T. Hibi
Nagoya Math. J.
Vol. 112 (1988), 1–24

CANONICAL IDEALS OF COHEN-MACAULAY PARTIALLY ORDERED SETS

TAKAYUKI HIBI

Introduction

Our dream is to revive the ideal theory in partially ordered sets from a viewpoint of commutative algebra.

Historically, the concept of ideals in commutative algebra was first studied by Dedekind, who considered the ring of algebraic integers in an algebraic number field.

Let S be a set and S^* the set whose elements are the various subsets of S. Then S^* turns out to be a lattice ordered by inclusion. On the other hand, we may regard S^* to be a commutative ring with identity if we define addition and multiplication in S^* as follows: $A + B := (A - B) \cup (B - A)$, $A \cdot B := A \cap B$. A subset I of the ring S^* is an ideal of S^* if and only if (i) $A \in I$, $B \in S^*$ and $B \subset A$ together imply $B \in I$ and (ii) $A \cup B \in I$ for any A, $B \in I$.

So, in Stone [Sto], a subset I of an arbitrary lattice L is called an ideal of L if (i) $\alpha \in I$, $\xi \in L$ and $\xi \leq \alpha$ together imply $\xi \in I$ and (ii) $\alpha \vee \beta \in I$ for any α, $\beta \in I$. Later, Frink [Fri] extended Stone's definition to partially ordered sets, abbreviated as posets. Here, ignoring the condition (ii) of Stone's definition, we call a subset I of an arbitrary poset Q a poset ideal of Q if $\alpha \in I$, $\beta \in Q$ and $\beta \leq \alpha$ together imply $\beta \in I$.

Recently, some remarkable works between commutative algebra and combinatorics have been accomplished ([Hoc_1], [Rei], [Sta_5], [Sta_9], [Sta_{13}]). One of the main topics in this area is the concept of Cohen-Macaulay posets, see [Bac], [Bjö], [H_3] and [Sta_8]. We now pay attention to poset ideals of Cohen-Macaulay posets to obtain certain ring-theoretical information.

Let $R = \bigoplus_{n \geq 0} R_n$ be an ASL (algebra with straightening laws [Eis]) domain on a Cohen-Macaulay poset Q over a field $R_0 = k$. Then by what means

Received December 15, 1986.

can we describe the canonical module K_R of R explicitly? Roughly speaking, as we can see in Stanley's paper [Sta_7], the canonical module K_R of a Cohen-Macaulay domain $R = \bigoplus_{n\geq 0} R_n$ is controlled by the numerical condition, i.e., the behavior of its Poincaré series $F(R, \lambda) := \sum_{n=0}^{\infty} (\dim_k R_n)\lambda^n$. In general, given a noetherian graded ring $R = \bigoplus_{n\geq 0} R_n$ defined over a field $R_0 = k$, it is difficult to check whether R is Cohen-Macaulay and to calculate its Poincaré series. However, as soon as R turns out to be an ASL on a poset Q over a field k, the desired information can be obtained easily from the combinatorics of the poset Q.

In this paper, we introduce the concept of "canonical ideals" of Cohen-Macaulay posets (cf. (1.1)). If $R = \bigoplus_{n\geq 0} R_n$ is an ASL domain on a Cohen-Macaulay poset Q, which possesses a canonical ideal I, over a field $R_0 = k$, then the canonical module K_R of R is isomorphic to the ideal $I \cdot R$ of R as graded R-modules up to shift in grading. This is a ring-theoretical background to define canonical ideals of Cohen-Macaulay posets.

Many interesting and important questions now occur. Among them, one of the fundamental problems is to classify all Cohen-Macaulay posets which possess canonical ideals. Our main result (3.12), in which the distributive lattices with canonical ideals are classified, is a starting point of this classification problem. We hope that the structure of Cohen-Macaulay posets with canonical ideals will be clear in our further study.

The author would like to thank Prof. Hideyuki Matsumura for providing stimulating atomosphere in which this paper was written, and to thank Prof. Kei-ichi Watanabe for suggesting the effective lemma (1.7). Also, special thanks are due to Prof. Richard P. Stanley for many exciting discussions on the topic of Cohen-Macaulay posets during the conference on Commutative Algebra and Combinatorics at Kyoto (August, 1985) and the conference on Graph Theory at Hakone (June, 1986).

§ 1. The what and why of canonical ideals

The purpose of this section is, first, to introduce the concept of canonical ideals of Cohen-Macaulay posets and, secondly, to state a ring-theoretical background of this notion.

To begin with, we summarize basic definitions and terminologies on combinatorics.

Every partially ordered set (*poset* for short) to be considered is finite, unless otherwise stated.

The *length* of a chain (totally ordered set) X is $\sharp(X) - 1$, where $\sharp(X)$ is the cardinality of X as a set.

The *rank* of a poset Q, denoted by $\operatorname{rank}(Q)$, is the supremum of lengths of chains contained in Q.

A poset Q is called *pure* if the length of any maximal chain of Q is equal to $\operatorname{rank}(Q)$.

The *height* (resp. *depth*) of an element α of a poset Q is the supremum of lengths of chains descending (resp. ascending) from α, and written as $\operatorname{height}_Q(\alpha)$ (resp. $\operatorname{depth}_Q(\alpha)$).

A *poset ideal* of a poset Q is a subset I such that $\alpha \in I$, $\beta \in Q$ and $\beta \leq \alpha$ together imply $\beta \in I$.

We say that a multichain $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_p$ of a poset Q *belongs to* a poset ideal I if $\alpha_i \in I$ for some i.

A *lattices* is a poset L any two of whose elements α and β have a greatest lower bound or "meet" denoted by $\alpha \wedge \beta$, and a least upper bound or "join" denoted by $\alpha \vee \beta$. A subposet P of a lattice L is called a *sublattice* of L if both $\alpha \wedge \beta$ and $\alpha \vee \beta$ in L are contained in P for all $\alpha, \beta \in P$.

Let $\mathbb{N}$ be the set of non-negative integers and $\mathbb{Z}$ the set of integers. A *weighted poset* (Q, ω) is a couple of a poset Q and a map ω, called a *weight* on Q, from Q to $\mathbb{N} - \{0\}$.

The weight of a multichain $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_p$ of a weighted poset (Q, ω) is defined to be $\sum_{1 \leq i \leq p} \omega(\alpha_i)$. For any non-negative integer n, let $c_n = c_n(Q, \omega)$ be the number of multichains of weight n. Thus in particular $c_0 = 1$. Then define the *Poincaré series* $F_{(Q,\omega)}(\lambda)$ of (Q, ω) to be the generating function

$$F_{(Q,\omega)}(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n \in \mathbb{Z}[[\lambda]]$$

of the sequence $\{c_n\}_{n \geq 0}$, which will turn out to be a rational function of the indeterminate λ.

Let I be a poset ideal of a poset Q and ω a weight on Q. For any positive integer n, write $c_n^I = c_n^I(Q, \omega)$ for the number of multichains of (Q, ω) of weight n which belong to I. The Poincaré series $F^I_{(Q,\omega)}(\lambda)$ of I in (Q, ω) is defined by

$$F^I_{(Q,\omega)}(\lambda) = \sum_{n=1}^{\infty} c_n^I \lambda^n \in \mathbb{Z}[[\lambda]] .$$

Let Q be a poset and $A = k[X_\alpha; \alpha \in Q]$ the polynomial ring in $\sharp(Q)$-variables over a field k. Also, let I_Q be the ideal of A generated by all monomials of the form $X_\alpha X_\beta$ such that α and β in Q are incomparable. Set $k[Q] := A/I_Q$, which is called the *Stanley-Reisner ring* of Q over k after the famous works [Sta_5] and [Rei].

A poset Q is called *Cohen-Macaulay* (resp. *Gorenstein*) over a field k if the Stanley-Reisner ring $k[Q]$ is Cohen-Macaulay (resp. Gorenstein).

Many interesting and important works of Cohen-Macaulay and Gorenstein posets are accomplished. Consult [Hoc_2] and [Sta_{11}] for further information.

We have now finished the preliminary steps for the definition of canonical ideals of Cohen-Macaulay posets.

(1.1) DEFINITION. Let Q be a Cohen-Macaulay poset of rank $d-1$ with a unique minimal element $-\infty$, and ω a weight on Q. Then a non-empty poset ideal I of Q is called a *canonical ideal* of the weighted poset (Q, ω) if the following conditions are satisfied:

(1.2) $F_{(Q,\omega)}(\lambda^{-1}) = (-1)^d \lambda^{-a} F^I_{(Q,\omega)}(\lambda)$ for some $a \in \mathbb{Z}$.

