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# MULTIPLICITY AND $t$-ISOMULTIPLE IDEALS

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## Introduction

Let $V$ be an irreducible non degenerate variety in $\boldsymbol{P}^n$; a classical geometric result says that degree $(V) \geq$ codim $V + 1$ and, if equality holds, $V$ is said to be of minimal degree. Varieties of minimal degree has been classified by Del Pezzo and Bertini and they all are intersections of quadrics. The local version of this result is due to J. Sally who proved that if $(A, \mathfrak{N})$ is a regular local ring and $(R = A/I, \mathfrak{M} = \mathfrak{N}/I)$ is a Cohen-Macaulay local ring of minimal multiplicity, according to the bound $e(R) \geq \text{height}\,(I) + 1$ given by Abhyankar, then the tangent cone $\mathrm{gr}_{\mathfrak{M}}(R)$ of $R$ is intersection of quadrics and it is Cohen-Macaulay.

On the other hand if $I \subset \mathfrak{N}^2$ and $S_R(\mathfrak{M})$ is the symmetric algebra of the $R$-module $\mathfrak{M}$, then by a result of A. Micali we know that $S_R(\mathfrak{M})$ is not a domain; however J. Risler proved that, if $R$ is reduced, then $S_R(\mathfrak{M})$ is reduced if and only if $\mathrm{gr}_{\mathfrak{M}}(R)$ is intersection of quadrics.

Recently J. Elias considered the case $I$ is a perfect codimension 2 ideal of the regular local ring $(A, \mathfrak{N})$; if $v = v(I)$ is the minimal number of generators of $I$, he proved that $e(A/I) \geq \binom{v}{2}$ and, if equality holds, $\mathrm{gr}_{\mathfrak{M}}(R)$ is intersection of hypersurfaces of degree $v - 1$.

Further if one tries to extend the theory of normal flatness along permissible ideals to the non regular case, then it is natural to consider ideals whose corresponding tangent cone is intersection of hypersurfaces of the same degree $t$ (see [Br]).

We say that an ideal $I$ is $t$-isomultiple if $\mathrm{gr}_{\mathfrak{M}}(R)$ is defined by equations of the same degree $t$; this means that $I$ has a standard base of elements of order $t$. As it turns out by the preceding examples, very often ideals with "minimal" multiplicity are $t$-isomultiple. In this paper we pursue this line in order to identify some interesting classes of $t$-isomultiple ideals.

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In section 1 we consider a complete intersection codimension $h$ ideal $I \subset \mathfrak{N}^t$ and prove that $I$ is $t$-isomultiple if and only if $e(A/I) = t^h$ (see Theorem 1.8). The main tool to prove this result is to investigate the condition $e(A/xA) = te(A)$, where $t$ is the order of $x$ and $(A, \mathfrak{N})$ is a local ring not necessarily regular. If we assume $\mathrm{gr}_{\mathfrak{N}}(A)$ to be Cohen-Macaulay, then we can prove that $e(A/xA) = te(A)$ if and only if the initial form of $x$ in $\mathrm{gr}_{\mathfrak{N}}(A)$ is a non zero divisor (see Corollary 1.6).

The main result of section 2 is Theorem 2.1, which gives us the possibility to reduce our problems to the 0-dimensional case and also throws light on the relationship between $t$-isomultiple ideals and the Cohen-Macaulay property of $\mathrm{gr}_{\mathfrak{M}}(R)$. We are dealing with a perfect codimension $h$ ideal $I \subset \mathfrak{N}^t$ of the regular local ring $(A, \mathfrak{N})$. Hence the ring $(R = A/I,\ \mathfrak{M} = \mathfrak{N}/I)$ is Cohen-Macaulay of dimension say $d$ and we can consider a minimal reduction $J = (x_1, \cdots, x_d)$ modulo $I$. Then we have that $(I + J)/J$ is $t$-isomultiple if and only if $I$ is $t$-isomultiple and $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay. It would be interesting to know whether the condition $I$ $t$-isomultiple implies $\mathrm{gr}_{\mathfrak{M}}(R)$ to be Cohen-Macaulay.

Now if $I$ is a perfect codimension $h$ ideal such that $I \subset \mathfrak{N}^t$ with $t \geq 3$, it is clear that the bound $e(A/I) \geq h + 1$ is not sharp. One can prove $e(A/I) \geq \binom{h+t-1}{h}$ and thus it is natural to consider ideals for which equality holds.

In section 3 we call these ideals $t$-extremal and prove that $I$ is $t$-extremal if and only if $I$ is $t$-isomultiple, $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay and $v(I) = \binom{h+t-1}{t}$ (see Theorem 3.2). This result extends to a considerable extent theorems of Sally and Elias and also explains the connection between the notion of $t$-extremal ideals and that of $t$-extremal Cohen-Macaualy graded algebras introduced by P. Schenzel in [Sch].

Perfect ideals $I$ with $e(A/I) = h + 2$ have been extensively studied by Sally in [$\mathrm{Sa}_2$]; here we say that the perfect codimension $h$ ideal $I$ is almost $t$-extremal if $I \subset \mathfrak{N}^t$ and $e(A/I) = \binom{h+t-1}{h} + 1$. In the second part of section 3 we prove that $I$ is almost $t$-extremal and $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay if and only if the Poincare series of $R$ is $P(R, z) = (1 - z)^{-1}\left[\sum_{i=0}^{t=1} \binom{h+i-1}{i} z^i + z^t\right]$ (see Proposition 3.8). Further $I$ is $t$-isomultiple, $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay and $v(I) = \binom{h+t-1}{t} - 1$ if and only if $I$ is almost $t$-extremal and $\tau(R) < \binom{h+t-2}{t-1}$, where $\tau(R)$ is the Cohen-Macaulay type of the local ring $R$ (see Theorem 3.10). Again this theorem not only extends the main result of Sally in [Sa$_2$] from the case $t = 2$ to the general case, but also gives a complete picture of the problem.

Section 4 is devoted to study perfect ideals which can be obtained as the specialization of $t$-isomultiple generic ideals (see Definition 4.1). Here we prove that for such an ideal $I$, the multiplicity of the local ring $A/I$ is bigger or equal to the multiplicity of the generic case and, if equality holds, then $I$ is $t$-isomultiple (see Theorem 4.2). As a corollary we get again Elias result on perfect codimension two ideals, but also a very easy proof that if $I$ is a Gorenstein codimension three ideal then $e(A/I) \geq [v(I)^3 - v(I)]/24$ and, if equality holds, $I$ is $[(v(I)-1)/2]$-isomultiple and $\mathrm{gr}_{\mathfrak{M}}(R)$ is Gorenstein. The first assertion is the main result in [E-I], while the second gives a positive answer to a conjecture stated in the same paper. Other interesting applications are given.

In the last section of the paper we prove two main results. The first gives an upper bound for the multiplicity of the local ring $A/I$, when $I$ is a 2-isomultiple codimension $h$ ideal such that $h \leq 6$ (see Theorem 5.9). Suitable examples show that the bound is sharp and suggest that for a $t$-isomultiple codimension $h$ ideal $I$ the following inequality could hold: $e(A/I) \leq t^{h-2}(t^2 - t + 1)$.

The second, see Theorem 5.15, gives a lower bound for the multiplicity of the local ring $A/I$, when $I$ is a $t$-isomultiple codimension $h$ ideal such that $v(I) = h + 1$ and $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay. It is perhaps worthy to remark that, without any assumption on $v(I)$, the trivial bound $e(A/I) \geq \binom{h+t-1}{h}$ is sharp. Here we prove that $e(A/I) \geq d_s$ where, given the integers $h$ and $t$, we define $s$ to be the integer part of $[(h+1)(t-1)]/2$ and we let $(\sum_{i=0}^{t-1} z^i)^{h+1} = \sum_i d_i z^i$.

## § 1.

Let $(A, \mathfrak{N})$ be a noetherian local ring with an infinite residue field $k = A/\mathfrak{N}$; let $I \subset \mathfrak{N}$ be an ideal in $A$ of height $h$ such that $R = A/I$ is a local ring of dimension $d$ and maximal ideal $\mathfrak{M} = \mathfrak{N}/I$.

A system of elements $f_1, \cdots, f_r$ in $I$ is called a standard base of $I$ if the initial forms $f_i^*$ in $\mathrm{gr}_{\mathfrak{N}}(A) = \bigoplus_{p \geq 0} (\mathfrak{N}^p/\mathfrak{N}^{p+1})$ generate the ideal $I^*$ of initial forms of $I$. This is equivalent to saying that $I\cap\mathfrak{N}^p=\sum_{i=1}^r f_i\mathfrak{N}^{p-v_i}$ holds for all $p\geq 0$, where $\mathfrak{N}^s=A$ if $s\leq 0$ and $v_i=v(f_i)$ is the largest integer $t$ such that $f_i\in\mathfrak{N}^t$, the order of $f_i$. It is clear that if $f_1,\cdots,f_r$ is a standard base of $I$, then $\mathrm{gr}_{\mathfrak{M}}(A/I)=\mathrm{gr}_n(A)/(f_1^*,\cdots,f_r^*)$. Further one can prove that $f_1^*,\cdots,f_r^*$ is a regular sequence in $\mathrm{gr}_{\mathfrak{N}}(A)$ if and only if $f_1,\cdots,f_r$ is a regular sequence in $A$ and a standard base of $(f_1,\cdots,f_r)$ (see [V-V]). The ideal $I$ is said to be *$t$-isomultiple* if $I$ has a standard base $f_1,\cdots,f_r$ of elements of the same order $t$.

It is clear that $I$ is $t$-isomultiple if and only if $\mathfrak{N}^{p+t}\cap I=\mathfrak{N}^pI$ for all $p\geq 0$.

Also if $I$ is $t$-isomultiple, then $I$ and $I^*$ have the same minimal number of generators. For example if $k$ is a field and $J\subset P=k[X_1,\cdots,X_n]$ an ideal generated by homogeneous elements of the same degree $t$, then the ideal $I=JA$ is $t$-isomultiple in the ring $A=k[\![X_1,\cdots,X_n]\!]$ or $A=k[X_1,\cdots,X_n]_{(X_1,\cdots,X_n)}$. The following result aroused our interest in the study of $t$-isomultiple ideals; a proof has been given in [R$_1$], but it is rather involved. We insert here an easy proof, also for the sake of completeness.

