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MODULAR FORMS AND THE AUTOMORPHISM GROUP OF LEECH LATTICE

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Dedicated to Professor Michio Kuga on his 60th birthday

This is a continuation of my previous papers [2], [3], [4] concerning to the monstrous moonshine.

The automorphism group $\cdot O$ of the Leech lattice L plays an important role in the study of moonshine. Especially it is important to study theta functions associated with quadratic sublattices of L consisting of fixed vectors of elements of $\cdot O$. In this paper, we discuss the properties that these functions are expected to satisfy in the relation to the monstrous moonshine.

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§1.

1.1. Throughout this paper, we use the same notation as in [4] and [5]. We recall them first.

$\cdot O$ has a natural 24-dimensional representation over $\boldsymbol{Q}$ induced by the action on the Leech lattice. So each element π of $\cdot O$ is described by Frame shape with respect to this representation:

$$\pi = \prod_{1 \leqq t} t^{r_t}, \qquad r_t \in \boldsymbol{Z}.$$

Then $\deg \pi = \sum t \cdot r_t = 24$. Let $\mathrm{wt}\, \pi = \frac{1}{2} \sum r_t$. We classify every elements of $\cdot O$ into the following 3 types:

(1.1) π is called type C if $r_t \geqq 0$ for all $t \geqq 1$.

(1.2) π is called type E if $\mathrm{wt}\, \pi$ is positive but there exists some t such that $r_t < 0$.

(1.3) π is called type F if $\mathrm{wt}\, \pi = 0$.

For each π, we can suitably choose a positive integer N which is a

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multiple of $\operatorname{ord}\pi =$ l.c.m. of all t with $r_t \neq 0$. Let Q be any Hall divisor of N, i.e. Q is a positive divisor of N and $(Q, N/Q) = 1$. Then we define Atkin-Lehner's involution $W_{Q,N}$ as follows:

$$\pi \circ W_{Q,N} = \prod_{\substack{t \\ s=(t,Q)}} \left(\frac{t \cdot Q}{s^2}\right)^{r_t}.$$

We denote by $S(\pi)$ the set of all distinct images of π by $W_{Q,N}$ for all Hall divisors of N. When π is of type C, we know that $S(\pi) = \{\pi\}$. When π is of type E, for any $\pi' \in S(\pi)$, we know that

(1.4) π' is a Frame shape of a certain element of $\cdot O$ or $\deg \pi' = 0$.

Moreover, for π of type E, we call π self-conjugate if the following condition is satisfied:

(1.5) If $\pi' \in S(\pi)$ is not equal to π, then $\deg \pi' = 0$.

If π does not satisfy the above condition, we call π non-self-conjugate.

For each π, we consider two kinds of modular forms $\eta_\pi(z)$ and $\vartheta_\pi(z)$ defined as follows:

$$\eta_\pi(z) = \prod_t \eta(tz)^{r_t},$$
$$\vartheta_\pi(z) = \theta(z; L^\pi) = \sum_{x \in L^\pi} e^{\pi i z \langle x, x \rangle}$$

where $\eta(z)$ is the Dedekind η-function and $L^\pi = \{x \in L; \pi \cdot x = x\}$. These are modular forms of weight $\operatorname{wt}\pi$.

Concerning to these modular forms, the following problem is very important.

PROBLEM 1.1. Does there exist a theta function $f(z)$ satisfying that there exists an element g of F_1 such that

$$\frac{f(z)}{\eta_\pi(z)} = T_g(z) + c, \qquad c\colon \text{constant},$$

where $T_g(z)$ denotes the Thompson series assigned to g in (1). In Section 4 of [4], we discussed this problem in enlarging the choices of the functions of right hand side.

Concerning to this problem, there is a remarkable conjecture by Conway and Norton

CONJECTURE 1.1. *The notation being as above, and let π be an element of $\cdot O$ of type C or of self-conjugate type E. Then there exists an* *element g of F_1 such that*

$$\frac{\vartheta_\pi(z)}{\eta_\pi(z)} = T_g(z) + c\,, \qquad c\colon \text{a constant}.$$

Namely, $\vartheta_\pi(z)$ is a solution for $\eta_\pi(z)$ in Problem 1.1.

$\vartheta_\pi(z)$ are given explicitly when π is of type C or π is of self-conjugate type E which belongs to the monomial subgroup $2^{12}\cdot M_{24}$ by Kondo-Tasaka [6] and Kondo [7]. For these cases, it is seen that Conjecture 1.1 is true by [4], [6].

However, when π is of non-self-conjugate type E, we can not expect the above conjecture is true. In this case, $\dfrac{\vartheta_\pi(z)}{\eta_\pi(z)}$ have other good property in the relation to the moonshine, and moreover, some elements of self-conjugate type E are also seen to have this property; this is a main theme of this paper.

1.2. We mainly consider modular functions written by $\dfrac{\vartheta_\pi(z)}{\eta_\pi(z)}$ guided by a suggestion of Conway and Norton [1] in the following sections, but, modular functions of a form $\dfrac{\eta_\pi(z)}{\eta_{\pi'}(z)}$ are also relevant to the Thompson series. In this paragraph, we study these modular functions.

Let $\pi = \prod t^{r_t}$ and $\pi' = \prod t^{r'_t}$ be generalized permutations. We define the quotient of π by π' as follows:

$$\frac{\pi}{\pi'} = \prod t^{r_t - r'_t}.$$

This is also a generalized permutation.

Let π be any element of $\cdot O$ such that $S(\pi)$ contains an element π' of degree 0. Then the quotient $\dfrac{\pi}{\pi'}$ is of degree 24.