(1.3) The subposet $Q - I$ is Cohen-Macaulay with $\mathrm{rank}(Q - I) = d - 2$.

(1.4) EXAMPLE. a) First, consider the following Cohen-Macaulay poset

$Q =$

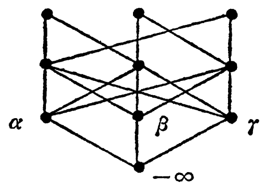


Fig. 1.

Let ω be the natural weight on Q, i.e., $\omega(x) = 1$ for any $x \in Q$. Then the Poincaré series of the weighted poset (Q, ω) is

$$F_{(Q,\omega)}(\lambda) = \frac{1 + 6\lambda + 9\lambda^2 + 2\lambda^3}{(1-\lambda)^4}.$$

Let $I_\alpha = \{-\infty, \alpha\}$. Since $Q - I_\alpha$ is Cohen-Macaulay and the Poincaré series of I_α in (Q, ω) is

$$F^{I_\alpha}_{(Q,\omega)}(\lambda) = \frac{2\lambda + 9\lambda^2 + 6\lambda^3 + \lambda^4}{(1-\lambda)^4},$$

the poset ideal I_α is a canonical ideal of (Q, ω). Of course, $I_\beta = \{-\infty, \beta\}$, $I_\gamma = \{-\infty, \gamma\}$ are also canonical ideals of (Q, ω). Thus a canonical ideal of a weighted poset is not necessarily unique even if it exists.

b) Secondly, let n be a positive integer and ω the weight on the Cohen-Macaulay poset

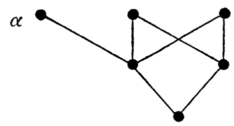
$Q =$ 

Fig. 2.

defined by $\omega(x) = 1$ if $x \neq \alpha$ and $\omega(\alpha) = n$. Then

$$F_{(Q,\omega)}(\lambda) = \frac{(1+\lambda)^2(1+\lambda+\lambda^2+\cdots+\lambda^{n-1})+\lambda^n}{(1-\lambda)^2(1-\lambda^n)}.$$

Hence, it can be checked that (Q, ω) has a canonical ideal if and only if $n = 1$.

c) Finally, let Q be the following Cohen-Macaulay poset

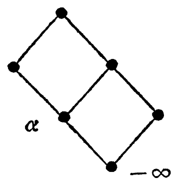
$Q =$ 

Fig. 3.

We write ω for the natural weight on Q and denote by ω' the weight on Q defined by $\omega'(x) = 1$ if $x \neq \alpha, \beta$ and $\omega'(\alpha) = \omega'(\beta) = 2$. Then $I = \{-\infty, \beta\}$ is a canonical ideal of (Q, ω), while $I = \{-\infty\}$ is a canonical ideal of (Q, ω').

A weighted poset (Q, ω) is called *numerically Gorenstein* over a field k if Q is Cohen-Macaulay over k and

$$F_{(Q,\omega)}(\lambda^{-1}) = (-1)^d \lambda^{-a} F_{(Q,\omega)}(\lambda)$$

for some $a \in \mathbb{Z}$, where $d = \operatorname{rank}(Q) + 1$.

If Q is a Cohen-Macaulay poset with a unique minimal element $-\infty$ and ω is a weight on Q, then the weighted poset (Q, ω) is numerically Gorenstein if and only if $I = \{-\infty\}$ is a canonical ideal of (Q, ω).

(1.5) Proposition. *Let (Q, ω) be a Cohen-Macaulay weighted poset with a canonical ideal I. Then $(Q - I, \omega)$ is numerically Gorenstein.*

Proof. Since

$$F_{(Q,\omega)}(\lambda) = F^I_{(Q,\omega)}(\lambda) + F_{(Q-I,\omega)}(\lambda),$$

we have the equalities

$$\begin{aligned} F_{(Q-I,\omega)}(\lambda^{-1}) &= F_{(Q,\omega)}(\lambda^{-1}) - F^I_{(Q,\omega)}(\lambda^{-1}) \\ &= (-1)^d \lambda^{-a} F^I_{(Q,\omega)}(\lambda) - F^I_{(Q,\omega)}(\lambda^{-1}) \\ &= (-1)^{d-1} \lambda^{-a} [(-1)^d \lambda^a F^I_{(Q,\omega)}(\lambda^{-1}) - F^I_{(Q,\omega)}(\lambda)] \\ &= (-1)^{d-1} \lambda^{-a} F_{(Q-I,\omega)}(\lambda) \end{aligned}$$

for some $a \in \mathbb{Z}$ and $d = \operatorname{rank}(Q) + 1$. Hence $(Q - I, \omega)$ is numerically Gorenstein. Q.E.D.

Next, let us recall the definition and some basic results on algebras with straightening laws from [D-E-P] and [Eis].

Suppose that R is a commutative ring and Q, a subset of R, is a poset. A *monomial* is a product of the form $\alpha_1\alpha_2 \cdots \alpha_p$, where $\alpha_i \in Q$. A monomial $\alpha_1\alpha_2 \cdots \alpha_p$ is called *standard* if $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_p$. Now, let k be a field, R a k-algebra and Q a poset contained in R which generates R as a k-algebra. Then we call R an *algebra with straightening laws* (abbreviated as ASL) on Q over k if the following conditions are satisfied:

(ASL-1) The set of standard monomials is a basis of the algebra R as a vector space over k.

(ASL-2) If α and β in Q are incomparable (written as $\alpha \not\sim \beta$) and if

$$\alpha\beta = \sum_i r_i \gamma_{i1}\gamma_{i2} \cdots \gamma_{ip_i},$$

where $0 \neq r_i \in k$ and $\gamma_{i1} \leq \gamma_{i2} \leq \cdots$, is the unique expression, called the *straightening relation*, for $\alpha\beta$ in R as a linear combination of distinct standard monomials guaranteed by (ASL-1), then $\gamma_{i1} \leq \alpha, \beta$ for every i.

Note that the right-hand side of the straightening relation in (ASL-2) is allowed to be the empty sum ($= 0$), but that, though 1 is a standard monomial, no $\gamma_{i1}\gamma_{i2} \cdots \gamma_{ip_i}$ can be 1.

It can be checked that the dimension of R as a k-algebra coincides with $\operatorname{rank}(Q) + 1$.

Let (Q, ω) be a weighted poset. An ASL R on Q over k is called an ASL on (Q, ω) if there is a grading $R = \bigoplus_{n \geq 0} R_n$ such that $R_0 = k$ and $\alpha \in R_{\omega(\alpha)}$ for every $\alpha \in Q$.

Since the Stanley-Reisner ring $k[Q]$ is the simplest ASL on Q over k, we also call $k[Q]$ the *discrete* ASL on Q over k. For any weight ω on Q, $k[Q]$ is an ASL on the weighted poset (Q, ω).

Let (Q, ω) be a weighted poset and R an ASL on (Q, ω) over k. Then R is Cohen-Macaulay (resp. Gorenstein) if the poset Q is Cohen-Macaulay (resp. Gorenstein) over k. This result is called a fundamental theorem in the theory of ASL, see [D-E-P]. Thanks to this fundamental theorem, we can obtain many information about any ASL on Q from the combinatorics of the poset Q.

The following lemma is quite essential in our work.

(1.6) LEMMA ([D-E-P]). *Let R be an ASL on a poset Q over a field k. If I is a poset ideal, then the set of standard monomials belonging to I is a basis of the ideal $I \cdot R$ of R as a vector space over k and the quotient ring $R/I \cdot R$ is an ASL on the subposet $Q - I$ over k.*

It is natural to ask why we present the concept of canonical ideals of Cohen-Macaulay posets. So, we now turn to the statement of a ring-theoretical background of canonical ideals of Cohen-Macaulay posets.

Let $R = \bigoplus_{n \geq 0} R_n$ be a noetherian graded ring defined over a field $R_0 = k$, and $M = \bigoplus_{n \in \mathbf{Z}} M_n$ a finitely generated graded R-module. The *Hilbert function* of M is defined by

$$H(M, n) = \dim_k M_n, \qquad \text{for } n \in \mathbb{Z}.$$

Thus in particular $H(M, n) = 0$ for $n \ll 0$. Define the Poincaré series of M to be

$$F_M(\lambda) = \sum_{n=-\infty}^{\infty} H(M, n)\lambda^n \in \mathbb{Z}[[\lambda]][\lambda^{-1}].$$

It is a consequence of the Hilbert syzygy theorem that $F_M(\lambda)$ is a rational function of λ.

The theory of canonical modules of noetherian graded rings is developed in [Sta$_7$] and [G-W]. We here summarize fundamental results from [H-K] and [Sta$_7$].