Let us assume $A$ regular and $I\subset\mathfrak{N}^2$; we let $S_R(\mathfrak{M})$ be the Symmetric algebra of $\mathfrak{M}$ over $R$. By a result of Micali (see [M]) we know that $S_R(\mathfrak{M})$ is not a domain since $R$ is not regular. As for reduceness one can prove the following.

PROPOSITION 1.1. *If $R$ is reduced, then $S_R(\mathfrak{M})$ is reduced if and only if $I$ is 2-isomultiple.*

*Proof.* By the universal property of the Symmetric algebra, we have $S_R(\mathfrak{M})=\bigoplus_{p\geq 0}(\mathfrak{N}^p/I\mathfrak{N}^{p-1})$. Since the ring $\mathrm{gr}_n(A)=\bigoplus_{p\geq 0}(\mathfrak{N}^p/\mathfrak{N}^{p+1})$ is a domain and the Rees algebra $\boldsymbol{R}_R(\mathfrak{M})=\bigoplus_{p\geq 0}(\mathfrak{N}^p/I\cap\mathfrak{N}^p)$, which is a subring of $R[T]$, is reduced, we get that the ideals $\bigoplus_{p\geq 0}\mathfrak{N}^{p+1}$ and $\bigoplus_{p\geq 0}(I\cap\mathfrak{N}^p)$ are radical ideals in the ring $\bigoplus_{p\geq 0}\mathfrak{N}^p$. We claim that $\mathrm{Rad}(\bigoplus_{p\geq 0}I\mathfrak{N}^{p-1})=(\bigoplus_{p\geq 0}(I\cap\mathfrak{N}^p))\cap(\bigoplus_{p\geq 0}\mathfrak{N}^{p+1})$ and from this the conclusion follows. Now $I\mathfrak{N}^{p-1}\subset(I\cap\mathfrak{N}^p)\cap\mathfrak{N}^{p+1}$; on the other hand, by the Artin Rees lemma, there exists a positive integer $r$ such that $I\cap\mathfrak{N}^{r+k}\subset I\mathfrak{N}^k$ for all $k\geq 0$. Thus if $x\in I\cap\mathfrak{N}^{p+1}$, then $x^{r-1}\in I\cap\mathfrak{N}^{(p+1)(r-1)}\subset I\mathfrak{N}^{p(r-1)-1}$; this proves the other inclusion and the proposition.

In the following for a local ring $R$, $e(R)$ denotes the multiplicity of $R$.

EXAMPLE 1.2. If $R$ is a reduced hypersurface ring then $S_R(\mathfrak{M})$ is reduced if and only if $e(R) = 2$.

Moreover, if $I$ is a complete intersection ideal of height $h$ and $S_R(\mathfrak{M})$ is reduced or, which is the same by the above proposition, if $I$ is 2-isomultiple, then $e(R) = 2^h$ (see [$R_2$]).

It is suggested in [B] that the converse is a corollary of the following exercise: If $(A, \mathfrak{N})$ is a local ring such that $\mathrm{gr}_{\mathfrak{N}}(A)$ is a complete intersection and if $f_1, \cdots, f_r$ are elements of $\mathfrak{N}$ of order $v_1, \cdots, v_r$, then $e(A/f_1, \cdots, f_r)) = e(A) \prod_{i=1}^{r} v_i$ if and only if $f_1^*, \cdots, f_r^*$ is a regular sequence in $\mathrm{gr}_{\mathfrak{N}}(A)$ (see [B], ex. 4 pg. 104).

Unfortunately this is not true even if $A$ is a regular local ring.

EXAMPLE 1.3. Let $A = k[\![X, Y, Z]\!]$, $I = (X^2, XY, XZ - Y^7)$. Then $e(A/I) = 8$ but $f_1^* = X^2$, $f_2^* = XY$, $f_3^* = XZ$ is not a regular sequence in $\mathrm{gr}_{\mathfrak{N}}(A)$.

However, if we assume that $r = 1$ or that $f_1, \cdots, f_r$ is a regular sequence in $A$, then the above result holds even with the weaker assumption that $\mathrm{gr}_{\mathfrak{N}}(A)$ is Cohen-Macaulay.

In the following if $M$ is a finitely generated $A$-module $l(M)$ will denote its length.

PROPOSITION 1.4. *Let $(A, \mathfrak{N})$ be a local ring of dimension $d$ and $x$ a parameter in $A$ with $t = v(x)$. Then*

i) $e(A/xA) \geq te(A)$

ii) *If $x^*$ is a non zero divisor in* $\mathrm{gr}_{\mathfrak{N}}(A)$, *then* $e(A/xA) = te(A)$

iii) *If $e(A/xA) = te(A)$, then $x^*$ is a parameter in* $\mathrm{gr}_{\mathfrak{N}}(A)$.

*Proof.* We have exact sequences:

$$0 \longrightarrow A/\mathfrak{N}^n : x \longrightarrow A/\mathfrak{N}^n \longrightarrow A/\mathfrak{N}^n + x \longrightarrow 0$$

$$0 \longrightarrow \mathfrak{N}^n : x/\mathfrak{N}^{n-t} \longrightarrow A/\mathfrak{N}^{n-t} \longrightarrow A/\mathfrak{N}^n : x \longrightarrow 0$$

from which we get:

$$l(A/\mathfrak{N}^n) = l(A/\mathfrak{N}^n + x) + l(A/\mathfrak{N}^{n-t}) - l(\mathfrak{N}^n : x/\mathfrak{N}^{n-t}).$$

Since $\dim A/xA = d - 1$, we get from this that $l(\mathfrak{N}^n : x/\mathfrak{N}^{n-t})$ is for all large $n$ a polynomial $f(n)$ of degree $d - 1$ with leading coefficient $[e(A/xA) - te(A)]/(d - 1)!$. This proves i) and also ii) since if $x^*$ is a non zero divisor in $\mathrm{gr}_{\mathfrak{N}}(A)$, then $x$ is a non zero divisor in $A$ and $\mathfrak{N}^n \cap (xA) = \mathfrak{N}^{n-t}(xA)$. Further $e(A/xA) = te(A)$ if and only if $\deg(f(n)) \leq d - 2$.

Now it is clear that $\dim_k(0\colon x^*)_{n-t-1} = l((\mathfrak{N}^n\colon x)) \cap \mathfrak{N}^{n-t-1}/\mathfrak{N}^{n-t})$ hence, if $e(A/xA) = te(A)$ then, for all large $n$, $\dim_k(0\colon x^*)_{n-t-1}$ is a polynomial in $n$ of degree less or equal to $d-2$. This proves that the $\mathrm{gr}_n(A)$-module $0\colon x^*$ has Krull dimension less or equal to $d-1$, or, which is the same, that the ring $\mathrm{gr}_{\mathfrak{N}}(A)/(0\colon (0\colon x^*))$ has Krull dimension less or equal to $d-1$. Now it is easy to see that this implies $x^*$ is a parameter in $\mathrm{gr}_{\mathfrak{N}}(A)$.

*Remark.* The assertions i) and ii) are well known (see [B]), while iii) has been proved in [S] in the case $t=1$. Here we used many of the central ideas of the original proof.

Since we need to cover also the case where $x$ is not a parameter, we add the following result which is more or less known.

PROPOSITION 1.5. *Let $(A, \mathfrak{N})$ be a local ring of dimension $d$ and $x$ an element in $A$ with $t = v(x)$, such that $\dim A/xA = d$. Then we have:*
 i) $e(A/xA) = e(A)$ *if* $\dim(A/0\colon x) < d$
 ii) $e(A/xA) = e(A) - e(A/0\colon x)$ *if* $\dim(A/0\colon x) = d$

*Proof.* We have an exact sequence

$$0 \longrightarrow xA \longrightarrow A \longrightarrow A/xA \longrightarrow 0\,.$$

Since $xA \simeq A/0\colon x$ the result follows by the additivity of the multiplicity.

COROLLARY 1.6. *Let $(A, \mathfrak{N})$ be a local ring such that $\mathrm{gr}_{\mathfrak{N}}(A)$ is Cohen-Macaulay. If $x$ is an element in $A$ of order $t$, then the following conditions are equivalent:*
 i) $e(A/xA) = te(A)$
 ii) *$x^*$ is not a zero divisor in* $\mathrm{gr}_{\mathfrak{N}}(A)$

*Proof.* If $x^*$ is not a zero divisor in $\mathrm{gr}_{\mathfrak{N}}(A)$, then $X$ is not a zero divisor in $A$, hence $e(A/xA) = te(A)$ by Proposition 1.4. Conversely if $x$ is a non zero divisor in $A$, by Proposition 1.4 we get that $x^*$ is a parameter in $\mathrm{gr}_{\mathfrak{N}}(A)$, hence a non zero divisor. If $x$ is a zero divisor, since $A$ is Cohen-Macaulay, $0\colon x$ is contained in a minimal prime $\mathfrak{p}$ of $A$, hence $\dim A/0\colon x \geq \dim A/\mathfrak{p} = \dim A$ which implies, by Proposition 1.5., $e(A/xA) < e(A)$.

The following example shows that in the above Corollary we cannot delete the condition $\mathrm{gr}_{\mathfrak{N}}(A)$ is Cohen-Macaulay.

EXAMPLE 1.7. Let $A = k[\![t^6, t^7, t^{15}]\!] = k[\![X, Y, Z]\!]/(X^5 - Z^2, Y^3 - XZ)$, then $A$ is a complete intersection domain of multiplicity 6 and $x$ a non zero divisor in $A$. Since $\mathrm{gr}_{\mathfrak{N}}(A)$ is not Cohen-Macaulay and has dimension 1, $x^*$ is a zero divisor in $\mathrm{gr}_{\mathfrak{N}}(A)$ and $e(A/xA) = 6 = e(A)$.

