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ON SOME DIMENSION FORMULA FOR AUTOMORPHIC FORMS OF WEIGHT ONE III

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Dedicated to Professor Michio Kuga on his 60th birthday

Let Γ be a fuchsian group of the first kind and assume that Γ does not contain the element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $S_1(\Gamma)$ be the linear space of cusp forms of weight 1 on the group Γ and denote by d_1 the dimension of the space $S_1(\Gamma)$. When the group Γ has a compact fundamental domain, we have obtained the following (Hiramatsu [3]):

$$(*) \qquad d_1 = \frac{1}{2} \operatorname*{Res}_{s=0} \zeta^*(s)\,,$$

where $\zeta^*(s)$ denotes the Selberg type zeta function defined by

$$\zeta^*(s) = \sum_{\alpha=1}^{\infty} \sum_{k=1}^{\infty} \frac{(\operatorname{sgn} \lambda_{0,\alpha})^k \log |\lambda_{0,\alpha}|}{|\lambda_{0,\alpha}^k - \lambda_{0,\alpha}^{-k}|} |\lambda_{0,\alpha}^k + \lambda_{0,\alpha}^{-k}|^{-s}\,.$$

Here, $\lambda_{0,\alpha}$ denotes the eigenvalue ($|\lambda_{0,\alpha}| > 1$) of representative P_α of the primitive hyperbolic conjugacy classes $\{P_\alpha\}$ in Γ.

In this paper we give a formula of the dimension d_1 for a general discontinuous group Γ of finite type such that $\Gamma \not\ni \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, by using the Selberg trace formula (Selberg [5], Kubota [4]).[1)]

The notation used here will generally be those of [1].

§ 1. The Selberg eigenspace $\mathscr{M}(1, -\frac{3}{2})$, Eisenstein series and continuous spectrum

1.1. Let Γ be a fuchsian group of the first kind not containing the element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and suppose that Γ has a non-compact fundamental

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1) For the dimension d_1 in the case of $\Gamma \ni \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, refer to Hiramatsu [2].

domain in the upper half plane S. Let T be the real torus and put $\tilde{S} = S \times T$. Let $L^2(\Gamma\backslash\tilde{S})$ be the space of functions $f(z, \phi)$ on $\tilde{S}$ satisfying the conditions:

1) $f(z, \phi)$ is a measurable function on $\tilde{S}$;
2) $f(g(z, \phi)) = f(z, \phi)$ for $g \in \Gamma$;
3) $\int_{\Gamma\backslash\tilde{S}} |f(z, \phi)|^2 d(z, \phi) < \infty$.

Moreover we denote by $\mathscr{M}_\Gamma(k, \lambda) = \mathscr{M}(k, \lambda)$ the set of functions $f(z, \phi)$ satisfying the following conditions:

(i) $f(z, \phi) \in L^2(\Gamma\backslash\tilde{S})$;
(ii) $\tilde{\Delta} f(z, \phi) = \lambda f(z, \phi)$, $(\partial/\partial\phi) f(z, \phi) = -\sqrt{-1}\, k f(z, \phi)$,

where
$$\tilde{\Delta} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{5}{4}\frac{\partial^2}{\partial \phi^2} + y\frac{\partial}{\partial \phi}\frac{\partial}{\partial x}\,.$$

Then the following equality holds (Hiramatsu [2]):

THEOREM 1. $\mathscr{M}(1, -\frac{3}{2}) = \{e^{-\sqrt{-1}\phi} y^{1/2} F(z) : F(z) \in S_1(\Gamma)\}$; *and hence*

$$d_1 = \dim S_1(\Gamma) = \dim \mathscr{M}\left(1, -\frac{3}{2}\right). \tag{1}$$

1.2. We consider an invariant integral operator on the Selberg eigenspace $\mathscr{M}(k, \lambda)$ defined by a point-pair invariant kernel

$$\omega_\delta(z, \phi; z', \phi') = \left|\frac{(yy')^{1/2}}{(z - \bar{z}')/2\sqrt{-1}}\right|^\delta \frac{(yy')^{1/2}}{(z - \bar{z}')/2\sqrt{-1}} e^{-\sqrt{-1}(\phi - \phi')}\,, \qquad (\delta > 1)\,.$$

