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THE HILBERT SERIES OF RINGS OF MATRIX CONCOMITANTS

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Introduction

Throughout this paper, K will be a field of characteristic zero. Let $K\langle x_1, \cdots, x_m\rangle$ be the K-algebra in m variables $x_1, \cdots, x_m$ and $I_{m,n}$ the T-ideal consisting of all polynomial identities satisfied by m n by n matrices. The ring $R(n, m) = K\langle x_1, \cdots, x_m\rangle/I_{m,n}$ is called the ring of m generic n by n matrices.

This ring can be described as follows. Let $X_1, \cdots, X_m$ be m generic n by n matrices over the field K. That is $X_k = (x_{ij}(k))$, $1 \leq i, j \leq n$, $1 \leq k \leq m$, where the $x_{ij}(k)$ are independent commutative variables over K. Then $R(n, m)$ is the K-algebra generated by $X_1, \cdots, X_m$. We denote by $K[x_{ij}(k)]$, $1 \leq i, k \leq n$, $1\leq k \leq m$, the commutative polynomial ring generated by the entries of generic n by n matrices $X_1, \cdots, X_m$. The subring of $K[x_{ij}(k)]$ generated by all the traces of monomials in $R(n, m)$ is called the ring of invariants of m generic n by n matrices and will be denoted by $C(n, m)$. The subring of $M_n(K[x_{ij}(k)]$ generated by $R(n, m)$ and $C(n, m)$ is called the trace ring of m generic n by n matrices and will be denoted by $T(n, m)$.

The functional equation of the Hilbert series of the ring $T(n, m)$ is proved by Le Bruyn [L1] for $n = 2$ and by Formanek [F2] for $m \geq n^2$. We prove the functional equation in a more general situation (4.3. Theorem).

Our method is as follows. The trace ring $T(n, m)$ is a fixed ring of $GL(n, K)$ and hence the Hilbert series has a integral expression by a classical result of Molien-Weyl. This formula reduce the problem to a problem of relative invariants for a torus group. By using a theorem of Stanley [S], we can prove the desired functional equation.

The rest of this paper was motivated by a result of Le Bruyn [L2], who treats trace ring of 2 by 2 generic matrices and proved, among other things, that the trace ring $T(2, m)$ is a Cohen-Macaulay module over its

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center $C(n, m)$. Giving an explicit form of a homogeneous system of parameters for $C(2, m)$, we show that $T(2, m)$ is a free module of rank $\frac{1}{m-1}\binom{2m-2}{m-1}2^m$ over the polynomial ring $B(2, m)$ generated by elements of the homogeneous system of parameters for $C(2, m)$ (8.2. Theorem). As an example we give an explicit description of $C(2, 4)$ and $T(2, 4)$ (9.1. Theorem).

Procesi [P2] gave an explicit presentation of the Hilbert series of $T(2, m)$ and observed a close relation between the Hilbert series of $T(2, m)$ and that of the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}(2, m)$ (see [L2]). Then 8.2. Theorem together with Procesi's observation above suggest that there is a canonical free basis of $T(2, m)$ over the polynomial ring $B(2, m)$.

§1. Matrix invariants and concomitants

Let G be a classical group in $GL(n, K)$. That is one of the groups,

$$SL(n, K)\,, \quad SO(n, K)\,, \quad Sp(n, K)\,.$$

Let $V(G, m)$ be the vector space $\oplus^m \mathrm{Lie}(G)$, where $\mathrm{Lie}(G)$ denotes the Lie algebra of G. The group G acts rationally on $V(G, m)$ according to the formula:

If $g \in G$, $(A_1, \cdots, A_m) \in V(G, m)$,
then $g(A_1, \cdots, A_m) = (\mathrm{Ad}(g)A_1, \cdots, \mathrm{Ad}(g)A_m)$,
where $\mathrm{Ad}(g)$ denotes the adjoint representation of G.

We denote by $K[V(G, m)]$ the ring of polynomial functions on $V(G, m)$ and by $C(G, m)$ the ring of polynomial G-invariants of $K[V(G, m)]$. Let $K[V(G, m)]_d$ be the K-subspace of $K[V(G, m)]$ consisting of polynomials of multi-degree $d = (d_1, \cdots, d_m) \in N^m$. The rings $K[V(G, m)]$ and $C(n, m)$ are graded rings:

$$K[V(G, m)] = \bigoplus_{d \in N^m} K[V(G, m)]_d\,,$$

and

$$C(G, m) = \bigoplus_{d \in N^m} C(G, m)_d$$

where

$$C(G, m) = K[V(G, m)]_d \cap C(G, m)\,.$$

A polynomial map $f\colon V(G, m) \to \mathrm{Lie}(G)$ is called a polynomial concomitant if that is compatible with the action of G i.e., $f(g\cdot v) = \mathrm{Ad}(g)f(v)$ for any $g \in G$ and $v \in V(G, m)$.