THEOREM 1.8. *Let $(A, \mathfrak{N})$ be a local ring such that $\mathrm{gr}_{\mathfrak{N}}(A)$ is Cohen-Macaulay; if $f_1, \cdots, f_r$ is a regular sequence in $A$ and $v_1, \cdots, v_r$ are positive integers such that $v(f_i) \geq v_i$, then with $R = A(f_1, \cdots, f_r)$, the following conditions are equivalent*:

i) $e(R) = e(A) \prod_{i=1}^{r} v_i$

ii) *$f_1^*, \cdots, f_r^*$ is a regular sequence in $\mathrm{gr}_{\mathfrak{N}}(A)$ and $v(f_i) = v_i$ for all $i = 1, \cdots, r$.*

*Proof.* After Corollary 1.6 we need only to prove that i) implies ii). If $r = 1$, we have $v(f_1)e(A) \leq e(R) = e(A)v_1 \leq e(A)v(f_1)$, hence $v_1 = v(f_1)$ and we can apply Corollary 1.6. We argue by induction on $r$; let $J = (f_1, \cdots, f_{r-1})$, $B = A/J$, $\mathfrak{M} = \mathfrak{N}/J$ and $f = \bar{f}_r$. Then we have:

$$e(R) = e(A) \prod_{i=1}^{r} v_i = e(B/f) \geq v(f)e(B) \geq v(f)e(A) \prod_{i=1}^{r-1} v(f_i)$$
$$\geq e(A) \prod_{i=1}^{r} v(f_i) \geq e(A) \prod_{i=1}^{r} v_i;$$

hence $v_i = v(f_i)$ for all $i$ and $e(B) = e(A) \prod_{i=1}^{r-1} v_i$ which implies by inductive assumption that $f_1^*, \cdots, f_{r-1}^*$ is a regular sequence in $\mathrm{gr}_{\mathfrak{N}}(A)$. Further $\mathrm{gr}_{\mathfrak{M}}(B) = \mathrm{gr}_{\mathfrak{N}}(A)/(f_1^*, \cdots, f_{r-1}^*)$, hence $\mathrm{gr}_{\mathfrak{M}}(B)$ is Cohen-Macaulay; since $e(B/f) = v(f)e(B)$ we get that $f^*$ is a non zero divisor in $\mathrm{gr}_{\mathfrak{M}}(B)$. But since $v(f) = v(f_r)$ this implies $f_r^*$ is a non zero divisor modulo $(f_1^*, \cdots, f_{r-1}^*)$ and the conclusion follows.

## § 2. Reduction to the Artinian case

One of the main tool in the following sections is the reduction to the 0-dimensional case. Thus we are led to consider the problem of lifting a standard base from a quotient ring to the ring itself and conversely. Some results on this topics have been obtained in [R-V]; we recall here what we need in the following.

Let $(A, \mathfrak{N})$ be a local ring, $I$ and $J$ ideals of $A$, and denote by "$-$" reduction modulo $J$ and by "$\sim$" reduction modulo $I$. Let $I = (f_1, \cdots, f_r)$ and $v_i = v(f_i)$.

1. If $f_1, \cdots, f_r$ is a standard base of $I$ and $v_i = v(\bar{f}_i)$ for all $f_i \notin J$, then the following conditions are equivalent:

i) $\bar{f}_1, \cdots, \bar{f}_r$ is a standard base of $\bar{I}$

ii) $\mathfrak{N}^p \cap (I+J) = \sum_i \mathfrak{N}^{p-v_i} f_i + \mathfrak{N}^p \cap J$ for all $p \geq 0$.

iii) There exist elements $x_1, \cdots, x_d$ in $J$ such that $\tilde{x}_1, \cdots, \tilde{x}_d$ is a standard base of $\tilde{J}$ and $v(\tilde{x}_i) = v(x_i)$ for $i = 1, \cdots, d$.

2. Assume that $\bar{f}_1, \cdots, \bar{f}_r$ is a standard base of $\bar{I}$ and $v(\bar{f}_i) = v_i$ for $i = 1, \cdots, r$. If there exists a minimal base $x_1, \cdots, x_d$ of $J$ such that

i) $x_1^*, \cdots, x_d^*$ is a regular sequence in $\mathrm{gr}_{\mathfrak{N}}(A)$

ii) $\tilde{x}_1, \cdots, \tilde{x}_d$ is a regular sequence in $\tilde{A}$

then $f_1, \cdots, f_r$ is a standard base of $I$.

For the proof of these two facts see [R-V], Theorems 2.2 and 2.6. For the rest of the paper $(A, \mathfrak{N})$ *is a regular local ring with an infinite residue field* $k$, $I \subset \mathfrak{N}^2$ *a codimension* $h$ *ideal of* $A$ *such that* $(R = A/I$, $\mathfrak{M} = \mathfrak{N}/I)$ *is a Cohen-Macaulay local ring of dimension* $d$.

Let $x_1, \cdots, x_d$ be a minimal reduction modulo $I$ and $J = (x_1, \cdots, x_d)$, then $\tilde{x}_1, \cdots, \tilde{x}_d$ is a regular sequence in $R$ and $e(R) = e(R/\tilde{J}) = l(R/\tilde{J})$. Further since $J \cap I = IJ$, we have a canonical isomorphism of $k$-vector spaces $I/I\mathfrak{N} \simeq \bar{I}/\bar{I}\bar{\mathfrak{N}}$. We denote by $\bar{R}$ the ring $\bar{A}/\bar{I} = A/I + J = R/\tilde{J}$ and call it an artinian reduction of $R$.

3. $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay if and only if $\tilde{x}_1, \cdots, \tilde{x}_d$ is a standard base of $\tilde{J}$. A proof of this result can be found in [Ro], Proposition 2.4. However it is also a trivial consequence of the fact that for every minimal reduction $a_1, \cdots, a_d$ of the maximal ideal of the Cohen-Macaulay ring $(R, \mathfrak{M})$ one has $\mathfrak{M}^{n+1} = (a_1, \cdots, a_d)\mathfrak{M}^n$ for some $n \geq 0$, hence $a_1^*, \cdots, a_d^*$ is a system of parameters in $\mathrm{gr}_{\mathfrak{M}}(R)$.

Collecting all these facts one can prove the following theorem which will be used extensively for the rest of the paper. As before we denote by "$-$" reduction modulo $J$ and by "$\sim$" reduction modulo $I$.

THEOREM 2.1. *If for some integer* $t \geq 2$ *we have* $I \subset \mathfrak{N}^t$, *then the following conditions are equivalent*:

i) *$I$ is $t$-isomultiple and* $\mathrm{gr}_{\mathfrak{M}}(R)$ *is Cohen-Macaulay*

ii) *$\bar{I}$ is $t$-isomultiple.*

*Proof.* Since $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay, by 3 $\tilde{x}_1, \cdots, \tilde{x}_d$ is a standard base of $\tilde{J}$ and of course $v(\tilde{x}_i) = v(x_i) = 1$. Hence condition iii) of 1 holds and it suffices now to check that if $f \in I$ and $v(f) = t$ then $v(\bar{f}) = t$. But if $f \in \mathfrak{N}^{t+1} + J$, then $f = \sum_i a_i x_i + b$ with $b \in \mathfrak{N}^{t+1}$; it follows that $\sum_i \tilde{a}_i \tilde{x}_i \in \tilde{J} \cap \tilde{\mathfrak{N}}^{t+1} = \tilde{J}\tilde{\mathfrak{N}}^t$, hence $\sum_i \tilde{a}_i \tilde{x}_i = \sum_i \tilde{b}_i \tilde{x}_i$ with $\tilde{b}_i \in \tilde{\mathfrak{N}}^t$. Since

$\tilde{x}_1, \cdots, \tilde{x}_d$ is a regular sequence in $\tilde{A} = R$, this implies for all $i = 1, \cdots, d$, $a_i - b_i = \sum_{j \neq i} c_{ij} x_j + d_i$ with $c_{ij} = -c_{ji}$ and $d_i \in I$. Thus $\sum_i a_i x_i - \sum_i b_i x_i \in I\mathfrak{N}$, hence $\sum_i a_i x_i \in \mathfrak{N}^{t+1}$ which implies $f \in \mathfrak{N}^{t+1}$, a contradiction.

Conversely let $\bar{f}_1, \cdots, \bar{f}_r$ be a standard base of $\bar{I}$ with $t = v(\bar{f}_i)$, $i = 1, \cdots, r$. This implies $t = v(f_i)$, $i = 1, \cdots, r$; since $x_1, \cdots, x_d$ is a minimal base of $J$ such that $x_1^*, \cdots, x_d^*$ is a regular sequence in $\operatorname{gr}_{\mathfrak{N}}(A)$ and $\tilde{x}_1, \cdots, \tilde{x}_d$ is a regular sequence in $\tilde{A} = R$, we can apply 2 and get that $f_1, \cdots, f_r$ is a standard base of $I$. Now, again by 1, we get $\mathfrak{N}^p \cap (I + J) = \sum_i \mathfrak{N}^{p-v_i} f_i + \mathfrak{N}^p \cap J = \sum_i \mathfrak{N}^{p-v_i} f_i + \mathfrak{N}^{p-1} J$ for all $p \geq 0$; hence $(I + J) \cap (\mathfrak{N}^p + I) = I + \mathfrak{N}^{p-1} J$ which means that $\tilde{x}_1, \cdots, \tilde{x}_d$ is a standard base of $\tilde{J}$ and $\operatorname{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay.

*Remark* 2.2. It is clear that if $\operatorname{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay, the ideal $I$ does not need to be $t$-isomultiple for some $t$. For example if $A = k[\![X, Y, Z]\!]$ and $I = (X^3 - YZ, Y^3 - X^2Z, Z^2 - XY^2)$, then $R = k[\![t^4, t^5, t^7]\!]$ and thus $\operatorname{gr}_{\mathfrak{M}}(R) = k[X, Y, Z]/(YZ, Y^3 - X^2Z, Z^2)$ is Cohen-Macaulay, but $I^*$ is not generated by elements of the same degree.

*Remark* 2.3. We don't know if the condition "$I$ is $t$-isomultiple" implies that $\operatorname{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay. In [H-R-V] we gave an example of a 3-isomultiple ideal such that $R$ and $\operatorname{gr}_{\mathfrak{M}}(R)$ have different Betti numbers (here and in the following the Betti numbers of the $A$-module $M$ are the ranks of the free modules in a minimal free resolution of $M$). However, in this example, $\operatorname{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay (see Remark 4.11).