Let $R = \bigoplus_{n \geq 0} R_n$ be a Cohen-Macaulay graded ring defined over a field $R_0 = k$, and K_R the canonical module of R. Then the Poincaré series $F_{K_R}(\lambda)$ of K_R coincides with $(-1)^d F_R(\lambda^{-1})$, where $d = \dim(R)$.

If R is Gorenstein, then $F_R(\lambda^{-1}) = (-1)^d \lambda^{-a} F_R(\lambda)$ for some $a \in \mathbb{Z}$. Moreover, if R is a Cohen-Macaulay integral domain, then R is Gorenstein if and only if $F_R(\lambda^{-1}) = (-1)^d \lambda^{-a} F_R(\lambda)$ for some $a \in \mathbb{Z}$.

If R is a Cohen-Macaulay integral domain, then the canonical module K_R of R is isomorphic to a graded ideal I of R as graded R-modules up to shift in grading. In this case, if $I \neq R$, then R/I is Gorenstein and $\dim(R/I) = \dim(R) - 1$.

(1.7) LEMMA. *Let $R = \bigoplus_{n \geq 0} R_n$ be a Cohen-Macaulay graded domain defined over a field $R_0 = k$ with $\dim(R) = d$. Assume that I is a graded ideal of R which satisfies the following conditions:*

(i) *$F_R(\lambda^{-1}) = (-1)^d \lambda^{-a} F_I(\lambda)$ for some $a \in \mathbb{Z}$.*

(ii) *R/I is Cohen-Macaulay and $\dim(R/I) = d - 1$.*

Then the canonical module K_R of R is isomorphic to I as graded R-modules up to shift in grading.

Proof. Since

$$(*) \qquad 0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

is an exact sequence of graded R-modules, we have the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \underline{\mathrm{Hom}}_R(R/I, K_R) \longrightarrow \underline{\mathrm{Hom}}_R(R, K_R) \longrightarrow \underline{\mathrm{Hom}}_R(I, K_R) \\ &\longrightarrow \underline{\mathrm{Ext}}^1_R(R/I, K_R) \longrightarrow \underline{\mathrm{Ext}}^1_R(R, K_R) \longrightarrow \underline{\mathrm{Ext}}^1_R(I, K_R) \\ &\longrightarrow \underline{\mathrm{Ext}}^2_R(R/I, K_R) \longrightarrow \underline{\mathrm{Ext}}^2_R(R, K_R) \longrightarrow \underline{\mathrm{Ext}}^2_R(I, K_R) \\ &\longrightarrow \underline{\mathrm{Ext}}^3_R(R/I, K_R) \longrightarrow \cdots. \end{aligned}$$

On the other hand, $\underline{\mathrm{Hom}}_R(R/I, K_R) = 0$, $\underline{\mathrm{Ext}}^1_R(R/I, K_R) \simeq K_{R/I}$, $\underline{\mathrm{Hom}}_R(R, K_R) \simeq K_R$ and $\underline{\mathrm{Ext}}^1_R(R, K_R) = 0$ by (ii), see [G-W, (2.1.6)] and [G-W, (2.2.9)]. Thus we have the exact sequence

$$(**) \qquad 0 \longrightarrow K_R \longrightarrow \underline{\mathrm{Hom}}_R(I, K_R) \longrightarrow K_{R/I} \longrightarrow 0.$$

By (i) we have the equality $F_{K_R}(\lambda)$ $(= (-1)^d F_R(\lambda^{-1})) = \lambda^{-a} F_I(\lambda)$, and by the same method as in the proof of (1.5) we can check $F_{K_{R/I}}(\lambda)$ $(= (-1)^{d-1} F_{R/I}(\lambda^{-1})) = \lambda^{-a} F_{R/I}(\lambda)$. Hence, thanks to (*) and (**), we obtain

$$F_{\underline{\mathrm{Hom}}_R(I, K_R(-a))}(\lambda) = F_R(\lambda),$$

where $K_R(-a)$ is a shift in grading of K_R. Thus there exists a degree

preserving R-homomorphism $\varphi\colon I \to K_R(-a)$. Since R is an integral domain and $K_R(-a)$ is a fractional ideal of R, the map φ must be a multiplication by a homogeneous element of degree zero of the quotient field of R, hence φ is injective. Thus $I \simeq K_R(-a)$ since $F_I(\lambda) = \lambda^a F_{K_R}(\lambda) = F_{K_R(-a)}(\lambda)$.
Q.E.D.

The above result (1.7) is false if we drop the assumption that R is an integral domain. For example, let A be the polynomial ring $k[X_1, X_2, X_3, X_4, X_5]$ with the natural grading, i.e., $\deg(X_i) = 1$, R the quotient ring $A/(X_1X_2, X_3X_4)$ of A and I the ideal (X_1, X_2X_5) of R. Then R is reduced, $K_R \simeq X_5 \cdot R$, and $R/I \simeq R/(X_5)$ is Gorenstein, however, $I \not\simeq K_R$.

Let $R = \bigoplus_{n\geq 0} R_n$ be an ASL on a weighted poset (Q, ω) over a field $R_0 = k$. Then, by (ASL-1), the Poincaré series $F_R(\lambda)$ of R coincides with the Poincaré series $F_{(Q,\omega)}(\lambda)$ of (Q, ω). Moreover, if I is a poset ideal of Q, then $F_{I\cdot R}(\lambda) = F^I_{(Q,\omega)}(\lambda)$ by (1.6).

Hence, by virtue of a fundamental theorem of ASL and (1.6), we obtain the following result as a corollary to (1.7).

(1.8) Corollary. *Let (Q, ω) be a Cohen-Macaulay weighted poset which possesses a canonical ideal I, and $R = \bigoplus_{n\geq 0} R_n$ an ASL domain on (Q, ω). Then the canonical module K_R of R is isomorphic to the ideal $I\cdot R$ of R as graded R-modules up to shift in grading.*

This is the reason why we introduce the concept of canonical ideals of Cohen-Macaulay posets.

A weighted poset (Q, ω) is called *weakly Gorenstein* over a field k if Q is Cohen-Macaulay over k and there exists a Gorenstein ASL on (Q, ω) over k. If Q is Gorenstein over k, then (Q, ω) is weakly Gorenstein over k for any weight ω on Q. Also, a weakly Gorenstein weighted poset is automatically numerically Gorenstein. See [H_1], [H_2], [H_6], [H_7] and [Wat] for some results on Gorenstein posets.

A weighted poset (Q, ω) is called *integral* over a field k if there exists an ASL domain on (Q, ω) over k. Refer to [H_2], [H_4], [H_5], [H_8], [H-W] and [Wat] for some information on integral posets.

We close this section with the following

(1.9) Proposition. *Let (Q, ω) be a Cohen-Macaulay weighted poset which possesses a canonical ideal I. If (Q, ω) is integral then the weighted poset $(Q - I, \omega)$ is weakly Gorenstein.*

§ 2. Edge-labelings of partially ordered sets

This section is a fundamental work which is indispensable for tho classification (3.12) of distributive lattices with canonical ideals. More systematic study related with this section will appear in [H_9].

Given a poset P, we write $P^\wedge$ for the poset obtained by adjoining a *new* pair of elements, written as $0^\wedge$ and $1^\wedge$, to P such that $0^\wedge < x < 1^\wedge$ for all $x \in P$. If we only require that $0^\wedge$ or $1^\wedge$ be adjoined, we write $P_{0^\wedge}$ or $P^{1^\wedge}$ respectively. We use the convention that $0^\wedge$ or $1^\wedge$ is *never* an element of P.

The symbol "$<\cdot$" denotes the covering relation, that is to say, $x <\cdot y$ means that $x < y$ and $x < z < y$ for no z. For any poset P, we write $\mathscr{C}(P)$ for its covering relation

$$\mathscr{C}(P) = \{(x, y) \in P \times P;\ x <\cdot y\}.$$

Thus, roughly speaking, $\mathscr{C}(P)$ is the set of "edges" in the Hasse diagram of the poset P.

An *edge-labeling* of P is a map $\delta\colon \mathscr{C}(P) \to \mathbb{N}$. Thus an edge-labeling corresponds to an assignment of non-negative integers to the edges of the Hasse diagram of P. The technique of edge-labelings originated in Stanley's work [Sta_3] and was developed by Björner [Bjö].

An edge-labeling δ of a poset P is called *positive* (resp. *non-zero*) if $\delta(x, y) > 0$ for any (resp. some) $x <\cdot y$ of P.