THEOREM 1.1. *For any π and π' as above, the quotient $\dfrac{\pi}{\pi'}$ is also a Frame shape of an element of $\cdot O$.*

Proof. By speculations on the tables in [5].

Since $\eta_{\pi/\pi'}(z) = \dfrac{\eta_\pi(z)}{\eta_{\pi'}(z)}$, $\dfrac{\eta_{\pi'}(z)}{\eta_\pi(z)}$ is equal to a Thompson series for some element of F_1 up to a constant term.

This result will be useful in the following sections.

§ 2.

To state our result, we classify elements π of self-conjugate type E more in detail:

(2.1) π is called type E_1 if $S(\pi) = \{\pi\}$.

(2.2) π is called type E_2 if $S(\pi) \neq \{\pi\}$.

We shall see that all π of non-self-conjugate type E or type E_2 have other good property which is explained a little bit later. We shall state our results in the following 3 cases:

Case 1. We discuss all π which are of type E_2 and $S(\pi)$ consists of two elements.

Case 2. We discuss all π which are of type E_2 but are not contained in Case 1.

Case 3. We discuss all π which are of non-self-conjugate type E. We give the tables of π, N, and $S(\pi)$ contained in these cases.

Table 1. π in Case 1 (19 elements).

π	N	$S(\pi)\backslash\{\pi\}$	π	N	$S(\pi)\backslash\{\pi\}$
$\frac{2^{16}}{1^8}$	2	$\frac{1^{16}}{2^8}$	$\frac{2\cdot 3^3\cdot 12^3}{1\cdot 4\cdot 6^3}$	12	$\frac{1^3\cdot 4^3\cdot 6}{2^3\cdot 3\cdot 12}$
$\frac{3^9}{1^3}$	3	$\frac{1^9}{3^3}$	$\frac{4^2\cdot 12^2}{2\cdot 6}$	12	$\frac{1^2\cdot 3^2}{2\cdot 6}$
$\frac{4^8}{2^4}$	4	$\frac{1^8}{2^4}$	$\frac{1^3\cdot 12^3}{2\cdot 3\cdot 4\cdot 6}$	12	$\frac{3^3\cdot 4^3}{1\cdot 2\cdot 6\cdot 12}$
$\frac{2^6\cdot 4^4}{1^4}$	4	$\frac{2^6\cdot 1^4}{4^4}$	$\frac{2^2\cdot 14^2}{1\cdot 7}$	14	$\frac{1^2\cdot 7^2}{2\cdot 14}$
$\frac{5^5}{1}$	5	$\frac{1^5}{5}$	$\frac{1^2\cdot 15^2}{3\cdot 5}$	15	$\frac{3^2\cdot 5^2}{1\cdot 15}$
$\frac{2^4\cdot 6^4}{1^2\cdot 3^2}$	6	$\frac{1^4\cdot 3^4}{2^2\cdot 6^2}$	$\frac{1\cdot 2\cdot 10\cdot 20}{4\cdot 5}$	20	$\frac{2\cdot 4\cdot 5\cdot 10}{1\cdot 20}$
$\frac{3^3\cdot 6^3}{1\cdot 2}$	6	$\frac{1^3\cdot 2^3}{3\cdot 6}$	$\frac{2^2\cdot 5\cdot 20}{1\cdot 4}$	20	$\frac{1\cdot 4\cdot 10^2}{5\cdot 20}$
$\frac{8^4}{4^2}$	8	$\frac{1^4}{2^2}$	$\frac{1\cdot 4\cdot 6\cdot 24}{3\cdot 8}$	24	$\frac{2\cdot 3\cdot 8\cdot 12}{1\cdot 24}$
$\frac{2^3\cdot 4\cdot 8^2}{1^2}$	8	$\frac{1^2\cdot 2\cdot 4^3}{8^2}$	$\frac{2\cdot 3\cdot 4\cdot 24}{1\cdot 8}$	24	$\frac{1\cdot 6\cdot 8\cdot 12}{3\cdot 24}$
$\frac{9^3}{3}$	9	$\frac{1^3}{3}$			

Table 2. π in Case 2 (2 elements)

π	N	$S(\pi)\backslash\{\pi\}$		
$\frac{1\cdot 6^6}{2^2\cdot 3^3}$	6	$\frac{2\cdot 3^6}{1^2\cdot 6^3}$,	$\frac{3\cdot 2^6}{1^3\cdot 6^2}$,	$\frac{1^6\cdot 6}{2^3\cdot 3^2}$
$\frac{2^3\cdot 6\cdot 12^2}{1\cdot 3\cdot 4^2}$	12	$\frac{2^3\cdot 3^2\cdot 6}{1^2\cdot 4\cdot 12}$,	$\frac{2\cdot 4^2\cdot 6^3}{1\cdot 3\cdot 12^2}$,	$\frac{1^2\cdot 2\cdot 6^3}{3^2\cdot 4\cdot 12}$

Table 3. All π of type E_1 (9 elements)

$\frac{1^8\cdot 4^8}{2^8}$,	$\frac{2^4\cdot 8^4}{4^4}$,	$\frac{1^4\cdot 8^4}{2^2\cdot 4^2}$,	$\frac{1^3\cdot 9^3}{3^2}$,	$\frac{1^2\cdot 3^2\cdot 4^2\cdot 12^2}{2^2\cdot 6^2}$,	$\frac{2^2\cdot 16^2}{4\cdot 8}$,	$\frac{1^2\cdot 16^2}{2\cdot 8}$
$\frac{2\cdot 6\cdot 8\cdot 24}{4\cdot 12}$,	$\frac{1\cdot 4\cdot 7\cdot 28}{2\cdot 14}$					

2.1. *Case* 1. For each π, we can associate two kinds of modular functions which appear in the monstrous moonshine [1].

