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THE POSSIBLE COHOMOLOGY OF CERTAIN TYPES OF TAUT SUBMANIFOLDS

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Introduction

The first purpose of this paper is to exhibit several families of compact manifolds that do not admit taut embeddings into any sphere. The second is to enumerate the possible Z_2-cohomology rings of those compact manifolds which do admit a taut embedding and whose cohomology rings satisfy certain degeneracy conditions. The first purpose is easily attained once the second has been accomplished, for it is a simple matter to present families of spaces whose cohomology rings satisfy the required degeneracy conditions, but are not on the list of those admitting a taut embedding.

A key ingredient in the creation of this list (Theorem 9) is Proposition 5. This proposition provides a criterion under which the cohomology ring of a taut submanifold M is expressed as a subring of the cohomology ring of some sphere-bundle P over another submanifold Λ. It turns out that all possible cohomology rings for Λ, and hence for P, (and thus eventually for M), can be determined when M satisfies the degeneracy conditions. Indeed, the determination of the possible cohomology for Λ relies upon Münzner's work on isoparametric hypersurfaces and enters in the proof of Theorem 6.

Recall that an embedding of a compact, connected manifold M into the unit n-dimensional sphere S^n is said to be taut if for every closed metric ball B in S^n, the inclusion map $j: M \cap B \to M$ induces injective homomorphisms $j_*: H_*(M \cap B) \to H_*(M)$ of Čech homology groups with Z_2 coefficients [3], [9].

Every homology and cohomology group occuring in this paper is assumed to have coefficients in the field Z_2.

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§ 1. The multiplicity function and links

Let M be a compact, connected, m-dimensional manifold admitting a taut embedding into some sphere. Throughout this section we will be working with a fixed taut embedding of M into S^n. In order to facilitate the following study of links, it will be convenient to suppose further that this embedding is substantial, which is to say that M is not contained in any round hypersphere of S^n. This presents no lack of generality because every taut submanifold is substantial and taut in the smallest round subsphere containing it. We will also temporarily eliminate from study the cases when M is either a sphere or a point. This also is no important restriction because the taut embeddings of spheres are well-known. They are precisely the round subspheres in S^n. However, this restriction together with substantiality clearly allows us to assume $m < n$. This assumption is implicit in the discussion that follows.

Let N_1 be the total space of the unit normal sphere bundle of the embedding of M into S^n, and let $\pi: N_1 \to M$ be the projection map. The multiplicity function is an upper semi-continuous, positive integer-valued function on N_1 defined by setting $\mathrm{mult}(X)$ equal to the multiplicity of the first focal point along the M-geodesic whose initial tangent vector is $X \in N_1$. This in turn is equal to the multiplicity of the largest eigenvalue of the shape operator A_X. Thus $\mathrm{mult}(X) + \mathrm{mult}(-X) \leq m$, because the multiplicity of the largest eigenvalue of A_{-X} is equal to the multiplicity of the smallest eigenvalue of A_X (and the smallest and largest eigenvalues of A_X are not equal because M is taut and substantial). The element X in N_1 is said to be regular if mult is continuous at X and singular otherwise. The set of regular points is an open and dense subset of N_1 [8].

It is well known that tautness implies that mult can only take values equal to the dimensions in which the homology of M does not vanish. Adding substantiality prevents mult from attaining the value $m = \dim(M)$ by Proposition 3.2 of [8].

Let $C(M)$ be the cut locus of M in S^n. For each $q \in C(M)$, define the link $\Lambda(q) \subset N_1$ to the collection of all initial tangent vectors to the M-geodesics that minimize distance from M to q. In [8], it is proved that the restriction $\pi | \Lambda(q)$ is a homeomorphism of $\Lambda(q)$ onto its image, and that the image $\pi(\Lambda(q))$ is a spherical top-set. Since spherical top-sets are a particular kind of connected critical set of a distance function, Ozawa's Theorem [12] implies that $\pi(\Lambda(q))$ is a smooth taut submanifold in S^n.

Thus $\pi | \Lambda(q)$ gives a taut embedding of $\Lambda(q)$ into S^n. Ozawa's Theorem also implies $\text{mult}(X) = \dim(\Lambda(q))$ for all X in $\Lambda(q)$. This affirmatively answers a question raised in [8].

