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TORUS-EQUIVARIANT VECTOR BUNDLES ON PROJECTIVE SPACES

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Introduction

For a free $\boldsymbol{Z}$-module N of rank n, let $T = T_N$ be an n-dimensional algebraic torus over an algebraically closed field k defined by N. Let $X = T_N \operatorname{emb}(\Delta)$ be a smooth complete toric variety defined by a fan Δ (cf. [6]). Then T acts algebraically on X. A vector bundle E on X is said to be an equivariant vector bundle, if there exists an isomorphism $f_t\colon t^*E \to E$ for each k-rational point t in T, where $t\colon X \to X$ is the action of t. Equivariant vector bundles have T-linearizations (see Definition 1.2 and [2], [4]), hence we consider T-linearized vector bundles.

The n-dimensional projective space $\boldsymbol{P}^n$ has a natural action of T and can be regarded as a toric variety. In [4], we classified indecomposable equivariant vector bundles of rank two on $\boldsymbol{P}^2$. When $n > 2$, Hartshorne [3] constructed vector bundles of rank two from codimension two subschemes satisfying certain conditions. Bertin and Elencwajg [2] then used this method to construct equivariant vector bundles of rank two on $\boldsymbol{P}^n$ and showed that there exist no indecomposable equivariant vector bundles of rank two on $\boldsymbol{P}^n$ which are obtained in this way.

In this paper, we generalize our method in [4] to show that there exist no indecomposable equivariant vector bundles of rank r $(1 < r < n)$ on $\boldsymbol{P}^n$ (Corollary 3.5) and that indecomposable equivariant vector bundles of rank n on $\boldsymbol{P}^n$ are isomorphic to $E(d)$ or $E^*(d)$ for some integer d, where E is defined by an exact sequence

$$0 \longrightarrow \mathcal{O}_{\boldsymbol{P}^n} \longrightarrow \bigoplus_{i=0}^{n} \mathcal{O}_{\boldsymbol{P}^n}(a_i) \longrightarrow E \longrightarrow 0$$

for positive integers a_i.

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§ 1. Preliminaries

Let N be a free $\boldsymbol{Z}$-module of rank n. Let M be the dual $\boldsymbol{Z}$-module of N. Then there is a natural $\boldsymbol{Z}$-bilinear map

$$\langle\ ,\ \rangle\colon M \times N \longrightarrow \boldsymbol{Z}.$$

It can naturally be extended to $M_{\boldsymbol{R}} \times N_{\boldsymbol{R}} \to \boldsymbol{R}$, where $M_{\boldsymbol{R}} = M \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ and $N_{\boldsymbol{R}} = N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$. We denote $\varphi(\xi) = \langle \xi, \varphi \rangle$ for ξ in $M_{\boldsymbol{R}}$ and φ in $N_{\boldsymbol{R}}$. Let $T = T_N$ be an n-dimensional algebraic torus defined by N over an algebraically closed field k. Then we can identify M with the additive group of characters of T. Let $X = T_N \operatorname{emb}(\Delta)$ be a smooth complete toric variety of dimension n defined by a fan Δ of N for which the reader is referred to [6].

Definition 1.1. An equivariant vector bundle E on X is a vector bundle on X such that there exists an isomorphism $f_t\colon t^*E \to E$ for every k-rational point t in T, where $t\colon X \to X$ is the action of t on X.

Definition 1.2. An equivariant vector bundle $E = (E, f_t)$ is said to be T-linearized if $f_{tt'} = f_{t'} \circ t'^*(f_t)$ holds for every pair of k-rational points t, t' of T, where

$$f_{tt'} = f_{t'} \circ t'^*(f_t)\colon (tt')^*E \xrightarrow{t'^*(f_t)} t'^*E \xrightarrow{f_{t'}} E.$$

In [4], we showed that an equivariant vector bundle necessarily has a T-linearization. We also studied how to describe T-linearized vector bundles in terms of fans, as we now recall.

We denote by $\Delta(l)$ the set of l-dimensional cones in Δ. For C in $\Delta(l)$, there exists a finite subset $\{\varphi_1, \cdots, \varphi_l\}$ of N such that $C = \boldsymbol{R}_0\varphi_1 + \cdots + \boldsymbol{R}_0\varphi_l$, where $\boldsymbol{R}_0$ is the set of non-negative real numbers. We say that $\{\varphi_1, \cdots, \varphi_l\}$ is the fundamental system of generators of C if φ_i are primitive, *i.e.*, φ_i is not a non-trivial integral multiple of any element of N. The fundamental system of generators $\{\varphi_1, \cdots \varphi_l\}$ of C is uniquely determined by C and is denoted by $|C|$. We consider the following:

(I) $m\colon \{|C'| \mid C' \in \Delta(1)\} \longrightarrow \boldsymbol{Z}^{\oplus r}$
sending φ to $m(\varphi) = (m(\varphi)_1, \cdots, m(\varphi)_r)$, and for every C in $\Delta(n)$

$$m_C\colon |C| \longrightarrow \boldsymbol{Z}^{\oplus r}$$

so that there exists a permutation $\tau = \tau_C$ such that

$$m_C(\varphi) = (m_C(\varphi)_1, \cdots, m_C(\varphi)_r) = (m(\varphi)_{\tau(1)}, \cdots, m(\varphi)_{\tau(r)})$$

for every φ in $|C|$.

Let C be an n-dimensional cone in $\Delta(n)$. Then we have a set of characters $\{\xi(C)_1, \cdots, \xi(C)_r\}$ in M by solving, for each $1 \leq i \leq r$, the equations $\varphi(\xi(C)_i) = m_C(\varphi)_i$ for every φ in $|C|$. Then it is easy to see that (I) is equivalent to the following:

(I′) $\xi\colon \Delta(n) \longrightarrow M^{\oplus r}$

sending C to $\xi(C) = (\xi(C)_1, \cdots, \xi(C)_r)$ such that there exists a permutation $\tau = \tau_{C,C'}$ for every pair of cones C and C' in $\Delta(n)$, so that $\varphi(\xi(C)_i) = \varphi(\xi(C')_{\tau(i)})$ for every i and every φ in $|C| \cap |C'|$.