Then, the integral operator ω_δ vanishes on $\mathscr{M}(k, \lambda)$ for all $k \neq 1$. It is easy to see that the integral

$$\int_{\tilde{D}} \sum_{M \in \Gamma} \omega_\delta(z, \phi; M(z, \phi)) d(z, \phi) \qquad (\tilde{D} = \Gamma\backslash\tilde{S})$$

is uniformly bounded at a neighborhood of each irregular cusp of Γ. We also see that by the Riemann-Roch theorem, the number of regular cusps of Γ is even. In the following we assume that κ_1, κ_2 is a maximal set of regular cusps of Γ which are not equivalent with respect to Γ. Let Γ_i be the stabilizer in Γ of κ_i, and fix an element $\sigma_i \in SL(2, \boldsymbol{R})$ such that $\sigma_i\infty = \kappa_i$ and such that $\sigma_i^{-1}\Gamma_i\sigma_i$ is equal to the group $\Gamma_0 = \left\{\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \boldsymbol{Z}\right\}$. Then the Eisenstein series attached to the regular cusp κ_i is defined by

$$(2)\qquad E_i(z,\phi;s)=\sum_{\substack{\sigma\in\Gamma_i\backslash\Gamma\\ \sigma_i^{-1}\sigma=\binom{*\ *}{c\ d}}}\frac{y^s}{|cz+d|^{2s}}e^{-\sqrt{-1}(\phi+\arg(cz+d))}\qquad (i=1,2),$$

where $s=t+\sqrt{-1}r$ with $t>1$. It is easy to check that

(i) $E_i(M(z,\phi);s)=E_i(z,\phi;s)$ for $M\in\Gamma$;

(ii) $\tilde{\Delta}E_i(z,\phi;s)=\{s(s-1)-\frac{5}{4}\}E_i(z,\phi;s)$;

(iii) $(\partial/\partial\phi)E_i(z,\phi;s)=-\sqrt{-1}E_i(z,\phi;s)$.

By the above (i), $E_i(z,\phi;s)$ has the Fourier expansion at κ_j in the form

$$E_i(\sigma_j(z,\phi);s)=\sum_{m=-\infty}^{\infty}a_{ij,m}(y,\phi;s)e^{2\pi\sqrt{-1}mx}.$$

The constant term $a_{ij,0}(y,\phi;s)$ is given by

$$\begin{aligned}e^{\sqrt{-1}\phi}a_{ij,0}(y,\phi;s)&=a_{ij,0}(y;s)\\&=\delta_{ij}y^s-\sqrt{-1}\sqrt{\pi}\frac{\Gamma(s)}{\Gamma(s+\frac{1}{2})}\varphi_{ij,0}(s)y^{1-s},\end{aligned}$$

where $\delta_{ij}=1$ or 0 according to $i=j$ or not, and

$$\varphi_{ij,0}(s)=\sum_{c\neq 0}\frac{(\mathrm{sgn}\,c)\cdot N_{ij}(c)}{|c|^{2s}}$$

with $N_{ij}(c)=\#\left\{0\leqq d<|c|:\begin{pmatrix}* & *\\ c & d\end{pmatrix}\in\sigma_i^{-1}\Gamma\sigma_j\right\}$. We put

$$\varphi_{ij}(s)=-\sqrt{-1}\sqrt{\pi}\frac{\Gamma(s)}{\Gamma(s+\frac{1}{2})}\varphi_{ij,0}(s),$$

and $\Phi(s)=(\varphi_{ij}(s))$. Then it is easy to see that the Eisenstein matrix $\Phi(s)$ is a skew-symmetric matrix. The Eisenstein series $E_i(z,\phi;s)$ has no constant terms in its Fourier expansion at any irregular cusp of Γ.