LEMMA 1. *$(\pi | \Lambda(q))^*: H^*(M) \to H^*(\Lambda(q))$ is a surjective ring homomorphism.*

Proof. The inclusion of a spherical top-set into M induces an injective homomorphism in Z_2-homology [3], [9]. (Singular rather than Čech homology can be used because the spaces in question are manifolds.) Thus $\pi | \Lambda(q)$ induces an injective map in homology. By the dual pairing of cohomology and homology when coefficients are in a field, $\pi | \Lambda(q)$ induces a surjective ring homomorphism in cohomology.

For each $X \in N_1$, let $\xi(X)$ be the unique $q \in C(M)$ such that $X \in \Lambda(q)$. Thus $\xi(X)$ is the cut point along the M-geodesic with initial tangent vector X. Clearly the mapping $\xi: N_1 \to C(M)$ is continuous, onto, and proper (i.e. the inverse images of compacta are compact). The following is also proved in [13].

PROPOSITION 2. *Let $U \subset N_1$ be a connected component of the set of regular points, and set $V = \xi(U) \subset C(M)$. Then V is an open subset of $C(M)$ forming an $(n-k-1)$-dimensional submanifold of S^n, and $\xi: U \to V$ is a locally trivial bundle with fibers which are links diffeomorphic to S^k where k is the constant value of mult on U.*

Proof. Since mult is integer-valued and semi-continuous, it is locally constant where it is continuous. Thus mult has a constant value k on U.

Since N is a compact Hausdorff space, $C(M)$ has the identification topology under ξ. Thus V is open in $C(M)$ because $\xi^{-1}(V) = U$ is open in N_1.

Now $\xi | U$ is a locally trivial bundle with a typical fiber $\Lambda(q)$ for $q \in V$ because it is proper and a smooth submersion. To see it is a submersion, notice that $\xi = \exp \circ \tau$ where $\exp: N \to S^n$ is the exponential map on the normal bundle N of M, and $\tau: N_1 \to Q$ is defined by setting $\tau(X)$ equal to the first focal point along the ray through X in the focal locus $Q \subset N$. Clearly, τ carries U diffeomorphically onto an open subset of the regular focal locus which is a smooth hypersurface in N [6], and exp is a submersion on $\tau(U)$ because the kernel of $\exp_*$ is tangent to Q and of constant dimension k on $\tau(U)$. The rank theorem can then be used to put a smooth $(n-k-1)$-dimensional manifold structure on V.

If $q \in V$ and $X \in \Lambda(q)$, then the cohomology of $\Lambda(q)$ with Z_2 coefficients can be computed by

$$\begin{aligned} H^i(\Lambda(q), X) &\cong \check{H}^i(\Lambda(q), X) \cong H_{n-1-i}(C(M), C(M) - q) \\ &\cong H_{n-1-i}(\boldsymbol{R}^{n-k-1}, \boldsymbol{R}^{n-k-1} - 0) . \end{aligned}$$

Here one uses in succession (1) the identification of singular with Čech cohomology for manifolds, (2) Theorem 1.4 and the following remark of [7], and (3) excision since q has a neighborhood in $C(M)$ homeomorphic to $\boldsymbol{R}^{n-k-1}$. Consequently, $\Lambda(q)$ and S^k have the same Z_2-cohomology. Thus $\Lambda(q)$ is diffeomorphic to S^k because it is well known that a taut submanifold having the cohomology of S^k is diffeomorphic to S^k. This completes the proof that $\xi \mid U$ is a locally trivial bundle with fiber S^k.

COROLLARY 3. *Let* $X \in N_1$, $\operatorname{mult}(X) = k$, *and* $\xi(X) = q$. *Then* X *is regular if and only if* $\Lambda(q)$ *is homeomorphic to* S^k.

Proof. If X is regular, then the result follows from Proposition 2. If X is singular, then there are points of N_1 arbitrarily close to X having multiplicity strictly less than k. Thus by Lemmas 3.4 and 3.5 of [8], $\Lambda(q)$ has a non-zero homology group in some positive dimension less than k. Hence $\Lambda(q)$ is not homeomorphic to S^k.

There is a simple cohomology condition that prevents the existence of certain singular points.

LEMMA 4. *For a given positive integer* k, *suppose that the cup product* $xy = 0$ *whenever* $x \in H^i(M)$, $y \in H^j(M)$ *with* $i, j > 0$ *and* $i + j = k$. *Then there can be no singular* X *in* N_1 *with* $\operatorname{mult}(X) = k$.