(II) $P\colon \Delta(n) \times \Delta(n) \longrightarrow GL_r(k)$

sending (C, C') to $P(C, C') = (P(C, C')_{ij})$ such that $P(C, C')_{ij} \neq 0$ only if $\varphi(\xi(C)_i) \geq \varphi(\xi(C')_j)$ for every φ in $|C| \cap |C'|$ and that

$$P(C, C')P(C', C'') = P(C, C'')$$

for every C, C', C'' in $\Delta(n)$.

For (m, P) defined by (I) and (II), we denote by $E(m, P)$ the T-linearized vector bundle obtained from (m, P). We refer the reader to [4] as for the construction of the T-linearized vector bundle $E(m, P)$.

(III) Two pairs (m, P) and (m', P') defined by (I) and (II) are said to be equivalent if there exists a permutation $\tau = \tau_C$ for every C in $\Delta(n)$ such that

$$(m_C(\varphi)_1, \cdots, m_C(\varphi)_r) = (m'_C(\varphi)_{\tau(1)}, \cdots, m'_C(\varphi)_{\tau(r)})$$

for every φ in $|C|$ and if there exists

$$\sigma\colon \Delta(n) \longrightarrow GL_r(k)$$

such that $\sigma(C)_{ij} \neq 0$ only if $\varphi(\xi(C)_i) \geq \varphi(\xi(C)_j)$ for every φ in $|C|$ and such that

$$P'(C, C') = \sigma(C)^{-1}P(C, C')\sigma(C')$$

holds for every C and C' in $\Delta(n)$.

THEOREM 1.3 (cf. ([4]). *Let $X = T_N \operatorname{emb}(\Delta)$ be a smooth complete toric variety defined by a fan Δ. Then the set of T-linearized vector bundles of rank r up to T-isomorphism corresponds bijectively to the set of* (I) (*or* (I′)) *and* (II) *up to the equivalence* (III).

Remark 1.4 Let D_φ be the divisor corresponding to the cone $\boldsymbol{R}_0\varphi$ in $\Delta(1)$. Put $m_\varphi = m(\varphi)$ where m is defined by (I) in the case $r = 1$. Let

$P(C, C') = 1$ for every C and C' in $\Delta(n)$. Then the T-linearized vector bundle $E(m, P)$ is the line bundle $\mathcal{O}_X(-\sum m_\varphi D_\varphi)$, where the summation is taken over φ in $\{|C| \mid C \in \Delta(1)\}$.

Remark 1.5. Let $E = E(m, P)$ be the T-linearized vector bundle of rank r defined by (m, P). Then $E \otimes \mathcal{O}_X(-\sum m_\varphi D_\varphi)$ is T-isomorphic to $E(m', P)$, where

$$m'(\varphi) = (m(\varphi)_1 + m_\varphi, \cdots, m(\varphi)_r + m_\varphi)$$

for every φ in $\{|C| \mid C \in \Delta(1)\}$. The dual vector bundle E^* is T-isomorphic to $E(-m, {}^tP^{-1})$, where

$${}^tP^{-1}(C, C') = {}^tP(C, C')^{-1},$$

and

$$-m(\varphi) = (-m(\varphi)_1, \cdots, -m(\varphi)_r)$$

for every φ in $\{|C| \mid C \in \Delta(1)\}$.

§ 2. Some lemmas

LEMMA 2.1. *Let C and C' be two cones in $\Delta(n)$. Suppose $P(C, C')_{ii} \neq 0$ holds for every i. Then $\varphi(\xi(C)_i) = \varphi(\xi(C')_i)$ holds for every i and every φ in $|C| \cap |C'|$.*

Proof. Since $P(C, C')_{ii} \neq 0$ we have $\varphi(\xi(C)_i) \geq \varphi(\xi(C')_i)$ for every i and every φ in $|C| \cap |C'|$. Hence

$$\varphi(\xi(C)_1) + \cdots + \varphi(\xi(C)_r) \geq \varphi(\xi(C')_1) + \cdots + \varphi(\xi(C')_r).$$

Since the two sets $\{\varphi(\xi(C)_1), \cdots, \varphi(\xi(C)_r)\}$ and $\{\varphi(\xi(C')_1), \cdots, \varphi(\xi(C')_r)\}$ are the same sets by (I′), we have

$$\varphi(\xi(C)_1) + \cdots + \varphi(\xi(C)_r) = \varphi(\xi(C')_1) + \cdots + \varphi(\xi(C')_r).$$

Therefore $\varphi(\xi(C)_1) = \varphi(\xi(C')_1), \cdots, \varphi(\xi(C)_r) = \varphi(\xi(C')_r)$ for every φ in $|C| \cap |C'|$.

LEMMA 2.2. *Let C and C' be two cones in $\Delta(n)$ such that $C \cap C'$ is in $\Delta(n-1)$. Then, by rearranging $\{\xi(C)_1, \cdots, \xi(C)_r\}$ and $\{\xi(C')_1, \cdots, \xi(C')_r\}$ and replacing (m, P) by an equivalent pair, we can reduce the matrix $P(C, C')$ to an upper triangular matrix.*

Proof. Put $P = P(C, C')$. Since $\det(P) \neq 0$, we may assume that $P_{ii} \neq 0$ for every i by rearranging $\{\xi(C)_i\}$ and $\{\xi(C')_i\}$. Suppose that $P_{hk} \neq 0$

and $P_{kh} = 0$ for some h, k $(h > k)$. Then, by interchanging $\xi(C)_h$ with $\xi(C)_k$ and $\xi(C')_h$ with $\xi(C')_k$ we have $P_{hk} = 0$, $P_{kh} \neq 0$. So we may further assume that if $P_{hk} \neq 0$ for $h > k$ then $P_{kh} \neq 0$.