With $T(G, m)$ we will denote the set of polynomial concomitants. Then $T(G, m)$ is a $C(G, m)$-module. Let $P(G, m)$ denote the set of polynomial maps from $V(G, m)$ to $\mathrm{Lie}(G)$ and define the action of G on $P(G, m)$ by

$$(g\cdot f)(v) = \mathrm{Ad}(g)f(g^{-1}v)\,, \qquad \text{if } g \in G,\ f \in P(G, m)\,.$$

Then $T(G, m)$ is the fixed space of $P(G, m)$ under the action of G.

Let $X_1, \cdots, X_m$ be generic matrices in $\mathrm{Lie}(G)$. Then, for each i, X_i is identified with the i-coordinate map

$$(A_1, \cdots, A_m) \longrightarrow A_i\,, \qquad (A_1, \cdots, A_m) \in V(G, m)\,.$$

The following theorem is a direct consequence from some result of Procesi [P1].

1.1. THEOREM. *The ring $C(G, m)$ is generated by factors of polynomials of the form $\mathrm{Tr}(X_{i_1}X_{i_2}\cdots X_{i_j})$, where $X_{i_1}\cdots X_{i_j}$ runs over all possible (non-commutative) monomials in m generic matrices $X_1, \cdots, X_m$ in* $\mathrm{Lie}(G)$.

§ 2. Molien-Weyl formula

Let G be a semi-simple linear algebraic group over the complex number field C and V a G-module. We denote by $K[V]$ the polynomial ring on V. The action of G on the vector space V can be extended on $K[V]$ by a canonical way. Let $K[V]^G$ be the subring of $K[V]$ consisting of G-invariant polynomials. Then $K[V]^G$ is a graded ring:

$$K[V]^G = \bigoplus_{d\in N} K[v]_d^G$$

where $K[v]_d^G$ is the K-vector space of G-invariant polynomials of degree d.

The Hilbert series for the graded ring $K[V]^G$ is the formal power series defined by

$$\chi(K[V]^G, t) = \sum_{d\in N} \dim K[V]_d^G t^d\,.$$

The Molien-Weyl formula gives an integral expression for the Hilbert series $\chi(K[V]^G, t)$.

2.1. PROPOSITION. *Let T be a maximal torus of a maximal compact subgroup K of G. If $|t| < 1$, then*

$$\chi(K[V]^G, t) = \frac{1}{|W|}\int_T \frac{(1-\alpha_1(g))\cdots(1-\alpha_N(g))}{\det(1-tg)}dg$$

where W is the Weyl group of G and $\alpha_1, \cdots, \alpha_N$ is the set of roots of G with respect to T and dg is the normalized Haar-measure on T.

Let $V_1, \cdots, V_m$ be G-modules and set $V = \oplus_{i=1}^m V_i$. Then by defining $\deg t_i$, $1 \leq i \leq m$, *is to be* $(0, \cdots, 1, \cdots, 0)$, *where i-th coordinate is* 1, $K[V]$ *is an N^m-graded ring*

$$K[V] = \bigoplus_{d \in N^m} K[V]_d .$$

Corresponding to this decomposition of $K[V]$, we have

$$K[V]^G = \oplus K[V]_d^G , \qquad K[V]_d^G = K[V]^G \cap K[V]_d .$$

The multi-valued Hilbert series in m variables $\boldsymbol{t} = (t_1, \cdots, t_m)$ is defined by

$$\chi(K[V]^G, \boldsymbol{t}) = \sum_d \dim K[V]_d^G \boldsymbol{t}^d ,$$

where if $d = (d_1, \cdots, d_m) \in N^m$, $\boldsymbol{t}^d = \prod t_i^{d_i}$.

The Molien-Weyl formula in this case is only a slight modification of 2.1. Proposition.

2.2. Proposition. *Notations being as avobe, if $|t_1| < 1, \cdots, |t_m| < 1$,*

$$\chi(K[V]^G, \boldsymbol{t}) = \frac{1}{|W|}\int_T \frac{(1-\alpha_1(g))\cdots(1-\alpha_N(g))}{\prod_i \det(1-t_i g)}dg .$$

2.3. Corollary.

$$\chi(C(G, m), \boldsymbol{t}) = \frac{1}{|W|}\prod_{i=1}^m (1-t_i)^{-r}\int_T \frac{(1-\alpha_1(g))\cdots(1-\alpha_N(g))}{\prod_i \prod_j (1-t_i\alpha_j(g))}dg ,$$

where r = rank of G.