We use the following convention like in [1]: for each element g in F_1, $T_g(z)$ denotes the Thompson series given in [1] which is of the form $T_g(z) = q^{-1} + 0 + \sum_{n=1} H_n(g)q^n$. We put $t_{g,c} = T_g(z) + c$, c: *a* constant, and if we don't need to specify the constant term c, we simply write t_g instead of $t_{g,c}$.

THEOREM 2.1. *Let π be in Case* 1. *Then there exists a unique modular form $\theta_\pi(z) = \sum_{n=1}^{\infty} a_n(\pi)q^n$ satisfying the following conditions.*

(2.3) *$\theta_\pi(z)$ is a theta function of some even integral, positive definite quadratic lattice.*

(2.4) $a_1(\pi) = 0$.

(2.5) *There exists an element g in F_1 such that*

$$\frac{\theta_\pi(z)}{\eta_\pi(z)} = t_g(z)\,.$$

(2.6) *There exists an element g' in F_1 such that*

$$\frac{\theta_\pi(z)}{\eta_{\pi'}(z)} = 1 + \beta_\pi t_{g'}^{-1}\,, \qquad \beta_\pi\text{: } a \text{ constant.}$$

(2.7) *Let m denote the order of g in the above condition* (2.5). *Then there exists a constant α_π such that*

$$\frac{\theta_\pi(z)}{\eta_\pi(z)} + \alpha_\pi \frac{\theta_\pi(z)}{\eta_{\pi'}(z)} = t_{m+}(z) + c_{\pi'}\,,$$

where $S(\pi) = \{\pi, \pi'\}$ *and* $m+$ *denote the element of* F_1 *given in* [1].

Proof. In Theorem 3.2 in [4], we gave all automorphic forms $\theta_\pi(z)$ satisfying (2.3), (2.4), and (2.5). Then, by checking the condition (2.7) for each case, we can find a unique solution which is given by the following table. Each element in F_1 is written by the Atlas name in [1] like mA, mB, $\cdots$. Here m denote the order of the element. By using the formula of symmetrizations in [1], we see that (2.7) implies (2.6).

Table 4.

π	g	α_π	$\theta_\pi(z)$
$\frac{2^{16}}{1^8}$	$2B$	2^8	$\theta(2z; E_8)$
$\frac{3^9}{1^3}$	$3B$	3^4	$q^0(1, 0, 54, 72, 0, 432, 270, 0, \cdots)$
$\frac{4^8}{2^4}$	$4C$	2^5	$\theta\left(2z; \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}\right)$
$\frac{2^6\cdot 4^4}{1^4}$	$4C$	2^6	$q^0(1, 0, 12, 64, 60, 0, 160, 384, \cdots)$
$\frac{5^5}{1}$	$5B$	5^2	$q^0(1, 0, 10, 20, 0, 20, 0, \cdots)$
$\frac{2^4\cdot 6^4}{1^2\cdot 3^2}$	$6C$	2^4	$\theta\left(2z; \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)^2$
$\frac{3^3\cdot 6^3}{1\cdot 2}$	$6D$	3^3	$\theta\left(3z; \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}\right)$
$\frac{8^4}{4^2}$	$8E$	2^3	$\theta(4z)^2$
$\frac{2^3\cdot 4\cdot 8^2}{1^2}$	$8E$	2^4	$q^0(1, 0, 2, 8, 6, 16, 12, 0, \cdots)$
$\frac{9^3}{3}$	$9B$	3^2	$\theta\left(3z; \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$
$\frac{2\cdot 3^3\cdot 12^3}{1\cdot 4\cdot 6^3}$	$12B$	3	$\theta\left(2z; \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$

π	g	α_π	$\theta_\pi(z)$
$\frac{4^2\cdot 12^2}{2\cdot 6}$	$12B$	2^3	$\theta\left(4z;\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)$
$\frac{1^3\cdot 12^3}{2\cdot 3\cdot 4\cdot 6}$	$12H$	-1	$\theta\left(2z;\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)$
$\frac{2^2\cdot 14^2}{1\cdot 7}$	$14B$	2^2	$\theta\left(2z;\begin{bmatrix}2 & 1\\ 1 & 4\end{bmatrix}\right)$
$\frac{1^2\cdot 15^2}{3\cdot 5}$	$15C$	1	$\theta\left(z;\begin{bmatrix}4 & 1\\ 1 & 4\end{bmatrix}\right)$
$\frac{1\cdot 2\cdot 10\cdot 20}{4\cdot 5}$	$20F$	-1	$\theta\left(z;\begin{bmatrix}4 & 2\\ 2 & 6\end{bmatrix}\right)$
$\frac{2^2\cdot 5\cdot 20}{1\cdot 4}$	$20C$	5	$\theta\left(5z;\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\right)$
$\frac{1\cdot 4\cdot 6\cdot 24}{3\cdot 8}$	$24I$	1	$\theta\left(z;\begin{bmatrix}4 & 0\\ 0 & 6\end{bmatrix}\right)$
$\frac{2\cdot 3\cdot 4\cdot 24}{1\cdot 8}$	$24C$	3	$\theta\left(3z;\begin{bmatrix}2 & 0\\ 0 & 4\end{bmatrix}\right)$

Remark 2.1. In [4], we could not prove that $\theta_\pi(z)$ for $\pi = \frac{3^9}{1^3}$ in Theorem 1.2 is a theta function. Here we show that this is true. We consider $\theta_\pi(z) \mid W_{3,3}$, where $W_{3,3}$ denote the Atkin-Lehner's involution. Then we see that this coincides with $\theta(z; E_6)$ up to a constant factor, where E_6 denotes the even integral, positive definite quadratic lattice obtained from Lie algebra E_6. This also shows that $\theta_\pi(z)$ itself is a theta function.