## § 3. *t*-extremal and almost *t*-extremal ideals

As before $(A, \mathfrak{N})$ is a regular local ring and $I$ a perfect ideal of $A$ of codimension $h$ such that $(R = A/I, \mathfrak{M} = \mathfrak{N}/I)$ has dimension $d$. It is well known that $e(R) \geq h + 1$ and, if the equality holds, then $I$ is 2-isomultiple and $\operatorname{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay (see [Sa$_1$]).

If $I \subset \mathfrak{N}^t$ with $t \geq 3$, then this bound is not sharp. If $x_1, \cdots, x_d$ is a minimal reduction modulo $I$, then we know that $e(R) = e(\bar{R}) = l(\bar{A}/\bar{\mathfrak{N}}^t) + l(\bar{\mathfrak{N}}^t/\bar{I}) = \binom{h+t-1}{h} + l(\bar{\mathfrak{N}}^t/\bar{I})$; thus $e(R) \geq \binom{h+t-1}{h}$ and the equality holds if and only if $\bar{I} = \bar{\mathfrak{N}}^t$. Of course this bound is sharp since for all $t$, the ring $R = k[\![X_1, \cdots, X_h]\!]/(X_1, \cdots, X_h)^t$ has multiplicity $\binom{h+t-1}{h}$.

Definition 3.1. We say that the perfect codimension $h$ ideal $I$ is *t-extremal* if $I \subset \mathfrak{N}^t$ and $e(R) = \binom{h+t-1}{h}$.

Thus Sally's theorem says that 2-extremal ideals are 2-isomultiple. On the other hand if $h = 2$ and $v = v(I)$ is the minimal number of generators of $I$, then by the Hilbert-Burch theorem we have $I \subset \mathfrak{N}^{v-1}$. Recently Elias proved that if $I$ is $(v-1)$-extremal, then $I$ is $(v-1)$-isomultiple (see [E]).

Both these results are particular cases of the following general result which clarifies also the connection between the notion of $t$-extremal ideal and that of $t$-extremal Cohen-Macaulay graded ring introduced by Schenzel in [Sch].

Recall that if $G = k[X_1, \cdots, X_n]/J$ is a Cohen-Macaulay standard $k$-algebra of dimension $d$ and codimension $h$, we denote by $H(G, n) = \dim_k G_n$ and $h(G, n)$ respectively the Hilbert function and the Hilbert polynomial of $G$. We define the *index of regularity of* $G$ *as*:

$$i(G) = \max\{n \in Z / H(G, n) \neq h(G, n)\} + 1 .$$

Further let $t$ be the initial degree of $J$, which is the minimum degree of the generators of J. Schenzel proved that $i(G) + d \geq t$.

DEFINITION 3.2. (see [Sch]) We say that $G$ is $t$-extremal if $i(G) + d = t$.

It follows from the paper of Schenzel that $G$ is $t$-extremal if and only if $G$ is Cohen-Macaulay and $P(G, z) = (1-z)^{-d} \sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i$ where $P(G, z)$ is the Poincare series of $G$, which is by definition the series $\sum_{n \geq 0} H(G, n) z^n$.

*Remark.* We note that $G$ is a $t$-extremal graded Cohen-Macaulay ring if and only if the function $H_G^{-d}$ is maximal according to the definition given by Orecchia in [O]. Also $G$ is a $t$-extremal graded Cohen-Macaulay ring if and only if $G$ is compressed of type $\binom{h+t-2}{t-1} z^{t-1}$, according to the definition given by Fröberg and Laksov in [F-L] (see also [I]).

THEOREM 3.3. *For a perfect codimension $h$ ideal $I$ of $A$, the following conditions are equivalent*:

i) *$I$ is $t$-extremal*

ii) *$I$ is $t$-isomultiple*, $\mathrm{gr}_{\mathfrak{M}}(R)$ *is Cohen-Macaulay and* $v(I) = \binom{h+t-1}{t}$

iii) $P(R, z) = (1 - z)^{-d} \sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i$

iv) $\mathrm{gr}_{\mathfrak{M}}(R)$ *is t-extremal*

*Proof.* Let $I$ be $t$-extremal; then $\bar{I} = \bar{\mathfrak{N}}^t$, hence $\bar{I}$ is $t$-isomultiple and $v(\bar{I}) = \binom{h+t-1}{t} = v(I)$. Using Theorem 2.1, we get that $I$ is $t$-isomultiple and $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay.

If condition ii) holds we have $\mathrm{gr}_{\mathfrak{M}}(R) = k[X_1, \cdots, X_{d+h}]/I^*$ is Cohen-Macaulay, hence we may assume $X_{h+1}, \cdots, X_{d+h}$ is a regular sequence modulo $I^*$. Now the condition $v(I) = \binom{h+t-1}{t}$ implies

$$\mathrm{gr}_{\mathfrak{M}}(R)/(\bar{X}_{h+1}, \cdots, \bar{X}_{d+h}) \simeq k[X_1, \cdots, X_h]/(X_1, \cdots, X_h)^t$$

hence

$$P(R, z) = P(\mathrm{gr}_{\mathfrak{M}}(R), z) = (1 - z)^{-d} \sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i .$$

On the other hand if

$$P(R, z) = (1 - z)^{-d} \sum_{i=0}^{t-1} \binom{h+i-1}{i} z_i$$

then

$$e(R) = \sum_{i=0}^{t-1} \binom{h+i-1}{i} = \binom{h+t-1}{h};$$

but $\sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i \equiv (1-z)^{-h} \bmod (z^t t)$, hence $P(R, z) \equiv P(A, z) \bmod (z^t)$ which implies $I \subset \mathfrak{N}^t$ and $I$ is $t$-extremal, so that iii) implies i). Using Schenzel's results, we conclude the proof of the theorem.

*Remarks.* 1. It is clear from the proof of the theorem that if $I$ is $t$-extremal, then $R$ and $\mathrm{gr}_{\mathfrak{M}}(R)$ have the same Betti numbers, namely the Betti numbers of $k[X_1, \cdots, X_h]/(X_1, \cdots, X_h)^t$.

2. In condition ii) we dannot delete the hypothesis on the number of generators of $I$: if $A = [\![X, Y]\!]$ and $I = (X^3, X^2Y, Y^3)$, then $h = 2$, $I$ is 3-isomultiple but $e(R) = 7$.

3. The equivalence between i) and ii) has been proved by Orecchia in [O] in the case $d = 1$.

4. If $h = 2$, $t = v(I) - 1$ and $I$ is $t$-isomultiple then, by the main result of [R-V], $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay (see Remark 2.3 and [E]).

5. As remarked in [O], the above theorem applies to various classes of affine space curves, locally requiring an arbitrary large number of

generators (see [Mc], [Mo], [Ma]). It turns out that the tangent cone at the origin of these curves is projectively Cohen-Macaulay.

6. It is worthy to remark that if a graded $k$-algebra $G$ has Poincare series $P(G, z) = (1-z)^{-d} \sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i$, then $G$ is not necessarily Cohen-Macaulay. In fact let $G = k[X, Y, Z]/(X^2, XY, XZ, Y^3)$, then $P(G, z) = (1+2z)(1-z)^{-1}$ but $G$ is not Cohen-Macaulay.

7. It is clear that if $I$ is a principal ideal such that $I \subset \mathfrak{N}^t$, then $e(R) = t$. We remark that for a perfect ideal $I$ also the converse holds; in fact if $I \subset \mathfrak{N}^t$ with $t \geq 2$, $e(R) = t$ and $h \geq 2$, then $t \geq \binom{h+t-1}{h} \geq t(t+1)/2$, so that $t \leq 1$, a contradiction. Thus $h = 1$ and $I$ is principal.

The case of local rings $R$ with multiplicity $h+2$ has been extensively studied by Sally in [$\mathrm{Sa}_2$]. She proved that if the Cohen-Macaulay type of $R$ is strictly less than $h$, then $I$ is 2-isomultiple and $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay. This suggested us to consider the following class of perfect ideals.

DEFINITION 3.4. We say that the perfect codimension $h$ ideal $I$ is *almost t-extremal* if $I \subset \mathfrak{N}^t$ and $e(R) = \binom{h+t-1}{h} + 1$.

EXAMPLE 3.5. (Sally) Let $R = k[\![t^4, t^5, t^{11}]\!]$ then

$$\mathrm{gr}_{\mathfrak{M}}(R) = k[X, Y, Z)/(Z^2, XZ, YZ, Y^4)$$

so that $I$ is almost 2-extremal but $\mathrm{gr}_{\mathfrak{M}}(R)$ is not Cohen-Macaulay.

EXAMPLE 3.6. (Sally) Let $R = k[\![t^5, t^6, t^8, t^9]\!]$ then

$$\mathrm{gr}_{\mathfrak{M}}(R) = k[X, Y, Z, W]/(XW - YZ, ZW, Z^2, W^2, YW, X^2Z - Y^3).$$

In this case $I$ is almost 2-extremal, $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay but $I$ is not 2-isomultiple.

In the next lemma we collect some properties of almost $t$-extremal ideals. We denote by $\tau(R)$ the Cohen-Macaulay type of $R$.

