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NONCOMMUTATIVE CLASSICAL INVARIANT THEORY

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§1. Introduction

Let K be a field of characteristic zero, V a finite dimensional vector space and G a subgroup of $GL(V)$. The action of G on V is extended to the symmetric algebra on V over K,

$$K[V] = K \oplus V \oplus S^2(V) \oplus \cdots \oplus S^n(V) \oplus \cdots$$

and the tensor algebra on V over K,

$$K\langle V\rangle = K \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots .$$

Here $S^n(V)$ and $V^{\otimes n}$ denote the n-th symmetric power and n-th tensor power of V respectively.

We denote by $K[V]^G$ and $K\langle V\rangle^G$ the invariant ring of G acting on $K[V]$ and $K\langle V\rangle$, respectively. A main result of invariant theory says that, if G is linearly reductive, $K[V]^G$ is finitely generated. On the other hand Dicks and Formanek [2] proved that, if G is a finite group and not scalar, $K\langle V\rangle^G$ is not finitely generated. Lane [4] and Kharchenko [3] independently proved that, for arbitrary subgroup G of $GL(V)$, $K\langle V\rangle^G$ is a free associative K algebra.

In classical invariant theory one deals with the special linear group $SL(n)$. Consider the general n-ary form of degree r

$$f = \sum \frac{r!}{r_1! \cdots r_n!} a_{r_1 \cdots r_n} x_1^{r_1} \cdots x_n^{r_n}, \qquad r_1 + \cdots + r_n = r,$$

with coefficients $a_{r_1,\ldots,r_n}$ which are indeterminates over K.

If, for a linear transformation with determinant one, $x_1, \cdots, x_n$ undergo a linear transformation $x_i = \sum_j g_{ji} x'_j$, $g = (g_{ji}) \in SL(n)$, f is transformed into f of the form $f' = \sum r!/r_1! \cdots r_n!\, a'_{r_1 \cdots r_n}, x_1'^{r_1} \cdots x_n'^{r_n}$. The mapping $a_{r_1 \cdots r_n} \mapsto g(a_{r_1 \cdots r_n}) = a'_{r_1 \cdots r_n}$ defines a representation of $SL(n)$ on the vector space spanned by $a_{r_1 \cdots r_n}$'s over K.

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A homogeneous polynomial $J(a_{r_1\cdots r_n})$ in the indeterminates is called an invariant if $(*)$ $J(g(a_{r_1\cdots r_n}))=J(a_{r_1\cdots r_n})$ holds for all $g\in SL(n)$. Let $K[a_{r_1\cdots r_n}]^{SL(n)}$ denote the ring of invariants. The main problem in classical invariant theory is to determine the structure of $K[a_{r_1\cdots r_n}]^{SL(n)}$. In 1890 Hilbert proved that $K[a_{r_1\cdots r_n}]$ is finitely generated by using Hilbert's basis theorem and Cayley's Ω process. All invariants of arbitrary n-ary form are written down by famous Clebsch-Gordan's symbolic method. But the explicit structure of $K[a_{r_1\cdots r_n}]^{SL(n)}$ is not known except special cases ([7], [9]).

Let us consider $a_{r_1\cdots r_n}$ as noncommutative variables over K. Let $K\langle a_{r_1\cdots r_n}\rangle$ be the free associative algebra generated by $a_{r_1\cdots r_n}$'s. A homogeneous element $J(a_{r_1\cdots r_n})$ of degree d in the noncommutative graded ring $K\langle a_{r_1\cdots r_n}\rangle$ is called a noncommutative invariant of degree d if it satisfies $(*)$ for any $g\in SL(n)$. We denote by $K\langle a_{r_1\cdots r_n}\rangle^{SL(n)}$ the ring of noncommutative invariants. For a nonnegatve integer d, we write $K\langle a_{r_1\cdots r_n}\rangle_d^{SL(n)}$ for the vector space of invariants of degree d. Let V be a vector space of dimension n, that is the standard $SL(b)$-module, and $\alpha_1,\cdots,\alpha_n$ be a basis of V. Then by the mapping $a_{r_1\cdots r_n}\mapsto\alpha_1^{r_1}\cdots\alpha_n^{r_n}$, $K[a_{r_1\cdots r_n}]$ is, as an $SL(n)$-module, isomorphic to $K[S^r(V)]$ and $K\langle a_{r_1\cdots r_n}\rangle$ is, as a $SL(n)$-module, isomorphic to $K\langle S^r(V)\rangle$.

We write $c(n,r,d)$ and $\bar{c}(n,r,d)$ for dimension of $K[a_{r_1\cdots r_n}]_d^{SL(n)}$ and $K\langle a_{r_1\cdots r_n}\rangle_d^{SL(n)}$ respectively. In section 2, some notations from representation theory of the general linear group are introduced and we give combinatorial formulas for $c(n,r,d)$ and $\bar{c}(n,r,d)$. In section 3, we give explicitly a free generating set of $K\langle a_{r_1\cdots r_n}\rangle^{SL(n)}$. The basic idea is to use the (noncommutative) symbolic method, by which all noncommutative invariants are written down explicitly. In section 4 we give dimension formulas of invariants. In section 5, instead of the usual Hilbert series $\sum c(n,r,d)t^d$, we shall investigate a formal power series

$$F_{n,d}(t)=\sum_{r\in N}c(n,r,d)t^{dr}\,.$$

We shall prove that, if $d\geqq 2n-1$, $F_{n,d}(t)$ satisfies the following functional equation

$$F_{n,d}(1/t)=(-1)^{nd-n-d}t^{nd}F_{n,d}(t)\,.$$

In the last section we shall investigate the ring of invariants of skew symmetric tensors.