PROPOSITION 2.1. *The notation being the same as above, we have* $\theta_\pi(z) - \eta_{\pi'}(z) = b_\pi \eta_\pi(z)$, *where* b_π *are non zero constants.*

Proof. By comparing Theorem 1.1 and (2.5) in Theorem 2.1, we see that $\frac{\eta_{\pi'}(z)}{\eta_\pi(z)}$ and $\frac{\theta_\pi(z)}{\eta_\pi(z)}$ correspond to the Thompson series of the same element in F_1; this implies Proposition 2.1.

Remark 2.2. By comparing (2.6) in Theorem 2.1 and Proposition 2.1, we see that g' in (2.6) is the same as g in (2.5).

2.2. *Case* 2. In Table 2, we put $S(\pi)\backslash\{\pi\} = \{\pi_1, \pi_2, \pi_3\}$ in order.

When $\pi = \frac{1\cdot 6^6}{2^2\cdot 3^3}$, put $\theta_\pi(z) = \theta\left(2z;\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)$ and when

$$\pi = \frac{2^3 \cdot 6 \cdot 12^2}{1 \cdot 3 \cdot 4^2}, \text{ put } \theta_\pi(z) = \theta\left(3z; \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right).$$

PROPOSITION 2.2. *The notation being the same as above, we have* $\theta_\pi(z) - \eta_{\pi'}(z) = b_{\pi'}\eta_\pi(z)$ *with some non zero constants* $b_{\pi'}$ *for any* $\pi' \in \{\pi_1, \pi_2, \pi_3\}$.

Proof. For each π, the levels, characters and weights of $\eta_\pi(z)$, $\eta_{\pi'}(z)$ and $\theta_\pi(z)$ are the same, so we get the proof by computing a few Fourier coefficients of these modular forms.

THEOREM 2.2. *The notation being the same as above, we have the following formula.*

(2.8) *When* $\pi = \dfrac{1 \cdot 6^6}{2^2 \cdot 3^3}$, *we have*

$$\frac{\theta_\pi(z)}{\eta_\pi(z)} = t_{6E,1}, \qquad \frac{\theta_\pi(z)}{\eta_{\pi_1}(z)} = 1 - 3t^{-1}_{6E,4},$$

$$\frac{\theta_\pi(Z)}{\eta_{\pi_2}(z)} = 1 - 2t^{-1}_{6E,3}, \qquad \frac{\theta_\pi(z)}{\eta_{\pi_3}(z)} = 1 + 6t^{-1}_{6E,-5}.$$

(2.9) *When* $\pi = \dfrac{2^3 \cdot 6 \cdot 12^2}{1 \cdot 3 \cdot 4^2}$, *we have*

$$\frac{\theta_\pi(z)}{\eta_\pi(z)} = t_{12I,-1}, \qquad \frac{\theta_\pi(z)}{\eta_{\pi_1}(z)} = 1 - 2t^{-1}_{12I,1},$$

$$\frac{\theta_\pi(z)}{\eta_{\pi_2}(z)} = 1 - \ t^{-1}_{12I,0}, \qquad \frac{\theta_\pi(z)}{\eta_{\pi_3}(z)} = 1 + 2t^{-1}_{12I,-3}.$$

Proof. The statement for $\dfrac{\theta_\pi(z)}{\eta_\pi(z)}$ can proved by using Theorem 3.2 in [4]. Other statements are proved by combining this result and Theorem 1.1 and Proposition 2.2.

We put

$$j_\pi(z) = \frac{\theta_\pi(z)}{\eta_\pi(z)}, \qquad j_{\pi_i}(z) = \frac{\theta_\pi(z)}{\eta_{\pi_i}(z)} \qquad \text{for } i = 1, 2, 3.$$

Then we have COROLLARY 2.1.

COROLLARY 2.1.

(2.10) *When* $\pi = \dfrac{1 \cdot 6^6}{2^2 \cdot 3^3}$, *we have*

$$j_\pi(z) - 3j_{\pi_1}(z) + 4j_{\pi_2}(z) + 12j_{\pi_3}(z) = t_{6\tau,14}.$$

(2.11) *When* $\pi = \dfrac{2^3 \cdot 6 \cdot 12^2}{1 \cdot 3 \cdot 4^2}$, *we have*

$$j_\pi(z) - 2j_{\pi_1}(z) + 3j_{\pi_2}(z) + 6j_{\pi_3}(z) = t_{12+,6}\,.$$

Proof. By combining (2.8) with the formula of symmetrizations in [1], we can see that the left hand side of (2.10) is equal to

$$t_{6E,1} - 3 - (t_{6C} - t_{6E}) + 4 + (t_{6D} - t_{6E}) + 12 + (t_{6B} - t_{6E})\,.$$

Then, by using the formula in p. 319 in [1], we get the proof. (2.11) can be proved by the similar method.

As is seen in Theorem 3.2 in [4], $\theta\left(2z; \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$ and $\theta\left(3z; \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)$ satisfy the conditions (2.3) and (2.4) in Theorem 2.1 for the above π.

