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QUASI-ANOSOV DIFFEOMORPHISMS AND PSEUDO-ORBIT TRACING PROPERTY

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Let M be a compact boundaryless C^∞-manifold, and let $\mathrm{Diff}(M)$ be the space of C^1-diffeomorphisms of M endowed with the C^1-topology. An Axiom A diffeomorphism is said to satisfy the *strong transversality condition* if for every $x \in M$, $T_xM = T_xW^s(x) + T_xW^u(x)$. For an Axiom A diffeomorphism, the strong transversality is a sufficient condition to be structurally stable (i.e. there is a neighbourhood $\mathscr{U} \subset \mathrm{Diff}(M)$ of f such that for every $g \in \mathscr{U}$, there is a homeomorphism h on M satisfying $f \circ h = h \circ g$). We say that $f \in \mathrm{Diff}(M)$ is *topologically stable* if for every $\varepsilon > 0$, there is a neighbourhood $\mathscr{U}_\varepsilon$ of f in the set of homeomorphisms of M with the C^0-topology such that for every $g \in \mathscr{U}_\varepsilon$, there is a continuous surjection h on M satisfying $f \circ h = h \circ g$ and $d(h(x), x) < \varepsilon$ for $x \in M$ (here d denotes a metric compatible with the topology of M).

Let g: $X \to X$ be a homeomorphism of a compact metric space (X, d). A sequence of points $\{x_i\}_{i=a}^{b}$ $(-\infty \leqq a < b \leqq \infty)$ in X is called a *δ-pseudo-orbit* of g if $d(g(x_i), x_{i+1}) < \delta$ for $a \leqq i \leqq b-1$. A sequence $\{x_i\}$ is called to be *ε-traced* by $x \in X$ if $d(g^i(x), x_i) < \varepsilon$ holds for $a \leqq i \leqq b$. We say that g has *pseudo-orbit tracing property* (abbrev. POTP) if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit of g can be ε-traced by some point in X. We say that g is *expansive* if there exists $c > 0$ such that $d(g^n(x), g^n(y)) \leqq c$ for every $n \in Z$ implies $x = y$. Such a number c is called an *expansive constant* for g. For the materials of topological dynamics on compact manifolds, see Morimoto [4].

It is well known that every homeomorphism on M with expansivity and POTP is topologically stable, and that every topologically stable homeomorphism on M of dimension $\geqq 2$ has POTP (see [4]). Every Axiom A diffeomorphism f satisfying the strong transversality condition is topologically stable (thus every Anosov diffeomorphism is topologically

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stable) and so f has POTP.

We say that $f \in \mathrm{Diff}(M)$ is *quasi-Anosov* if for every $0 \neq v \in TM$, the set $\{\|(Tf)^n(v)\|: n \in Z\}$ is unbounded. A quasi-Anosov diffeomorphism is equivalent to an Axiom A diffeomorphism satisfying $T_xW^s(x) \cap T_xW^u(x) = \{0_x\}$ for every $x \in M$ ([3]). Obviously every Anosov diffeomorphism is quasi-Anosov and its converse is true if $\dim M = 2$ ([3]). But it is known ([1]) that the converse is not true on a 3-dimensional manifold. Mañé proved the following

THEOREM ([3]). *For $f \in \mathrm{Diff}(M)$ the following conditions are mutually equivalent;*

(i) *f is Anosov,*

(ii) *f is quasi-Anosov and satisfies the strong transversality condition,*

(iii) *f is quasi-Anosov and structurally stable.*

The aim of this note is to prove the following theorem related to the above results.

THEOREM. *Every quasi-Anosov diffeomorphism with POTP must be an Anosov diffeomorphism.*

First of all we prepare a lemma that we need.

LEMMA. *Let M be as before and let $f \in \mathrm{Diff}(M)$ be quasi-Anosov. Then there are an integer $m > 0$ and a neighbourhood $\mathscr{V} \subset \mathrm{Diff}(M)$ of f such that for every $g \in \mathscr{V}$ and every $0 \neq v \in TM$, $\|(Tg)^n(v)\| \geqq 2\|v\|$ for some n with $|n| = m$.*

Proof. Since f is quasi-Anosov, it is easy to see that there is $N > 0$ such that for every $0 \neq v \in TM$, $\|(Tf)^n(v)\| \geqq 3\|v\|$ for some n with $|n| < N$. Thus following the proof of Lemma 2.3 of [2] we see that there is $m > 0$ such that for every $0 \neq v \in TM$, $\|(Tf)^n(v)\| \geqq 3\|v\|$ for some n with $|n| = m$. Thus if we choose a neighbourhood $\mathscr{V} \subset \mathrm{Diff}(M)$ of f such that for every $g \in \mathscr{V}$, $\|(Tg)^n - (Tf)^n\| < 1$ for $|n| = m$, then the conclusion of this lemma is obtained.