The edge-labeling which we are interested in is the following

(2.1) Definition. An edge-labeling δ of a poset P is called *path-free* if, for any two unrefinable chains

$$x = x_0 <\cdot x_1 <\cdot \cdots <\cdot x_n = y$$

and

$$x = y_0 <\cdot y_1 <\cdot \cdots <\cdot y_m = y$$

of P combining x with y, we have the equality

$$(***) \qquad \sum_{i=0}^{n-1} \delta(x_i, x_{i+1}) = \sum_{j=0}^{m-1} \delta(y_j, y_{j+1}).$$

Let P be an arbitrary poset. We denote by $\mathscr{D}(P^\wedge)$ the set of path-free edge-labelings of $P^\wedge$. Define the partial order in $\mathscr{D}(P^\wedge)$ as follows: $\delta \le \delta'$ if $\delta(x, y) \le \delta'(x, y)$ for every $x <\cdot y$ of $P^\wedge$. Also, let $\mathscr{D}_*(P^\wedge)$ (resp. $\mathscr{D}_+(P^\wedge)$) be the subposet of $\mathscr{D}(P^\wedge)$ which consists of all path-free positive (resp. non-zero) edge-labelings of $P^\wedge$. Let $\mathscr{M}_*(P^\wedge)$ (resp. $\mathscr{M}_+(P^\wedge)$) be the set of minimal elements of the poset $\mathscr{D}_*(P^\wedge)$ (resp. $\mathscr{D}_+(P^\wedge)$). Though we mainly consider finite posets only in this paper, we here study the infinite poset $\mathscr{D}(P^\wedge)$ exceptionally.

We make $\mathscr{D}(P^\wedge)$ an additive semigroup with identity by $(\delta + \delta')(x, y) := \delta(x, y) + \delta'(x, y)$. Note that if $\delta \leq \delta'$ then the edge-labeling $\delta' - \delta$, which is defined by $(\delta' - \delta)(x, y) := \delta'(x, y) - \delta(x, y)$, is contained in $\mathscr{D}(P^\wedge)$.

On the other hand, we naturally associate $\mathscr{D}(P^\wedge)$ (resp. $\mathscr{D}_*(P^\wedge)$) with the set of solutions in non-negative (resp. positive) integers to the system (***) of linear equations.

(2.2) Example. The set $\mathscr{D}(P^\wedge)$ of

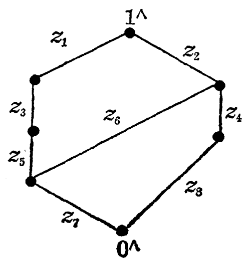
$P^\wedge =$ 

Fig. 4.

corresponds to the set of solutions in non-negative integers to the system

$$\begin{cases} z_1 + z_3 + z_5 = z_2 + z_6 \\ z_4 + z_8 = z_6 + z_7 \end{cases}$$

of linear equations.

For any poset ideal I, including $I = \emptyset$, of P, we denote by δ_I the path-free edge-labeling of $P^\wedge$ defined by

$$\delta_I(x, y) = \begin{cases} 1 & \text{if } x \in I \cup \{0^\wedge\} \text{ and } y \notin I \cup \{0^\wedge\} \\ 0 & \text{otherwise}. \end{cases} \tag{2.3}$$

(2.4) Example. Consider the following poset

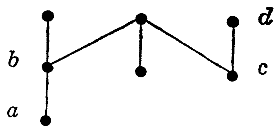
$P =$ 

Fig. 5.

and the poset ideal $I = \{a, b, c, d\}$. Then the path-free edge-labeling δ_I of $P^\wedge$ looks like

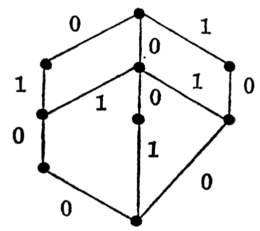


Fig. 6.

Now, it is natural to ask what the set $\mathscr{M}_+(P^\wedge)$ is.

(2.5) PROPOSITION. *A path-free edge-labeling δ of $P^\wedge$ is contained in $\mathscr{M}_+(P^\wedge)$ if and only if $\delta = \delta_I$ for some poset ideal I of P.*

Proof. We easily see that $\delta_I \in \mathscr{M}_+(P^\wedge)$ for any poset ideal I of P. Conversely, let $\delta \in \mathscr{D}_+(P^\wedge)$ and I the poset ideal of P consisting of all elements x of P with the following property: For some (or equivalently, any) unrefinable chain

$$x = x_0 <\cdot\, x_1 <\cdot \cdots <\cdot\, x_n = 1$$

of $P^{1\wedge}$, we have

$$\sum_{i=0}^{n-1} \delta(x_i, x_{i+1}) > 0\,.$$

We claim $\delta_I \leq \delta$. Let $(x, y) \in \mathscr{C}(P^\wedge)$ with $\delta_I(x, y) = 1$ and

$$(0^\wedge \leq) \quad x <\cdot\, y = y_0 <\cdot\, y_1 <\cdot \cdots <\cdot\, y_m = 1^\wedge$$

one of the unrefinable chains of $P^\wedge$ combining x with $1^\wedge$. Then we have the inequailty

$$\delta(x, y) + \sum_{j=0}^{m-1} \delta(y_j, y_{j+1}) > 0$$

since $x \in I \cup \{0^\wedge\}$ and $\delta \in \mathscr{D}_+(P^\wedge)$. On the other hand, the equality $\sum_{0 \leq j \leq m-1} \delta(y_j, y_{j+1}) = 0$ holds since $y \notin I$, thus $\delta(x, y) > 0$ as desired. Q.E.D.

Remark. If I and I', $I \neq I'$, are poset ideals of P, then δ_I and $\delta_{I'}$ are incomparable in the poset $\mathscr{D}(P^\wedge)$.

We now study $\mathscr{D}_*(P^\wedge)$ and $\mathscr{M}_*(P^\wedge)$.

(2.6) PROPOSITION. *$\sharp(\mathcal{M}_*(P^\wedge)) = 1$ if and only if P is pure.*

Proof. To begin with, define $\delta^{[d]} \in \mathcal{D}_*(P^\wedge)$ to be

$$\delta^{[d]}(x, y) = \operatorname{depth}_{P^\wedge}(x) - \operatorname{depth}_{P^\wedge}(y) \tag{2.7}$$

for any $x <\!\cdot\, y$ of $P^\wedge$. We claim $\delta^{[d]} \in \mathcal{M}_*(P^\wedge)$. If

$$0^\wedge = x_0 <\!\cdot\, x_1 <\!\cdot\, \cdots <\!\cdot\, x_r = 1^\wedge \qquad (r = \operatorname{rank}(P^\wedge))$$

is a maximal chain of $P^\wedge$, whose length is equal to $\operatorname{rank}(P^\wedge)$, then $\delta^{[d]}(x_i, x_{i+1}) = 1$ for every i. If $\delta \in \mathcal{D}(P^\wedge)$ and $\delta < \delta^{[d]}$, then we have the inequality

$$\sum_{i=0}^{r-1} \delta(x_i, x_{i+1}) < \sum_{i=0}^{r-1} \delta^{[d]}(x_i, x_{i+1}),$$

thus $\delta(x_i, x_{i+1}) = 0$ for some i, $0 \le i < r$. Hence $\delta \notin \mathcal{D}_*(P^\wedge)$, and therefore $\delta^{[d]} \in \mathcal{M}_*(P^\wedge)$.

Now, the "if" part is easy. In fact, if P is pure then $\delta^{[d]}(x, y) = 1$ for any $x <\!\cdot\, y$ of $P^\wedge$. Obviously, $\delta^{[d]} \le \delta$ for any $\delta \in \mathcal{D}_*(P^\wedge)$. Hence $\mathcal{M}_*(P^\wedge) = \{\delta^{[d]}\}$.

To see why the "only if" part is true, assume that P is not pure. Then $\delta^{[d]}(\alpha, \beta) > 1$ for some $\alpha <\!\cdot\, \beta$ of $P^\wedge$. We consider the map $d'\colon P^\wedge \to \mathbb{N}$ defined by

$$d'(x) = \begin{cases} \operatorname{depth}_{P^\wedge}(x) + \delta^{[d]}(\alpha, \beta) - 1 & \text{if } \alpha \ne x \le \beta \\ \operatorname{depth}_{P^\wedge}(x) & \text{otherwise}. \end{cases}$$

By means of this map d', we define $\delta' \in \mathcal{D}_*(P^\wedge)$ to be $\delta'(x, y) := d'(x) - d'(y)$. Then $\delta'(\alpha, \beta) = 1$, thus $\delta^{[d]} \not\le \delta'$, hence $\sharp(\mathcal{M}_*(P^\wedge)) > 1$. Q.E.D.