Suppose that P is not an upper triangular matrix. Then we take minimal k such that $P_{hk} \neq 0$ for some $h > k$. Then $P_{ij} = 0$ for $j < k$ and $i > j$. Since $P_{hk} \neq 0$ and $P_{kh} \neq 0$ by assumption, we have

$$\varphi(\xi(C)_h) \geq \varphi(\xi(C')_k) \quad \text{and} \quad \varphi(\xi(C)_k) \geq \varphi(\xi(C')_h)$$

for every φ in $|C| \cap |C'|$. Consequently, for φ in $|C| \cap |C'|$, we have $\varphi(\xi(C)_h) > \varphi(\xi(C')_h)$ if $\varphi(\xi(C)_h) > \varphi(\xi(C)_k)$, while $\varphi(\xi(C)_k) > \varphi(\xi(C')_k)$ if $\varphi(\xi(C)_h) < \varphi(\xi(C)_k)$, a contradiction by Lemma 2.1. Therefore we have

$$\varphi(\xi(C)_h) = \varphi(\xi(C)_k) \qquad \text{for every } \varphi \text{ in } |C| \cap |C'|\,.$$

Since $C \cap C'$ is in $\Delta(n-1)$, put $|C| - |C| \cap |C'| = \{\psi\}$. Suppose $\psi(\xi(C)_h) < \psi(\xi(C)_k)$. Then we interchange $\xi(C)_h$ and $\xi(C)_k$. This procedure interchanges P_{hi} with P_{ki} for each $1 \leq i \leq r$. Therefore the minimality of k is preserved. Hence we may assume that

$$\varphi(\xi(C)_h) \geq \varphi(\xi(C)_k) \qquad \text{for every } \varphi \text{ in } |C|\,.$$

Now we define $\sigma(C) = (\sigma(C)_{ij})$ by

$$\sigma(C)_{ij} = \begin{cases} 1 & \text{for } i = j\,, \\ c \neq 0 & \text{for } i = h \quad \text{and} \quad j = k\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and replace (m, P) by an equivalent pair using this $\sigma(C)$. This is allowed by what we have just seen. In this way, we can reduce ourselves to the case $P_{hk} = 0$. Hence we have $P_{ik} = 0$ for all i $(i > k)$. After this replacement, however. P_{ii} may be zero for $i > k$. By rearranging $\{\xi(C)_{k+1}, \cdots, \xi(C)_r\}$ and $\{\xi(C')_{k+1}, \cdots, \xi(C')_r\}$, we may assume that $P_{ii} \neq 0$ for every i. So we can repeat the same procedure, which will terminate after finitely many steps and leads to an upper triangular matrix P.

LEMMA 2.3. *Let C, C', C'' be three distinct cones in $\Delta(n)$ such that $C' \cap C''$ is in $\Delta(n-1)$. Then, by rearranging $\{\xi(C)_i\}$, $\{\xi(C')_i\}$ and $\{\xi(C'')_i\}$ and replacing (m, P) by an equivalent pair, we can reduce ourselves to the situation where $P(C', C'')$ is an upper triangular matrix and $P(C, C')_{ii} \neq 0$ for every i.*

Proof. By Lemma 2.2, $P(C', C'')$ is first reduced to an upper triangular matrix. Since $\det(P(C, C')) \neq 0$, we have $P(C, C')_{ii} \neq 0$ by rearranging $\{\xi(C)_1, \cdots, \xi(C)_r\}$.

COROLLARY 2.4. *Let C, C', C'' be three distinct cones in $\Delta(n)$ such that $C' \cap C''$ is in $\Delta(n-1)$. Then by rearranging $\{\xi(C)_i\}$, $\{\xi(C')_i\}$ and $\{\xi(C'')_i\}$ and replacing (m, P) by an equivalent pair, we may assume that*

$$\varphi(\xi(C)_i) = \varphi(\xi(C')_i) \qquad \text{for every } \varphi \text{ in } |C| \cap |C'|$$

and

$$\varphi(\xi(C')_i) = \varphi(\xi(C'')_i) \qquad \text{for every } \varphi \text{ in } |C'| \cap |C''|$$

hold for every i.

LEMMA 2.5. *Let C, C', C'' be three distinct cones in $\Delta(n)$ such that $C \cap C'$ is in $\Delta(n-1)$. Suppose that $P(C', C'') = I$ is the identity matrix. Then, by rearranging $\{\xi(C)_i\}$, $\{\xi(C')_i\}$ and $\{\xi(C'')_i\}$ and replacing (m, P) by an equivalent pair, we can reduce ourselves to the situation where $P(C, C')$ is an upper triangular matrix and $P(C', C'') = I$.*

Proof. By Lemma 2.2, $P(C, C')$ is reduced to an upper triangular matrix. In this case, $\{\xi(C')_1, \cdots, \xi(C')_r\}$ is only rearranged. If we rearrange $\{\xi(C'')_1, \cdots, \xi(C'')_r\}$ exactly as $\{\xi(C')_1, \cdots, \xi(C')_r\}$ is rearranged, then $P(C', C'')$ remains the identity matrix.

LEMMA 2.6. *Let C and C' be two cones in $\Delta(n)$. Suppose that $P = P(C, C')$ is an upper triangular matrix. Let φ be in $|C| \cap |C'|$. Then, by rearranging $\{\xi(C)_i\}$ and $\{\xi(C')_i\}$, we may assume that $P(C, C')$ is an upper triangular matrix and that*

$$\varphi(\xi(C)_1) \geq \varphi(\xi(C)_2) \geq \cdots \geq \varphi(\xi(C)_r)$$

and

$$\varphi(\xi(C')_1) \geq \varphi(\xi(C')_2) \geq \cdots \geq \varphi(\xi(C')_r)$$

hold.