Proof. Suppose there is a singular X with $\operatorname{mult}(X) = k$. Let $\Lambda(q)$ be the link containing X. Then $\Lambda(q)$ is a k-dimensional manifold which is not a homology sphere by Corollary 3. By Poincaré duality, there exist cohomology classes $u \in H^i(\Lambda(q))$, $v \in H^j(\Lambda(q))$ with $i + j = k$, $i, j > 0$ such that $uv \neq 0$. By Lemma 1, there exist $x \in H^i(M)$, $y \in H^j(M)$ such that $\pi^*(x) = u$, $\pi^*(y) = v$. Hence $\pi^*(xy) = uv \neq 0$. Thus $xy \neq 0$. This contradicts the hypothesis. Therefore no such X exists.

The mapping which sends $X \in \Lambda(q)$ to $\xi(-X) \in C(M)$ is one-to-one, and therefore is a diffeomorphism of $\Lambda(q)$ onto its image which will be denoted $\Omega(q)$. For if there were $X \neq Y$ in $\Lambda(q)$ with $\xi(-X) = \xi(-Y) = p$, then the extensions of the M-geodesics with initial tangents X and Y in the directions $-X$ and $-Y$ respectively would produce two distinct minimizing geodesics in S^n joining p to q. Thus p and q would be antipodal points in S^n. And so, if r_1 and r_2 were the respective minimal distances from q and p to M, then $r_1 + r_2 = \operatorname{diam}(S^n)$, and M would be contained in the metric sphere of radius r_1 about q (which is also the metric sphere of radius r_2 about p) in contradiction to substantiality.

If for every $X \in \Lambda(q)$, $-X$ is a regular point of N_1, then, by Proposition 2, the set $P = \xi^{-1}(\Omega(q)) = \bigcup\{\Lambda(p): p \in \Omega(q)\}$ has the structure of a sphere bundle over $\Omega(q)$, because it is the restriction to $\Omega(q)$ of a sphere bundle over some space V containing $\Omega(q)$. Thus, under the diffeomorphism of $\Omega(q)$ with $\Lambda(q)$, there is a mapping $\bar{\xi}: P \to \Lambda(q)$ making P into a S^k-bundle for some k. There is a section $\sigma: \Lambda(q) \to P$ given by $\sigma(X) = -X$ for $X \in \Lambda(q)$.

Because $\bar{\xi}$ admits a section, the Gysin sequence decomposes into short exact sequences,

$$\begin{tikzcd} 0 \arrow[r] & H^i(\Lambda(q)) \arrow[r, shift left, "\bar{\xi}^*"] \arrow[d] & H^i(P) \arrow[l, shift left, dashed, "\sigma^*"] \arrow[r, "\delta"] \arrow[d] & H^{i-k}(\Lambda(q)) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^i(*) \arrow[r] & H^i(S^k) \arrow[r] & H^{i-k}(*) \arrow[r] & 0 . \end{tikzcd}$$

The bottom row represents the Gysin sequence of P restricted to a point $* \in \Lambda(q)$. There clearly exists a $u \in H^k(P)$ such that $\delta(u) = 1 \in H^0(\Lambda(q))$ and $\sigma^*(u) = 0$. The above diagram shows that u restricts to the generator of $H^k(S^k)$ for each fiber S^k of P. Thus, by the Leray-Hirsch Theorem, $\bar{\xi}^*: H^*(\Lambda(q)) \to H^*(P)$ turns $H^*(P)$ into a free $H^*(\Lambda(q))$-module with basis $\{1, u\}$. Suppose $u^2 = \bar{\xi}^*(z)u + \bar{\xi}^*(y)$ for some $z, y \in H^*(\Lambda(q))$. Then $0 = \sigma^*(u^2) = \sigma^*(\bar{\xi}^*(z)u + \bar{\xi}^*(y^*)) = y$. Thus $u^2 = \bar{\xi}^*(z)u$. Therefore $H^*(P) \approx H^*(\Lambda(q))[u]/\langle u^2 + zu\rangle$ for some $z \in H^k(\Lambda(q))$. In this representation, $\bar{\xi}^*(x) = x$ and $\sigma^*(xu + y) = y$ for $x, y \in H^*(\Lambda(q))$.

Let $f: P \to M$ be defined by $f(X) = \pi(X)$.