From the above construction of $\delta' \in \mathcal{D}_*(P^\wedge)$, we immediately see the following

(2.8) COROLLARY. *For each covering relation $x <\!\cdot\, y$ of $P^\wedge$ there exists $\delta \in \mathcal{M}_*(P^\wedge)$ with $\delta(x, y) = 1$.*

(2.9) DEFINITION. Let P be an arbitrary poset and $\mathcal{I}$ a collection of poset ideals of P. Then $\mathcal{I}$ is called *basic* if the following conditions are satisfied:

(2.10) The empty set $\varnothing$ is contained in $\mathcal{I}$.

(2.11) If I and J are poset ideals such that $I \in \mathcal{I}$ and $J \subset I$, then $J \in \mathcal{I}$.

(2.12) There exists $\delta_* \in \mathscr{D}(P^\wedge)$, called the *shifting* of $\mathscr{I}$, such that $\mathscr{M}_*(P^\wedge) = \{\delta_I + \delta_*; I \in \mathscr{I}\}$.

For which posets P does there exist a basic set $\mathscr{I}$?

(2.13) PROPOSITION. *Let P be an arbitrary poset. Then $\mathscr{I} = \{\emptyset\}$ is a basic set of P if and only if P is pure.*

Proof. Thanks to the proof of (2.6), if P is pure then $\mathscr{M}_*(P^\wedge) = \{\delta^{[d]}\}$, where $\delta^{[d]}$ is the edge-labeling (2.7). Let $\delta_* := \delta^{[d]} - \delta_\phi \in \mathscr{D}(P^\wedge)$. Then $\mathscr{I} = \{\emptyset\}$ is a basic set of P with the shifting δ_*.

On the other hand, if $\mathscr{I} = \{\emptyset\}$ is a basic set of P, then $\sharp(\mathscr{M}_*(P^\wedge)) = 1$, hence P is pure by (2.6). Q.E.D.

(2.14) LEMMA. *Let $\mathscr{I}$ be a basic set of a poset P and $\delta_* \in \mathscr{D}(P^\wedge)$ the shifting of $\mathscr{I}$. Then $\delta_*(x, y) \leq 1$ for each covering relation $x <\cdot\, y$ of $P^\wedge$. Moreover, $\delta_*(x, y) = 1$ if $0^\wedge < x <\cdot\, y$ $(\leq 1^\wedge)$.*

Proof. Thanks to (2.8), $\delta_*(x, y) \leq 1$ for each covering relation $x <\cdot\, y$ of $P^\wedge$. Also, since $\emptyset \in \mathscr{I}$, $\delta_\phi + \delta_*$ must be positive, hence $\delta_*(x, y) > 0$ if $0^\wedge < x <\cdot\, y$ $(\leq 1^\wedge)$. Q.E.D.

By the path-free property of the shifting δ_*, we obtain

(2.15) COROLLARY. *Assume that a poset P possesses a basic set $\mathscr{I}$. Then, for any element α of P, the interval*

$$[\alpha, 1^\wedge) := \{x \in P^{1^\wedge}; \alpha \leq x < 1^\wedge\}$$

of $P^{1^\wedge}$ is pure.

(2.16) LEMMA. *Assume that, for each element α of a poset P, the interval $[\alpha, 1^\wedge)$ of $P^{1^\wedge}$ is pure. Then, for any maximal chain of $P^\wedge$ of the form*

$$0^\wedge = x_0 <\cdot\, x_1 <\cdot \cdots <\cdot\, x_{\mathrm{rank}(P^\wedge)} = 1^\wedge,$$

and for any $\delta \in \mathscr{M}_*(P^\wedge)$, we have $\delta(x_i, x_{i+1}) = 1$ for every $0 \leq i < \mathrm{rank}(P^\wedge)$.

Proof. Let $\delta \in \mathscr{M}_*(P^\wedge)$. We define the map $d_\delta \colon P^\wedge \to \mathbb{N}$ as follows. If $x \in P_{0^\wedge}$ and

$$x = x_0 <\cdot\, x_1 <\cdot \cdots <\cdot\, x_n = 1^\wedge$$

is one of the unrefinable chains of $P^\wedge$ combining x with $1^\wedge$, then

$$(2.17) \qquad d_\delta(x) := \sum_{i=0}^{n-1} \delta(x_i, x_{i+1}),$$

and $d_\delta(\hat{1}) = 0$. Then $d_\delta(x) \geq \mathrm{depth}_{\hat{P}}(x)$ for every $x \in \hat{P}$. To obtain the conclusion, we have only to prove $d_\delta(\hat{0}) = \mathrm{rank}(\hat{P})$.

So, assume that $d_\delta(\hat{0}) > \mathrm{rank}(\hat{P})$. Let $\mathfrak{A}$ be the subset of P consisting of all elements $x \in P$ with $\mathrm{depth}_{\hat{P}}(x) < d_\delta(x)$, and $\mathfrak{B} = P - \mathfrak{A}$. Since the interval $[\alpha, \hat{1})$ of $P^{\hat{1}}$ is pure for any $\alpha \in P$, the subset $\mathfrak{A}$ of P is a poset ideal of P. Also, if $x \in \mathfrak{A} \cup \{\hat{0}\}$, $y \in \mathfrak{B} \cup \{\hat{1}\}$ and $x <\cdot\, y \in \mathscr{C}(\hat{P})$, then $d_\delta(x) - d_\delta(y) > 1$. We now define another map $d^{\#}\colon \hat{P} \to \mathbb{N}$ to be

$$d^{\#}(x) = \begin{cases} d_\delta(x) - 1 & \text{if } x \in \mathfrak{A} \cup \{\hat{0}\} \\ d_\delta(x) & \text{if } x \in \mathfrak{B} \cup \{\hat{1}\}, \end{cases}$$

and, by using this map $d^{\#}$, define $\delta^{\#} \in \mathscr{D}_*(\hat{P})$ to be $\delta^{\#}(x, y) := d^{\#}(x) - d^{\#}(y)$. Then $\delta^{\#} < \delta$ in $\mathscr{D}_*(\hat{P})$, which contradicts $\delta \in \mathscr{M}_*(\hat{P})$. Q.E.D.

(2.18) PROPOSITION. *A poset P possesses a basic set if and only if the following conditions are satisfied:*

(2.19) *For any element α of P, the interval $[\alpha, \hat{1})$ of $P^{\hat{1}}$ is pure.*

(2.20) *The inequality* $\mathrm{rank}(\hat{P}) - \mathrm{depth}_{\hat{P}}(\beta) \leq 2$ *holds for any element $\beta \in P$ with $\hat{0} <\cdot\, \beta$ in $\hat{P}$.*

Proof. First, we shall prove the "only if" part. Thanks to (2.15), the condition (2.19) holds. Let δ_* be the shifting of a basic set $\mathscr{I}$ of P and $\delta^{[d]} \in \mathscr{M}_*(\hat{P})$ the edge-labeling (2.7). Then $\delta^{[d]} = \delta_I + \delta_*$ for some $I \in \mathscr{I}$, hence

$$\begin{aligned} \mathrm{rank}(\hat{P}) - \mathrm{depth}_{\hat{P}}(\beta) &= \delta^{[d]}(\hat{0}, \beta) \\ &= \delta_I(\hat{0}, \beta) + \delta_*(\hat{0}, \beta) \leq 2 \end{aligned}$$

by (2.14) if $\hat{0} <\cdot\, \beta$ in $\hat{P}$.

Conversely, to prove the "if" part, let $\mathfrak{C}$ be the set of minimal elements of P and, for $i = 1, 2$,

$$(2.21) \qquad \mathfrak{C}_i = \{x \in \mathfrak{C};\ \mathrm{rank}(\hat{P}) - \mathrm{depth}_{\hat{P}}(x) = i\}.$$

Let $\mathscr{I}$ be the set of poset ideals I of P with $I \cap \mathfrak{C}_1 = \varnothing$. Also, let $\delta_* \in \mathscr{D}(\hat{P})$ be the edge-labeling defined by

$$\delta_*(x, y) = \begin{cases} 0 & \text{if } x = \hat{0} \text{ and } y \in \mathfrak{C}_1 \\ 1 & \text{otherwise}. \end{cases}$$

We claim $\mathscr{I}$ is a basic set of P with the shifting δ_*.