Proof. Suppose $\varphi(\xi(C)_h) < \varphi(\xi(C)_{h+1})$. Since $\varphi(\xi(C)_i) = \varphi(\xi(C')_i)$ for every i by Lemma 2.1, we have $P_{h,h+1} = 0$. Hence we have $P_{h,h} \neq 0$, $P_{h+1,h+1} \neq 0$, $P_{h,h+1} = 0$, $P_{h+1,h} = 0$. By interchanging the order of $\xi(C)_h$ and $\xi(C)_{h+1}$ as well as $\xi(C')_h$ and $\xi(C')_{h+1}$, we have $\varphi(\xi(C)_h) > \varphi(\xi(C)_{h+1})$. After a finite repetition of this process we are done.

COROLLARY 2.7. *Let C, C' be two cones in $\Delta(n)$. Suppose that $P(C, C')$ is an upper triangular matrix. Let $\varphi_1, \cdots, \varphi_l$ be elements in $|C| \cap |C'|$. Then, by rearranging $\{\xi(C)_i\}$ and $\{\xi(C')_i\}$, we may assume that $P(C, C')$ is an upper triangular matrix and that for every pair of h and k $(h > k)$, one of the following conditions holds*:

(a) $\varphi_i(\xi(C)_k) = \varphi_i(\xi(C)_{k+1}) = \cdots = \varphi_i(\xi(C)_h) \quad$ *for* $1 \leq i \leq l$.

(b) *There exists v $(\leq l)$ such that*

$$\varphi_i(\xi(C)_k) = \varphi_i(\xi(C)_{k+1}) = \cdots = \varphi_i(\xi(C)_h) \quad \text{for } 1 \leq i < v$$

and

$$\varphi_v(\xi(C)_k) > \varphi_v(\xi(C)_h) .$$

Proof. We first apply Lemma 2.6 to φ_1. If $\varphi_1(\xi(C)_k) = \cdots = \varphi_1(\xi(C)_h)$, then we further apply Lemma 2.6 to φ_2 with respect to $\{\xi(C)_k, \cdots, \xi(C)_h\}$ only. Repeating this procedure, we are done.

§ 3. The case of P^n

From this section on, we restrict ourselves to the case $X = \boldsymbol{P}^n$ and consider a T-linearized vector bundle $E = E(m, P)$ of rank r $(r \geq 2)$ on $\boldsymbol{P}^n$. When $n = 1$, a vector bundle on $\boldsymbol{P}^1$ is split. Hence we assume $n \geq 2$. Let $\{\varphi_1, \cdots, \varphi_n\}$ be a $\boldsymbol{Z}$-base of N and let $\varphi_0 = -\varphi_1 - \varphi_2 - \cdots - \varphi_n$. Let Δ be the fan defined by $\{\varphi_0, \varphi_1, \cdots, \varphi_n\}$, *i.e.*, $\Delta(n)$ consists of $C_i = \sum_{j \neq i} \boldsymbol{R}_0 \varphi_j$ $(i = 0, 1, \cdots, n)$. Then $\boldsymbol{P}^n = T_N \operatorname{emb}(\Delta)$ and $U_{C_i} = \{X_i \neq 0\}$ is the affine open set in $\boldsymbol{P}^n$ corresponding to C_i, where $X_0, \cdots, X_n$ are homogeneous coordinates. In this case we note that $C \cap C'$ is an $(n-1)$-dimensional cone in Δ for every pair of cones C and C' in $\Delta(n)$.

PROPOSITION 3.1. *Suppose that $P(C', C'') = I$ is the identity matrix for some C' and C'' in $\Delta(n)$ with $C' \neq C''$. Then the T-linearized vector bundle E is a direct sum of T-linearized line bundles, hence, in particular, decomposable.*

Proof. Let C be another cone in $\Delta(n)$. By Lemma 2.5, we may assume that $P = P(C, C')$ is an upper triangular matrix and $P(C', C'') = I$. We show that P can be reduced to the identity matrix. Suppose $P_{hk} \neq 0$ for some h and k $(h < k)$. Then $\varphi(\xi(C)_h) \geq \varphi(\xi(C')_k)$ for every φ in $|C| \cap C'|$, while $\varphi(\xi(C)_k) = \varphi(\xi(C')_k)$ by Lemma 2.1. Hence we have $\varphi(\xi(C)_h) \geq \varphi(\xi(C)_k)$ for every φ in $|C| \cap |C'|$.

Put $\{\psi\} = |C| - |C| \cap |C'|$. Then ψ is in $|C| \cap |C''|$ by the definition of Δ. Assume now that $\psi(\xi(C)_h) < \psi(\xi(C)_k)$. Since $P(C, C'') = P(C, C')$ $P(C', C'') = P$ we have $P(C, C'')_{hk} = P_{hk} \neq 0$, and $\varphi(\xi(C)_h) \geq \varphi(\xi(C'')_k)$ for every φ in $|C| \cap |C''|$. Therefore, since ψ is in $|C| \cap |C''|$, we have $\psi(\xi(C)_k) > \psi(\xi(C)_h) \geq \psi(\xi(C'')_k)$. This is a contradiction to Lemma 2.1 since $P(C, C'') = P$ is an upper triangular matrix and $\det(P) \neq 0$. So we have $\psi(\xi(C)_h) \geq \psi(\xi(C)_k)$. Since $|C| = (|C| \cap |C'|) \cup \{\psi\}$ we have $\varphi(\xi(C)_h) \geq \varphi(\xi(C)_k)$ for every φ in $|C|$. Now we define $\sigma(C) = (\sigma(C)_{ij})$ by