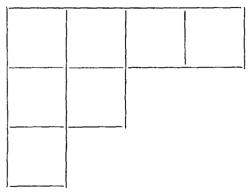
NOTATION

N	nonnegative integers
$\boldsymbol{Q}$	rational integers
$\langle a\rangle_n$	for $a \in N$, the vector $(a, \cdots, a) \in N^n$
$\lvert X\rvert$	for a set, cardinarity of X.
$a \mid b$	for $a, b \in N$, a divides b.

§ 2. Representation of the general linear groups

In this section we summarize the results on the representations of the general linear group $GL(n)$ which we will use later.

A Young diagram λ with n rows is a nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \cdots, \lambda_n)$, $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n$. We think of λ as a sequence of rows of "boxes" of length $\lambda_1, \lambda_2, \cdots, \lambda_n$. For example

$\lambda = (4, 2, 1) =$  .

A Young tableau is a numbering of the boxes of a Young diagram with integers $1, 2, \cdots$. If a Young tableau has $i_1$1's, $i_2$2's, $\cdots$, the sequence $(i_1, i_2, \cdots)$ is called the weight of a Young tableau.

Let V be a vector space of dimension n. There is a 1–1 correspondence between Young diagrams with m boxes and $\leqq n$ rows and irreducible $GL(n)$-submodules of $V^{\otimes m}$. We write $[\lambda]$ for the irreducible $GL(n)$-module in $V^{\otimes m}$ corresponding to λ. Then one dimensional irreducible $GL(n)$ submodule of $V^{\otimes mn}$ corresponds to the rectangular Young diagram with n rows and m columns. The following lemma is a special case of the Littlewood-Richardoson rule [5].

LEMMA 2.1. *Let μ be a Young diagram. Then*

$$[\mu] \otimes \Big[\overbrace{\square\square\square\square}^{r}\Big] = \bigotimes_{\lambda} [\lambda]$$

where λ ranges all Young diagrams which can be built by the addition of r boxes to the Young diagram μ, no two added boxes appearing in the same column of λ.

A Young tableau Y is called column strict if the entries of Y increase down columns and do not decrease across rows. Figure 1 gives an example of column strict tableau with weight (2.2.3).

1	1	2	3
2	3		
3			

Figure 1

A rectangular Young tableau Y with n rows is called column strict Young tableau of degree d if Y is column strict and has weight $\langle r\rangle_d$. We denote by $K(n, r, d)$ the set of all column strict Young tableaux of degree d. By Lemma 2.1, we obtain:

PROPOSITION 2.2.

$$\dim K\langle a_{r_1\cdots r_n}\rangle_d^{SL(n)} = |K(n, r, d)|.$$

EXAMPLE 2.3. Figure 2 gives the column strict Young tableau of degree 4 for $(n, r, d) = (2, 2, 4)$.

$$Y_1 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 3 & 3 & 4 & 4 \\ \hline\end{array} \qquad Y_2 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 4 & 4 \\ \hline\end{array} \qquad Y_3 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 4 & 4 \\ \hline\end{array}$$

Figure 2

By Proposition 2.2, noncommutative invariants of degree 4 for $(n, r) = (2, 2)$ has 3 linearly independent invariants.

§ 3. A free generating set of the noncommutative ring of invariants

In this section we will construct a free generating set of the ring of invariants $K\langle a_{r_1\cdots r_n}\rangle^{SL(n)}$. Let V be an n dimensional vector space and $\alpha_1, \cdots, \alpha_n$ a fixed basis of V. Consider the $SL(n)$ equivariant isomorphism $\varphi\colon K < S^r(V) > \to K\langle a_{r_1\cdots r_n}\rangle$ obtained from the mapping $\alpha_1^{r_1}\cdots\alpha_n^{r_n} \mapsto a_{r_1\cdots r_n}$. For positive integers $k_1, \cdots, k_n$, and d, $(k_1 < k_2 < \cdots < k_n \leqq d$, let $\langle k_1, \cdots, k_n\rangle$ be the tensor in $\otimes^d K[V]$ defined by

$$\langle k_1, \cdots, k_n \rangle \\ = \sum_\sigma \operatorname{sgn} \sigma 1 \otimes \cdots \otimes \alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(2)} \otimes \cdots \otimes \alpha_{\sigma(n)} \otimes \cdots \otimes 1 .$$

Here the sum ranging over all permutations on n letters $1, \cdots, n$ and $\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(n)}$ appear in k-th, $\cdots$, k_n-th places in the tensor product and other factors are all 1.

Obviously $\langle k_1, \cdots, k_n \rangle$ is invariant under the action of $SL(n)$. Let

i_1	j_1	$\cdot$	m_1
i_2	j_2	$\cdot$	m_2
$\cdot$	$\cdot$	$\cdot$	$\cdot$
i_n	j_n	$\cdot$	m_n

be a column strict Young tableau of degree d. Since each number out of $1, 2, \cdots, d$ appears r times in Y,

$$\langle i_1, \cdots, i_n \rangle \langle j_1, \cdots, j_n \rangle \cdots \langle m_1, \cdots, m_n \rangle$$

is a $SL(n)$-invariant tensor in $\otimes^d S^r(V)$. We set

$$F(Y, a) = \varphi(\langle i_1, \cdots, i_n \rangle \langle j_1, \cdots, j_n \rangle \cdots \langle m_1, \cdots, m_n \rangle) .$$

Then $F(Y, a)$ is a noncommutative invariant. We say that $F(Y, a)$ is an invariant associated with a column strict Young tableau Y of degree d.