Let $\delta \in \mathscr{M}_*(P^\wedge)$. Then $\delta - \delta_* \in \mathscr{D}_+(P^\wedge)$. Hence, thanks to (2.5), $\delta_I \leq \delta - \delta_*$ for some poset ideal I of P. If $x_1 \in \mathfrak{C}_1$ and

$$0^\wedge = x_0 <\cdot\, x_1 <\cdot \cdots <\cdot\, x_{\mathrm{rank}(P^\wedge)} = 1^\wedge$$

is a maximal chain of $P^\wedge$, then, by (2.16), $\delta(x_i, x_{i+1}) = 1$ for every i, $0 \leq i < \mathrm{rank}(P^\wedge)$. Thus $(\delta - \delta_*)(x_i, x_{i+1}) = 0$ if $i \geq 1$. So, $\delta_I(x_i, x_{i+1}) = 0$ if $i \geq 1$, hence $x_1 \notin I$, and therefore $I \in \mathscr{I}$. Now, $\delta_I + \delta_* \in \mathscr{D}_*(P^\wedge)$, $\delta \in \mathscr{M}_*(P^\wedge)$ and $\delta_I + \delta_* \leq \delta$ in $\mathscr{D}_*(P^\wedge)$ together imply $\delta_I + \delta_* = \delta$. Hence $\mathscr{M}_*(P^\wedge) \subset \{\delta_I + \delta_*; I \in \mathscr{I}\}$. On the other hand, $\delta_I + \delta_* \in \mathscr{D}_*(P^\wedge)$ if $I \in \mathscr{I}$. Thus, thanks to the remark after the proof of (2.5), $\mathscr{M}_*(P^\wedge) = \{\delta_I + \delta_*; I \in \mathscr{I}\}$.

Q.E.D.

(2.22) Corollary. *A basic set $\mathscr{I}$ of a poset P is unique if it exists.*

Proof. If P is pure, then $\sharp(\mathscr{M}_*(P^\wedge)) = 1$ by (2.6), hence $\mathscr{I} = \{\varnothing\}$ is a unique basic set of P.

Assume that P is not pure and that P satisfies the conditions (2.19) and (2.20). Let $\mathfrak{C}_i$ $(i = 1, 2)$ be the sets (2.21) and $\mathscr{I}$ a basic set consisting of all poset ideals I of P with $I \cap \mathfrak{C}_1 = \varnothing$. Let $\mathscr{I}'$ be another basic set of P and δ'_* the shifting of $\mathscr{I}'$. If a poset ideal $I' \in \mathscr{I}'$ contains an element $y \in \mathfrak{C}_1$, then $\delta_{I'}(0^\wedge, y) = 0$, thus $\delta'_*(0^\wedge, y) = 1$. Since $\delta'_*(x, y) = 1$ if $0^\wedge < x <\cdot\, y \leq 1^\wedge$ by the latter half of (2.14) and δ'_* is path-free, we have $\delta'_*(0^\wedge, x) = 2$ if $x \in \mathfrak{C}_2$, which contradicts the first half of (2.14). Thus $\mathscr{I}' \subset \mathscr{I}$. Since $\sharp(\mathscr{I}') = \sharp(\mathscr{I}) = \sharp(\mathscr{M}_*(P^\wedge))$, we have $\mathscr{I}' = \mathscr{I}$. Q.E.D.

So, from now on, we call *the* basic set $\mathscr{I}$ of a poset P.

§ 3. Which distributive lattices possess canonical ideals?

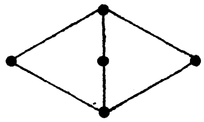
We now consider the problem of finding all distributive lattices which possess canonical ideals.

First, recall the Birkhoff's fundamental structure theorem [Bir, p. 59] for finite distributive lattices.

A lattice L is called *distributive* if the distributive laws

$$\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$$
$$\alpha \vee (\beta \wedge \gamma) = (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$$

hold for all $\alpha, \beta, \gamma \in L$. A lattice L is distributive if and only if L contains neither



nor

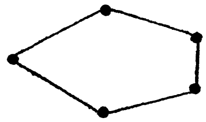


Fig. 7. Fig. 8.

as a sublattice.

Let P be an arbitrary poset and $\mathscr{J}(P)$ the poset consisting of all poset ideals of P, ordered by inclusion. Then it can be checked immediately that $\mathscr{J}(P)$ is a distributive lattice. A classical fundamental structure theorem of Birkhoff guarantees the converse, that is to say, for any finite distributive lattice D, there exists a unique poset P such that $D = \mathscr{J}(P)$.

An element α of a lattice is called *join-irreducible* if $\alpha = \beta \vee \gamma$ implies $\alpha = \beta$ or $\alpha = \gamma$. Let D be a distributive lattice and P the subposet consisting of all join-irreducible elements of D. Then $D = \mathscr{J}(P)$. For example, if

$D =$

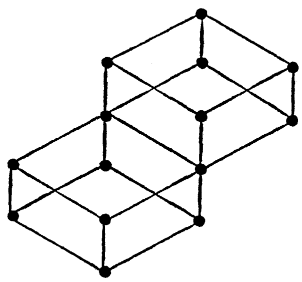


Fig. 9.

then

$P =$

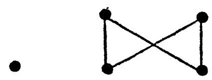


Fig. 10.

Next, let us consider the solutions in non-negative integers to a system of linear equations over $\mathbb{Z}$.

Let $\Phi = (a_{ij})_{\substack{1 \leq i \leq r \\ 1 \leq j \leq n}}$ be an $r \times n$ $\mathbb{Z}$-matrix and

$$E_\Phi := \left\{\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}^n;\ \sum_{j=1}^{n} a_{ij}\beta_j = 0,\ 1 \leqq i \leqq r\right\}$$

the set of solutions in non-negative integers to the system of linear equations $\sum_{j=1}^{n} a_{ij}x_j = 0$ ($i = 1, 2, \cdots, r$) over $\mathbb{Z}$. Clearly, E_Φ is an additive semigroup with identity.

An element $\beta \in E_\Phi$ is called *fundamental* if $\beta = \gamma + \delta$ ($\gamma, \delta \in E_\Phi$) implies $\gamma = \beta$ or $\delta = \beta$. We denote by FUND_Φ the set of non-zero fundamental elements of E_Φ. It is a consequence in classical invariant theory that there are only finitely many non-zero fundamental elements of E_Φ. Thus E_Φ is finitely generated as an additive semigroup, in other words, E_Φ is an affine semigroup. Consult [Sta_2], [Sta_4], [Sta_6] and [Sta_{10}] for further information.

Let k be a field and $k[X_1, X_2, \cdots, X_n]$ the polynomial ring in n-variables over k and $R_\Phi := k[E_\Phi]$ the affine semigroup ring

$$k[X^\beta, \beta \in E_\Phi] \qquad (\subset k[X_1, X_2, \cdots, X_n])$$

of E_Φ over k, where $X^\beta = X_1^{\beta_1} X_2^{\beta_2} \cdots X_n^{\beta_n}$ if $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$. By virtue of [$\mathrm{Hoc}_1$], R_Φ is Cohen-Macaulay. Note that R_Φ is generated by $\{X^\beta; \beta \in \mathrm{FUND}_\Phi\}$ as a k-algebra. We call an element $\beta = (\beta_1, \beta_2, \cdots, \beta_n) \in E_\Phi$ *positive* if $\beta_i > 0$ for every $1 \leq i \leq n$. Let E_Φ^* be the set of positive elements of E_Φ and

$$k[E_\Phi^*] := k[X^\beta; \beta \in E_\Phi^*],$$

which is an ideal of R_Φ. Without loss of generality, we may assume that the set E_Φ^* is non-empty.

Assume, for the moment, that R_Φ is endowed a structure of a graded ring $\bigoplus_{n \geq 0} (R_\Phi)_n$ over $(R_\Phi)_0 = k$ such that each monomial X^β, $\beta \in E_\Phi$, is contained in $(R_\Phi)_n$ for some n $(= n_\beta) > 0$. Let $F_{R_\Phi}(\lambda)$ (resp. $F_{k[E_\Phi^*]}(\lambda)$) be the Poincaré series of the graded ring R_Φ (resp. the graded ideal $k[E_\Phi^*]$ of R_Φ). Then

(3.1) LEMMA. $F_{R_\Phi}(\lambda^{-1}) = (-1)^d F_{k[E_\Phi^*]}(\lambda)$, *where* $d = \dim(R_\Phi)$.

Proof. Consult [Sta_2, (23)] and [Sta_2, (26)]. Q.E.D.

(3.2) COROLLARY. *The canonical module* K_{R_Φ} *of* $R_\Phi = \bigoplus_{n \geq 0} (R_\Phi)_n$ *coincides with* $k[E_\Phi^*]$.

Proof. Consult [Sta_7, (6.7)]. Q.E.D.