PROPOSITION 5. *Suppose for every* $X \in \Lambda(q)$ *that* $-X$ *is a regular point with* $\operatorname{mult}(-X) = k = m - \dim(\Lambda(q))$. *Then* $\operatorname{degree}(f) \equiv 1 \pmod 2$. *Thus* $f^*: H^*(M) \to H^*(P)$ *is an injective ring homomorphism.*

Proof. For each point $p \in \Omega(q)$, the top-sets $\pi(\Lambda(q))$ and $\pi(\Lambda(p))$ intersect in the single point $\pi(X)$ where X is the unique element of $\Lambda(q)$ such that $\xi(-X) = p$. Clearly $\pi(X) \in \pi(\Lambda(q)) \cap \pi(\Lambda(p))$. Moreover, there is at most one point in the intersection. For if r_1 and r_2 are the respective distances from q and p to M, then, by substantiality, $\mathrm{dist}(p, q) = r_1 + r_2 < \mathrm{diam}(S^n)$. Thus the metric ball of radius r_1 about q (which contains the set $\pi(\Lambda(q))$) intersects the metric ball of radius r_2 about p (which contains $\pi(\Lambda(p))$) in a single point.

Next fix $X \in \Lambda(q)$, and set $y = \pi(X) \in \pi(\Lambda(q)) \subset M$. Set $\xi(Y) = p$, then $y \in \pi(\Lambda(q)) \cap \pi(\Lambda(p))$. Thus there exists a unique $Z \in \Lambda(q)$ such that $\xi(-Z) = p$ and $\pi(Z) = y$. Since $\pi \mid \Lambda(q)$ is one-to-one, $Z = X$. Since $\xi \mid -\Lambda(q)$ is one-to-one, $-Z = Y$. Therefore $Y = -X$. This shows $f^{-1}(y) = \{-X\}$. Observe that $\{-X\}$ is the transverse intersection in P of $\Lambda(p)$ and $-\Lambda(q)$, for π carries $\Lambda(p)$ and $-\Lambda(q)$ diffeomorphically onto the transverse submanifolds $\pi(\Lambda(p))$ and $\pi(\Lambda(q))$ of M, whose tangent spaces at y are respectively the eigenspaces of the smallest and largest eigenvalues of the shape operator A_X. Transversality is immediate because the sum of the dimensions of these two eigenspaces equals $\dim(\Lambda(p)) + \dim(\Lambda(q)) = m = \dim(M) = \dim(P)$ by hypothesis. This also shows that the differential f_* is an isomorphism at $-X$. Thus y is a regular value of f, and $f^{-1}(y)$ has only one point. Therefore the mod 2 degree of f is 1 [10].

Now $f^! f^*$ is multiplication by $1 = \mathrm{degree}_2(f)$ where $f^!: H^*(P) \to H^*(M)$ is the cohomology transfer [4]. Thus $f^*: H^*(M) \to H^*(P)$ is injective.

Remark. Since $f \circ \sigma = \pi \mid \Lambda(q)$, Lemma 1 implies that $H^*(M)$ is a subring of $H^*(P)$ which σ^* takes onto $H^*(\Lambda(q))$.

EXAMPLE. Consider the Veronese embedding which is a substantial taut embedding of $M = \boldsymbol{R}P^2$ into S^4. Here for every $q \in C(M)$, $\Lambda(q) \approx S^1$. The manifold P is a S^1-bundle over S^1. Since $H^*(\boldsymbol{R}P^2)$ injects into $H^*(P)$, P is a Klein bottle. Note f is not a homeomorphism.

A similar example is given by the Veronese embedding of $M = \boldsymbol{R}P^3$ into S^8. This is also taut and substantial. Here there exists a $q \in C(M)$ with $\Lambda(q) \approx \boldsymbol{R}P^2$. For each $X \in \Lambda(q)$, $-X$ is a regular point of multiplicity one. Thus P is a (non-trivial) S^1-bundle over $\boldsymbol{R}P^2$. The other Veronese embeddings give like examples [3].

§2. Münzner's list

If M admits a taut embedding into S^n such that every point of N_1 is regular, then the hypothesis of Theorem 3.1 in [6] is satisfied. Consequently, $C(M)$ is a submanifold of S^n (consisting of two components when M is a hypersurface and one otherwise), and S^n decomposes as a topological union of a disk bundle over M with a disk bundle over $C(M)$ identified along the boundaries of the disk bundles. Münzner essentially determined all possible Z_2-cohomology rings of M in this situation.

THEOREM 6. *Suppose M^m admits a taut embedding into a sphere such that every point of N_1 is regular, then $H^*(M)$ must be isomorphic to a ring in one of the following classes of graded algebras over Z_2.*

♯0. Z_2 *(In this case M is a point).*

♯1. $Z_2[x]/\langle x^2\rangle$ *where* degree $x = m$. *(In this case, M is a sphere, for it is a taut homology sphere.)*

♯2. $Z_2[x, y]/\langle x^2, y^2\rangle$ *where* degree x + degree $y = m$.