$$\sigma(C)_{ij} = \begin{cases} 1 & \text{for } i = j\,, \\ c \neq 0 & \text{for } i = h \quad \text{and} \quad j = k\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and replace (m, P) by an equivalent pair using this $\sigma(C)$. Then we can reduce ourselves to the case $P(C, C')_{hk} = 0$. If this process is repeated for all $P(C, C')_{hk} \neq 0$ $(h \neq k)$, then finally $P(C, C')$ will become a diagonal matrix. By taking $\sigma'(C) = (\sigma'(C)_{ij})$ with $\sigma'(C)_{ij} = P(C, C')_{ij}$ and replacing (m, P) by an equivalent pair using this $\sigma'(C)$, we may assume $P(C, C') = I$. Hence $P(C, C'') = I$.

Furthermore, if $n \geq 3$, let C^* be another cone in $\Delta(n)$. We apply the same process to $P(C^*, C')$. Then we may assume $P(C^*, C') = I$. Hence

$$P(C, C^*) = P(C, C')P(C^*, C')^{-1} = I\,.$$

Therefore all $P(C, C^*)$ can be reduced to the identity matrix. This means that the T-linearized vector bundle E is a direct sum of T-linearized line bundles by the very construction of $E(m, P)$.

PROPOSITION 3.2. *Suppose that $P(C, C')$, $P(C', C'')$ and $P(C'', C)$ are upper triangular matrices for some triple of pairwise distinct cones C, C' and C'' in $\Delta(n)$. Then the T-linearized vector bundle E is a direct sum of T-linearized line bundles.*

Proof. Concerning the first row of P for (m, P), we suppose that there exists $s > 1$ such that

$$P(C, C')_{1j} = 0\,, \quad P(C', C'')_{1j} = 0\,, \quad P(C'', C)_{1j} = 0 \qquad \text{for } 1 < j < s$$

and that

$$P(C, C')_{1s} \neq 0\,, \qquad P(C'', C)_{1s} \neq 0\,.$$

Then, since $P(C, C')_{1s} \neq 0$ and $P(C'', C)_{1s} \neq 0$, we have

$$\varphi(\xi(C)_1) \geq \varphi(\xi(C')_s) \qquad \text{for every } \varphi \text{ in } |C| \cap |C'|$$

and

$$\varphi(\xi(C'')_1) \geq \varphi(\xi(C)_s) \qquad \text{for every } \varphi \text{ in } |C''| \cap |C| .$$

Since, by Lemma 2.1,

$$\varphi(\xi(C)_s) = \varphi(\xi(C')_s) \qquad \text{for every } \varphi \text{ in } |C| \cap |C'|$$

and

$$\varphi(\xi(C)_1) = \varphi(\xi(C'')_1) \qquad \text{for every } \varphi \text{ in } |C''| \cap |C| ,$$

we have

$$\varphi(\xi(C)_1) \geq \varphi(\xi(C)_s) \qquad \text{for every } \varphi \text{ in } |C| \cap |C'|$$

and

$$\varphi(\xi(C)_1) \geq \varphi(\xi(C)_s) \qquad \text{for every } \varphi \text{ in } |C''| \cap |C| .$$

Since $|C| = (|C| \cap |C'|) \cup (|C| \cap |C''|)$ we have

$$\varphi(\xi(C)_1) \geq \varphi(\xi(C)_s) \qquad \text{for every } \varphi \text{ in } |C| .$$

We define $\sigma(C) = (\sigma(C)_{ij})$ by

$$\sigma(C)_{ij} = \begin{cases} 1 & \text{for } i = j , \\ c \neq 0 & \text{for } i = 1 \text{ and } j = s , \\ 0 & \text{otherwise} , \end{cases}$$

and replace (m, P) by an equivalent pair using this $\sigma(C)$. Then we can reduce ourselves to the case $P(C, C')_{1s} = 0$. Furthermore if $P(C'', C)_{1s} \neq 0$ or $P(C', C'')_{1s} \neq 0$ we do the same. Then we are reduced to the case $P(C'',C)_{1s} = 0$, hence $P(C', C'')_{1s} = 0$. Thus we may assume that

$$P(C, C')_{1j} = 0 , \quad P(C', C'')_{1j} = 0 , \quad P(C'', C)_{1j} = 0 \qquad \text{for } j \neq 1 .$$

Repaet the same process to the other rows successively. Then $P(C, C')$, $P(C', C'')$ and $P(C'', C)$ become diagonal matrices. We define $\sigma(C) = (\sigma(C)_{ij})$ by $\sigma(C)_{ij} = P(C, C')_{ij}$ and replace (m, P) by an equivalent pair using this $\sigma(C)$. Then we are reduced to the case $P(C, C') = I$. Therefore, by Proposition 3.1, the T-linearized vector bundle E is a direct sum of T-linearized line bundles.

COROLLARY 3.3. *Suppose that $P(C, C')$ and $P(C', C'')$ are upper triangular martices for some triple of pairwise distinct cones C, C' and C'' in*

$\Delta(n)$. Then the T-linearized vector bundle E is a direct sum of T-linearized line bundles.