LEMMA 3.7. *Let $I$ be an almost $t$-extremal ideal such that* $\dim R = \dim A/I = 0$. *Then*

i) $l(\mathfrak{N}^t/I) = 1$

ii) $\mathfrak{N}^{t+1} \subset I$

iii) $\tau(R) \leq \binom{h+t-2}{t-1}$ *and equality holds if and only if* $l(\mathfrak{N}^{t-1}/I: \mathfrak{N}) = 1$

iv) $\tau(R) = \tau(\mathrm{gr}_{\mathfrak{M}}(R))$

*Proof.* i) We have $e(R) = l(A/I) = l(A/\mathfrak{N}^t) + l(\mathfrak{N}^t/I) = \binom{h+t-1}{h} + l(\mathfrak{N}^t/I)$, hence $l(\mathfrak{N}^t/I) = 1$.

ii) We have $\mathfrak{N}^t \supset \mathfrak{N}^{t+1} + I \supset I$, hence $1 = l(\mathfrak{N}^t/I) = l(\mathfrak{N}^t/\mathfrak{N}^{t+1} + I) + l(\mathfrak{N}^{t+1} + I/I)$. Since, by Nakayama, $\mathfrak{N}^t \neq \mathfrak{N}^{t+1} + I$, we must have $\mathfrak{N}^{t+1} \subset I$.

iii) We have $\mathfrak{N}^{t-1} \supset I: \mathfrak{N} \supset I$ and $I \subset \mathfrak{N}^t \subset \mathfrak{N}^{t-1}$, hence $\binom{h+t-2}{t-1} + 1 = l(\mathfrak{N}^{t-1}/I: \mathfrak{N}) + \tau(R)$. Since $\mathfrak{N}^t \not\subset I$, we have $\mathfrak{N}^{t-1} \neq I; \mathfrak{N}$, hence the conclusion follows.

iv) It is clear that we always have $\tau(R) \leq \tau(\mathrm{gr}_{\mathfrak{M}}(R))$. On the other hand if $\tilde{x}^* \in \mathfrak{M}^p/\mathfrak{M}^{p+1}$ is an element of $0: \mathrm{gr}_{\mathfrak{M}}(R)_+$, then $\tilde{x}\mathfrak{M} \subset \mathfrak{M}^{p+2}$, hence $x\mathfrak{N} \subset I + \mathfrak{N}^{p+2}$. Now if $p < t-1$ then $p+2 \leq t$, hence $I \subset \mathfrak{N}^{p+2}$ which implies $x \in \mathfrak{N}^{p+1}$, a contradiction. Thus $p \geq t-1$ and we get $\mathfrak{N}^{p+2} \subset \mathfrak{N}^{t+1} \subset I$, so that $x\mathfrak{N} \subset I$ and $\tilde{x} \in 0: \mathfrak{M}$. This gives the conclusion.

PROPOSITION 3.8. *Let $I$ be a perfect, codimension $h$ ideal in $A$*

$$P(R, z) = (1-z)^{-d}\left[\sum_{i=0}^{t-1}\binom{h+i-1}{i}z^i + z^t\right]$$

*if and only if $I$ is almost $t$-extremal and $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay.*

*Proof.* Let $\mathrm{gr}_{\mathfrak{M}}(R)$ be Cohen-Macaulay, $J = (x_1, \cdots, x_d)$ a minimal reduction modulo $I$. Then we have

$$\begin{aligned} P(R, z) &= P(\mathrm{gr}_{\mathfrak{M}}(R), z) = (1-z)^{-d}P(\mathrm{gr}_{\mathfrak{M}}(R)/(\tilde{x}_1^*, \cdots, \tilde{x}_d^*), z) \\ &= (1-z)^{-d}P(\mathrm{gr}_{\mathfrak{M}/\tilde{J}}(R/\tilde{J}), z), \end{aligned}$$

hence we may assume $\dim R = 0$. But then, if $I$ is almost $t$-extremal, we have, by the above lemma, $\mathfrak{N}^{t+1} \subset I$ and the conclusion follows. Conversely, if

$$P(R, z) = (1-z)^{-d}\left[\sum_{i=0}^{t-1}\binom{h+i-1}{i}z^i + z^t\right],$$

then $e(R) = \sum_{i=0}^{t-1}\binom{h+i-1}{i} + 1 = \binom{h+t-1}{h} + 1$. Also $P(R, z) \equiv P(A, z) \bmod (z^t)$, which implies $I \subset \mathfrak{N}^t$, so that $I$ is almost $t$-extremal. In order to prove $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay, we need to show $\mathfrak{M}^p \cap \tilde{J} = \mathfrak{M}^{p-1}\tilde{J}$ for all $p \geq 1$. Now if $p \leq t$, then $I \subset \mathfrak{N}^t \subset \mathfrak{N}^p$ and we have $\mathfrak{M}^p \cap \tilde{J} = \mathfrak{N}^p \cap (I + J)/I = (I + \mathfrak{N}^p \cap J)/I = (I + \mathfrak{N}^{p-1}J)/I = \mathfrak{M}^{p-1}\tilde{J}$. On the other hand we have $\mathfrak{M} \supset \tilde{J} \supset \tilde{J}\mathfrak{M}^t$ and $\tilde{J}\mathfrak{M}^s \subset \mathfrak{M}^{t+1} \subset \mathfrak{M}$, hence $l(\mathfrak{M}^{t+1}/\tilde{J}\mathfrak{M}^t) = l(\mathfrak{M}/\tilde{J}) + l(\tilde{J}/\tilde{J}\mathfrak{M}^t) - l(\mathfrak{M}/\mathfrak{M}^{t+1})$. Now

$$\begin{aligned} l(\tilde{J}/\tilde{J}\mathfrak{M}^t) &= l(J + I/J\mathfrak{N}^t + I) = l(J/(J\mathfrak{N}^t + I \cap J)) = l(J/J\mathfrak{N}^t) \\ &= \textstyle\sum_{i=0}^{t-1} \dim \mathrm{gr}_{\mathfrak{N}}(J)_i = \sum_{i=0}^{t-1} \dim \mathrm{gr}_{\mathfrak{N}}(J)(-1)_{i+1} \\ &= \text{(by [H-R-V], Lemma 3)} \end{aligned}$$

$$\textstyle\sum_{i=0}^{t-1} \dim (J^*)_{i+1} = \sum_{i=0}^{t} \dim (J^*)_i = [(1/(1-z)^{d+h}) - (1/(1-z)^h)]_0^t$$

where if $F(z) = \sum_{i \geq 0} a_i z^i$ is a power series, we let $[F(z)]_0^t = \sum_{i=0}^t a_i$. Thus if $f(z) = \sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i + z^t$ we get

$$\begin{aligned} l(\mathfrak{M}^{t+1}/\tilde{J}\mathfrak{M}^t) &= [f(z)]_1^t + [(1/(1-z)^{d+h}) - (1/(1-z)^h)]_0^t - [f(z)/(1-z)^d]_1^t \\ &= [(f(z)(1-z)^{d+h} + 1 - (1-z)^d - f(z)(1-z)^h)/(1-z)^{d+h}]_0^t \\ &= [(1-(1-z)^d)(1-f(z)(1-z)^h)/(1-z)^{d+h}]_0^t. \end{aligned}$$

Now $f(z) = 1/(1-z)^h \bmod (z^t)$, hence $(1-(1-z)^d)(1-f(z)(1-z)^h) \equiv 0 \bmod (z^{t+1})$, from which we deduce $l(\mathfrak{M}^{t+1}/\tilde{J}\mathfrak{M}^t) = 0$. This implies $\mathfrak{M}^{r+1} = \tilde{J}\mathfrak{M}^r$ for all $r \geq t+1$ and the conclusion follows.

*Remark* 3.9. It is worthy to remark that if a graded $k$-algebra $G$ of dimension $d$ and codimension $h$ has Poincare series

$$P(G, z) = (1-z)^{-d}\left[\textstyle\sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i + z^t\right],$$

then $G$ is not necessarily Cohen-Macaulay. In fact let

$$G = k[X, Y, Z]/(X^2, XY, XZ^2, Y^4),$$

then $G$ has codimension 2, dimension 1 and $P(G, z) = (1 + 2z + z^2)/(1-z)$ but $G$ is not Cohen-Macaulay.

THEOREM 3.10. *For a perfect codimension $h$ ideal $I$ of $A$, the following conditions are equivalent*:

i) *$I$ is almost $t$-extremal and* $\tau(R) < \binom{h+t-2}{t-1}$

ii) *$I$ is $t$-isomultiple*, $\mathrm{gr}_{\mathfrak{M}}(R)$ *is Cohen-Macaulay and*

$$v(I) = \binom{h+t-1}{t} - 1$$

iii) $P(R, z) = (1 - z)^{-d}\left[\sum_{i=0}^{t-1}\binom{h+i-1}{i}z^i + z^t\right]$ *and*

$$\tau(R) < \binom{h+t-2}{t-1}.$$

*Further if one of the above equivalent conditions holds, then* $\tau(R) = \tau(\mathrm{gr}_{\mathfrak{M}}(R))$

*Proof.* We prove i) implies ii). If $I$ is almost $t$-extremal, then $\bar{I}$ is almost $t$-extremal hence, using again Theorem 2.1 and the equalities $v(I) = v(\bar{I})$ and $\tau(R) = \tau(\bar{R})$, we may assume $\dim R = 0$. Let $\mathfrak{N} = (x_1, \cdots, x_h)$; since $l(\mathfrak{N}^t/I) = 1$ there exists a monomial $m$ of degree $t$ in $x_1, \cdots, x_h$ such that $m \notin I$, hence $\mathfrak{N}^t = (I, m)$.

We claim that if $\tau(R) < \binom{h+t-2}{t-1}$, then $m\mathfrak{N} \subset I\mathfrak{N}$; this would imply $\mathfrak{N}^{t+1} = I\mathfrak{N}$, from which it follows that $\mathfrak{N}^{t+k} \cap I = \mathfrak{N}^k I$ for every $k \geq 0$ and also $v(I) = l(I/I\mathfrak{N}) = l(\mathfrak{N}^t/\mathfrak{N}^{t+1}) - l(\mathfrak{N}^t/I) = \binom{h+t-1}{t} - 1$ which is exactly what we need for our implication.

We prove the claim by contradiction along the following line:

$$mx_1 \notin I\mathfrak{N} \Longrightarrow x_1^{t+1} \notin I\mathfrak{N} \Longrightarrow \tau(R) = \binom{h+t-2}{t-1}$$

As for the first implication, it is enough to prove that if $m = x_j p$ and $mx_1 \notin I\mathfrak{N}$, then $px_1^2 \notin I\mathfrak{N}$. But we have $m$, $x_1 p \in \mathfrak{N}^t \backslash I$, hence for some $c \notin \mathfrak{N}$, $m - cx_1 p \in I$; thus $cx_1^2 p = x_1 m - x_1(m - cx_1 p) \notin I\mathfrak{N}$ as required.

Now let $x_1^{t+1} \notin I\mathfrak{N}$; after reordering the $x_i$'s, we may assume $x_1^t, x_1^{t-1}x_2, \cdots, x_1^{t-1}x_s \notin I$ and $x_1^{t-1}x_{s+1}, \cdots, x_1^{t-1}x_h \in I$ for some $s \geq 1$. We are going to prove that $\mathfrak{N}^{t-1} = I : \mathfrak{N} + (x_1^{t-1})$ which implies by Lemma 3.7 iii), that $\tau(R) = \binom{h+t-2}{t-1}$.

*Step* 1. If $p \in \mathfrak{N}^t \backslash I$, then $px_j \notin I\mathfrak{N}$ for every $j \leq s$.