Given two a_r and a_r ($r = (r_1, \cdots, r_n)$, $s = (s_1, \cdots, s_n) \in N^n$ with $|r| = |s| = r$), we say that a_r is bigger or equal than a_s, if $r = (r_1, \cdots, r_n)$ is lexicographically bigger or equal than $s = (s_1, \cdots, s_n)$. Moreover, given two noncommutative monomial $M_1 = a_{r(1)} \cdots a_{r(d)}$ and $M_2 = a_{s(1)} \cdots a_{s(d)}$, we say that M_1 is lexicographically bigger or equal than M_2, if $M_1 = M_2$ or, for the first factors $a_{r(j)}$, $a_{s(j)}$ such that $a_{r(j)} \neq a_{s(j)}$, $a_{r(j)}$ is bigger or equal than $a_{s(j)}$. Suppose that each number i, $1 \leqq i \leqq d$, appears i_1 times in the first row of a given Young tableau Y of degree d and i_2 times in the second row of Y, etc. We set $[\mathrm{i}] = (i_1, \cdots, i_n)$. Then it follows from the construction of $F(Y, a)$ that the highest term in the monomials of $F(Y, a)$ is $a_{[i]} \cdots a_{[d]}$. For example, in Example 2, the highest terms of Y_1, Y_2 and Y_3 are $a_{20}^2 a_{02}^2$, $a_{20} a_{11}^2 a_{02}$ and $a_{20} a_{02} a_{20} a_{02}$, respectively. In particular, different column strict Young tableaux Y_1 and Y_2 have different highest terms. Gathering up we have proved the following

THEOREM 3.1. *To each column strict Young tableau Y of degree d, associate the noncommutative invariant $F(Y, a)$. Then $F(Y, a)$'s constitute a free basis of the vector space $K\langle a_{r_1\cdots r_n}\rangle_d^{SL(n)}$.*

In particular noncommutative invariants associated with all column strict Young tableaux of degree d ($d \in N$) generate the ring of invariants $K\langle a_{r_1\cdots r_n}\rangle^{SL(n)}$.

EXAMPLE 3.2 (Almkvist, Dicks, and Formanek [1]). Consider the binary form of degree n:

$$f = \sum \tbinom{n}{k} a_k x_1^k x_2^{n-k}\,.$$

The column strict Young tableau of degree 2 is

$$Y = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & \cdot & \cdot & \cdot & 1 \\ \hline 2 & 2 & \cdot & \cdot & \cdot & 2 \\ \hline \end{array}\,.$$

The associated invariant of degee 2 is

$$\begin{aligned} F(Y, a) &= \varphi(\alpha_1 \otimes \alpha_2 - \alpha_2 \otimes \alpha_1)^n \\ &= \sum (-1)^{n-k} \tbinom{n}{k} a_k a_{n-k}\,. \end{aligned}$$

If n is even, say $2s$, there exist unique column strict Young tableau of degree 3,

$$Y = \begin{array}{|c|c|c|c|c|c|c|c|c|} \multicolumn{6}{c}{\overbrace{\hspace{6em}}^{n}} & \multicolumn{3}{c}{\overbrace{\hspace{3em}}^{s}} \\ \hline 1 & \cdot & \cdot & \cdot & \cdot & 1 & 2 & \cdot & 2 \\ \hline 2 & \cdot & 2 & 3 & \cdot & \cdot & \cdot & \cdot & 3 \\ \hline \multicolumn{3}{c}{\underbrace{\hspace{3em}}_{s}} & \multicolumn{6}{c}{\underbrace{\hspace{6em}}_{n}} \end{array}\,.$$

The associated invariant of degree 3 is

$$\begin{aligned} F(Y, a) &= \varphi(\alpha_1 \otimes \alpha_2 \otimes 1 - \alpha_2 \otimes \alpha_1 \otimes 1)^s (\alpha_1 \otimes 1 \otimes \alpha_2 - \alpha_1 \otimes 1 \otimes \alpha_2)^s \\ &\quad \times (1 \otimes \alpha_1 \otimes \alpha_2 - 1 \otimes \alpha_2 \otimes \alpha_1)^s \\ &= \sum_{i=0}^{s} \sum_{j=0}^{s} \sum_{k=0}^{s} (-1)^{i+j+k} \binom{s}{i} \binom{s}{j} \binom{s}{k} a_{s-i+j} a_{s-j+k} a_{s-k+i}\,. \end{aligned}$$

Let Y_1 and Y_2 be column strict Young tableaux of degree d_1 and d_2 respectively. We write $Y_1 \hat{\oplus} Y_2$ for the Young tableau which is obtained from Y_1 and Y_2 as follows: after adding the Young tableau Y_2 from right to Y_1, replace each entry, say i, in Y_2 by $i + d_1$. For instance, if

$$Y_1 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 4 & 4 \\ \hline \end{array}\,, \quad Y_2 = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & 3 \\ \hline \end{array}\,,$$

$Y_1 \hat{\oplus} Y_2$ is given by

$$Y_3 = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 5 & 5 & 6 \\ \hline 2 & 3 & 4 & 4 & 6 & 7 & 7 \\ \hline \end{array}\,.$$

Obviously $Y_1 \oplus Y_2$ is a column strict Young tableau of degree $d_1 + d_2$, and $F(Y_1 \hat{\oplus} Y_2, a) = F(Y_1, a)F(Y_2, a)$. Since $(Y_1 \hat{\oplus} Y_2) \hat{\oplus} Y_3 = Y_1 \hat{\oplus} (Y_2 \hat{\oplus} Y_3)$, we may write $Y_1 \hat{\oplus} Y_2 \hat{\oplus} Y_3$ for $(Y_1 \hat{\oplus} Y_2) \hat{\oplus} Y_3$ or $Y_1 \hat{\oplus} (Y_2 \hat{\oplus} Y_3)$.