♯3. $Z_2[x]/\langle x^3\rangle$ *where* degree $x = 1, 2, 4$ *or* 8, *and* $m = 2$ degree x.

♯4. $Z_2[x, y]/\langle x^2 + y^2 + xy, y^3\rangle$ *where* degree x = degree $y = 1, 2, 4$ *or* 8, *and* $m = 3$ degree x.

♯5. $Z_2[x, y, z]/\langle x^2, y^2, z^2\rangle$ *where* degree x + degree y = degree z, *and* $m = 2$ degree z.

♯6. $Z_2[x]/\langle x^4\rangle$ *where* degree $x = 1, 2$ *or* 4, *and* $m = 3$ degree x.

♯7. $Z_2[x, y]/\langle x^4, y^2\rangle$ *where* degree x = degree $y = 1, 2$ *or* 4, *and* $m = 4$ degree x.

♯8. $Z_2[x, y]/\langle x^3, y^2\rangle$ *where* degree x *is either* 1 *or even,* degree $y = 3$ degree x, *and* $m = 5$ degree x.

♯9. $Z_2[x, y, z]/\langle x^2 + y^2 + xy, y^3, z^2\rangle$ *where* degree x = degree y *is either* 1 *or even,* degree $x = 3$ degree x, *and* $m = 6$ degree x.

Proof. This list is extracted from Münzner's work by the following argument. First assume M is a hypersurface. Then, by means of Satz C and Satz 8 in [11], one has for Münzner's case $g = 1$, that $H^*(M)$ is isomorphic to a ring in class ♯1; for $g = 2$, a ring in class ♯2; for $g = 3$, a ring in class ♯4; for $g = 4$, either class ♯5 or ♯7, depending upon whether case 1 or case 2 in the proof of Satz 8 [11] is fulfilled; and for $g = 6$, class ♯9. Next assume M is not a hypersurface, then $H^*(N_1)$ is among classes ♯1, ♯2, ♯4, ♯5, ♯7, ♯9. By Satz 5 and Satz 6 in [11], the homomorphism $\pi^*: H^*(M) \to H^*(N_1)$ is injective, and makes $H^*(N_1)$ into a free $H^*(M)$-module on two generators. Hence by calculating, in each case, just what subrings of $H^*(N_1)$ turn it into such a free module, the possibilities for $H^*(M)$ are obtained. In detail, if $H^*(N_1)$ is in class ♯1, then

$H^*(M)$ is in class #0; if #2, then #1; if #4, then #3; if #5, then #2; if #7, then either #2 or #6; and if #9, then #8. Then restrictions on the degrees of the generators in #3, #4, #6 and #7 are not in Münzner, but come from Theorem of Adams [1] and Adem [2], for in these cases there must be a manifold whose Z_2-cohomology is a truncated polynomial ring in one generator.

Examples may be found in [3], [5], [14]. These examples show that the topological type of M is not necessarily determined by its cohomology ring. For instance, some rings in class #2 occur both for a product of two spheres and for certain Stiefel manifolds.

None of the rings in this list can be the cohomology ring of a connected sum of two compact m-dimensional manifolds which are not Z_2-homology spheres. For by means of the Mayer-Vietoris sequences, if $M = M_1 \# M_2$, then $H^*(M)$ has a presentation as an amalgamated direct sum of $H^*(M_1)$ and $H^*(M_2)$ in which the units in $H^0(M_1)$ and $H^0(M_2)$, are amalgamated to form the unit $1 \in H^0(M)$, and the generators of $H^m(M_1)$ and $H^m(M_2)$ are amalgamated to form the generator of $H^m(M)$. It is a simple (though tedious) exercise to show case by case that none of the rings in the list admits such a presentation.

§ 3. Applications and examples

PROPOSITION 7. *Suppose M^m admits a taut embedding in some sphere. Assume that the cup product $xy = 0$ whenever $x \in H^i(M)$, $y \in H^j(M)$ for $i, j > 0$ with $i + j < m$. Then the Z_2-cohomology ring $H^*(M)$ is in one of the classes #0, #1, #2, or #3.*

Proof. Without loss of generality one may assume M is taut and substantially embedded. By Lemma 4 every point of N_1 is regular. By Theorem 6, $H^*(M)$ is one of the rings on the list. Classes #4 through #9 can be eliminated because they do not satisfy the condition on the cup product.