§ 3. Linear diophantine equation

Let $a_1, \cdots, a_m$ and b be fixed column vectors in V, and set

$$E(A, b) = \{x = (x_1, \cdots, x_m) \in N^m,\ a_1x_1 + \cdots + a_mx_m = b\} ,$$

where A is the r by m matrix defined by

$$A = [a_1, \cdots, a_m] .$$

Let $F(A, b, \boldsymbol{t})$ be the formal power series in m variables $\boldsymbol{t} = (t_1, \cdots, t_m)$ defined by

$$F(A, b, \boldsymbol{t}) = \sum_{\alpha \in E(A,b)} \boldsymbol{t}^{\alpha},$$

where if $\alpha = (\alpha_1, \cdots, \alpha_m)$ then $\boldsymbol{t}^{\alpha} = t_1^{\alpha_1} \cdots t_m^{\alpha_m}$.

R. Stanley proved the following

3.1. THEOREM ([S]). *Suppose that the system of linear equations $a_1x_1 + \cdots + a_mx_m = b$ has a rational solution $\alpha = (\alpha_1, \cdots, \alpha_m) \in Q^m$ with $-1 < \alpha_i \leq 0$ for all i and $\mathbf{1} \in E(A, \mathbf{0})$. Then $F(A, b, \boldsymbol{t})$ is a rational function in $\boldsymbol{t} = (t_1, \cdots, t_m)$ which satisfies the functional equation*

$$F(A, b, \boldsymbol{t}^{-1}) = (-1)^d \quad t_1 \cdots t_m F(A, -b, \boldsymbol{t}).$$

where $\boldsymbol{t}^{-1} = (t_1^{-1}, \cdots, t_m^{-1})$.

The next lemma will be used to prove the functional equation of the ring of polynomial concomitants.

3.2. LEMMA ([T1] Lemma 1.1). *If $|t_1| < 1, \cdots, |t_m| < 1$,*

$$F(A, b, \boldsymbol{t}) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^r \int_T \frac{\varepsilon^{-b}}{\prod_{i=1}^{m}(1 - \varepsilon^{a_i} t_i)} \frac{d\varepsilon_1 \cdots d\varepsilon_r}{\varepsilon_1 \cdots \varepsilon_r}$$

where the integral is taken over the r-dimensional torus T and, if $a = (a_1, \cdots, a_r) \in Z^r$, $E^a = \prod_i \varepsilon_i^{a_i}$.

§ 4. The functional equation of the Hilbert series $(T(G, m), t)$

We return to the situation in section 1. Let $X_1, \cdots, X_m$ be generic matrices of Lie(G). Define $\deg X_i$ to be the i-th unit vector $(0, \cdots, 1, \cdots, 0) \in N^m$.

The Hilbert series $\chi(T(G, m), \boldsymbol{t})$ for the N^m-graded module $T(G, m)$ is defined by

$$\chi(T(G, m), \boldsymbol{t}) = \sum_{d \in N^m} \dim T(G, m)_d \boldsymbol{t}^d.$$

Let X_{m+1} be a new generic matrix in Lie(G). Since the trace Tr (X, Y), X, $Y \in$ Lie(G), is a nondegenerate bilinear form on Lie$(G) \times$ Lie(G), it follows that Tr$(X\ X_{m+1})$, $X \in T(G, m)$ defines an injection from $T(G, m)$ onto the subspace of $C(G, m + 1)$ consisting of invariants of degree one in X_{m+1}. Then by 2.3. Corollary we have

4.1. PROPOSITION. *With notations as 2.3. Corollary, the Hilbert series* $\chi(T(G, m), \boldsymbol{t})$ *has the following expression*

$$\chi(T(G, m), \boldsymbol{t}) = \frac{1}{|W|} \prod (1 - t_i)^{-r} \int_T \frac{(r + \sum \alpha_j(g)) \prod (1 - \alpha_j(g))}{\prod_{i,j} (1 - t_i \cdot \alpha_j(g))} dg .$$

By the theorem of Hochster-Roverts [H-R], $C(G, m)$ is a Cohen-Macaulay domain which is Gorenstein. It follows from a theorem of Stanley [S] that the Hilbert series satisfies a functional equation of the form

$$\chi(C(G, m), \boldsymbol{t}^{-1}) = \pm (t_1, \cdots, t_m)^a \chi(C(G, m), \boldsymbol{t}) ,$$

for some $a \in Z$. Here $\boldsymbol{t}^{-1} = (t_1^{-1}, \cdots, t_m^{-1})$.

In our case, we can determine the integer a.

4.2. THEOREM ([T]). *If* $m \geq 2$, *the Hilbert series for the ring* $C(G, m)$ *satisfies the functional equation*

$$\chi(C(G, m), \boldsymbol{t}^{-1}) = (-1)^d (t_1, \cdots, t_m)^a \chi(C(G, m), \boldsymbol{t}) ,$$

where $d = (m - 1) \dim G$ *and* $a = \dim G$.

We prove the same functional equation for $T(G, m)$.