We now find a free generating set of $K\langle a_{r_1\cdots r_n}\rangle^{SL(n)}$. To do so, we introduce a terminology: a column strict Young tableau Y of degree d is called decomposable if there are two column strict Young tableaux Y_1 and Y_2 of degree d_1 and d_2 respectively such that $Y = Y_1 \hat{\oplus} Y_2$, $d = d_1 + d_2$. A column strict Young tableau is called indecomposable if it is not decomposable. For example, in Example 2.3, Y_1 and Y_2 are indecomposable but Y_3 is decomposable. A free generating set of the ring $K\langle a_{r_1\cdots r_n}\rangle^{SL(n)}$ is given by the following

THEOREM 3.3. *The set of noncommutative invariants associated with all indecomposable Young tableaux is a free generating set of the noncommutative invariant ring* $K\langle a_{r_1\cdots r_n}\rangle^{SL(n)}$.

Proof. If a column strict Young tableau is decomposable, the associated invariant is a product of two invariants, neither of them are not constants. Therefore the set of noncommutative invariants associated with all indecomposable Young tableaux is a generating set of $K\langle a_{r_1\cdots r_n}\rangle^{SL(n)}$. It remains to show that this set is free. Assume contrary that some invariants $F(Y_1, a), \cdots, F(Y_k, a)$, associated with indecomposable Young tableaux $Y_1, \cdots, Y_k$ are not free. Let

$$(*) \qquad \sum_{1 \leq i_1, \cdots, i_p \leq k} c_{i_1\cdots i_p} F(Y_{i_1}, a) \cdots F(Y_{i_p}, a)\,, \qquad c_{i_1\cdots i_p} \in K\,,$$

be a nontrivial relation among $F(Y_1, a), \cdots, F(Y_k, a)$. Then we have

$$\sum_{1 \leq i_1, \cdots, i_p \leq k} c_{i_1\cdots i_p} F(Y_{i_1} \hat{\oplus} \cdots \hat{\oplus} Y_{i_p}, a) = 0\,.$$

Now without loss of generality we can assume that degrees of $Y_{i_1} \hat{\oplus} \cdots \hat{\oplus} Y_{i_p}$ are all equal. Then it is easy to see that, for two column strict Young tableaux with same degree, $Y_{i_1} \hat{\oplus} \cdots \hat{\oplus} Y_{i_p}$ and $Y_{j_1} \hat{\oplus} \cdots \hat{\oplus} Y_{j_q}$,

$$Y_{i_1} \hat{\oplus} \cdots \hat{\oplus} Y_{i_p} = Y_{j_1} \hat{\oplus} \cdots \hat{\oplus} Y_{j_q}$$

if and only if $p = q$ and $Y_{i_\ell} = Y_{j_\ell}$, $1 \leqq \ell \leqq p$. Hence it follows from Theorem 3.1 that the relation (*) is a trivial relation. This contradicts to our assumption and hence the theorem is proved.

THEOREM 3.4. *The ring of noncommutative invariants $K\langle a_{r_1 \cdots r_n}\rangle^{SL(n)}$ is not finitely generated.*

Proof. By Theorem 3.3, it is enough to show that there are infinitely many indecomposable Young tableaux. This is obvious since, for any positive integer d such that $n | rd$, the column strict Young tableaux of degree d given by

$1 \cdots 1$	$2 \cdots 2$	$\cdots$	$s \cdots s$
$s+1 \cdots s+1$	$s+2 \cdots s+2$	$\cdots$	$2s \cdots 2s$
$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$d \cdots d$

, $\quad s = rd/n$,

is indecomposable.

For a $GL(n)$-module M, we denote by $\chi(M, \varepsilon)$, $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$, the character of M. Let, for a Young diagram, denote by $m(M, \lambda)$ the multiplicity of the irreducible $GL(n)$-module $[\lambda]$ in the irreducible decomposition:

$$M = \sum m(M, \lambda)[\lambda] .$$

The character of $[\lambda]$, $\lambda = (\lambda_1, \cdots, \lambda_n)$, is given by the Schur function,

$$\chi([\lambda], \varepsilon) = \begin{vmatrix} \varepsilon_1^{\nu_1} \cdots \varepsilon_1^{\nu_n} \\ \cdots \\ \varepsilon_n^{\nu_1} \cdots \varepsilon_n^{\nu_n} \end{vmatrix} \Bigg/ \begin{vmatrix} \varepsilon_1^{n-1} \cdots 1 \\ \cdots \\ \varepsilon_n^{n-1} \cdots 1 \end{vmatrix}, \qquad \nu_i = \lambda_i + n - i \ (1 \leqq i \leqq n),$$

and hence we have

$$\chi(M, \varepsilon) \begin{vmatrix} \varepsilon_1^{n-1}, \cdots, 1 \\ \cdots \\ \varepsilon_n^{n-1}, \cdots, 1 \end{vmatrix} = \sum m(V, \lambda) \begin{vmatrix} \varepsilon_1^{\nu_1}, \cdots, \varepsilon_1^{\nu_n} \\ \cdots \\ \varepsilon_n^{\nu_1}, \cdots, \varepsilon_n^{\nu_n} \end{vmatrix} .$$