THEOREM 3.4. *Let $r > 1$ and $n \geq 2$. For a pair (m, P), suppose the corresponding T-linearized vector bundle $E = E(m, P)$ on $\boldsymbol{P}^n$ of rank r is indecomposable. Put $|C| = \{\varphi_1, \cdots, \varphi_n\}$ for a cone C in $\Delta(n)$. Then, for any pair of distinct integers s and t ($1 \leq s,\ t \leq n$), there exist two integers h and k such that*

$$\varphi_s(\xi(C)_h) < \varphi_s(\xi(C)_k)\,, \qquad \varphi_t(\xi(C)_h) > \varphi_t(\xi(C)_k)$$

and

$$\varphi_i(\xi(C)_h) = \varphi_i(\xi(C)_k) \qquad \textit{for } i \neq s,\ t\,.$$

Proof. We prove the assertion only when $s = 1$ and $t = 2$ since the proof in general is the same. Since we are working on $\boldsymbol{P}^n$, there exist C' and C'' in $\Delta(n)$ such that

$$|C'| = \{\varphi_0, \varphi_2, \varphi_3, \cdots, \varphi_n\}\,, \qquad |C''| = \{\varphi_0, \varphi_1, \varphi_3, \cdots, \varphi_n\}$$

for some $\varphi_0 \in N$. By Lemma 2.3, we assume that $P(C, C')_{ii} \neq 0$ for every i and that $P(C', C'')$ is an upper triangular matrix. Hence by Lemma 2.1, we get

$$\varphi(\xi(C)_i) = \varphi(\xi(C')_i) \qquad \text{for every } \varphi \text{ in } |C| \cap |C'| = \{\varphi_2, \varphi_3, \cdots, \varphi_n\}$$

and

$$\varphi(\xi(C')_i) = \varphi(\xi(C'')_i) \qquad \text{for every } \varphi \text{ in } |C'| \cap |C''| = \{\varphi_1, \varphi_3, \cdots, \varphi_n\}\,.$$

Therefore

$$\varphi(\xi(C)_i) = \varphi(\xi(C')_i) = \varphi(\xi(C'')_i)$$

for every φ in $|C| \cap |C'| \cap |C''| = \{\varphi_3, \varphi_4, \cdots, \varphi_n\}$. If $n = 2$ then this amounts to nothing since $|C| \cap |C'| \cap |C''| = \emptyset$. When $n \geq 3$, we further apply Corollary 2.7 to $\varphi_3, \cdots, \varphi_n$ and $P(C', C'')$ and may assume that $P(C', C'')$ is an upper triangular matrix and that for every pair h and k ($h > k$), one of the following conditions holds:

(a) $\varphi_i(\xi(C)_k) = \varphi_i(\xi(C)_{k+1}) = \cdots = \varphi_i(\xi(C)_h)$ for $3 \leq i \leq n$.

(b) There exists v such that

$$\varphi_i(\xi(C)_k) = \varphi_i(\xi(C)_{k+1}) = \cdots = \varphi_i(\xi(C)_h) \qquad \text{for } 3 \leq i < v$$

and

$$\varphi_v(\xi(C)_k) > \varphi_v(\xi(C)_h)\,.$$

$P(C, C')$ cannot be reduced to an upper triangular matrix, since, otherwise, E would be decomposable by Corollary 3.3 contradicting our assumption. Thus there exist integers $h > k$ such that $P(C, C')_{hk} \neq 0$. Then we have $\varphi(\xi(C)_h) \geq \varphi(\xi(C')_k)$ for every φ in $|C| \cap |C'|$. Since $\varphi(\xi(C)_k) = \varphi(\xi(C')_k)$ for every φ in $|C| \cap |C'|$, we have $\varphi(\xi(C)_h) \geq \varphi(\xi(C)_k)$ for every φ in $|C| \cap |C'|$, hence for every φ in $\{\varphi_3, \cdots, \varphi_n\} = |C| \cap |C'| \cap |C''|$. This means, by (a) and (b) above, that

$$\varphi(\xi(C)_k) = \varphi(\xi(C)_{k+1}) = \cdots = \varphi(\xi(C)_h)$$

for every φ in $\{\varphi_3, \cdots, \varphi_n\}$. Since φ_2 is in $|C| \cap |C'|$, we have $\varphi_2(\xi(C)_h) \geq \varphi_2(\xi(C)_k)$ as we saw above. Hence we have the following four possibilities:

1. $\varphi_1(\xi(C)_h) \geq \varphi_1(\xi(C)_k)$ and $\varphi_2(\xi(C)_h) = \varphi_2(\xi(C)_k)$,
2. $\varphi_1(\xi(C)_h) < \varphi_1(\xi(C)_k)$ and $\varphi_2(\xi(C)_h) = \varphi_2(\xi(C)_k)$,
3. $\varphi_1(\xi(C)_h) \geq \varphi_1(\xi(C)_k)$ and $\varphi_2(\xi(C)_h) > \varphi_2(\xi(C)_k)$,
4. $\varphi_1(\xi(C)_h) < \varphi_1(\xi(C)_k)$ and $\varphi_2(\xi(C)_h) > \varphi_2(\xi(C)_k)$.

We now show that the case 4 happens for some h and k $(h > k)$ such that $P(C, C')_{hk} \neq 0$. Suppose that the case 4 does not happen for any such h, k. Then, by interchanging $\xi(C)_h$ and $\xi(C)_k$ if the case 2 happens, we have

$$\varphi(\xi(C)_h) \geq \varphi(\xi(C)_k) \qquad \text{for every } \varphi \text{ in } |C|\,.$$