(3.3) EXAMPLE. Let $\Phi = [1\ 1\ -2]$, so we study the solutions in non-negative integers to the linear equation $x + y - 2z = 0$ over $\mathbb{Z}$. Then $\mathrm{FUND}_\Phi = \{(2, 0, 1), (0, 2, 1), (1, 1, 1)\}$ and $R_\Phi = k[X^2Z, Y^2Z, XYZ]$. Then R_Φ is considered as a graded ring $\bigoplus_{n \geq 0} (R_\Phi)_n$ with $\deg(X^2Z) = p$, $\deg(Y^2Z) = q$ and $\deg(XYZ) = r$ if and only if $p + q = 2r$. Under the assumption

$p + q = 2r$, we have

$$F_{R_\Phi}(\lambda) = \frac{1 + \lambda^r}{(1 - \lambda^p)(1 - \lambda^q)}, \qquad F_{k[E_\Phi^*]}(\lambda) = \frac{\lambda^r(1 + \lambda^r)}{(1 - \lambda^p)(1 - \lambda^q)}.$$

Thus $F_{R_\Phi}(\lambda^{-1}) = F_{k[E_\Phi^*]}(\lambda)$.

Let $D = \mathscr{J}(P)$ be a distributive lattice and $\mathscr{D}(P^\wedge)$ the affine semigroup considered in the previous section. Also, let k be a field and $k[\mathscr{D}(P^\wedge)]$ the affime semigroup ring of $\mathscr{D}(P^\wedge)$ over k.

(3.4) LEMMA. *If we embed $D = \mathscr{J}(P)$ into $k[\mathscr{D}(P^\wedge)]$ by the injective map $\psi: D\ (= \mathscr{J}(P)) \to k[\mathscr{D}(P^\wedge)]$ defined by*

$$\psi(I) = X^{\delta_I} \in k[\mathscr{D}(P^\wedge)] \qquad (I \in \mathscr{J}(P)),$$

then $k[\mathscr{D}(P^\wedge)]$ is an ASL on D over k.

Proof. By means of the map d_δ defined in (2.17), it is easy to see that the affine semigroup $\mathscr{D}(P^\wedge)$ is isomorphic to $\mathscr{S}(D)$ of [H_2, (3.2)]. Hence the conclusion follows from [H_2] immediately. Also, see [Gar]. Q.E.D.

Note that

$$\delta_I + \delta_J = \delta_{I\cap J} + \delta_{I\cup J} \tag{3.5}$$

for all $I, J \in \mathscr{J}(P)$, and that

$$k[\mathscr{D}(P^\wedge)] \simeq k[X_\alpha;\ \alpha \in D]/(X_\alpha X_\beta - X_{\alpha\wedge\beta}X_{\alpha\vee\beta};\ \alpha \not\sim \beta). \tag{3.6}$$

Hence,

(3.7) COROLLARY. *Let $D = \mathscr{J}(P)$ be a distributive lattice and ω a weight on D. Then the k-algebra $k[\mathscr{D}(P^\wedge)]$ is an ASL on the weighted poset (D, ω), with respect to the embedding ψ, over k if and only if ω satisfies the equality*

$$\omega(\alpha) + \omega(\beta) = \omega(\alpha \wedge \beta) + \omega(\alpha \vee \beta) \tag{3.8}$$

for any $\alpha, \beta \in D$.

Before studying the k-algebra $k[\mathscr{D}(P^\wedge)]$ further, we recall the concept of "wonderful posets".

A poset Q is called *wonderful* (or locally semimodular) if the following condition holds in the poset $Q^\wedge$: If $y_1, y_2 < z$ are covers of an element x, then there is an element $y \leq z$ which is a cover of both y_1 and y_2.

(3.9) LEMMA. *A wonderful poset is Cohen-Macaulay over an arbitrary field.*

For the proof, consult [D-E-P, (8.1)] and [Bjö, (6.1)]. Also, see [H_3, (4.4)].

Without difficulty, we can check that every distributive lattice is wonderful.

(3.10) LEMMA ([D-E-P, Lemma 8.2]). *Let Q be a wonderful poset and $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ a collection of minimal elements of Q. Define the poset ideal I of Q to be*

$$I = \{x \in Q;\ x \ngeq \alpha_i \text{ for all } i\}.$$

Then the subposet $Q - I$ is wonderful.

Now, our main result is

(3.11) THEOREM. *Let $D = \mathscr{J}(P)$ be a distributive lattice and ω a weight on D satisfying the condition (3.8). Then a poset ideal $\mathscr{I}$ of $\mathscr{J}(P)$ is a canonical ideal of the weighted poset (D, ω) if and only if $\mathscr{I}$ is the basic set of P.*

Proof. First, to prove the "if" part, assume that a poset ideal $\mathscr{I}$ of $\mathscr{J}(P)$ is a basic set of P with the shifting $\delta_* \in \mathscr{D}(P^\wedge)$. Let $\mathscr{I} \cdot k[\mathscr{D}(P^\wedge)]$ be the ideal of $k[\mathscr{D}(P^\wedge)]$ generated by $\{X^{\delta_I};\ I \in \mathscr{I}\}$. Since

$$\mathscr{M}_*(P^\wedge) = \{\delta_I + \delta_*;\ I \in \mathscr{I}\},$$

we have

$$k[\mathscr{D}_*(P^\wedge)] = X^{\delta_*}(\mathscr{I} \cdot k[\mathscr{D}(P^\wedge)]).$$

Hence, if we consider $k[\mathscr{D}(P^\wedge)]$ to be a graded ring $\bigoplus_{n \geq 0} (k[\mathscr{D}(P^\wedge)])_n$ over $(k[\mathscr{D}(P^\wedge)])_0 = k$ with $\deg(X^{\delta_I}) = \omega(\alpha)$, $\alpha(\in D) = I(\in \mathscr{J}(P))$, for any $\alpha \in D$, then

$$F_{k[\mathscr{D}_*(P^\wedge)]}(\lambda) = \lambda^{-a} F_{\mathscr{I} \cdot k[\mathscr{D}(P^\wedge)]}(\lambda),$$

where $a = -\deg(X^{\delta_*})$. Thus, since $k[\mathscr{D}(P^\wedge)]$ is an ASL on (D, ω) over k,

$$(-1)^d F_{(D,\omega)}(\lambda^{-1}) = \lambda^{-a} F^{\mathscr{I}}_{(D,\omega)}(\lambda)$$

by (3.1), where $d = \operatorname{rank}(D) + 1$.

The remains of our work is to prove the subposet $D - \mathscr{I}$ is Cohen-Macaulay with $\operatorname{rank}(D - \mathscr{I}) = \operatorname{rank}(D) - 1$. Let $\mathfrak{C}_i$ $(i = 1, 2)$ be the subsets (2.21) of P. Then, by the latter half of the proof of (2.18), we obtain

$$\mathscr{I} = \{\alpha \in D(=\mathscr{J}(P));\ \alpha \not\geq \{x\} \in \mathscr{J}(P) \text{ for all } x \in \mathfrak{C}_1\}.$$

Hence, we have $\operatorname{rank}(D - \mathscr{I}) = \operatorname{rank}(D) - 1$ and, thanks to (3.10), $D - \mathscr{I}$ is wonderful.

Now, we shall prove the "only if" part. Since $\mathscr{I}$ is a canonical ideal of (D, ω), the ideal $\mathscr{I} \cdot k[\mathscr{D}(P^\wedge)]$ of the ASL $k[\mathscr{D}(P^\wedge)]$ on the weighted poset (D, ω) over k is isomorphic to the canonical module $K_{k[\mathscr{D}(P^\wedge)]} = k[\mathscr{D}_*(P^\wedge)]$ of $k[\mathscr{D}(P^\wedge)]$ as graded $k[\mathscr{D}(P^\wedge)]$-modules up to shift in grading. Let

$$b = \min\{n \in \mathbb{N};\ (k[\mathscr{D}_*(P^\wedge)])_n \neq 0\}$$

and

$$c = \min\{n \in \mathbb{N};\ (\mathscr{I} \cdot k[\mathscr{D}(P^\wedge)])_n \neq 0\}.$$

Also, let f be a homogeneous element of degree $b - c$ of the quotient field of $k[\mathscr{D}(P^\wedge)]$ such that an isomorphism, up to shift in grading, from $\mathscr{I} \cdot k[\mathscr{D}(P^\wedge)]$ to $k[\mathscr{D}_*(P^\wedge)]$ is obtained by the multiplication of f. Let $X^{\delta_{I'}}$ $(I' \in \mathscr{I})$ be a monomial with $\deg(X^{\delta_{I'}}) = c$. Also, let

$$\mathscr{N} = \{\delta \in \mathscr{M}_*(P^\wedge);\ \deg(X^\delta) = b\}.$$

If $f \cdot X^{\delta_{I'}}$ is the linear combination

$$f \cdot X^{\delta_{I'}} = \sum_{\delta \in \mathscr{N}} c_\delta X^\delta \qquad (c_\delta \in k),$$

then, since $\delta_{I'} < \delta$ in $\mathscr{D}(P^\wedge)$ for any $\delta \in \mathscr{N}$, we obtain

$$f = \sum_{\delta \in \mathscr{N}} c_\delta X^{\delta - \delta_{I'}} \in k[\mathscr{D}(P^\wedge)].$$