Moreover, when π is of type E_1, we showed in Theorem 1.2 in [4] that there exists a unique solution $\theta_\pi(z)$ satisfying the above conditions (2.3), (2.4) and (2.5). Of course, we should remark that some of them are not yet proved to be theta functions. Then, to strengthen Conjecture 1.1, we propose the following

CONJECTURE 2.1. *The notation being as above, for any* π *of self-conjugate type E, we have*

$$\theta_\pi(z) = \theta(z; L^\pi)\,. \tag{2.12}$$

By the result of Kondo [7], we know that this conjecture is true if π belongs to the monomial subgroup $2^{12} \cdot M_{24}$ of $\cdot O$.

§ 3.

Case 3. In this case, we know that there exists no solution of Problem 1.1. So there is no clue to search for $\theta(z; L^\pi)$ through considering Problem 1.1.

On the other hand, Prof. Kondo [7] informed us the explicit description of $\theta(z; L^\pi)$ for all π in $2^{12} \cdot M_{24}$. To be lucky enough, $2^{12} \cdot M_{24}$ contains several elements of non-self-conjugate type E. His result is as follows:

Table 5.

π	$\theta(z; L^\pi)$
$\dfrac{1^4 \cdot 2 \cdot 6^5}{3^4}$	$-\dfrac{9}{5}E^{(3)}_{1,6} + \dfrac{81}{5}E^{(3)}_{6,1} - \dfrac{81}{5}E^{(3)}_{3,2} + \dfrac{9}{5}E^{(3)}_{2,3}$

π	$\theta(z; L^{\pi})$
$\frac{2^5\cdot 3^4\cdot 6}{1^4}$	$-\frac{9}{5}E^{(3)}_{1,6}+\frac{9}{5}E^{(3)}_{6,1}-\frac{9}{5}E^{(3)}_{3,2}+\frac{9}{5}E^{(3)}_{2,3}$
$\frac{1^2\cdot 2\cdot 10^3}{5^2}$	$-\frac{5}{3}E^{(2)}_{1,10}+\frac{25}{3}E^{(2)}_{10,1}-\frac{25}{3}E^{(2)}_{5,2}+\frac{5}{3}E^{(2)}_{2,5}$
$\frac{2^3\cdot 5^2\cdot 10}{1^2}$	$-\frac{5}{3}E^{(2)}_{1,10}+\frac{5}{3}E^{(2)}_{10,1}-\frac{5}{3}E^{(2)}_{5,2}+\frac{5}{3}E^{(2)}_{2,5}$
$\frac{1\cdot 6\cdot 10\cdot 15}{3\cdot 5}$	$\theta\left(2z; \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\right)$
$\frac{2\cdot 3\cdot 5\cdot 30}{1\cdot 15}$	$\theta\left(2z; \begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix}\right)$
$\frac{1^2\cdot 4\cdot 6^2\cdot 12}{3^2}$	$-E^{(2)}_{1,\psi}+3E^{(2)}_{\psi,1}-3E^{(2)}_{3,4}+E^{(2)}_{4,3}$
$\frac{2^2\cdot 3^2\cdot 4\cdot 12}{1^2}$	$-E^{(2)}_{1,\psi}+E^{(2)}_{\psi,1}+E^{(2)}_{3,4}-E^{(2)}_{4,3}$

Here we use the same notation as in Appendix of [4]. ψ denotes the Dirichlet character modulo 12 defined by $\psi(d)=(-1)^{(d-1)/2}\left(\frac{d}{3}\right)$. Fourier coefficients for small n of these forms are given in Table 8.

But we can not see the relation between $\frac{\theta(z; L^{\pi})}{\eta_{\pi}(z)}$ and modular functions appeared in monstrous moonshine directly. So, like in Section 2, we had better to consider linear sums of these functions.

There are 15 elements of non-self-conjugate type E. Some of them are transformed into each other by the actions of Atkin-Lehner's involutions, and there exist only 5 different $S(\pi)$ as follows:

Table 6.

π	N	$S(\pi)=\{\pi_1, \pi_2, \pi_3, \pi_4\}$			
$\frac{2^5\cdot 3^4\cdot 6}{1^4}$	6	$\frac{2^5\cdot 3^4\cdot 6}{1^4}$,	$\frac{1^4\cdot 2\cdot 6^5}{3^4}$,	$\frac{1^5\cdot 3\cdot 6^4}{2^4}$,	$\frac{1\cdot 2^4\cdot 3^5}{6^4}$
$\frac{2^3\cdot 5^2\cdot 10}{1^2}$	10	$\frac{2^3\cdot 5^2\cdot 10}{1^2}$,	$\frac{1^2\cdot 2\cdot 10^3}{5^2}$,	$\frac{1^3\cdot 5\cdot 10^2}{2^2}$,	$\frac{1\cdot 2^2\cdot 5^3}{10^2}$
$\frac{1\cdot 6\cdot 10\cdot 15}{3\cdot 5}$	30	$\frac{1\cdot 6\cdot 10\cdot 15}{3\cdot 5}$,	$\frac{2\cdot 3\cdot 5\cdot 30}{1\cdot 15}$,	$\frac{2\cdot 3\cdot 5\cdot 30}{6\cdot 10}$,	$\frac{1\cdot 6\cdot 10\cdot 15}{2\cdot 30}$