Now we take the smallest k such that $P(C, C')_{hk} \neq 0$ for some $h > k$. We define $\sigma(C) = (\sigma(C)_{ij})$ by

$$\sigma(C)_{ij} = \begin{cases} 1 & \text{for } i = j\,, \\ c \neq 0 & \text{for } i = h \text{ and } j = k\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and replace (m, P) by an equivalent pair using this $\sigma(C)$. Then we can reduce ourselves to the case $P(C, C')_{hk} = 0$. Repeating the same procedure for every h such that $h > k$ and $P(C, C')_{hk} \neq 0$, we have $P(C, C')_{ik} = 0$ for all $i > k$. After this procedure, $P(C, C')_{ii}$ may be zero for some i, but, by rearranging the order of $\{\xi(C)_{k+1}, \cdots, \xi(C)_r\}$, we have $P(C, C')_{ii} \neq 0$ for all $i > k$. So we can successively apply the same procedure, and $P(C, C')$ is finally reduced to an upper triangular matrix. By Corollary 3.3, this is a contradiction to the indecomposability of E. Therefore there exist h and k such that

$$\varphi_1(\xi(C)_h) < \varphi_1(\xi(C)_k)\,, \qquad \varphi_2(\xi(C)_h) > \varphi_2(\xi(C)_k)$$

and

$$\varphi_i(\xi(C)_h) = \varphi_i(\xi(C)_k) \qquad \text{for } i \geq 3\,.$$

COROLLARY 3.5. *Suppose* $1 < r \leq n$ $(n \geq 2)$ *and let* E *be an indecomposable T-linearized vector bundle of rank* r *on* $\boldsymbol{P}^n$. *Then we have*:

(1) $r = n$.

(2) *For every* $C \in \Delta(n)$ *and every* φ *in* $|C|$, *all except one of* $\{\varphi(\xi(C)_1), \cdots, \varphi(\xi(C)_n)\}$ *are the same integers.*

(3) *Let* $C = \boldsymbol{R}_0\varphi_1 + \cdots + \boldsymbol{R}_0\varphi_n$ *be in* $\Delta(n)$. *We can tensor a suitable T-linearized line bundle to E and rearrange the order of* $\{\xi(C)_1, \cdots, \xi(C)_n\}$, *so that the following hold for every* $i = 1, \cdots, n$:

$$\varphi_i(\xi(C)_i) = a_i \quad \text{and} \quad \varphi_i(\xi(C)_j) = 0 \qquad \text{for any } j \neq i\,.$$

In this case, $a_1, \cdots, a_n$ *are all positive or all negative.*

Proof. Let $|C| = \{\varphi_1, \cdots, \varphi_n\}$ for a $C \in \Delta(n)$, and apply Theorem 3.4 to C. For each s, we first see that $\varphi_s(\xi(C)_1), \cdots, \varphi_s(\xi(C)_r)$ cannot be all equal, since we can pick $t \neq s$ and apply Theorem 3.4 to the pair (s, t).

Clearly, Theorem 3.4 gives a one-to-one map from the set $\{(s, t) \mid 1 \leq s < t \leq n\}$ to the set of pairs $\{h, k\}$ of distinct integers between 1 and r. Thus $n(n-1)/2 \leq r(r-1)/2$. Since $r \leq n$ by assumption, we have $r = n$, and the above map must be a bijection.

We can so rearrange $\xi(C)_1, \cdots, \xi(C)_n$ that for each i, the pair $(1, i)$ is sent to the pair $\{1, i\}$ by the above map. Then for each i, we see that $\varphi_i(\xi(C)_j)$ are equal for all $j \neq i$.

In view of Remark 1.5, we may tensor a T-linearized line bundle to E so that the following holds for each i:

$$\varphi_i(\xi(C)_i) = a_i \quad \text{and} \quad \varphi_i(\xi(C)_j) = 0 \qquad \text{for any } j \neq i\,.$$

By Theorem 3.4, we see that a_s, a_t should have the same sign for all $s \neq t$.

§4. Determination of $P(C, C')$

In this section we consider $P(C, C')$ for an indecomposable T-linearized vector bundle of rank n on $\boldsymbol{P}^n$ $(n \geq 2)$. By Corollary 3.5, we may assume that, for every C and every φ in $|C|$, we have $\varphi(\xi(C)_i) \geq 0$ and $\varphi(\xi(C)_i) = 0$ except for one i. For C, C', $C'' \in \Delta(n)$, let

$$|C| = \{\varphi_1, \varphi_2, \varphi_3, \cdots, \varphi_n\},$$
$$|C'| = \{\varphi_0, \varphi_2, \varphi_3, \cdots, \varphi_n\},$$
$$|C''| = \{\varphi_0, \varphi_1, \varphi_3, \cdots, \varphi_n\}.$$

By changing the order of $\{\xi(C)_i\}$, $\{\xi(C')_i\}$ and $\{\xi(C'')_i\}$, we assume that

$$\begin{aligned}
(\varphi_0(\xi(C')_1), \cdots, \varphi_0(\xi(C')_n)) &= (\varphi_0(\xi(C'')_1), \cdots, \varphi_0(\xi(C'')_n)) \\
&= (a, 0, 0, \cdots, 0) \qquad (a > 0), \\
(\varphi_1(\xi(C)_1), \cdots, \varphi_1(\xi(C)_n)) &= (b, 0, 0, \cdots, 0), \\
(\varphi_1(\xi(C'')_1), \cdots, \varphi_1(\xi(C'')_n)) &= (0, b, 0, \cdots, 0) \qquad (b > 0), \\
(\varphi_2(\xi(C)_1), \cdots, \varphi_2(\xi(C)_n)) &= (\varphi_2(\xi(C')_1), \cdots, \varphi_2(\xi(C')_n)) \\
&= (0, c, 0, \cdots, 0) \qquad (c > 0),
\end{aligned}$$

and

$$\begin{aligned}
(\varphi_i(\xi(C)_1), \cdots, \varphi_i(\xi(C)_n)) &= (\varphi_i(\xi(C')_1), \cdots, \varphi_i(\xi(C')_n)) \\
&= (\varphi_i(\xi(C'')_1), \cdots, \varphi_i(\xi(C'')_n)) \\
&= (0, \cdots, 0, d_i, \cdots, 0)
\end{aligned}$$

for $i \geq 3$, where $d_i > 0$ is the i-th entry.