4.3. THEOREM. *With notations as before, if* $m \geq 3$ *then the Hilbert series for* $T(G, m)$ *satisfies the functional equation*

$$\chi(T(G, m), \boldsymbol{t}^{-1}) = (-1)^d (t_1, \cdots, t_m)^a \chi(T(G, m), \boldsymbol{t}) ,$$

where $d = (m - 1) \dim G$ *and* $a = \dim G$.

Proof. The maximal torus of G is isomorphic to the group

$$\begin{pmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_r \end{pmatrix}, \quad |\varepsilon_i| = 1, \ r = \text{rank of } T$$

and every root α_j of G with respect to T can be written as $\alpha_j = \varepsilon^{a_j}$ for some $a_j = (a_{j1}, \cdots, a_{jr}) \in Z^r$, where $\varepsilon^{a_j} = \varepsilon_1^{a_{i1}} \cdots \varepsilon_r^{a_{ir}}$.

By 4.1. Theorem, the Hilbert series $\chi(T(G, m), \boldsymbol{t})$ has the integral expression. We write the numerator

$$\sum_j \varepsilon^{a_j} \prod_j (1 - \varepsilon^{a_j})$$

in the integral as a linear combination of terms of the form ε^{-b}, where $-b$ is a vector in Z^r of the form

$$-b = a_{j_1} + \cdots + a_{j_k} + a_j \qquad (j_1 < j_2 < \cdots < j_k).$$

Then the integral is a linear combination of terms of the form

$$F(b, t) = (2\pi\sqrt{-1}^{-r}) \int_T \frac{\varepsilon^{-b}}{(1 - t_i \varepsilon^{a_j})(1 - t_i)^r} \frac{d\varepsilon_1 \cdots d\varepsilon_r}{\varepsilon_1 \cdots \varepsilon_r}.$$

By 3.2. Lemma, $F(b, t)$ is a Hilbert series associated with a system of linear diophantine equations. If $m \geq 3$, this system of linear equations satisfies the condition of Stenley's theorem (3.1. Theorem) because the vector b is a linear combination of roots a_j with nonnegative integer coefficients c such that $0 \leq c \leq 2$ for all j. Therefore we obtain the desired result because a is a root if and only if $-a$ is a root.

§ 5. The functional equation of trace rings

Let $X_1, \cdots, X_m$ be m generic n by n matrices. According to the decomposition of each matrix variable

$$X_i = \frac{1}{n} \operatorname{Tr}(X_i) + X_i^\circ,$$

where X_i° is a an n by n generic matrix in Lie $(SL(n, K))$, we have

$$T(n, m) = T(SL(n, K), m)[\operatorname{Tr}(X_1), \cdots, \operatorname{Tr}(X_m)] \oplus C(SL(n, k), m).$$

This remark, due to Procesi [P1], enables us translate the structure of the trace ring $T(n, m)$ into that of $T(SL(n, K), m)$.

5.1. Theorem. *If $n \geq 3$, $m \geq 2$ or $n = 2$, $m \geq 3$, the Hilbert series of the trace ring of m generic n by n matrices satisfies the functional equation*

$$\chi(T(n, m), t^{-1}) = (-1)^d (t_1, \cdots, t_m)^{n^2} \chi(T(n, m), t),$$

where $d = (m-1)n^2 + 1$.

Proof. If $m \geq 3$, this is a direct consequence from 4.3. Theorem. If $m = 2$, $n \geq 3$, it is easy to see that the proof of 4.3. Theorem holds good, and we obtain the desired result.

§ 6. Homogeneous coordinate rings of the Grassmannian Gr (2, *m*)

First we recall the definition of the homogeneous coordinate ring of the Grassmannian. Recall that if Ω denotes the set of all one dimensional linear subspaces in the $m-1$ dimensional complex projective space P^{m-1}, we have an explicit embedding $\Omega \to P^N$, where $N = \binom{m}{2} - 1$. Ω is called the Grassmannian and denoted by $\mathrm{Gr}(2, m)$. It is well known that dimension and degree of $\mathrm{Gr}(2, m)$, $m \geq 2$, as a projective variety are $2m-4$ and $\frac{1}{m-1}\binom{2m-4}{m-2}$ respectively.

Let $C[p_{ij}]$, $1 \leq i < j \leq m$, be the polynomial ring in the $\binom{m}{2}$ variables p_{ij}, which coordinatize P^N. Let I be the ideal of $C[p_{ij}]$ generated by all the polynomials of the form

$$p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk}\,, \qquad 1 \leq i < j < k < l \leq m\,.$$