In fact for some $a, b \notin \mathfrak{N}$ we have $x_1^t - ax_1^{t-1}x_j$, $x_1^t - bp \in I$, hence $x_1^{t+1} = x_1(x_1^t - ax_1^{t-1}x_j) + ax_j(x_1^t - bp) + abpx_j$. Since $x_1^{t+1} \notin I\mathfrak{N}$, this implies $px_j \notin I\mathfrak{N}$.

*Step* 2. If $\sum_{i=1}^h n_i = t$, then we have $p = x_1^{n_1} \cdots x_h^{n_h} \in I$ if and only if $\exists\, r > s$ with $n_r > 0$.

If $n_r > 0$ with $r > s$ and $p \notin I$, then by repeated use of step 1, we get $x_1^{t-1}x_r \notin I$, a contradiction. Conversely, if $n_r = 0$ for every $r > s$ and $n_k > 0$ with $1 \leq k \leq s$, since $x_1^{t-1}x_k \notin I$, we get, by repeated use of step 1, $p \notin I$.

*Step* 3. $\mathfrak{N}^{t-1} = I\colon \mathfrak{N} + (x_1^{t-1})$

If $m = x_1^{n_1} \cdots x_h^{n_h}$ with $\sum_i n_i = t - 1$ and $n_r > 0$ for some $r > s$, then $m \in I\colon \mathfrak{N}$ by step 2. Let $n_r = 0$ for every $r > s$; then, again by step 2, we have $x_1 m \notin I$, hence $x_1^t - a x_1 m \in I$ for some $a \notin \mathfrak{N}$. We claim that $x_1^{t-1} - am \in I\colon \mathfrak{N}$ as required. But, if for some $j$ we have $x_j(x_1^{t-1} - am) \notin I$, then for some $b \notin \mathfrak{N}$ we have $x_1^t - b x_j(x_1^{t-1} - am) \in I$, hence

$$x_1^{t+1} = x_1[x_1^t - b x_j(x_1^{t-1} - am)] + b x_j(x_1^t - a x_1 m) \in I\mathfrak{N},$$

a contradiction.

We prove now ii) implies iii).

If condition ii) holds, then $\mathrm{gr}_{\mathfrak{M}}(R) = k[X_1, \cdots, X_{d+h}]/I^*$ is Cohen-Macaulay, hence we may assume $X_{h+1}, \cdots, X_{h+d}$ is a regular sequence modulo $I^*$; thus we have $P(R, z) = P(\mathrm{gr}_{\mathfrak{M}}(R), z) = (1-z)^{-d} P(k[X_1, \cdots, X_h]/\mathfrak{A}, z)$ where $\mathfrak{A}$ is an homogeneous ideal of codimension $h$, generated by $\binom{h+t-1}{t} - 1$ elements of degree $t$ in $S = k[X_1, \cdots, X_h]$. If we prove that $S_{t+1} \subset \mathfrak{A}$, then we get $P(S/\mathfrak{A}, z) = \sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i + z^t$ as required.

Now we can find a term ordering on the set of monomials in $S$ and a suitable linear changing of coordinates, such that if $X_i < X_j$ for all $j > i$, then all the monomials of degree $t$ in $S$, save $X_1^t$, are in $M(\mathfrak{A})$, the ideal generated by the maximum monomials of elements of $\mathfrak{A}$. Now since $S/\mathfrak{A}$ and $S/M(\mathfrak{A})$ have the same Hilbert function, $\mathfrak{A}$ and $M(\mathfrak{A})$ have the same codimension, hence $M(\mathfrak{A})$ must contain some other monomials. By a theorem of Giusti (see [G], Theorem 2.6), we can prove that $X_1^{t+1} \in M(\mathfrak{A})$, hence $S_{t+1} = M(\mathfrak{A})_{t+1} = \mathfrak{A}_{t-1}$ as required.

In order to complete the proof of ii) implies iii), we must show that if $\mathfrak{A}$ is a 0-dimensional homogeneous ideal of $S = k[X_1, \cdots, X_h]$ generated by forms of degree $t$ such that $v(\mathfrak{A}) = \binom{h+t-1}{t} - 1$ and $P(S/\mathfrak{A}, z) = \sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i + z^t$, then $\tau(S/\mathfrak{A}) < \binom{h+t-2}{t-1}$. Since $\dim S_t/\mathfrak{A}_t = 1$ and the socle of $S/\mathfrak{A}$ is concentred in degree $t-1$ and $t$, it is enough to show that $\dim(\mathfrak{A}\colon S_1)_{t-1} < \binom{h+t-2}{t-1} - 1$. Let us assume, by contradiction, that $\dim(\mathfrak{A}\colon S_1)_{t-1} = \dim S_{t-1} - 1$.

*Step* 1. If $P$, $Q$ are monomials of degree $t-1$ such that for some $i$ and $j$, $X_i P$, $X_j Q \notin \mathfrak{A}$, then $X_j P \notin \mathfrak{A}$.

We have $P, Q \notin \mathfrak{A}: S_1$. Now if $P = Q$, then $X_jQ = X_jP \notin \mathfrak{A}$, otherwise $P + aQ \in \mathfrak{A}: S_1$ for some $a \in k$; hence $X_j(P + aQ) \in \mathfrak{A}$ and this implies $X_jP \notin \mathfrak{A}$.

*Step* 2. For some $i \geq 1$, $X_i^t \notin \mathfrak{A}$.

If $X_1^t, \cdots, X_h^t \in \mathfrak{A}$, then by step 1 we get $S_t = \mathfrak{A}_t$, a contradiction to $\dim S_t/\mathfrak{A}_t = 1$.

After reordering the $X_i$'s, we may assume $X_1^t, \cdots, X_s^t \notin \mathfrak{A}$, $X_{s+1}^t, \cdots, X_h^t \in \mathfrak{A}$ for some $s \geq 1$.

*Step* 3. If $M$ is a monomial of degree $t$, then $M \in \mathfrak{A}$ if and only if $M \in (X_r)$ for some $r > s$.

If $r > s$ and $M = X_rP \notin \mathfrak{A}$, then by step 1, $X_r^t \notin \mathfrak{A}$, a contradiction. Conversely if $M$ is monomial in $X_1, \cdots, X_s$, then by repeated use of step 1, it is clear that $M \notin \mathfrak{A}$.

Now for every $j = 2, \cdots, s$ we have $X_1^{t-2}X_j \notin (\mathfrak{A}: S_1)_{t-1}$ by step 3, hence we can find $c_1 = 1, c_2, \cdots, c_s \in k - \{0\}$ such that $X_1^{t-1} - c_jX_1^{t-2}X_j \in (\mathfrak{A}: S_1)_{t-1}$.

*Step* 4. Let $Q$ be a monomial of degree $t-2$ such that for some $c \in k$, $X_1^{t-1} - cX_1Q \in \mathfrak{A}: S_1$. Then, for every $j = 1, \cdots, s$, we have $X_1^{t-1} - cc_jX_jQ \in \mathfrak{A}: S_1$.

We have $X_j(X_1^{t-1} - cX_1Q) \in \mathfrak{A}$ and $X_1(X_1^{t-1} - c_jX_1^{t-2}X_j) \in \mathfrak{A}$, hence $X_1^t - cc_jX_1X_jQ \in \mathfrak{A}$. If $P = X_1^{t-1} - cc_jX_jQ \notin \mathfrak{A}: S_1$, $X_1P \in \mathfrak{A}$ and for some $d \in k$, $X_1^{t-1} - dP \in \mathfrak{A}: S_1$. Hence $X_1(X_1^{t-1} - dP) = X_1^t - dX_1P \in \mathfrak{A}$, which implies $X_1^t \in \mathfrak{A}$, a contradiction.

If $M = X_1^{n_1} \cdots X_s^{n_s}$ with $\sum_i n_i = t$ and $n_1 < t$, then we let $c_M = \prod_{i \geq 2} c_i^{n_i}$.

*Step* 5. $X_1^t - c_MM \in \mathfrak{A}$.

If $n_1 = t-1$ and $M = X_1^{t-1}X_j$, then

$$X_1^t - c_MM = X_1^t - c_jX_1^{t-1}X_j = X_1(X_1^{t-1} - c_jX_1^{t-2}X_j) \in \mathfrak{A}.$$

If $n_1 < t-1$ and $M = X_pX_qN$, then $X_1^{t-1} - c_qX_1^{t-2}X_q \in \mathfrak{A}: S_1$, hence by repeated use of step 4, we get $X_1^{t-1} - (c_M/c_p)X_qN \in \mathfrak{A}: S_1$. Thus $X_pX_1^{t-1} - (c_M/c_p)M \in \mathfrak{A}$ which implies $X_1^t - c_MM \in \mathfrak{A}$, since $c_pX_1^{t-1}X_p \equiv X_1^t \bmod \mathfrak{A}$.

*Step* 6. height $(\mathfrak{A}) \leq h-1$, a contradiction.

By step 5 we have $X_1^t - c_MM \in \mathfrak{A}$ for every $M$ of degree $t$ in $X_1, \cdots, X_s$, $M \neq X_1^t$, while, by step 3, all the other monomials of degre $t$ are in $\mathfrak{A}$.

Since $v(\mathfrak{A}) = \binom{h+t-1}{t} - 1$, this implies

$$\mathfrak{A} \subset (X_1^t - c_M M, X_{s+1}, \cdots, X_h) \subset (X_1 - c_2X_2, \cdots, X_1 - c_sX_s, X_{s+1}, \cdots, X_h)$$

hence

$$\begin{aligned} \text{height } (\mathfrak{A}) &\leq \text{height } (X_1 - c_2X_2, \cdots, X_1 - c_sX_s, X_{s+1}, \cdots, X_h) \\ &\leq h - s + s - 1 = h - 1 . \end{aligned}$$

Finally iii) implies i) by Proposition 3.8, ii).

As for the last assertion of the theorem, if one of the equivalent conditions holds then $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay, hence we may assume $\dim R = 0$ and apply Lemma 3.7., iv).

*Remarks* 1. The implication i) $\Rightarrow$ ii) has been proved by J. Sally in the case $t = 2$ (see [Sa$_2$]). The first example where our result applies is the following. Let $R = k[\![t^7, t^8, t^{10}]\!]$, then we have $I \subset \mathfrak{N}^3$, $e(R) = 7 = \binom{2+3-1}{2} + 1$ and $\tau(R) \leq 2$, hence $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay and $I$ is 3-isomultiple.