Some examples of manifolds satisfying the cup product condition in Proposition 7 are the projective planes FP^2, products of two spheres $S^k \times S^{m-k}$ (more generally, sphere bundles over spheres), and connected sums of these types as long as dimensions agree. For example $(S^1 \times S^4) \# (S^2 \times S^2) \# CP^2$, $HP^2 \# HP^2$, $(S^1 \times S^{14}) \# (S^4 \times S^{11}) \# (S^7 \times S^8)$. Another example is the 3-dimensional nil-manifold obtained by dividing the group of upper triangular 3×3 matrices by the group of such integral matrices. Its cohomology ring is the same as that of $(S^1 \times S^2) \# (S^1 \times S^2)$. The following theorem is an immediate corollary of Proposition 6. It generalizes results of Ozawa [12].

THEOREM 8. *No manifold having the same Z_2-cohomology ring as a connected sum of projective planes and/or products of two spheres admits a taut embedding into a sphere.*

§ 4. The main list

Assume now that ℓ is the least positive dimension in which the homology of M does not vanish. Thus $H_i(M) = 0$ for all $0 < i < \ell$.

If there is a link $\Lambda(q)$ of dimension $m - \ell$, then for every $X \in \Lambda(q)$, $\operatorname{mult}(-X) = \ell$ and $-X$ is regular. Hence $\Lambda(q)$ satisfies the hypothesis of Proposition 5. For if $X \in \Lambda(q)$, then $\ell \leq \operatorname{mult}(-X) \leq m - \operatorname{mult}(X) = \ell$ where the first inequality is valid because mult cannot take on any value between 0 and ℓ, and the second because $\operatorname{mult}(X) + \operatorname{mult}(-X) \leq m$. Since mult cannot take on any value less than ℓ, upper semi-continuity implies mult is continuous at all points where it equals ℓ. Note that $\Lambda(q) \cong \pi(\Lambda(q)) \subset S^n$ is a taut submanifold by Theorem 1.1 of [12].

THEOREM 9. *Suppose M^m admits a taut embedding into some sphere. Let ℓ be the least positive dimension in which the homology of M does not vanish, Assume that the cup product $xy = 0$ whenever $x \in H^i(M)$, $y \in H^j(M)$ for $i, j > 0$ with $i + j < m - \ell$. Then $H^*(M)$ is isomorphic to one of the following rings.*

(0) Z_2.

(1) $Z_2[x]/\langle x^2 \rangle$ *where* degree $x = m$.

(2) $Z_2[x, y]/\langle x^2, y^2 \rangle$ *where* degree x + degree $y = m$.

(3) $Z_2[x]/\langle x^3 \rangle$ *where* degree $x = 1, 2, 4$, *or* 8, *and* $m = 2$ degree x.

(4) $Z_2[x, y]/\langle x^2 + y^2 + xy, y^3 \rangle$ *where* degree x = degree $y = \ell = 1, 2, 4$ *or* 8, *and* $m = 3\ell$.

(5) $Z_2[x]/\langle x^4 \rangle$ *where* degree $x = \ell = 1, 2$ *or* 4, *and* $m = 3\ell$.

(6) $Z_2[x, u]/\langle x^3, u^2 \rangle$ *where* degree x = degree $u = \ell$, *and* $m = 3\ell$.

(7) $Z_2[x, u]/\langle x^3, u^2 + xu \rangle$ *where* degree x = degree $u = \ell = 1, 2, 4$ *or* 8, *and* $m = 3\ell$.

(8) $Z_2[x, y, u]/\langle x^2, y^2, u^2 \rangle$ *where* degree x = degree y = degree $u = \ell$, *and* $m = 3\ell$.

(9) $Z_2[x, y, u]/\langle x^2, y^2, u^2 + xu \rangle$ *where* degree x = degree y = degree $u = \ell$, *and* $m = 3\ell$.

Proof. First assume that M satisfies the more stringent cup product condition of Proposition 7. Then by that proposition we have the possibilities (0), (1), (2), or (3). Thus we may henceforth assume that there exist cohomology classes $x \in H^i(M)$, $y \in H^j(M)$ with $xy \neq 0$, $i, j > 0$, and $i + j = m - \ell$.