1.3. First we define the compact part of $E_i(z,\phi;s)$ by

$$E_i^Y(z,\phi;s)=\begin{cases}E_i(z,\phi;s)\cdot-a_{ij,0}(\mathrm{Im}(\sigma_j^{-1}z),\phi;s), & \text{if } \mathrm{Im}(\sigma_j^{-1}z)>Y,\\ E_i(z,\phi;s), & \text{otherwise,}\end{cases}$$

where Y denotes a sufficiently large number. Then, the following Maass-Selberg relation of our case may be obtained in a way similar to the proof of Theorem 2.3.2. in Kubota [4]:

$$(3)\qquad \frac{1}{2\pi}(E_i^Y(z,\phi;s),E_i^Y(z,\phi;\bar{s}'))=\frac{Y^{s+s'-1}-\varphi_{ij}(s)\overline{\varphi_{ij}(\bar{s}')}Y^{-s-s'+1}}{s+s'-1}\qquad (i\neq j).$$

We also see that the Eisenstein matrix $\Phi(s)$ converges to a unique unitary matrix $\Phi(s_0)$ when s tends to a point $s_0 = \frac{1}{2} + \sqrt{-1}r_0$. Therefore we have

$$\begin{aligned}\Phi(s_0)\Phi(1-s_0) &= \Phi(s_0)\Phi(\bar{s}_0) = -\ \Phi(s_0)\overline{\Phi(s_0)} \\ &= \Phi(s_0)^t\overline{\Phi(s_0)} = I\,;\end{aligned}$$

and hence each $E_i(z, \phi; s)$ has a meromorphic continuation to the whole s-plane, and the column vector $\mathscr{E}(z, \phi; s) = {}^t(E_1, E_2)$ satisfies the functional equation

$$\mathscr{E}(z, \phi; s) = \Phi(s)\mathscr{E}(z, \phi; 1-s)\,.$$

Since Γ is a discontinuous group of finite type, the integral operator defined by ω_δ is not generally completely continuous on $L^2(\Gamma\backslash\tilde{S})$ and the space $L^2(\Gamma\backslash\tilde{S})$ has the following spectral decomposition

$$L^2(\Gamma\backslash\tilde{S}) = L_0^2(\Gamma\backslash\tilde{S}) \oplus L_{sp}^2(\Gamma\backslash\tilde{S}) \oplus L_{\mathrm{cont}}^2(\Gamma\backslash\tilde{S})\,,$$

where L_0^2 is the space of non-analytic cusp forms, L_{sp}^2 is the discrete part of the orthogonal complement of L_0^2 and L_{cont}^2 is continuous part of the spectra. By using the meromorphic continuation of the Eisenstein series $E_i(z, \phi; s)$ defined by (2), we put

$$\tilde{H}_\delta(z, \phi; z', \phi') = \frac{1}{8\pi^2}\sum_{i=1}^{2}\int_{-\infty}^{\infty} h(r)E_i(z, \phi; \tfrac{1}{2} + \sqrt{-1}r)\overline{E_i(z', \phi'; \tfrac{1}{2} + \sqrt{-1}r)}\,dr\,.$$

Here $h(r)$ denotes the eigenvalue of ω_δ in $\mathscr{M}(1, \lambda)$ which is given by

$$(4)\qquad h(r) = 2^{2+\delta}\pi\frac{\Gamma(1/2)\Gamma((1+\delta)/2)}{\Gamma(\delta)\Gamma(1+\delta/2)}\Gamma\left(\frac{\delta}{2} + \sqrt{-1}r\right)\Gamma\left(\frac{\delta}{2} - \sqrt{-1}r\right)$$

with $\lambda = s(s-1) - \frac{5}{4}$ and $s = \frac{1}{2} + \sqrt{-1}r$. We put

$$K_\delta(z, \phi; z', \phi') = \sum_{M\in\Gamma}\omega_\delta(z, \phi; M(z', \phi'))\,,$$