π	N	$S(\pi)=\{\pi_1,\pi_2,\pi_3,\pi_4\}$			
$\frac{2^2\cdot 9\cdot 18}{1\cdot 6}$	18	$\frac{2^2\cdot 9\cdot 18}{1\cdot 6}$,	$\frac{1\cdot 2\cdot 18^2}{6\cdot 9}$,	$\frac{1^2\cdot 9\cdot 18}{2\cdot 3}$,	$\frac{1\cdot 2\cdot 9^2}{3\cdot 18}$
$\frac{2^2\cdot 3^2\cdot 4\cdot 12}{1^2}$	12	$\frac{2^2\cdot 3^2\cdot 4\cdot 12}{1^2}$,	$\frac{1^2\cdot 4\cdot 6^2\cdot 12}{3^2}$,	$\frac{1\cdot 2^2\cdot 3\cdot 12^2}{4^2}$,	$\frac{1\cdot 3\cdot 4^2\cdot 6^2}{12^2}$

Each $S(\pi)$ consists of four elements and we denote them by π_1, π_2, π_3 and π_4 in order. This ordering of elements of $S(\pi)$ has an important meaning which we explain now. They satisfy the following properties in common:

(3.1) π_1, π_2 and π_3 are elements of $\cdot O$ and $\deg \pi_4 = 0$.

(3.2) If one of elements in $S(\pi)$ belongs to the monomial subgroup $2^{12}\cdot M_{24}$, then only π_1 and π_2 belong to this subgroup.

Moreover, the first four $S(\pi)$ in Table 6 satisfy the following properties in common:

(3.3) Let π_1^2, π_2^2 and π_3^2 denote elements of $\cdot O$ by taking the second power. Then Frame shapes of π_1^2, and π_2^2 coincide with each other and its weight is not equal to that of π_1.

(3.4) The weight of Frame shape of π_3^2 is equal to that of π_3.

Table 7.

π	π_1^2	π_3^2
$\frac{2^5\cdot 3^4\cdot 6}{1^4}$	$1^6\cdot 3^6$	$\frac{3^9}{1^3}$
$\frac{2^3\cdot 5^2\cdot 10}{1^2}$	$1^4\cdot 5^4$	$\frac{5^5}{1}$
$\frac{1\cdot 6\cdot 10\cdot 15}{3\cdot 5}$	$1\cdot 3\cdot 5\cdot 15$	$\frac{1^2\cdot 15^2}{3\cdot 5}$
$\frac{2^2\cdot 9\cdot 18}{1\cdot 6}$	$\frac{1^3\cdot 9^3}{3^2}$	$\frac{9^3}{3}$
$\frac{2^2\cdot 3^2\cdot 4\cdot 12}{1^2}$	$1^2\cdot 2^2\cdot 3^2\cdot 6^2$	$\frac{1^5\cdot 3\cdot 6^4}{2^4}$

Table 8.

π	$\theta_{\pi_1}(\frac{1}{2}z)$ $\theta_{\pi_2}(\frac{1}{2}z)$
$\dfrac{2^5\cdot 3^4\cdot 6}{1^4}$	$q^0(1, 0, 54, 72, 0, 432, 270, 0, 918, \cdots)$ $q^0(1, 72, 270, 720, 936, 2160, 2214, 3600, 4590, \cdots)$
$\dfrac{2^3\cdot 5^2\cdot 10}{1^2}$	$q^0(1, 0, 10, 20, 0, 20, 0, 60, 50, \cdots)$ $q^0(1, 20, 30, 60, 60, 120, 40, 180, 150, \cdots)$
$\dfrac{1\cdot 6\cdot 10\cdot 15}{3\cdot 5}$	$\theta\left(z; \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\right)$ $\theta\left(z; \begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix}\right)$
$\dfrac{2^2\cdot 9\cdot 18}{1\cdot 6}$	$\theta\left(3z; \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$ $\theta\left(z; \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$

THEOREM 3.1. *Let π and $S(\pi)$ be in Table 6 except for the last one. The notation is the same as above. For such π, let $\theta_{\pi_1}(z)$ and $\theta_{\pi_2}(z)$ be given in Table 8. Put $j_{\pi_i}(z) = \dfrac{\theta_{\pi_i}(z)}{\eta_{\pi_i}(z)}$ for $i = 1, 2$ and $j_{\pi_3}(z) = \dfrac{\theta_{\pi_3}(z)}{\eta_{\pi_3}(z)}$, where $\theta_{\pi_3}(z) = \theta_{\pi_1}(\frac{1}{2}z)$. Then these modular functions satisfy the following properties:*

(3.5) *$\theta_{\pi_i}(z)$ for $i = 1, 2, 3$ are theta functions of some even integral, positive definite quadratic lattices.*

(3.6) *There exist some elements g and g' in F_1 such that*

$$j_{\pi_i}(z) = t_g + c_i\cdot t^{-1}_{g',b_i},$$

where b_i, c_i are constants depending on π_i but g and g' can be chosen to depend only on π.

(3.7) *There exist some elements g, g' and g_i for $i = 1, 2, 3$ in F_1 such that*

$$j_{\pi_i}(z) = t_g + t_{g'} - t_{g_i},$$

where g and g' can be chosen to depend only on π.

Proof. The proof of (3.6) is done by giving explicitly these elements in F_1 as follows:

Table 9.