Take $\delta' \in \mathscr{N}$ with $c_{\delta'} \neq 0$. Since $f \cdot X^{\delta_I} \in k[\mathscr{D}_*(P^\wedge)]$, $X^{\delta' - \delta_{I'}} \cdot X^{\delta_I}$ must be contained in $k[\mathscr{D}_*(P^\wedge)]$ for any $I \in \mathscr{I}$. Thus

$$X^{\delta' - \delta_{I'}}(\mathscr{I} \cdot k[\mathscr{D}(P^\wedge)]) \subset k[\mathscr{D}_*(P^\wedge)].$$

On the other hand, the Poincaré series of the ideal $X^{\delta' - \delta_{I'}}(\mathscr{I} \cdot k[\mathscr{D}(P^\wedge)])$ coincides with that of $k[\mathscr{D}_*(P^\wedge)]$. Hence

$$X^{\delta' - \delta_{I'}}(\mathscr{I} \cdot k[\mathscr{D}(P^\wedge)]) = k[\mathscr{D}_*(P^\wedge)],$$

thus

$$\mathscr{M}_*(P^\wedge) = \{\delta_I + (\delta' - \delta_{I'});\ I \in \mathscr{I}\}.$$

So, $\mathscr{I}$ is the basic set of P with the shifting $\delta_* = \delta' - \delta_{I'}$. Q.E.D.

(3.12) COROLLARY. *Let D be a distributive lattice, P the subposet consisting of all join-irreducible elements of D and ω a weight on D satisfying the condition* (3.8). *Then the weighted poset (D, ω) possesses a canonical ideal if and only if the poset P satisfies the conditions* (2.19) *and* (2.20).

Also, if P satisfies (2.19) *and* (2.20), *then the poset ideal*

$$\mathscr{I} = \left\{ \alpha \in D;\ \begin{array}{l} \alpha \not\geq \beta \text{ for any join-irreducible} \\ \text{element } \beta \text{ of } D \text{ with} \\ \mathrm{rank}\,(P^{\wedge}) - \mathrm{depth}_{P^{\wedge}}(\beta) = 1 \end{array} \right\}$$

of D is a canonical ideal of the weighted poset (D, ω).

Moreover, a canonical ideal $\mathscr{I}$ of (D, ω) is unique if it exists.

We should remark that the above corollary (3.12) is a somewhat surprising generalization of Stanley's famous result [Sta_{12}, Cor. 4.5.17 (b)].

REFERENCES

[Bac] K. Baclawski, Cohen-Macaulay ordered sets, J. Algebra, **63** (1980), 226–258.

[Bir] G. Birkhoff, "Lattice Theory", third ed., Amer. Math. Soc. Colloq. Publ., No. 25, Amer. Math. Soc., Providence, R.I., 1967.

[Bjö] A. Björner, Shellable and Cohen-Macaulay partially ordered sets, Trans. Amer. Math. Soc., **260** (1980), 159–183.

[D-E-P] C. De Concini, D. Eisenbud and C. Procesi, Hodge Algebras, Astérisque, **91** (1982).

[Eis] D. Eisenbud, Introduction to algebras with straightening laws, in "Ring Theory and Algebra III" (B. R. McDonald, ed.), Lect. Notes in Pure and Appl. Math., No. 55, Dekker, New York, 1980, pp. 243–268.

[Fri] O. Frink, Ideals in partially ordered sets, Amer. Math. Monthly, **61** (1954), 223–234.

[Gar] A. Garsia, Combinatorial methods in the theory of Cohen-Macaulay rings, Adv. in Math., **38** (1980), 229–266.

[G-W] S. Goto and K.-i. Watanabe, On graded rings, I, J. Math. Soc. Japan, **30** (1978), 179–213.

[H-K] J. Herzog and E. Kunz, Der Kanonische Modul eines Cohen-Macaulay-Rings, Lect. Notes in Math., No. 238, Springer-Verlag, Berlin/Heidelberg/New York, 1971.

[H_1] T. Hibi, For which finite groups G is the lattice $L(G)$ of subgroups Gorenstein ?, Nagoya Math. J., **105** (1987), 147–151.

[H_2] ——, Distributive lattices, affine semigroup rings and algebras with straightening laws, in "Commutative Algebra and Combinatorics" (M. Nagata and H. Matsumura, eds.), Advanced Studies in Pure Math., Vol. 11, North-Holland, Amsterdam, 1987, pp. 93–109.

[H_3] ——, Union and glueing of a family of Cohen-Macaulay partially ordered sets, Nagoya Math. J., **107** (1987), 91–119.

[H_4] ——, Level rings and algebras with straightening laws, J. Algebra, **117** (1988), 343–362.

[H_5] ——, Classification of integral trees, Order, **3** (1987), 383–389.

[H_6] ——, Plane graphs and Cohen-Macaulay posets, European J. Combin., to appear.

[H_7] ——, A numerical characterization of Gorenstein complexes, submitted.

[H_8] ——, Toroidal posets, preprint.

[H_9] ——, Linear diophantine equations and Stanley's P-partitions, in preparation.

[H_{10}] ——, The Ehrhart polynomial of a convex polytope, in preparation.

[H-W] T. Hibi and K.-i. Watanabe, Study of three-dimensional algebras with straightening laws which are Gorenstein domains I, Hiroshima Math. J., **15** (1985), 27–54.

[Hoc_1] M. Hochster, Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes, Ann. of Math., **96** (1972), 318–337.

[Hoc_2] ——, Cohen-Macaulay rings, combinatorics, and simplicial complexes, in "Ring Theory II" (B. R. McDonald and R. Morris, eds.), Lect. Notes in Pure and Appl. Math., No. 26, Dekker, New York, 1977, pp. 171–223.

[Rei] G. Reisner, Cohen-Macaulay quotients of polynomial rings, Adv. in Math., **21** (1976), 30–49.

[Sta_1] R. Stanley, Ordered structures and partitions, Mem. Amer. Math. Soc., **119** (1972).

[Sta_2] ——, Linear homogeneous diophantine equations and magic labelings of graphs, Duke Math. J., **40** (1973), 607–632.

[Sta_3] ——, Finite lattices and Jordan-Hölder sets, Alg. Univ., **4** (1974), 361–371.

[Sta_4] ——, Combinatorial reciprocity theorems, Adv. in Math., **14** (1974), 194–253.

[Sta_5] ——, The upper bound conjecture and Cohen-Macaulay rings, Stud. Appl. Math., **54** (1975), 135–142.

[Sta_6] ——, Magic labelings of graphs, symmetric magic squares, systems of parameters, and Cohen-Macaulay rings, Duke Math. J., **43** (1976), 511–531.

[Sta_7] ——, Hilbert functions of graded algebras, Adv. in Math., **28** (1978), 57–83.

[Sta_8] ——, Balanced Cohen-Macaulay complexes, Trans. Amer. Math. Soc., **249** (1979), 139–157.

[Sta_9] ——, The number of faces of a simplicial convex polytope, Adv. in Math., **35** (1980), 236–238.

[Sta_{10}] ——, Linear diophantine equations and local cohomology, Invent. Math., **68** (1982), 175–193.

[Sta_{11}] ——, "Combinatorics and Commutative Algebra", Progress in Math., Vol. 41, Birkhäuser, Boston/Basel/Stuttgart, 1983.

[Sta_{12}] ——, "Enumerative Combinatorics, Volume I", Wadsworth, Monterey, Calif., 1986.

[Sta_{13}] ——, On the number of faces of centrally-symmetric simplicial polytopes, Graphs Combin., **3** (1987), 55–66.

[Sto] M. H. Stone, The theory of representations for Boolean algebras, Trans. Amer. Math. Soc., **40** (1936), 37–111.

[Wat] K.-i. Watanabe, Study of algebras with straightening laws of dimension 2, in "Algebraic and Topological Theories—to the memory of Dr. Takehiko Miyata" (M. Nagata et al., eds.), Kinokuniya, Tokyo, 1985, pp. 622–639.

Department of Mathematics
Faculty of Science
Nagoya University
Chikusa-ku, Nagoya 464, Japan