Proof of Theorem

Let $f: M \to M$ be a quasi-Anosov diffeomorphism with POTP. If we establish that there is a neighbourhood $\mathscr{U} \subset \mathrm{Diff}(M)$ of f such that every $g \in \mathscr{U}$ has a common expansive constant, then the conclusion of our theorem is easily obtained as follows.

Since f is expansive and has POTP, f is topologically stable. Thus if we choose a small (C^1-) neighbourhood $\mathscr{U}' \subset \mathscr{U}$ of f, then for every $g \in \mathscr{U}'$, there is a continuous surjection $h: M \to M$ with $f \circ h = h \circ g$ and $d(h(x), x) < c/3$ for all $x \in M$. Since an expansive constant for g is the same as that of f, we see that h is injective. This implies that f is structurally stable, and so f is Anosov.

We denote by exp the exponential map from TM to M determined by a Riemannian metric $\|\cdot\|$ on TM. Let $m > 0$ and $\mathscr{V}$ be as in the lemma and put $K = \sup\{\|(Tg)_x\|: x \in M,\ g \in \mathscr{V}\}$. Take and fix $\varepsilon > 0$ such that $\varepsilon(1 + K + \cdots + K^{m-1}) < 1/2$. Then there are $c = c(\varepsilon, f) > 0$ and a neighbourhood $\mathscr{U}(\subset \mathscr{V})$ of f such that for every $g \in \mathscr{U}$,

$$\| \exp^{-1}_{g^\sigma(x)} \circ g^\sigma \circ \exp_x v - (Tg)^\sigma_x(v)\| \leqq \|v\|\varepsilon \qquad (x \in M)$$

if $\|v\| \leqq c$ ($\sigma = \pm 1$). To get the conclusion, it is enough to see that c is a common expansive constant for all $g \in \mathscr{U}$. If this is false, then there exist $x, y \in M$ ($x \neq y$) and $g \in \mathscr{U}$ such that $d(g^n(x), g^n(y)) \leqq c$ for $n \in Z$ (here d is the metric induced by the Riemannian metric). Let $c_1 = \sup\{d(g^n(x), g^n(y)): n \in Z\}$ and take δ with $0 < \delta \leqq c_1/4$. Obviously $c_1 - \delta < d(g^k(x), g^k(y)) \leqq c_1$ for some $k \in Z$. Let $z = g^k(x)$, $w = g^k(y)$ and $v = \exp_z^{-1} w$. Then $c_1 - \delta < \|v\| = d(z, w)$ and $\|(Tg)^n(v)\| \geqq 2\|v\|$ for some n with $|n| = m$. We deal with only the case $\|(Tg)^m(v)\| \geqq 2\|v\|$ (since the case $\|(Tg)^{-m}(v)\| \geqq 2\|v\|$ follows in a similar way). Since $\|v\| = d(z, w) \leqq c$ we have

$$\| \exp^{-1}_{g(z)} \circ g \circ \exp_z v - (Tg)_z(v)\| \leqq \|v\|\varepsilon\,,$$

and so $\|(Tg)_z(v)\| \leqq c_1(1 + \varepsilon)$ (since $\| \exp^{-1}_{g(z)} \circ g \circ \exp_z v\| = d(g(z), g(w)) \leqq c_1$). Moreover

$$\begin{aligned}
&\| \exp^{-1}_{g^2(z)} \circ g^2 \circ \exp_z v - (Tg)^2_z(v)\| \\
&\qquad \leqq \| \exp^{-1}_{g^2(z)} \circ g^2 \circ \exp_z v - (Tg)_{g(z)}(\exp^{-1}_{g(z)} \circ g \circ \exp_z v)\| \\
&\qquad\quad + \|(Tg)_{g(z)}(\exp^{-1}_{g(z)} \circ g \circ \exp_z v) - (Tg)^2_z(v)\| \\
&\qquad \leqq c_1\varepsilon + Kc_1\varepsilon = c_1\varepsilon(1 + K)
\end{aligned}$$

and hence

$$\| \exp^{-1}_{g^2(z)} \circ g^2 \circ \exp_z v\| = d(g^2(z), g^2(w)) \leqq c_1$$

implies

$$\|(Tg)^2_z(v)\| \leqq c_1\{1 + \varepsilon(1 + K)\}\,.$$

By induction we have

$$2\|v\| \leqq \|(Tg)_z^m(v)\| \leqq c_1\{1 + \varepsilon(1 + K + \cdots + K^{m-1})\}.$$

Thus $c_1 - \delta < \|v\| \leqq 3c_1/4$ and we have $c_1/4 < \delta$. This is a contradiction.

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