Therefore $m(M, \lambda)$ is the coefficients of $\varepsilon_1^{\nu_1} \cdots \varepsilon_n^{\nu_n}$ in the expression of

$$\chi(M, \varepsilon) \begin{vmatrix} \varepsilon_1^{n-1} & \cdots & 1 \\ \varepsilon_n^{n-1} & \cdots & 1 \end{vmatrix}.$$

Let, as before, V be the standard $GL(n)$-module. Then

$$\chi(S^r(V), \varepsilon) = \sum \varepsilon_1^{r_1} \cdots \varepsilon_n^{r_n}, \qquad r_1 + \cdots + r_n = r.$$

For positive integers d and r, let $\underline{m}$, $\underline{m}_1, \cdots, \underline{m}_d$ be vectors in N^ℓ. If $\underline{m} = \underline{m}_1 + \cdots + \underline{m}_d$, the set of vectors $\{\underline{m}_1, \cdots, \underline{m}_d\}$ is called a partition of m, here no account is taken of the order of the parts. The ordered sequence $(\underline{m}_1, \cdots, \underline{m}_n)$ is called on ordered partition of $\underline{m}$. We will denote by $A(\underline{m}, d, r)$ (resp. $\bar{A}(\underline{m}, d, r)$) the set of all partitions (resp. ordered partitions) of $\underline{m}$ into d parts of length r. For instance, $A((2, 2, 2), 3, 2) = 1$, $\bar{A}((2, 2, 2), 3, 2) = 6$.

PROPOSITION 4.1.

(1) $c(n, r, d) = \begin{cases} \sum_\sigma \operatorname{sgn} \sigma A(\langle dr/n\rangle_n + \delta - \sigma(\delta), d, r), & \textit{if } dr/n \in N, \\ 0 & \textit{otherwise} \end{cases}$

(2) $\bar{c}(n, r, d) = \begin{cases} \sum_\sigma \operatorname{sgn} \sigma \bar{A}(\langle dr/n\rangle_n + \delta - \sigma(\delta), d, r), & \textit{if } dr/n \in N, \\ 0 & \textit{otherwise} \end{cases}$

where σ ranges over all permutations on $(n-1, \cdots, 0)$, and $\delta = (n-1, \cdots, 0)$.

Proof. As is readily seen from the definitions of $A(m, d, r)$ and $\bar{A}(m, d, r)$,

$$\chi(K[a]_d, \varepsilon) = \chi(S^d(S^r(V), \varepsilon) = \sum A(\underline{m}, d, r)\varepsilon_1^{m_1} \cdots \varepsilon_n^{m_n}$$

and

$$\chi(K\langle a\rangle_d, \varepsilon) = \chi(\overset{d}{\otimes} (S^r(V)), \varepsilon) = \sum A(\underline{m}, d, r)\varepsilon_1^{m_1} \cdots \varepsilon_n^{m_n}.$$

Since $c(n, r, d)$ (resp. $\bar{c}(n, r, d)$) is the multiplicity of the irreducible module associated with the Young diagram $\langle dr/n\rangle_n$ in the irreducible decomposition of $K[a]_d$ (resp. $K\langle a\rangle_d$), we see that $c(n, r, d)$ is the coefficient of

$$(*) \qquad \varepsilon_1^{dr/n+n-1}\varepsilon_2^{dr/n+n-2} \cdots \varepsilon_n^{dr/n}$$

in the expression of

$$\chi(K[a]_d, \varepsilon) \begin{vmatrix} \varepsilon_1^{n-1} & \cdots & 1 \\ & & \\ \varepsilon_n^{n-1} & \cdots & 1 \end{vmatrix} = \sum A(m, d, r)\varepsilon_1^{m_1} \cdots \varepsilon_n^{m_n} \begin{vmatrix} \varepsilon_1^{n-1} & \cdots & 1 \\ & \cdots & \\ \varepsilon_n^{n-1} & \cdots & 1 \end{vmatrix}.$$

If we take the term $\operatorname{sgn}\sigma\varepsilon_1^{\sigma(n-1)}\cdots\varepsilon_n^{\sigma(0)}$ of the second factor we must select the term $A(\langle dr/n\rangle_n+\delta-\sigma(\delta))\prod_{i=1}^n\varepsilon_i^{dr/n+n-i-\sigma(n-i)}$ of the first factor in order to obtain the monomial (*). Thus (1) is proved. By the same way we can prove (2).

Remark. If $n=2$, (1) is the Cayley-Sylvester theorem in the classical invariant theory of binary forms and (2) is a result of Michel Brion [1].