Thus by Lemma 4, either every point of N_1 is regular, or there exists a singular $X \in N_1$ of multiplicity $m - \ell$. In the former case, the possibilities in Theorem 6 which are compatible with the conditions on the cup product are #4 and #6. These are (4) and (5). In the latter case, by Proposition 5, $H^*(M)$ is a subring of $H^*(P)$ where P is a certain S^ℓ-bundle over the link Λ through X. By Corollary 3, Λ is not a sphere. By the remark following Proposition 5, $(\pi | \Lambda)^*: H^*(M) \to H^*(\Lambda)$ is onto. Thus $xy = 0$ whenever $x \in H^i(\Lambda)$ and $y \in H^j(\Lambda)$ with $0 < i, j$, and $i + j < m - \ell = \dim(\Lambda)$. Therefore $H^*(\Lambda)$ is a ring in either class #2 or #3 by Proposition 7.

Suppose $H^*(\Lambda) = Z_2[x]/\langle x^3 \rangle$ and $H^*(P) = Z_2[x, u]/\langle x^3, u^2 + zu \rangle$ where degree $u = \ell$, degree $x = k$, $\ell \leq k$, $2k = m - \ell$, and $z \in H^\ell(\Lambda)$.

If $\dim(H^\ell(M)) = 1$ as a vector space over Z_2, then $\ell = k$. For if $\ell < k$, then $\dim(H^\ell(P)) = \dim(H^k(P)) = 1$. Since $H^*(M) \to H^*(\Lambda)$ is onto, $H^k(M) \neq 0$. Thus $x, u \in H^*(M)$. But then $xu \neq 0 \in H^*(M)$ is a contradiction because $k + \ell < 2k = m - \ell$. Therefore $\ell = k$, and $m = 3\ell$. Thus take the generator $\bar{x} \in H^\ell(M)$. It maps onto $x \in H^\ell(\Lambda)$. Then $\bar{x}^2 \neq 0$ since it maps onto $x^2 \neq H^{2\ell}(\Lambda)$. By Poincaré duality $\dim(H^{2\ell}(M)) = \dim(H^\ell(M)) = 1$. Thus $\bar{x}^2$ is the generator of $H^{2\ell}(M)$. Again by Poincaré duality, $\bar{x} \cdot \bar{x}^2 \neq 0$. Clearly $H^*(M) = Z_2[\bar{x}]/\langle \bar{x}^4 \rangle$. This is (5).

If $\dim(H^\ell(M)) = 2$, then $\dim(H^\ell(P)) = 2$. Thus $\ell = k$, and $m = 3\ell$. Also $x, u \in H^*(M)$. Therefore $H^*(M) = H^*(P)$. This gives (6) and (7).

Since $\dim(H^\ell(P)) \leq 2$, even if $\ell = k$, it is impossible for $\dim(H^\ell(M)) > 2$.

Now suppose $H^*(\Lambda) = Z_2[x, y]/\langle x^2, y^2 \rangle$ and $H^*(P) = Z_2[x, y, u]/\langle x^2, y^2, u^2 + zu \rangle$ where degree $u = \ell$, degree $x = k$, degree $y = r$, $\ell \leq k \leq r$, $k + r = m = \ell$, and $z \in H^\ell(\Lambda)$. Again $H^\ell(M) \neq 0$. Since $\dim(H^\ell(P) \leq 3$, even if $\ell = k = r$, $\dim(H^\ell(M)) \leq 3$.

If $\dim(H^\ell(M)) = 3$, then $H^*(M) = H^*(P)$. This gives (8) and (9).

If $\dim(H^\ell(M)) = 2$, then $\ell = k$. Now $r = \ell = k$, for if $r > \ell = k$, then $\dim(H^\ell(P)) = 2$. Thus $u, x \in H^*(M)$. This is a contradiction since $xu \neq 0$ and $2\ell = \ell + k < r + k = m - \ell$. Therefore $k = \ell = r$, and $m = 3\ell$. Now, there is an $\bar{x} \in H^\ell(M)$ such that $\bar{x}^2 \neq 0$. For if not, take $\bar{y} \in H^\ell(M)$ such that $\bar{x}, \bar{y}$ is a basis for $H^\ell(M)$. Then the element $\bar{x}\cdot\bar{y} \in H^{m-\ell}(M)$ is non-zero since it maps onto the generator of $H^{m-\ell}(\Lambda)$ and has no Poincaré dual because $(\bar{x}\cdot\bar{y})\bar{x} = 0 = (\bar{x}\cdot\bar{y})\bar{y}$. Thus there exists $\bar{x} \in H^\ell(M)$ satisfying $\bar{x}^2 \neq 0$. Thus $H^*(P) = Z_2[x, y, u]/\langle x^2, y^2, u^2 + xu\rangle$. Thus $w^3 = 0$ for every $w \in H^\ell(P)$, and every two-dimensional subspace of $H^\ell(P)$ contains an element $\bar{y}$ such that $\bar{y}^2 = 0$. Therefore, there is a basis $\bar{x}, \bar{y}$ of $H^\ell(M)$ satisfying $\bar{x}^2 \neq 0$, $\bar{y}^2 = 0$. Since $\bar{x}, \bar{y}$ maps onto a basis of $H^\ell(\Lambda)$, $\bar{x}\cdot\bar{y} \neq 0$. It is now clear that $H^*(M) = Z_2[\bar{x}, \bar{y}]/\langle\bar{x}^3, \bar{y}^2\rangle$. This is (6).