The quotient ring $C[p_{ij}]/I$ is called the homogeneous coordinate ring of $\mathrm{Gr}(2, m)$ and will be denoted by $C[\mathrm{Gr}(2, m)]$. It is convenient to define degree of p_{ij} is to be 2. Let R_{2d} $(d \in N)$ denote the vector space of $C[\mathrm{Gr}(2, m)]$ generated by all homogeneous polynomials of degree $2d$:

$$C[\mathrm{Gr}(2, m)] = \bigoplus_{d \in N} R_{2d}\,.$$

The Hilbert series for the graded ring $C[\mathrm{Gr}(2, m)]$ is calculated by Hilbert [H]:

$$\chi(C[\mathrm{Gr}(2, m)], t) = \sum_{d \in N} \frac{(d+1)(d+m-1)}{(m-1)!\,(m-2)!} \prod_{i=2}^{m-2} (d+1)^2 t^{2d}\,.$$

Set, for $k = 3, 4, \cdots, 2m-1$, $\theta_k = \sum_{i+j=k} p_{ij}$. Then it is well known and can be easily proved that $\theta_3, \cdots, \theta_{2m-1}$ is a homogeneous system of parameters of $C[\mathrm{Gr}(2, m)]$. Since $C[\mathrm{Gr}(2, m)]$ is a Cohen-Macaulay ring and degree of $\mathrm{Gr}(2, m)$ is $\frac{1}{m-1}\binom{2m-4}{m-2}$, we have

6.1. LEMMA. *The homogeneous coordinate ring* $C[\mathrm{Gr}(2, m)]$ *is a free module of rank* $\frac{1}{m-1}\binom{2m-4}{m-2}$ *over the polynomial ring* $C[\theta_3, \cdots, \theta_{2m-1}]$.

We give an integral expression for the Hilbert series of $C[\mathrm{Gr}(2, m)]$.

6.2. LEMMA. *The Hilbert series for the ring* $C[\mathrm{Gr}(2, m)]$ *has the fol-*

lowing integral expression

$$\chi(C[\mathrm{Gr}\,(2, m)], t) = \frac{1}{4\pi\sqrt{-1}} \int_{|\varepsilon|=1} \frac{(1-\varepsilon^2)(1-\varepsilon^{-2})}{(1-\varepsilon t)^m (1-\varepsilon^{-1}t)^m} \frac{d\varepsilon}{\varepsilon} .$$

Proof. Let us consider the polynomial ring $C[x_1, \cdots, x_m, y_1, \cdots, y_m]$ in $2m$ independent variables $x_1, \cdots, x_m, y_1, \cdots, y_m$. The group action of the special linear group $SL(2, C)$ on the polynomial ring is defined by

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} \longrightarrow g \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad g \in SL(2, C), \quad 1 \leq i \leq m .$$

Let R be the ring of invariant polynomials under the action of $SL(2, C)$. Then R is generated by all invariant polynomials of the form

$$a_{ij} = \det \begin{pmatrix} x_i & y_j \\ y_i & y_j \end{pmatrix}, \qquad 1 \leq i < j \leq m ,$$

and the map $\theta_{ij} \rightarrow a_{ij}$ defines a degree preserving ring isomorphism

$$C[\mathrm{Gr}\,(2, m)] \xrightarrow{\sim} R .$$

Then, by the Molien-Weyl formula, we have

$$\chi(R_m, t) = \frac{1}{4\pi\sqrt{-1}} \int_{|\varepsilon|=1} \frac{(1-\varepsilon^2)(1-\varepsilon^{-2})}{(1-\varepsilon t)^m (1-\varepsilon^{-1}t)^m} \frac{d\varepsilon}{\varepsilon}$$

which proves the lemma.

§ 7. Rings of invariants of generic 2 by 2 matrices

Let $X_1, \cdots, X_m$ be m generic 2 by 2 matrices. Let $p_3, \cdots, p_{2m-1}$ be elements of $C(2, m)$ defined by

$$p_k = \sum_{i+j=k} \mathrm{Tr}\,(X_i X_j) , \qquad 3 \leq k \leq 2m - 1 .$$

We denote by $B(2, m)$ the subring of $C(2, m)$ generated by invariants:

$$\mathrm{Tr}\,(X_i) , \quad \mathrm{Tr}\,(X_i^2) , \quad 1 \leq i \leq m , \quad p_3, \cdots, p_{2m-1} .$$

7.1. THEOREM. *Let $C(2, m)$ be the ring of invariants of m generic 2 by 2 matrices. If $m \geq 2$ then $C(2, m)$ is a free module of rank*

$$\frac{1}{m-1} \binom{2m-4}{m-2} 2^{m-2}$$

over the ring $B(2, m)$.