2. If $I$ is almost 2-extremal and $R$ is Gorenstein, then it is easy to see that $\mathrm{gr}_{\mathfrak{M}}(R)$ is compressed of type $z^2$, hence $R$ and $\mathrm{gr}_{\mathfrak{M}}(R)$ have the same Betti numbers (see [F-L]). We do not know if the same result holds for an almost $t$-extremal ideal $I$ such that $1 < \tau(R) < \binom{h+t-2}{t-1}$. We can only remark that if $\tau(R) > \binom{h+t-2}{t-1} - h + 1$ then $\mathrm{gr}_{\mathfrak{M}}(R)$ is not compressed.

## § 4. Deformation of isomultiple ideals and Gorenstein ideals

We have seen that if $I$ is a perfect codimension two ideal of the regular local ring $(A, \mathfrak{N})$ then $e(R) \geq \binom{v(I)}{2}$ and, if equality holds, $I$ is $(v(I) - 1)$-isomultiple and $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay. Recently, Elias and Iarrobino proved that if $I$ is a Gorenstein codimension three ideal of $A$, then $e(R) \geq [v(I)^3 - v(I)]/24$ (see [E-I]).

In both cases the lower bound for the multiplicity of $R$ is given by the multiplicity of the "generic" case. Thus we are led to consider ideals which can be obtained, in a sense easily made precise, as the specializaton of $t$-isomultiple "generic" ideals.

DEFINITION 4.1. Let $B$ be a regular local ring and $J$ an ideal of $B$. We say that the local ring $T = B/J$ is a deformation of $R = A/I$ if $R \simeq T/\underline{x}$ where $\underline{x} = x_1, \cdots, x_s$ is a regular sequence in $T$. If this is the case, then we say that $R$ is a specialization of $T$ and we get $e(R) \geq e(T)$.

THEOREM 4.2. *Let $T = B/J$ be a deformation of $R$ such that* $\mathrm{gr}(T)$ *is Cohen Macaulay. If $J$ is a $t$-isomultiple ideal and $e(T) = e(R)$, then $I$ is a $t$-isomultiple ideal and* $\mathrm{gr}(T)$ *is a deformation of* $\mathrm{gr}(R)$.

*Proof.* Let $X_1, \cdots, X_s$ be elements of $B$ such that $x_i \equiv X_i \bmod J$ with $R \simeq T/\underline{x}$. By Theorem 1.8 we have $v(x_i) = 1$ for every $i$ and $x_1^*, \cdots, x_s^*$ is a regular sequence in $\mathrm{gr}(T)$. This implies that $X_1, \cdots, X_s$ can be extended to a minimal base of the maximal ideal of $B$. Further it is clear that we may assume emb. dim $B$ = emb. dim $T$. Thus emb. dim $R$ = emb. dim $A$ = emb. dim $T - s$ = emb. dim $B - s$. If $n = \dim A$, then $\mathrm{gr}(R) = k[Y_1, \cdots, Y_n]/I^* \simeq \mathrm{gr}(T/\underline{x}) \simeq \mathrm{gr}(T)/(x_1^*, \cdots, x_s^*) \simeq \mathrm{gr}(B)/(J^*, X_1^*, \cdots, X_s^*) \simeq k[Y_1, \cdots, Y_n]/\mathfrak{A}$ where $\mathfrak{A}$ is an ideal generated by homogeneous elements of degree $t$. This implies that $I$ is a $t$-isomultiple ideal and $\mathrm{gr}(T)$ is a deformation of $\mathrm{gr}(R)$.

This result can be applied in the following case.

Let $I$ be the ideal of $A$ generated by the $r \times r$ minors of a matrix $M = (a_{ij})$, $1 \leq i \leq n$, $1 \leq j \leq m$, $r \leq n \leq m$, $a_{ij} \in \mathfrak{N}$; if $J \subset B = A[X_{ij}]_{(\mathfrak{N}, X_{ij})}$ is the corresponding ideal associated to the generic matrix $X = (X_{ij})$, it is clear that $R = A/I \simeq B/(J, X_{ij} - a_{ij})$, $1 \leq i \leq n$, $1 \leq j \leq m$.

Now it is easy to see that $\{X_{ij} - a_{ij}\}$ is a regular sequence in $B$ mod $J$, hence if $T = B/J$, then $T$ is Cohen-Macaulay, $\mathrm{gr}(T) \simeq T$ and $T$ is a deformation of $R$.

This gives the following interesting examples of isomultiple ideals.

EXAMPLE 4.3. Let $I$ be a perfect codimension 2 ideal. Then we have:

i) $e(R) \geq \binom{v(I)}{2}$

ii) The following conditions are equivalent:

a) $e(R) = \binom{v(I)}{2}$

b) $I$ is $(v(I) - 1)$-isomultiple

c) $I$ is $(v(I) - 1)$-isomultiple and $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay.

EXAMPLE 4.4. Let $I$ be a Gorenstein codimension 3 ideal. Then we have:

i) $e(R) \geq [v(I)^3 - v(I)]/24$

ii) The following conditions are equivalent:

a) $e(R) = [v(I)^3 - v(I)]/24$

b) $I$ is $[v(I) - 1)/2]$-isomultiple

c) $I$ is $[(v(I)-1)/2]$-isomultiple and $\mathrm{gr}_{\mathfrak{M}}(R)$ is Gorenstein.

EXAMPLE 4.5. Let $I$ be a Gorenstein codimension 4 ideal generated by the $n \times n$ minors of an $(n+1) \times (n+1)$ matrix of elements in $\mathfrak{R}$. Then we have:

i) $e(R) \geq [n(n+1)^2(n+2)]/12$

ii) If equality holds in i) then $\mathrm{gr}_{\mathfrak{M}}(R)$ is Gorenstein and $I$ is $n$-isomultiple.

EXAMPLE 4.6. Let $I$ be a perfect ideal generated by the $r \times r$ minors of an $r \times s$ matrix of elements in $\mathfrak{R}$. If $I$ has codimension $s-r+1$, the maximum possible, then we have:

i) $e(R) \geq \binom{s}{s-r+1}$

ii) If equality holds in i), then $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay and $I$ is $r$-isomultiple.

EXAMPLE 4.7. Let $I$ be a perfect ideal generated by the $(n-1) \times (n-1)$ minors of a symmetric $n \times n$ matrix of elements of $\mathfrak{R}$. If $I$ has codimension 3, then we have:

i) $e(R) \geq n(n^2-1)$

ii) If equality holds in i), then $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay and $I$ is $(n-1)$-isomultiple.

*Remarks* 1. The computation of the "generic" multiplicity for all these examples can be done using the nice formula given by Huneke and Miller in [H-M].

2. The inequality i) of Example 4.4 is the main result of [E-I]. As for ii), it gives a positive answer to the following conjecture stated by Elias and Iarrobino in the same paper: if $e(R) = [v(I)^3 - v(I)]/24$ is $\mathrm{gr}_{\mathfrak{M}}(R)$ Cohen-Macaulay?

3. In Examples 4.3 and 4.4 one can use Corollaries 4.4 and 5.5 in [R-V] to prove that b) implies a) and c). For the other examples we don't know if the same conclusion holds.

4. If $I$ is Gorenstein of codimension 3, one can prove that $e(R) = [v(I)^3 - v(I)]/24$ if and only if $R$ is an extremal Gorenstein ring (see [E-I]).

Here, a Gorenstein ring $R$ is said to be extremal if $j(R) = 2t-2$, where $t$ is the initial degree of $R$ and $j(R)$ is the socle degree of $R$ which is defined as the degree of $P(\bar{R}, z)$ for an artinian reduction $\bar{R}$ of $R$. In general one has $j(R) \geq 2t - 2$, an inequality proved by Schenzel in the graded case and by Elias and Iarrobino in the local case. We can reprove this result as a Corollary of the following theorem whose proof is the same as that of Corollary 4.13 in [H-R-V].

THEOREM 4.8. *Let $(A, \mathfrak{N})$ be a regular local ring and $I \subset \mathfrak{N}^t$ a Gorenstein ideal such that $R = A/I$ is artinian. Then $I\mathfrak{N}^{t-1} \colon \mathfrak{N} \not\subset I$.*

Now if $R = A/I$ is a Gorenstein local ring with $I \subset \mathfrak{N}^t$, and if $\bar{R} = \bar{A}/\bar{I}$ is an artinian reduction of $R$, we can find an element $\bar{a} \in \bar{I}\bar{\mathfrak{N}}^{t-1} \colon \bar{\mathfrak{N}}$, $\bar{a} \notin \bar{I}$. Hence $\bar{a}\bar{\mathfrak{N}} \subset \bar{I}\bar{\mathfrak{N}}^{t-1} \subset \bar{\mathfrak{N}}^{2t-1}$, which implies $\bar{a} \in \bar{\mathfrak{N}}^{2t-2}$. Thus $\bar{\mathfrak{N}}^{2t-2} \not\subset \bar{I}$ and $j(R) \geq 2t - 2$.

5. Since extremal graded Gorenstein algebras of codimension $h$ and initial degree $t$ have multiplicity $\binom{h+t-1}{h} + \binom{h+t-2}{h}$ (see [Sch]), one can state the following conjecture: if $I$ is a Gorenstein ideal of codimension $h$ and $I \subset \mathfrak{N}^t$, then $e(R) \geq \binom{h+t-1}{h} + \binom{h+t-2}{h}$ and, if equality holds, $I$ is $t$-isomultiple.

For example, let $h = 4$ and $t = 2$; then we have to prove $e(R) \geq 6$. But, as usual, we may assume $R$ is artinian, hence $e(R) = l(A/\mathfrak{N}^2) + l(\mathfrak{N}^2/I) = 5 + l(\mathfrak{N}^2/I) \geq 6$ since $I \neq \mathfrak{N}^2$. If $e(R) = 6$, then $l(\mathfrak{N}^2/I) = 1$; by Theorem 4.8 we can find an element $a \notin I$, $a \in \mathfrak{N}^2$ such that $a\mathfrak{N} \subset I\mathfrak{N}$. Hence $\mathfrak{N}^2 = (I, a)$ and $\mathfrak{N}^3 = I\mathfrak{N}$, which implies $I$ is 2-isomultiple.