π	g	g'	c	π	g	g'	c
	$6A, -4$	$6E, -5$	-72		$30B, 1$	$30G, 0$	-2
$\frac{2^5\cdot 3^4\cdot 6}{1^4}$	$6C$	$6E, 3$	-8	$\frac{1\cdot 6\cdot 10\cdot 15}{3\cdot 5}$	$30C$	$30G, -2$	2
	$6D$	$6E, 4$	9		$30F$	$30G, -1$	-1
	$6E$	$6B, 12$	1		$30G$	$30A, -3$	1
	$6A, 4$	$6E, 3$	8		$30B, -1$	$30G, -2$	-2
$\frac{1^4\cdot 2\cdot 6^5}{3^4}$	$6B$	$6E, 4$	9	$\frac{2\cdot 3\cdot 5\cdot 30}{1\cdot 15}$	$30A$	$30G, -1$	-1
	$6C$	$6E, -5$	72		$30C$	$30G, 0$	2
	$6E$	$6D, -4$	81		$30G$	$30F, 1$	1
	$6A, 5$	$6E, 4$	-9		$30B, 0$	$30G, -1$	1
$\frac{1^5\cdot 3\cdot 6^4}{2^4}$	$6B$	$6E, 3$	-8	$\frac{2\cdot 3\cdot 5\cdot 30}{6\cdot 10}$	$30A$	$30G, -2$	2
	$6D$	$6E, -5$	72		$30F$	$30G, 0$	2
	$6E$	$6C, -6$	64		$30G$	$30C, -1$	4
	$10A, -2$	$10E, -3$	-20		$18B, -1$	$18D, -2$	-6
$\frac{2^3\cdot 5^2\cdot 10}{1^2}$	$10B$	$10E, 1$	-4	$\frac{2^2\cdot 9\cdot 18}{1\cdot 6}$	$18A$	$18D, 1$	3
	$10C$	$10E, 2$	5		$18C$	$18D, 0$	-2
	$10E$	$10D, 6$	1		$18D$	$18E, 3$	1
	$10A, 2$	$10E, 1$	1		$18B, 1$	$18D, 0$	2
$\frac{1^2\cdot 2\cdot 10^3}{5^2}$	$10B$	$10E, -3$	20	$\frac{1\cdot 2\cdot 18^2}{6\cdot 9}$	$18C$	$18D, -2$	6
	$10D$	$10E, 2$	5		$18E$	$18D, 1$	3
	$10E$	$10C, -2$	25		$18D$	$18A, -1$	9
	$10A, 3$	$10E, 2$	-5		$18B, 2$	$18D, 1$	-3
$\frac{1^3\cdot 5\cdot 10^2}{2^2}$	$10C$	$10E, -3$	20	$\frac{1^2\cdot 9\cdot 18}{2\cdot 3}$	$18A$	$18D, -2$	6
	$10D$	$10E, 1$	-4		$18C$	$18D, 0$	-2
	$10E$	$10B, -4$	16		$18D$	$18C, -3$	4

For each π, each line means that $j_\pi = t_g + c\cdot t_{g'}^{-1}$. If g and g' have the same order as that of π, those given in the above are all solutions satisfying the equation in (3.6).

For each π, we see that the elements g and g' appeared in the first line are the same for all π_i, $i = 1, 2, 3$. So we get the proof of (3.6). For example, the first line shows that

$$j_{\frac{2^5\cdot 3^4\cdot 6}{1^4}}(z) = t_{6A,-4} - 72t_{6E,-5}^{-1}\,.$$

Here the constant term of $j_\pi(z)$ appears in the second column, we only write it once for each π.

From the above table with the formulae of symmetrizations, it follows (3.7) for the following data. For example; let $\pi = \frac{2^5\cdot 3^4\cdot 6}{1^4}$. Then $j_{\pi_1}(z) = t_{6A} - 72t_{6E,-5}^{-1}$. The symmetrization shows that $t_{6B} = t_{6E} + 72t_{6E,-5}^{-1}$, hence we have $j_{\pi_1}(z) = t_{6A} + t_{6E} - t_{6B}$. When we apply the same argument to the second line with the symmetrization $t_{6D} = t_{6E} - 8t_{6E,3}^{-1}$, we have $j_{\pi_1}(z) = t_{6C} + t_{6D} - t_{6E}$. So we get the relation between Thompson series $t_{6A} + 2t_{6E} - t_{6B} - t_{6C} - t_{6D} = 0$ which is already described at p. 319 in [1]. From the third and fourth line, we obtain the same equations as aobve.

Table 10.

π	g	g'	g_1	g_2	g_3
$\frac{2^5\cdot 3^4\cdot 6}{1^4}$	$6A$	$6E$	$6B$	$6D$	$6C$
$\frac{2^3\cdot 5^2\cdot 10}{1^2}$	10A	$10E$	$10D$	$10C$	$10B$
$\frac{1\cdot 6\cdot 10\cdot 15}{3\cdot 5}$	$30B$	$30G$	$30A$	$30F$	$30C$
$\frac{2^2\cdot 9\cdot 18}{6\cdot 9}$	$18B$	$18D$	$18E$	$18A$	$18C$
$\frac{2^2\cdot 3^2\cdot 4\cdot 12}{1^2}$	$12A$	$12I$	$12H$	$12B$	$12E$

Remark 3.1. Let m denote the order of π. Then the symbols of g and g' appeared in (3.6) and (3.7) are equal to $m+$ and $m-$ respectively, except for $30G$.

Remark 3.2. The following fact was remarked by Mr. Lang; since the weight of π_3 and π_3^2 are equal, we see that $L^{\pi_3} = L^{\pi_3^2}$, so $\theta(z; L^{\pi_3}) = \theta(z; L^{\pi_3^2})$. In the above cases, all π_3^2 are of self-conjugate type E, and we already obtained the conjectured form of its associated theta function. We can see that these results are compatible with the above remark.

In Theorem 3.1, we can not deal with the case $\pi = \frac{2^2\cdot 3^2\cdot 4\cdot 12}{1^2}$ in the same manner, because you see in Table 7 that π does not satisfy (3.9). But we can obtain the following similar result.