and

$$\tilde{K}_\delta = K_\delta - \tilde{H}_\delta\,.$$

The integral operator $\tilde{K}_\delta$ is uniformly bounded at a neighborhood of each irregular cusp of Γ. Therefore we may assume that κ_1, κ_2 is a maximal set of cusps of Γ. Then the integral operator $\tilde{K}_\delta$ is complete continuous on $L^2(\Gamma\backslash\tilde{S})$ and has all discrete spectra of K_δ. Furthermore, an eigenvalue of $f(z, \phi)$ in $L_0^2(\Gamma\backslash\tilde{S}) \oplus L_{sp}^2(\Gamma\backslash\tilde{S})$ for $\tilde{K}_\delta$ is equal to that for K_δ and the image of $\tilde{K}_\delta$ on it is contained in $L_0^2(\Gamma\backslash\tilde{S})$. Considering the trace of $\tilde{K}_\delta$ on $L_0^2(\Gamma\backslash\tilde{S})$, we now obtain the following modified trace formula (Selberg [5]):

$$\sum_{n=1}^{\infty} h(\lambda^{(n)}) = \int_{\tilde{D}} \tilde{K}_\delta(z, \phi; z, \phi) d(z, \phi)$$
$$= \int_{\tilde{D}} \{\sum_{M\in\Gamma} \omega_\delta(z, \phi; M(z, \phi)) - \tilde{H}_\delta(z, \phi; z, \phi)\} d(z, \phi) ,$$

where each of $\lambda^{(n)}$ denotes an eigenvalue corresponding to an orthogonal basis $\{f^{(n)}\}$ for $L_0^2(\Gamma\backslash\tilde{S})$.

§ 2. A formula for the dimension d_1

2.1. We put

$$\int_{\tilde{D}} \{\sum_{M\in\Gamma} \omega_\delta(z, \phi; M(z, \phi)) - \tilde{H}_\delta(z, \phi; z, \phi)\} d(z, \phi)$$
$$= J(I) + J(P) + J(R) + J(\infty) ,$$

where $J(I)$, $J(P)$, $J(R)$, and $J(\infty)$ denote respectively the identity component, the hyperbolic component, the elliptic component, and the parabolic component of the traces. Then the components $J(I)$, $J(P)$ and $J(R)$ were obtained already in [3] and in the following we shall calculate the component $J(\infty)$. Let $\tilde{D}_i$ be a fundamental domain of the stabilizer Γ_i of cusp κ_i in Γ and denote by σ_i a linear transformation such that $\sigma_i^{-1}\Gamma_i\sigma_i = \Gamma_0$. Then we have

$$J(\infty) = \lim_{Y\to\infty} \left\{\sum_{i=1}^{2}\int_{\tilde{D}_i^Y} \sum_{\substack{M\in\Gamma_i\\ M\neq 1}} \omega_\delta(z, \phi; M(z, \phi)) d(z, \phi) - \int_{\tilde{D}_Y} \tilde{H}_\delta(z, \phi; z, \phi) d(z, \phi)\right\} ,$$

where $\tilde{D}_i^Y$ denotes the domain consisting of all points (z, ϕ) in $\tilde{D}_i$ such that $\mathrm{Im}\,(\sigma_i^{-1}z) < Y$, and $\tilde{D}_Y$ the domain consisting of all $(z, \phi) \in \tilde{D} = \Gamma\backslash\tilde{S}$ such that $\mathrm{Im}\,(\sigma_i^{-1}z) < Y$ for $i = 1, 2$.

Making use of a summation formula due to Euler-MacLaurin, we have for the first half of $J(\infty)$ (c.f. [2]),

$$\int_{\tilde{D}_i^Y} \sum_{\substack{M\in\Gamma_i\\ M\neq 1}} \omega_\delta(z, \phi; M(z, \phi)) d(z, \phi) = 2^2\pi \frac{\Gamma(1/2)\Gamma((\delta+1)/2)}{\Gamma(1+\delta/2)} \log Y + \alpha(\delta) + o(1)$$

as $Y \to \infty$, where $\alpha(\delta)$ denotes a function of δ such that $\lim_{\delta\to 0} \delta\alpha(\delta) = 0$. For the second half of $J(\infty)$, we have the following by the Maass-Selberg relation (3):

$$\frac{1}{8\pi^2}\int_{\tilde{D}_Y}\int_{-\infty}^{\infty} h(r)E_i(z,\phi;\tfrac{1}{2}+\sqrt{-1}r)\overline{E_i(z,\phi;\tfrac{1}{2}+\sqrt{-1}r)}drd(z,\phi)$$