§ 5. A functional equation

In this section we shall prove the following

THEOREM 5.1. *Let $F_{n,d}(t)$ be the formal power series defined by*

$$F_{n,d}(t)=\sum_{r\in N}\bar{c}(n,r,d)t^{dr}.$$

Then (1) *$F_{n,d}(t)$ is a rational function.*

(2) *If $d\geqq 2n-1$,*

$$F_{n,d}(1/t)=(-1)^{nd-d-n}t^{nd}F_{n,d}(t).$$

To prove this theorem, we need a result of Stanley [7]. In general let n and m be positive integers. Let A be an $m\times n$ matrix with integer entries. For a vector $b\in Z^m$, set

$$E(r):=\{x\in N^n,\ Ax=b\},$$

and

$$\hat{E}(r):=\{x\in N^n,\ Ax=-b\},\qquad r=0,1,2,\cdots.$$

Let us consider the formal power series:

$$F(E,t)=\sum_{r\in N}|E(r)|t^r,$$

and

$$\hat{F}(E,t)=\sum_{r\in N}|\hat{E}(r)|t^r.$$

THEOREM 5.2. (Stanley [7])

(1) *$F(E,t)$ and $F(\hat{E},t)$ are rational functions.*

(2) *Suppose that the system of linear equations $Ax=b$ has a solution $x=(x_1,\cdots,x_n)\in Q^n$ such that $-1<x_i\leqq 0$, $1\leqq i\leqq n$. Then*

$$F(E,1/t)=(-1)^\alpha t^nF(\hat{E},t),$$

$\alpha=n-\text{rank of }A$, assuming that the system of linear equations $Ax=0$ has a solution $x=(x_1,\cdots,x_n)\in N^n$, $x_i>0$ for all i.

Proof of Theorem 5.1. For a $d \times n$-matrix X, let $r_i(X)$ be sum of entries of the i-th row vector of X, $1 \leqq i \leqq d$, and $c_j(X)$ sum of j-th column vector of X, $1 \leqq i \leqq n$. Consider, for a permutation σ on the set $\{n-1, \cdots, 0\}$, a system of linear equations E_σ:

$$r_i(X) - r_j(X) = 0\,, \qquad 1 \leqq i \leqq j \leqq d\,,$$

and

$$c_p(X) - c_q(X) = q - p + \sigma(n-q) - \sigma(n-p)\,, \qquad 1 \leqq p \leqq q \leqq n\,.$$

The number $\overline{A}(\underline{m}, d, r)$, $\underline{m} = (m_1, \cdots, m_n)$, is interpritated as the number of $d \times n$ matrices such that sum of entries of any row is r and sum of entries of the i-th column is m_i, $1 \leqq i \leqq n$. Therefore by Theorem 4, we have

$$\begin{aligned} F_{n,d}(t) &= \sum_{r \in N} \bar{c}(n, r, d)t \\ &= \sum_r \sum_\sigma \operatorname{sgn} \sigma \overline{A}(\langle dr/n \rangle_n + \delta - \sigma(\delta), d, r)t^{dr} \\ &= \sum_\sigma \operatorname{sgn} \sigma F(E_\sigma, t)\,. \end{aligned}$$

For a $d \times n$ matrix X, let $\hat{X}$ denote the $d \times n$ matrix obtained from X by replacing each j-th column vector with the $(n+1-j)$-th vector of X, $1 \leqq j \leqq n$. Then if X is a solution of E_σ, we have:

$$\begin{aligned} c_p(\hat{X}) - c_q(\hat{X}) &= c_{n+1-p}(X) - c_{n+1-q}(X) \\ &= p - q + \hat{\sigma}(n-p) - \hat{\sigma}(n-q)\,, \qquad 1 \leqq p \leqq q \leqq n \end{aligned}$$

and

$$r_i(\hat{X}) - r_j(\hat{X}) = r_i(X) - r_j(X)\,, \qquad 1 \leqq i \leqq j \leqq n\,,$$

where $\hat{\sigma}$ stands for the permutation on the set $\{n-1, \cdots, 0\}$ defined by $\hat{\sigma}(n-p) = n - 1 - \sigma(p-1)$, $1 \leqq p \leqq n$.

Since, for any permutation of $n-1, \cdots, 0$, $\operatorname{sgn} \sigma = \operatorname{sgn} \hat{\sigma}$, we have $N(E_\sigma, dr) = N(\hat{E}_{\hat{\sigma}}, dr)$ and hence

$$\sum_\sigma \operatorname{sgn} \sigma F(E_\sigma, t) = \sum_\sigma \operatorname{sgn} \sigma F(\hat{E}_\sigma, t)\,.$$

Let X_0 be a $d \times n$ matrix whose p-th column vector is $(g, \cdots, g)$, $g = -(p + \sigma(n-p))/d$. If $d \geqq 2n-1$, for any σ and p, we have $-1 \leqq g \leqq 0$, and obviously X_0 is a solution of E_σ. Therefore, for any σ, the system of linear equations E_σ satisfies the assumption of Stanley's theorem and we have

$$\begin{aligned} F_{n,d}(1/t) &= \sum_{\sigma} \operatorname{sgn} \sigma F(E_\sigma, 1/t) \\ &= \sum_{\sigma} (-1)^{nd-n-d} \operatorname{sgn} \sigma t^{nd} F(\hat{E}_\sigma, t) \\ &= (-1)^{nd-d-n} t^{nd} F_{n,d}(t) . \end{aligned}$$

This completes the proof.