Proof. Let $(A_1, \cdots, A_m)$ be a tuple of 2 by 2 matrices such that any invariant in $\mathrm{Tr}(X_i)$, $\mathrm{Tr}(X_i^2)$, $p_3, \cdots, p_{2m-1}$, $1 \leq i \leq m$ vanishes at $(A_1, \cdots, A_m)$. We first prove by induction on m that any invariant which is not constant vanishes at $(A_1, \cdots, A_m)$. If $A_1 = 0$, then our assertion is obvious by assumption of induction and hence we can assume that A_1 is not zero matrix. Note that $A_1, \cdots, A_m$ are nilpotent matrices since $\mathrm{Tr}(A_i) = \mathrm{Tr}(A_i^2) = 0$ for $i = 1, 2, \cdots, m$. Then by a suitable componentwise adjoint action of the group $GL(2, K)$ on the matrices $A_1, \cdots, A_m$, we can assume that A_1 has the form

$$A_1 = \begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix}, \qquad \text{for some } a_1 = 0\,.$$

In general, let $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ be a nilpotent 2 by 2 matrix which satisfies the equation $\mathrm{Tr}(A_1 B) = 0$. Then we have

$$\mathrm{Tr}\left(\begin{pmatrix} 0 & a_i \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\right) = a_1 b_3 = 0\,,$$

and hence $b_3 = 0$. Since B is a nilpotent matrix, B has the form

$$B = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}.$$

By using this fact and the equation $p_3 = \cdots = p_{2m-1} = 0$ successively, one observe that each matrix A_i has the form

$$A_i = \begin{pmatrix} 0 & a_i \\ 0 & 0 \end{pmatrix}, \qquad 1 \leq i \leq m\,.$$

This implies that $\mathrm{Tr}(A_{i_1}, \cdots, A_{i_k}) = 0$, for any monomial $A_{i_1}, \cdots, A_{i_k}$, and hence any invariant which is not constant vanishes at $(A_1, \cdots, A_m)$. Therefore it follows from a fundamental theorem of Hilbert [H] that $C(2, m)$ is integral over the polynomial ring $B(2, m)$. Since Krull dimension of $C(2, m)$ is $4m - 3$, it follows that $\mathrm{Tr}(X_i)$, $\mathrm{Tr}(X_i^2)$, $p_3, \cdots, p_{2m-1}$ is a homogeneous system of parameters of the ring $C(2, m)$. Then the Cohen-Macaulay property of the ring $C(2, m)$ implies that $C(2, m)$ is a free module over the polynomial ring $B(2, m)$. Then by [T2], rank of

$C(2, m)$ over $B(2, m)$ is $\frac{1}{m-1}\binom{2m-4}{m-2}2^{m-2}$.

§ 8. Trace rings of generic 2 by 2 matrices

We now turn to cosideration of trace rings of generic 2 by 2 matrices. Procesi [P2] proved a one-to-one correspondence between a K-basis of the ring $T(SL(2, K), m)$ and standard Young tableaux of shape $\sigma = 3^a 2^b 1^c$ for all $a, b, c \in N$.

Procesi's theorem in particular gives an explicit presentation of the Hilbert series of the trace ring $T(SL(2, K), m)$

$$(T(SL(2, K), m) = \sum_{a,b,c\in N} L_{a,b,c}\, t^{3a+2b+c}$$

where $L_{a,b,c}$ is the number of standard Young tableaux of shape $3^a 2^b 1^c$ filled with indices from 1 to m.

From this fact Procesi (see [L2]) observed the following proposition and gave an elegant combinatrial proof of the functional equation for the Hilbert series $\chi(T(2, m), t)$. We give here a simple direct proof of Procesi's observation.

8.1. PROPOSITION. *Let $\chi(T(2, m), t)$ be the usual Hilbert series in one variable t for the trace ring $T(2, m)$. Then we have*

$$\chi(T(2, m), t) = (1-t)^{-2m}\chi(C[\mathrm{Gr}\,(2, m)], t)\,.$$

Proof. By the Molien-Weyl formula for the trace ring $T(2, m)$ we have

$$\begin{aligned}\chi(T(2, m), t) &= \frac{1}{4\pi\sqrt{-1}(1-t)^{2m}}\int_{|\varepsilon|=1}\frac{(2+\varepsilon+\varepsilon^{-1})(1-\varepsilon)(1-\varepsilon^{-1})}{(1-t)^m(1-\varepsilon^{-1}t)^m}\frac{d\varepsilon}{\varepsilon}\\ &= (1-t)^{-2m}\chi(C[\mathrm{Gr}\,(2, m)], t)\,,\end{aligned}$$

by 6.1 Lemma.

8.2. COROLLARY.

$$\chi(T(2, m), t) = \frac{1}{(1-t)^{2m}}\sum_d \frac{(d+1)(d+m-1)}{(m-1)!\,(m-2)!}\prod_{i=2}^{m-2}(d+i)^2 t^{2d}\,.$$

The proposition above links the Hilbert series of the trace ring $T(2, m)$ with that of the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}\,(2, m)$.