Let $\{\pi_1, \pi_2, \pi_3, \pi_4\}$ be the same as in Table 6. For $i = 1, 2$, put $\theta_{\pi_i}(z) = \theta(z; L^{\pi_i})$ given in Table 5. Put

$$\theta_{\pi_3}(z) = \theta\left(z; \begin{bmatrix} 4 & 2 & 2 & 2 \\ 2 & 4 & 1 & 1 \\ 2 & 1 & 4 & 1 \\ 2 & 1 & 1 & 4 \end{bmatrix}\right),$$
$$= q^0(1, 0, 14, 12, 0, 40, 18, 0, 62, 40, \cdots),$$
$$= -E^{(2)}_{1,\psi} + 4E^{(2)}_{\psi,1} + E^{(2)}_{3,4} - 4E^{(2)}_{4,3},$$

and put $j_{\pi_i}(z) = \dfrac{\theta_{\pi_i}(z)}{\eta_{\pi_i}(z)}$ for $i = 1, 2, 3$.

THEOREM 3.2. *Under the notation as above, the same statements as* (3.5), (3.6) *and* (3.7) *are true.*

Proof. The proof is the same as in Theorem 3.1. We obtain the following table in this case like Table 9:

Table 11.

π	g	g'	c
	$12A, -2$	$12I, -3$	-12
$\dfrac{2^2\cdot 3^2\cdot 4\cdot 12}{1^2}$	$12B$	$12I, 0$	-3
	$12E$	$12I, 1$	4
	$12I$	$12H, 4$	1
	$12A, 2$	$12I, 1$	-4
$\dfrac{1^2\cdot 4\cdot 6^2\cdot 12}{3^2}$	$12E$	$12I, -3$	12
	$12H$	$12I, 0$	-3
	$12I$	$12B, -4$	9
	$12A, 1$	$12I, 0$	3
$\dfrac{1\cdot 2^2\cdot 3\cdot 12^2}{4^2}$	$12B$	$12I, -3$	12
	$12H$	$12I, 1$	4
	$12I$	$12E, -2$	16

Summing up the above results, we may conjecture the following

CONJECTURE 3.1. *Let π be any element of $\cdot O$ of non-self-conjugate type E. Under the notation as above, we have*

$$\theta_\pi(z) = \theta(z; L^\pi).$$

To describe completely the relation between Thompson series of F_1 and modular functions $j_\pi(z)$, we need to study $j_{\pi_4}(z)$ defined by the following:

When $\pi = \dfrac{2^2\cdot 3^2\cdot 4\cdot 12}{1^2}$, we put

$$\begin{aligned}\theta_{\pi_4}(z) &= q^0(1, 12, 18, 40, 60, 24, 70, \cdots)\\ &= -E^{(2)}_{1,\psi} + 12E^{(2)}_{\psi,1} - 3E^{(2)}_{3,4} + 4E^{(2)}_{4,3}\end{aligned}$$

For the remaining π, we put

$$\theta_{\pi_4}(z) = \theta_{\pi_2}\left(\frac{1}{2}z\right),$$

and define $j_{\pi_4}(z) = \dfrac{\theta_{\pi_4}(z)}{\eta_{\pi_4}(z)}$.

THEOREM 3.3. *Let π be any element of $\cdot O$ of non-self-conjugate type E, and let m denote the order of π. Then we have*

$$j_{\pi_4}(z) = t_g - t_{g'},$$

where g and g' are given in the Table 10.

Proof. By the direct computations.

Hence, we can describe Thompson series appeared in Theorem 3.1 as linear sums of $j_{\pi_i}(z)$, $i = 1, 2, 3, 4$. For example:

COROLLARY 3.1. *We have*

$$\sum_1^4 j_{\pi_i}(z) = 3t_{m+}.$$

§ 4. Concluding Remark

In Section 2 and Section 3, we describe $\theta(z; L^\pi)$ explicitly for all elements π of $\cdot O$ at least as conjectures. We study one more nice property that these modular functions satisfy.

Let π be an element of $\cdot O$. Then, as we see in Section 2, there exists a suitable integer N such that, for all Hall divisor Q of N, the image of π by the action of Atkin-Lehner's involution $W_{Q,N}$ becomes also an element of $\cdot O$ or has degree 0.

On the other hand, it is known that $W_{Q,N}$ acts on theta functions. Namely, we know that

$$\theta_\pi(z) \mid W_{Q,N} = c\Theta(z; \pi, Q, N),$$

where $\Theta(z; \pi, Q, N)$ is a theta function of some even integral, positive definite quadratic lattice and c is a constant.

So we ask naturally what is the relation between these actions of Atkin-Lehner's involutions. The answer is very simple:

THEOREM 4.1. *Suppose that* $\pi \cdot W_{Q,N}$ *is an element of* $\cdot O$. *Then*

$$\theta_{\pi \cdot W_{Q,N}}(z) = \Theta(z; \pi, Q, N).$$

Proof. $\theta_\pi(z)$ can be written as a linear sum of Eisenstein series and cusp forms. The action of Atkin-Lehner's involution on these modular forms can be easily seen. When π belongs to Cases 1 and 2, Propositions 2.1 and 2.2 are also useful. So comparing these results with the table of $\theta_\pi(z)$, we can obtain the proof.

It seems interesting to find an intrinsic proof of this theorem.

Added in proof. Recently, M. Lang [8] has succeeded in computing $\vartheta_\pi(z)$ for all π in $\cdot O$ and Conjectures 2.1 and 3.1 are solved affimatively. Theorem 4.1 also describes the property of $\vartheta_\pi(z)$.

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