_J lies in $EC_{E\langle x\rangle}$, each $f(a_J)$ lies in $HC_{H\langle x\rangle}$ according to Lemma 1. The $f(a_J)$ satisfy $B(x) = 0$, where B is a differential polynomial over $HC_{H\langle x\rangle}$: $B = \sum f(a_J)Y^J$. Since $R = \mathrm{rank}_{HC_{H\langle x\rangle}}(x)$ it follows that $a_J = f(a_J)$. The representation of a_J reads

$$a_I + a_I \sum_h b_{I,h} y_{I,h} - \sum_j a_{I,j} x_j = 0 .$$

Applying f to this equality we have

$$a_I + a_I \sum_h f(b_{I,h}) y_{I,h} - \sum_j f(a_{I,j}) x_j = 0 .$$

Hence

$$a_I \sum_h (b_{I,h} - f(b_{I,h})) y_{I,h} - \sum_j (a_{I,j} - f(a_{I,j})) x_j = 0 .$$

The minimality of d_{I,E_I} implies $b_{I,h} = f(b_{I,h})$ and therefore $a_{I,j} = f(a_{I,j})$ since (x_j) are linearly independent over H. Thus $a_{I,j}$ and $b_{I,h}$ are both common elements of E_I and F, and hence belong to K. This concludes a_I is in $K\langle x\rangle$. Since we have assumed I to be arbitrary, the differential field extension L of K generated with all a_J is included in $K\langle x\rangle$. The inclusion relation $EL \subset EC_{E\langle x\rangle}$ implies $EL = EC_{EL}$ according to Lemma 1, that is to say, L depends rationally on arbitrary constants. Let S be a maximal differential field extension of K which is inculded in $K\langle x\rangle$ and depends rationally on arbitrary constants. If M is a differential field extension of K which is included in $K\langle x\rangle$ and depends rationally on arbitrary constants, then MS depends rationally on arbitrary constants according to Lemma 2. The maximality of S implies $MS = S$, that is, M is included in S, particularly so is L. Hence R is not lower than $\mathrm{rank}_S(x)$. By the assumption on S there is a differential field extension V of K such that $VS = VC_{VS}$ and V and S are linearly disjoint over K. There exists a differential field extension W of K such that V and W are differentially isomorphic over K and $E\langle x\rangle$ and W are linearly disjoint over K. Since $VC_{VS} = VS$ is differentially isomorphic onto WS over S it follows $WS = WC_{WS}$, using again Lemma 1. The differential field extension EW of E is linearly disjoint from $K\langle x\rangle$ over K and hence $R = \mathrm{rank}_{EWC_{EW\langle x\rangle}}(x)$. The tower $S \subset EWS \subset EWC_{WS} \subset EWC_{EW\langle x\rangle}$ implies $\mathrm{rank}_S(x)$ is not lower than $\mathrm{rank}_{EWC_{EW\langle x\rangle}}(x) = R$, which concludes $R = \mathrm{rank}_S(x)$ and completes the proof of our theorem.

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