Le Bruyn [L2] proved that $T(2, m)$ is a Cohen-Macaulay module over the ring $C(2, m)$. Recall that $\mathrm{Tr}(X_i)$, $\mathrm{Tr}(X_i^2)$, $p_3, \cdots, p_{2m-1}$, $1 \leq i \leq m$, is a homogeneous system of parameters of the ring $C(2, m)$. Then the Cohen-Macaulay property of the trace ring $T(2, m)$ says that $T(2, m)$ is a free module over the polynomial ring $B(2, m)$. Therefore we obtain

8.3. THEOREM. *The trace ring* $T(2, m)$ $(m \geq 2)$ *is a free module of rank* $\frac{1}{m-1}\binom{2m-4}{m-2}2^m$ *over the polynomial ring* $B(2, m)$.

Proof. Note that the map $\theta_i \to p_i$ $(3 \leq i \leq 2m-1)$ defines a degree preserving isomorphism

$$K[\theta_3, \cdots, \theta_{2m-1}] \longrightarrow K[p_3, \cdots, p_{2m-1}]\,.$$

Then the theorem follows from 6.1. Lemma and 8.1. Proposition.

The following proposition gives relations in the ring $T(SL(2, K), m)$ corresponding to the Plücker relations

$$p_{i_1i_2}p_{i_3i_4} - p_{i_1i_3}p_{i_2i_4} + p_{i_1i_4}p_{i_2i_3}\,, \qquad 1 \leq i_1 < i_2 < i_3 < i_4 \leq m\,.$$

8.4. PROPOSITION. *Let* X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4} *be 2 by 2 matrices whose traces are all zeros. Then the following relation holds.*

$$\begin{aligned} X_{i_1}X_{i_2}X_{i_3}X_{i_4} &- \mathrm{Tr}(X_{i_1}X_{i_2})X_{i_3}X_{i_4} + \mathrm{Tr}(X_{i_3}X_{i_4})X_{i_1}X_{i_2} \\ &- \mathrm{Tr}(X_{i_1}X_{i_3})X_{i_2}X_{i_4} - \mathrm{Tr}(X_{i_2}X_{i_4})X_{i_1}X_{i_3} + \mathrm{Tr}(X_{i_1}X_{i_4})X_{i_2}X_{i_3} \\ &+ \mathrm{Tr}(X_{i_2}X_{i_3})X_{i_1}X_{i_4} + \frac{1}{2}\{\mathrm{Tr}(X_{i_1}X_{i_2})\,\mathrm{Tr}(X_{i_3}X_{i_4}) \\ &- \mathrm{Tr}(X_{i_1}X_{i_3})\,\mathrm{Tr}(X_{i_2}X_{i_4}) + \mathrm{Tr}(X_{i_1}X_{i_4})\,\mathrm{Tr}(X_{i_2}X_{i_3})\} = 0\,. \end{aligned}$$

Proof. Recall the multi-linear Caylery-Hamilton theorem for 2 by 2 matrices A and B:

$$AB + BA - \mathrm{Tr}(A)B - \mathrm{Tr}(B)A + \mathrm{Tr}(A)\,\mathrm{Tr}(B) - \mathrm{Tr}(AB) = 0\,.$$

Applying the multi-linear Cayley-Hamilton theorem, we have

$$\begin{aligned} X_{i_1}X_{i_2}X_{i_3}X_{i_4} + X_{i_3}X_{i_4}X_{i_1}X_{i_2} - \mathrm{Tr}(X_{i_1}X_{i_2})X_{i_3}X_{i_4} - \mathrm{Tr}(X_{i_3}X_{i_4})X_{i_1}X_{i_2} \\ + \mathrm{Tr}(X_{i_1}X_{i_2})\,\mathrm{Tr}(X_{i_3}X_{i_4}) - \mathrm{Tr}(X_{i_1}X_{i_2}X_{i_3}X_{i_4}) = 0\,, \end{aligned}$$

and

$$\begin{aligned} X_{i_3}X_{i_4}X_{i_1}X_{i_2} = X_{i_1}X_{i_2}X_{i_3}X_{i_4} &+ \mathrm{Tr}(X_{i_1}X_{i_3})X_{i_2}X_{i_4} + \mathrm{Tr}(X_{i_2}X_{i_4})X_{i_1}X_{i_3} \\ &- \mathrm{Tr}(X_{i_1}X_{i_4})X_{i_2}X_{i_3} - \mathrm{Tr}(X_{i_2}X_{i_3})X_{i_1}X_{i_4} \\ &- \mathrm{Tr}(X_{i_1}X_{i_3})\,\mathrm{Tr}(X_{i_2}X_{i_4}) + \mathrm{Tr}(X_{i_1}X_{i_4})\,\mathrm{Tr}(X_{i_2}X_{i_3})\,. \end{aligned}$$