Finally, $\dim(H^\ell(M)) = 1$ is impossible. Suppose $\dim(H^\ell(M)) = 1$. Then $\ell < r$, for otherwise $\dim(H^\ell(M)) \geq \dim(H^\ell(\Lambda)) = 2$. Also, $\ell = k$, for if $\ell < k$, then $u, x \in H^*(M)$ giving the contradiction $xu \neq 0$ since $\ell + k < r + k = m - \ell$. Thus $\ell = k < r$. Now $H^r(M) \neq 0$ because $H^r(\Lambda) \neq 0$. Let $\bar{x} \in H^\ell(M)$ be the generator. Then $\bar{x}^2 = 0$ since $\ell + \ell < k + \ell = m - \ell$. Since some class in $H^{m-\ell}(M)$ is a cup product of classes in lower degrees, the only possibility after consideration of the degrees in which $H^*(M)$ does not vanish is that there exists $\bar{y} \in H^r(M)$ such that $\bar{x}\cdot\bar{y} \neq 0$. But $\bar{x}(\bar{x}\cdot\bar{y}) = \bar{x}^2\bar{y} = 0$ is a contradiction to Poincaré duality.

§ 5. Further examples

Examples of spaces that satisfy the condition on the cup product in Theorem 9 are products of three spheres of the same dimension $S^\ell \times S^\ell \times S^\ell$, the projective 3-spaces $F^\ell P^3$ ($\ell = 1, 2, 4$), and connected sums of these. For example $(S^\ell \times S^\ell \times S^\ell)\# \cdots \#(S^\ell \times S^\ell \times S^\ell)$ (k times), $CP^3\# \cdots \#CP^3$ (k times), $(S^2 \times S^2 \times S^2)\#CP^3$. Also any connected sum of the form $M_1 \# M_2$ where M_1 satisfies the hypothesis of Theorem 9 and M_2 satisfies the hypothesis of Proposition 7 as long as the least dimension in which the homology of M_2 does not vanish is greater than or equal to ℓ. For example $(S^2 \times S^2 \times S^2) \# (S^2 \times S^4)$, $CP^3 \# (S^3 \times S^3) \# (S^3 \times S^3)$.

THEOREM 10. *No manifold, whose cohomology ring is isomorphic to that of a connected sum of products of three spheres of the same dimension and/or projective 3-spaces, has a taut embedding except possibly for $F^\ell P^3 \# F^\ell P^3$ ($\ell = 1, 2, 4$). Also, no manifold, whose cohomology ring is isomorphic to that of a connected sum of the form $M_1 \# M_2$ where M_1 satisfies the hypothesis Theorem 9, and M_2 satisfies the hypothesis of Proposition 7, and the least dimension in which the homology of M_2 does not vanish is greater than or equal to the least dimension in which the homology of M does not vanish, has a taut embedding.*

Proof. A case by case analysis of the list in Theorem 9 shows that none of the rings on the list can be the cohomology ring of a connected sum of two manifolds that are not Z_2-homology spheres, except for (7) which could be the cohomology ring of $F^\ell P^3 \# F^\ell P^3$ ($\ell = 1, 2, 4$).

The preceding results are well adapted to study the cohomology ring of a taut submanifold having four non-zero homology groups. For in this case, the hypothesis of Theorem 9 is satisfied. An examination of the possibilities from this theorem leads to the conclusion that the Euler characteristic $\chi(M)$ is non-negative, and the sum of the Z_2-Betti numbers $b(M)$ does not exceed 8.

THEOREM 11. *No manifold M having exactly four non-zero homology groups (in particular, no 3-dimensional manifold) has a taut embedding if either $\chi(M) < 0$ or $b(M) > 8$.*

This supports the following conjecture.

CONJECTURE. *Let M be a taut submanifold of a sphere. Then $\chi(M) \geq 0$ and $b(M) \leq 2^{r-1}$ where r is the number of non-zero homology groups of M.*

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