Then, by (II), we have

$$P(C, C') = \begin{pmatrix} 1 & 0 & 0 \cdots 0 \\ p_2 & 1 & 0 \cdots 0 \\ p_3 & 0 & 1 \cdots 0 \\ \vdots & \vdots & \vdots \ddots \vdots \\ p_n & 0 & 0 \cdots 1 \end{pmatrix}, \qquad P(C', C'') = \begin{pmatrix} 1 & q_1 & 0 \cdots 0 \\ 0 & 1 & 0 \cdots 0 \\ 0 & q_3 & 1 \cdots 0 \\ \vdots & \vdots & \vdots \ddots \vdots \\ 0 & q_n & 0 \cdots 1 \end{pmatrix},$$

$$P(C', C) = \begin{pmatrix} 0 & -1 & 0 \cdots 0 \\ 1 & r_2 & 0 \cdots 0 \\ 0 & r_3 & 1 \cdots 0 \\ \vdots & \vdots & \vdots \ddots \vdots \\ 0 & r_n & 0 \cdots 1 \end{pmatrix}.$$

Since $P(C, C')P(C', C'')P(C'', C) = I$ we have:

LEMMA 4.1. *$q_1 = 1$, $p_2 = -1$, $r_2 = 1$ and*

$$p_i = -q_i = r_i \quad \text{for } 3 \leq i \leq n.$$

LEMMA 4.2. *$p_i \neq 0$ for $2 \leq i \leq n$.*

Proof. Fix two cones C and C' and $P(C, C')$. We take another C'' successively and calculate in the above way. Then we have $p_i \neq 0$ for $2 \leq i \leq n$.

LEMMA 4.3. *We may assume that $p_i = 1$ for $1 < i \leq n$.*

Proof. Since $p_i \neq 0$ for $i \geq 2$, we take

$$\sigma(C) = \sigma(C') = \begin{pmatrix} 1 & & & 0 \\ & p_2 & & \\ & & \ddots & \\ 0 & & & p_n \end{pmatrix}$$

and replace (m, P) by an equivalent pair using these $\sigma(C)$, $\sigma(C')$. Then we have

$$P(C, C') = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & & 0 \\ \vdots & & \ddots & \\ 1 & 0 & & 1 \end{pmatrix}.$$

Hence we may assume that $p_i = 1$ for $1 < i \leq n$.

If $P(C, C')$ is in the above form, then $P(C', C'')$ and $P(C'', C)$ are naturally determined if m in (I) is given. Therefore P in (m, P) is determined for every pair of cones in $\Delta(n)$. Hence, for each indecomposable T-linearized vector bundle of rank n, P is unique up to equivalence (III). Therefore, for any given m in (I) which we know by (3) of Corollary 3.5, an indecomposable T-linearized vector bundle is uniquely determined if it exists.

THEOREM 4.4. *Let E be a T-linearized vector bundle defined by the sequence*

$$(*) \qquad 0 \longrightarrow \mathcal{O}_{P_n} \xrightarrow{f} \bigoplus_{i=0}^{n} \mathcal{O}_{P_n}(a_i) \longrightarrow E \longrightarrow 0$$

such that f sends 1 to $(X_0^{a_0}, X_1^{a_1}, \cdots, X_n^{a_n})$, where $X_0, \cdots, X_n$ are homogeneous coordinates of $\boldsymbol{P}^n$ and $a_0, \cdots, a_n$ are positive integers. Then E is an indecomposable vector bundle.

Proof. Suppose E is decomposable and let $E = E_1 \oplus E_2 \oplus \cdots \oplus E_l$ with $l \geq 2$ be a decomposition of E into indecomposable vector bundles. Every indecomposable component E_i is T-equivariant by virtue of the

Krull-Schmidt Theorem (see [1]). Since rank $(E_i) < n$, E_i is necessarily a line bundle by Corollary 3.5. Hence E is a direct sum of line bundles and we may let

$$E = \mathcal{O}_{P^n}(d_1) \oplus \mathcal{O}_{P^n}(d_2) \oplus \cdots \oplus \mathcal{O}_{P^n}(d_n)$$

for $d_1 \leq d_2 \leq \cdots \leq d_n$. We may assume that $a_0 \leq a_1 \leq \cdots \leq a_n$. By tensoring the sequence $(*)$ with $\mathcal{O}_{P^n}(-k)$ for $k > 0$ we have

$$h^0\left(\bigoplus_{i=0}^{n} \mathcal{O}_{P^n}(a_i - k)\right) = h^0(E(-k)).$$

We have a contradiction, if we take $k = a_n$ when $a_n > d_n$ while we take $k = d_n$ when $a_n < d_n$. Hence we have $a_n = d_n$. Similarly, we have $a_i = d_i$ for $1 \leq i \leq n$. By $(*)$, we have $\det(E) = \mathcal{O}_{P^n}(\sum_{i=0}^{n} a_i)$, which is equal to $\mathcal{O}_{P^n}(\sum_{i=1}^{n} d_i)$. Hence $a_0 = 0$ and $(*)$ is split, a contradiction. Therefore E is indecomposable.

If we take $a_0 = a_1 = \cdots = a_n = 1$ in Theorem 4.4, then the T-linearized vector bundle E is the tangent bundle T_{P^n} for P^n.

COROLLARY 4.5. *T_{P^n} is indecomposable.*

By short calculation, we have

$$m(\varphi_i) = (-a_i, 0, \cdots, 0) \qquad \text{for } 0 \leq i \leq n$$

for the T-linearized vector bundle E which is defined by $(*)$ in Theorem 4.4. Therefore we have:

THEOREM 4.6. *An indecomposable equivariant vector bundle of rank n on P^n $(n \geq 2)$ is isomorphic to $E(d)$ or $E^*(d)$ for some integer d, where E is defined by the sequence $(*)$ in Theorem 4.4 for some positive integers a_i $(0 \leq i \leq n)$.*

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