Hence we have

$$(*)\quad 2X_{i_1}X_{i_2}X_{i_3}X_{i_4} - \mathrm{Tr}\,(X_{i_1}X_{i_2})X_{i_3}X_{i_4} + \mathrm{Tr}\,(X_{i_3}X_{i_4})X_{i_1}X_{i_2}$$
$$- \mathrm{Tr}\,(X_{i_1}X_{i_3})X_{i_2}X_{i_4} + \mathrm{Tr}\,(X_{i_2}X_{i_4})X_{i_1}X_{i_3} + \mathrm{Tr}\,(X_{i_1}X_{i_4})X_{i_2}X_{i_3}$$
$$+ \mathrm{Tr}\,(X_{i_1}X_{i_2})\,\mathrm{Tr}\,(X_{i_3}X_{i_4}) - \mathrm{Tr}\,(X_{i_1}X_{i_3})\,\mathrm{Tr}\,(X_{i_2}X_{i_4})$$
$$+ \mathrm{Tr}\,(X_{i_1}X_{i_4})\,\mathrm{Tr}\,(X_{i_2}X_{i_3}) - \mathrm{Tr}\,(X_{i_1}X_{i_2}X_{i_3}X_{i_4}) = 0\,.$$

We claim that

$$2\,\mathrm{Tr}\,(X_{i_1}X_{i_2}X_{i_3}X_{i_4}) = \mathrm{Tr}\,(X_{i_1}X_{i_2})\,\mathrm{Tr}\,(X_{i_3}X_{i_4}) - \mathrm{Tr}\,(X_{i_1}X_{i_3})\,\mathrm{Tr}\,(X_{i_2}X_{i_4})$$
$$+ \mathrm{Tr}\,(X_{i_1}X_{i_4})\,\mathrm{Tr}\,(X_{i_2}X_{i_3})\,.$$

Since both sides of the equation above are linear with respect to matrices $X_{i_1}, \cdots, X_{i_4}$, the claim is true if it is true when each X_i is replaced by one of matrices consisting of a basis of Lie $(SL(2, m))$. This can be easily verified. Then the lemma follows from the relation (*) and the claim.

§ 9. An explicit description of $C(2, 4)$ and $T(2, 4)$

Explicit description of the rings of invariants and the trace rings of two and three generic 2 by 2 matrices are given in [F-H-L], [F1] and [L-V]. They showed:

(1) $C(2, 2) = B(2, 2)$ and $T(2, 2)$ is a free $C(2, 2)$ module with basis 1, X_1, X_2, X_1X_2, (see [F-H-L]).

(2) $C(2, 3)$ is a free $B(2, 3)$ module with basis 1, $\mathrm{Tr}\,(X_1X_2X_3)$ (see [F2]) and $T(2, 3)$ is a free $B(2, 3)$ module with basis 1, X_1, X_2, X_3, X_1X_2, X_1X_3, X_2X_3, $X_1X_2X_3$ (see [L-V]).

In this section we will give an explicit description of the ring of invariants and the trace ring of four generic 2 by 2 matrices.

9.1. THEOREM. (1) *$C(2, 4)$ is a free module over the polynomial ring $B(2, 4)$ with basis* 1, $\mathrm{Tr}\,(X_1X_4)$, $\mathrm{Tr}\,(X_1X_4)^2$, $\mathrm{Tr}\,(X_1X_4)^3$, $\mathrm{Tr}\,(X_1X_2X_3)$, $\mathrm{Tr}\,(X_1X_2X_4)$, $\mathrm{Tr}\,(X_1X_3X_4)$, $\mathrm{Tr}\,(X_2X_3X_4)$.

(2) *$T(2, 4)$ is a free module over the ring $B(2, 4)$ with basis* 1, X_i, X_iX_j, $X_iX_jX_k$, $X_1X_2X_3X_4$, $\mathrm{Tr}\,(X_1X_4)$, $\mathrm{Tr}\,(X_1X_4)X_i$, $\mathrm{Tr}\,(X_1X_4)X_iX_j$, $\mathrm{Tr}\,(X_1X_4)X_iX_jX_k$, $\mathrm{Tr}\,(X_1X_4)X_1X_2X_3X_4$, $1 \le i \le 4$, $1 \le i < j \le 4$, $1 \le i < j < k \le 4$.

Proof. Formanek [F2] calculated the multi-valued Hilbert series:

$$\chi(T(2, 4), t) = \frac{(1 + t)^4(1 + t^2)}{(1 - t)^4(1 - t^2)^9}$$

It is easy to prove (2) by using 8.3. Theorem, 8.4. Proposition and the formula above. The trace map T: $T(2, 4) \to C(2, 4)$ is surjective and hence (1) follows from (2), 7.1. Theorem and the following relation

$$2\,\mathrm{Tr}\,(X_1X_2X_3X_4) = \mathrm{Tr}\,(X_1X_2)\,\mathrm{Tr}\,(X_3X_4) - \mathrm{Tr}\,(X_1X_3)\,\mathrm{Tr}\,(X_2X_4) + \mathrm{Tr}\,(X_1X_4)\,\mathrm{Tr}\,(X_2X_3)\,,$$

where $\mathrm{Tr}\,(X_i) = 0$, for $1 \leq i \leq 4$.

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