$$=\frac{1}{8\pi^2}\lim_{t\to 1/2}\int_{\tilde{D}}\int_{-\infty}^{\infty} h(r)E_i^Y(z,\phi;t+\sqrt{-1}r)\overline{E_i^Y(z,\phi;t+\sqrt{-1}r)}drd(z,\phi)+o(1)$$

$$(\text{as } Y\to\infty)$$

$$=\frac{1}{4\pi}\lim_{t\to 1/2}\int_{-\infty}^{\infty} h(r)\frac{Y^{2t-1}-\varphi_{ij}(s)\overline{\varphi_{ij}(s)}Y^{1-2t}}{2t-1}dr+o(1) \qquad (\text{as } Y\longrightarrow\infty)$$

$$=2^2\pi\frac{\Gamma(1/2)\Gamma((\delta+1)/2)}{\Gamma(1+\delta/2)}\log Y$$

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\varphi'_{ij}(\tfrac{1}{2}+\sqrt{-1}r)\overline{\varphi_{ij}(\tfrac{1}{2}+\sqrt{-1}r)}dr+o(1)$$

as $Y\to\infty$ and $t\to\frac{1}{2}$, where $j\neq i$.

2.2. In the following we shall calculate the limit $\lim_{\delta\to 0}\delta J(\infty)$. By the expression (4) of $h(r)$, we have

$$h(r)\sim\frac{c(\delta)|r|^{\delta}}{|r|e^{\pi|r|}}$$

as $r\to\infty$, where $c(\delta)$ is independent of r.
On the other hand, we have $\varphi_{ij}(\frac{1}{2}+\sqrt{-1}r)\varphi_{ij}(\frac{1}{2}-\sqrt{-1}r)=-1$ by (3). Therefore

$$\frac{\varphi'_{ij}(\frac{1}{2}+\sqrt{-1}r)}{\varphi_{ij}(\frac{1}{2}+\sqrt{-1}r)}=\frac{\varphi'_{ij}(\frac{1}{2}-\sqrt{-1}r)}{\varphi_{ij}(\frac{1}{2}-\sqrt{-1}r)};$$

and hence

$$\int_{-\infty}^{\infty} h(r)\varphi'_{ij}(\tfrac{1}{2}+\sqrt{-1}r)\overline{\varphi_{ij}(\tfrac{1}{2}+\sqrt{-1}r)}dr=\int_{-\infty}^{\infty} h(r)\frac{\varphi'_{ij}}{\varphi_{ij}}(\tfrac{1}{2}+\sqrt{-1}r)dr.$$

Now, since the operator $\tilde{K}_\delta$ is complete continuous on $L^2(\Gamma\backslash\tilde{S})$, we have

$$\lim_{\delta\to+0}\delta\int_{-\infty}^{\infty} h(r)\frac{\varphi'_{ij}}{\varphi_{ij}}(\tfrac{1}{2}+\sqrt{-1}r)dr=0.$$

It is clear that the above result, combined with the formulas (*) and (1) obtained in [3] and [2] respectively, proves the following

$$d_1=\frac{1}{2}\operatorname*{Res}_{s=0}\zeta^*(s).$$

Now our main result can be stated as follows.

THEOREM 2. *Let Γ be a fuchsian group of the first kind not containing* *the element* $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ *and suppose that the number of regular cusps of* Γ *is two. Let* d_1 *be the dimension for the space consisting of cusp forms of weight* 1 *with respect to* Γ. *Then* d_1 *is given by*

$$d_1 = \frac{1}{2} \operatorname*{Res}_{s=0} \zeta^*(s)\,,$$

where $\zeta^*(s)$ *denotes the Selberg type zeta-function defined in the first part of this paper.*

Remark. Let Γ be a general discontinuous group of finite type not containing the element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then, using the properties of the Eisenstein series defined at each regular cusp of Γ, we can prove that the number of regular cusps of Γ is even. We can also prove that in the same way as in the above case, the contribution from parabolic classes to d_1 vanishes.

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