Remark. We record explicit forms of $F_{2,d}(t)$, for $d = 3, 4, 5$.

$$\begin{aligned} F_{2,3}(t) &= \frac{1}{1-t^6} \\ F_{2,4}(t) &= \frac{1+2t^4+t^8}{(1-t^8)^2} \\ F_{2,5}(t) &= \frac{1+3t^{10}+t^{20}}{(1-t^{10})^3} \end{aligned}$$

§6. The ring of invariants of skew symmetric tensors

Let V be a vector space over K of dimension n with a basis $\alpha_1, \cdots, \alpha_n$. For a positive integer r, $r < n$, let $\wedge^r V$ denote the r-times skew symmetric product of V. In this section, considering $\wedge^r V$ as an $SL(n)$-module, we shall construct a generating set of the noncommutative ring $K\langle \wedge^r V\rangle^{SL(n)}$. Let

$$f = \sum a_{i_1\cdots i_r} x_{i_1} \wedge \cdots \wedge x_{i_r}, \qquad 1 \leqq i_1 < \cdots < i_r \leqq n,$$

be the generic skew symmetric tensor of rank r. Here we consider $a_{i_1\cdots i_r}$ as independent variables. If, for any linear transformation with determinant one, $x_1, \cdots, x_n$ undergo a linear transformation

$$x_i = \sum_j g_{ji} x'_j, \qquad g = (g_{ji}) \in SL(n),$$

f is transformed into f' of the form

$$f' = \sum a'_{i_1\cdots i_r} x'_{i_1} \wedge \cdots \wedge x'_{i_r},$$

the mapping $a_{i_1\cdots i_r} \mapsto a'_{i_1\cdots i_r}$ defines a representation of $SL(n)$ on the vector space spanned by $a_{i_1\cdots i_r}$'s over K.

Let $K\langle a_{i_1\cdots i_r}\rangle$ be the free associative algebra generated by $a_{i_1\cdots i_s}$'s. Then the mapping $a_{i_1\cdots i_r} \mapsto \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_s}$ gives an isomorphisms as $SL(n)$-modules between $K\langle a_{i_1\cdots i_r}\rangle$ and $K\langle \wedge^r V\rangle$. We denote this isomorphism by φ.

A Young tableau Y is called row strict if the entries of Y do not decrease down columns and increase across rows. Figure 3 gives an example of row strict tableau of weight (2, 3, 1, 1).

1	2	3	4
1	2		
2			

Figure 3

A rectangular Young tableau with n rows is called a row strict Young tableau of degree d if it is row strict and has weight $\langle r \rangle_d$. We denote by $R(n, r, d)$ the set of all row strict Young tableaux of degree d. The following lemma and proposition are clear.

LEMMA 6.1. *Let μ be a Young diagram. Then*

$$[\mu] \otimes \left(\left. \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} \right\} r \right) = \bigoplus_{\lambda} [\lambda] ,$$

where λ ranges all Young diagrams which can built by the addition of r boxes to the Young diagram μ, no two added boxes appearing in the same row of λ.

PROPOSITION 6.2.

$$\dim K\langle a_{i_1 \cdots i_r} \rangle_d^{SL(n)} = |R(n, r, d)| .$$

Let $i_1, \cdots, i_n$, and d be positive integers, $i_1 \leqq \cdots \leqq i_n \leqq d$. Suppose the number 1 appears β_1 times in the sequence $(i_1, \cdots, i_n)$, the number 2 appears β_2 times in $(i_1, \cdots, i_n)$, $\cdots$, and the number d appears β_d times in $(i_1, \cdots, i_n)$. Let $i_1, \cdots, i_n$ be the tensor in $\otimes^d K\langle V \rangle$ defined by

$$\begin{aligned} \langle i_1, \cdots, i_n \rangle = \sum_{\sigma} \operatorname{sgn} \sigma \alpha_{\sigma(1)} & \cdots \alpha_{\sigma(\beta_1)} \\ & \otimes \alpha_{\sigma(\beta_1+1)} \cdots \alpha_{\sigma(\beta_1+\beta_2)} \\ & \vdots \\ & \otimes \alpha_{\sigma(\beta_1+\cdots+\beta_{d-1}+1)} \cdots \alpha_{\sigma(n)} \end{aligned}$$

where σ ranges all permutation on $\{1, \cdots, n\}$ and, if a number j does not appear in $(i_1, \cdots, i_n)$,

$$\alpha_{\sigma(\beta_1+\cdots+\beta_{j-1}+1)} \cdots \alpha_{\sigma(\beta_1+\cdots+\beta_j)}$$

should be 1.

Then $\langle i_1, \cdots, i_n\rangle$ is invariant under the action of $SL(n)$.

We consider a row strict Young tableau of degree d of the form

$$\begin{array}{|c|c|c|c|} \hline i_1 & j_1 & \cdot & m_1 \\ \hline i_2 & j_2 & \cdot & m_2 \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline i_n & j_n & \cdot & m_n \\ \hline \end{array} \; .$$

Then $\langle i_1, \cdots, i_n\rangle\langle j_1, \cdots, j_n\rangle\langle m_1, \cdots, m_n\rangle$ is a $SL(n)$-invariant tensor in $\otimes^d K\langle V\rangle$.

Let A be the projection operator from $K\langle V\rangle_r$ onto $\wedge^r V$, that is $A(v_1 \otimes \cdots \otimes V_r) = v_1 \wedge \cdots \wedge v_r$. We extend A to the mapping $\otimes^d K\langle V\rangle_r \to \otimes^d \wedge^r V$, denoted also by A. We set

$$F(Y, a) = \varphi(A\langle i_1, \cdots, i_n\rangle\langle j_1, \cdots, j_n\rangle \cdots \langle m_1, \cdots, m_n\rangle) .$$

Then $F(Y, a)$ is a noncommutative invariant of degree d in $K\langle a_{i_1\cdots i_r}\rangle$.

EXAMPLE 6.3. Let n be an even integer, say $2m$. There is one row strict Young tableau Y of degree d of the form

$$Y = \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline 2 \\ \hline 2 \\ \hline \cdot \\ \hline \cdot \\ \hline m \\ \hline m \\ \hline \end{array} \; .$$

Then we have

$$\langle 1, 1, 2, 2, \cdots, m, m\rangle = \sum \operatorname{sgn} \sigma \alpha_{\sigma(1)}\alpha_{\sigma(2)} \otimes \alpha_{\sigma(3)}\alpha_{\sigma(4)} \otimes \cdots \otimes \alpha_{\sigma(n-1)}\alpha_{\sigma(n)}$$

and hence the associated noncommutative invariant of degree m is given by

$$F(Y, a) = \sum_{\sigma} \operatorname{sgn} \sigma a_{\sigma(1)\sigma(2)} a_{\sigma(3)\sigma(4)} \cdots a_{\sigma(n-1)\sigma(n)} \,.$$

In this case $F(Y, a)$ is the (noncommutative) Pfaffian.

EXAMPLE 6.4 ($r = 3$, $n = 6$). In this case there are 4 row strict Young tableaux of degree 4:

$$Y_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 3 \\ \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 2 & 4 \\ \hline 3 & 4 \\ \hline \end{array}\,, \quad Y_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 2 \\ \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 3 & 4 \\ \hline 3 & 4 \\ \hline \end{array}\,, \quad Y_3 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 1 & 3 \\ \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 2 & 4 \\ \hline 2 & 4 \\ \hline \end{array}\,, \quad Y_4 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 2 \\ \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 3 & 4 \\ \hline 3 & 4 \\ \hline \end{array}\,.$$

We have

$$\langle 1, 1, 1, 2, 2, 3\rangle = \sum \operatorname{sgn} \sigma \alpha_{\sigma(1)}\alpha_{\sigma(2)}\alpha_{\sigma(3)} \otimes \alpha_{\sigma(4)}\alpha_{\sigma(5)} \otimes \alpha_{\sigma(6)} \otimes 1$$

$$\langle 2, 3, 3, 4, 4, 4\rangle = \sum \operatorname{sgn} \sigma 1 \otimes \alpha_{\sigma(1)} \otimes \alpha_{\sigma(2)}\alpha_{\sigma(3)} \otimes \alpha_{\sigma(4)}\alpha_{\sigma(5)}\alpha_{\sigma(6)}$$

and hence we obtain

$$F(Y_1, a) = \sum_{\sigma,\lambda} \operatorname{sgn}(\sigma\mu) a_{\sigma(1)(2)\sigma(3)} a_{\sigma(4)\sigma(5)\mu(1)} a_{\sigma(6)\mu(2)\mu(3)} a_{\mu(4)\mu(5)\mu(6)} \,.$$

It is known (See p. 81 [6]) that $K[\wedge^3 V]_4$ contains one invariant and

$$K[\overset{3}{\wedge} V]_2 = \left(\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline & \\ \hline \end{array}\right) + \left(\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\right).$$

Then, considering $a_{ijk}(i < j < k)$ as commutative variables, one sees that

$$\sum_{\sigma,\mu} \operatorname{sgn}(\sigma\mu) a_{\sigma(1)\sigma(2)\sigma(3)} a_{\sigma(4)\sigma(5)\sigma(1)} a_{\sigma(6)\sigma(2)\sigma(3)} a_{\sigma(4)\sigma(5)\sigma(6)}$$

is an invariant of degree 4 in the commutative ring $K[\wedge^3 V]$.

For two row strict Young tableaux Y_1 and Y_2 of degree d_1 and d_2 respectively, $Y_1 \hat{\oplus} Y_2$ is defined by the same way as in section 3. $Y_1 \hat{\oplus} Y_2$ is a row strict Young tableau of degree $d_1 + d_2$ and $F(Y_1 \oplus Y_2, a) = F(Y_1, a)F(Y_2, a)$.

THEOREM 6.5. *The set of noncommutative invariants associated with all row strict Young tableaux of degree d is a basis of the vector space* $K\langle a_{i_1\cdots i_r}\rangle_d^{SL(n)}$.

Proof. We define an ordering on the set of noncommutative monomials in $K\langle a_{i_1\cdots i_r}\rangle$ as in the proof of Theorem 3.1. Suppose that each number j, $1 \leqq j \leqq d$, appears in the j_1-th, j_2-th, and j_r-th rows in a row strict Young tableau Y of degree d. We set $[j] = (j_1, \cdots, j_r)$. Then the highest term in the monomials of $F(Y, a)$ is $\pm a_{[1]} \cdots a_{[d]}$. For example, if $(n, r) = (4, 2)$,

1	2	3
1	4	5
2	4	6
3	5	6

is a row strict Young tableau of degree 6 and the highest term in the monomials of $F(Y, a)$ is $\pm a_{12}a_{13}a_{14}a_{23}a_{24}a_{34}$. It is easily seen that, for different row strict Young tableaux Y_1 and Y_2 of the same degree, the highest terms of associated noncommutative invariants are linearly independent. Therefore combining with Proposition 6.2, the proof is completed.

The notion of decomposable or indecomposable Young tableau is defined by the same way as in section 3 and the following theorems are proved in the exactly same way as the corresponding theorems.

THEOREM 6.6. *The set of noncommutative invariants associated with all indecomposable Young tableaux is a free generating set of the noncommutative invariant ring* $K\langle a_{i_1\cdots i_r}\rangle^{SL(n)}$.

THEOREM 6.7. *The ring of noncommutative invariants* $K\langle a_{i_1\cdots i_r}\rangle^{SL(n)}$ *is not finitely generated.*

Remark. Considering $a_{i_1 \cdots i_r}$ as commutative variables, the commutative ring of invariants $K[\wedge^r V]^{SL(n)}$ is generated by invariants associated with all indecomposable Young tableaux.

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