6. Let $I$ be a Gorenstein codimension 3 ideal not a complete intersection; we have, as a corollary of Example 4.4 that $I$ is 2-isomultiple if and only if $e(R) = 5$.

If we let $I \subset \mathfrak{N}^t$ with $t \geq 3$, then things are not so easy. For example, for a 4-isomultiple ideal we can have $e(R) = 30, 40, 49$ but we don't know if other values are allowed.

EXAMPLE 4.9. Let $I_1$, $I_2$ be the ideals generated by the $4 \times 4$ pfaffians of the matrices:

$$M_1 = \begin{pmatrix} 0 & w^3 & x^4 & -z^3 & 0 \\ -w^3 & 0 & y^3 & 0 & -x^3 \\ -x^4 & -y^3 & 0 & w & -z \\ z^3 & 0 & -w & 0 & y \\ 0 & x^3 & z & -y & 0 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & w^2 & 0 & -z^2 & 0 \\ -w^2 & 0 & y^2 & 0 & 0 \\ 0 & -y^2 & 0 & w^2 & y^2 \\ z^2 & 0 & -w^2 & 0 & z^2 \\ 0 & 0 & -z^2 & -y^2 & 0 \end{pmatrix}$$

Then $e(R_1) = 49$ and $R_1$ has socle degree $j(R_1) = 8$; but $e(R_2) = 40$ and $j(R_2) = 7$. Further $I_1$ and $I_2$ are 4-isomultiple ideals. Of course $e(R) = 30$ is given by the generic $9 \times 9$ matrix for which the socle degree is 6.

For a 3-isomultiple codimension 3 Gorenstein ideal with $\mathrm{gr}_{\mathfrak{M}}(R)$ Cohen-Macaulay, we have $v(I) = 3, 5, 7$. If $v(I) = 3$, then $e(R) = 27$; if $v(I) = 7$, then $e(R) = 14$ (see Example 4.4). If $v(I) = 5$ we have $e(R) = 19$ as the following Proposition shows.

PROPOSITION 4.10. *Let $I$ be a Gorenstein codimension* 3 *ideal such that* $v(I) = 5$ *and* $I \subset \mathfrak{N}^3$. *Then we have*:

i) $e(R) \geq 19$

ii) *If $I$ is* 3*-isomultiple and* $\mathrm{gr}_{\mathfrak{M}}(R)$ *is Cohen-Macaulay, then* $e(R) = 19$.

*Proof.* As usual we may assume $R$ is artinian. If $\mathfrak{N}^5 \subset I$, then $j(R) = 4$, hence by Theorem 2 in [E-I], we get

$$\begin{aligned} P(R, z) &= \textstyle\sum_{i=0}^{2} \binom{i+2}{2} z^i + \sum_{i=3}^{4} \binom{4-i+2}{2} z^i \\ &= 1 + 3z + 6z^2 + 3z^3 + z^4 . \end{aligned}$$

This implies $\dim_k I_3^* = 7$ which is a contradiction to $v(I) = 5$ and $I \subset \mathfrak{N}^3$. Hence $j(R) \geq 5$ and $H(R, 5) > 0$. Thus $e(R) > \sum_{i=0}^{4} H(R, i) = 1 + 3 + 6 + (10 - \dim_k I_3^*) + (15 - \dim_k I_4^*)$. Let $s = \dim_k I_3^*$; since $I_3^* = I/I \cap \mathfrak{N}^4$ we get $\dim (I \cap \mathfrak{N}^4)/I\mathfrak{N} = 5 - s$. Also $I = (f_1, \cdots, f_5)$ where $f_1, \cdots, f_s$ have order 3, and $f_{s+1}, \cdots, f_5$ have order $\geq 4$. Let $J = (f_1, \cdots, f_s)$ and $\mathfrak{A} = (f_{s+1}, \cdots, f_5)$. We have

$$I\mathfrak{N} \subset \mathfrak{A}\mathfrak{N} + J \cap \mathfrak{N}^4 \subset I \cap \mathfrak{N}^4 .$$

Since $I \cap \mathfrak{N}^4/\mathfrak{A}\mathfrak{N} + J \cap \mathfrak{N}^4 = \mathfrak{A}/\mathfrak{A}\mathfrak{N}$ we get $I\mathfrak{N} = \mathfrak{A}\mathfrak{N} + J \cap \mathfrak{N}^4$, from which it follows $J \cap \mathfrak{N}^4 = J\mathfrak{N}$. Thus $I_4^* = (I \cap \mathfrak{N}^4) + \mathfrak{N}^5/\mathfrak{N}^5 = (\mathfrak{A} + J\mathfrak{N}) + \mathfrak{N}^5/\mathfrak{N}^5$. If $s \leq 4$, then we easily get $e(R) \geq 19$. If $s = 5$, then $I \cap \mathfrak{N}^4 = I\mathfrak{N}$, hence $I_4^* = I\mathfrak{N} + \mathfrak{N}^5/\mathfrak{N}^5$ and we have a surjection of $k$-vector spaces $\varphi\colon (\mathfrak{N}/\mathfrak{N}^2)^5 \to I_4^*$ which is given by $\varphi(\bar{a}_1, \cdots, \bar{a}_5) = \sum_i \bar{a}_i \bar{f}_i$ and whose kernel we denote by $V$. Now it is clear that $\dim V \geq 3$, otherwise at least three rows (and hence three columns) of the skew-symmetric matrix whose pfaffians generate $I$, must have entries of order $\geq 2$ which implies that at least two generators of $I$ have order $\geq 4$, a contradiction to $\dim_k I_3^* = v(I) = 5$. Hence in any case $e(R) \geq 19$.

Now let us assume $I$ 3-isomultiple and $\mathrm{gr}_{\mathfrak{M}}(R)$ Cohen-Macaulay. As usual we may assume $\dim R = 0$. If $J$ is the ideal generated by a maximal regular sequence of elements of $I$, then it is well known (see for example [H-R-V] Proposition 4.2) that $\mathfrak{N}^7 \subset J$, hence, since $I/J$ is a non zero ideal in the Gorenstein ring $A/I$, we get $\mathfrak{N}^6 \subset J\colon \mathfrak{N} \subset I$ which implies $j(R) = 5$. Thus again by Theorem 2 in [E-I], we have $P(R, z) \leq 1 + 3z + 6z^2 + 6z^3 + 3z^4 + z^5$. But $H(R, 3) = 10 - \dim_k I_3^* = 5$, hence $e(R) \leq 19$ and since the other inequality holds, the conclusion follows.

*Remark* 4.11. Using the structure of the Gorenstein codimension three ideal $I$, we can prove that if $e(R) = 19$ then $I$ is 3-isomultiple and $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay. For example if $A/I = k[\![t^{19}, t^{26}, t^{34}, t^{40}]\!]$, then $I$ is 3-isomultiple and $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay, not Gorenstein (see [H-R-V], Remark after Lemma 3.3).

## § 5. Upper and lower bounds for the multiplicity of isomultiple ideals

Let $I$ be a perfect codimension $h$ ideal of the regular local ring $(A, \mathfrak{N})$ such that $(R, \mathfrak{M}) = (A/I, \mathfrak{N}/I)$ has dimension $d$. If $I \subset \mathfrak{N}^t$ we have seen that $e(R) \geq \binom{h+t-1}{h}$ and, if equality holds, $I$ is $t$-isomultiple and $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay.

We start this section with a slight modification of this result.

LEMMA 5.1. *Let $I$ be a perfect codimension $h$ ideal of $A$ such that $I \subset \mathfrak{N}^t$. Then $e(R) \geq \binom{h+t}{h} - v(I)$ and, if equality hold, $I$ is $t$-isomultiple and* $\mathrm{gr}_{\mathfrak{M}}(R)$ *is Cohen-Macaulay.*

*Proof.* Let $x_1, \cdots, x_d$ be a minimal reduction modulo $I$; then we have $e(R) = e(\bar{R})$ and $v(I) = v(\bar{I})$. Hence, using Theorem 2.1, we may assume $\dim R = 0$. Now we have

$$\begin{aligned} e(R) = l(R) &= l(R/\mathfrak{N}^t) + l(\mathfrak{N}^t/I) = l(R/\mathfrak{N}^t) + l(\mathfrak{N}^t/\mathfrak{N}^{t+1}) + l(\mathfrak{N}^{t+1}/I\mathfrak{N}) - v(I) \\ &= \binom{h+t}{h} - v(I) + l(\mathfrak{N}^{t+1}/I\mathfrak{N}). \end{aligned}$$

Hence $e(R) \geq \binom{h+t}{h} - v(I)$ and, if equality holds, then $\mathfrak{N}^{t+1} = I\mathfrak{N}$; this implies $\mathfrak{N}^p = I\mathfrak{N}^{p-t}$ for all $p \geq t+1$, hence $I$ is $t$-isomultiple and the conclusion follows.

It is clear that if $I$ is a complete intersection 3-isomultiple ideal of codimension two, we have $e(R) = 9 \neq \binom{h+t}{h} - v(I)$. However, in some particular case, we have a partial converse to the above result. The following lemma is possibly well known, but we insert here a proof for the sake of completeness.

LEMMA 5.2. *Let $J = (f_1, \cdots, f_h)$ be a t-isomultiple complete intersection ideal of dimension zero in $A$ and let $(R, \mathfrak{M}) = (A/J, \mathfrak{N}/J)$. Then with $p = h(t-1)$, we have:*

i) *There exists an element $x \in A$, such that $\mathfrak{N}^p = (J\mathfrak{N}^{p-t}, x)$*

ii) $0\colon \mathfrak{M}^* = (0\colon \mathfrak{M})^*$

iii) $\mathfrak{M}^n\colon \mathfrak{M} = (0\colon \mathfrak{M}) + \mathfrak{M}^{n-1}$ *for every $n \geq 1$*

iv) $\mathfrak{M}^p\colon \mathfrak{M}^s = (0\colon \mathfrak{M}) + \mathfrak{M}^{p-s}$ *for every $s$, $1 \leq s \leq p$*

v) $J\colon \mathfrak{N}^{p-s} = (J, \mathfrak{N}^{s+1})$ *for every $s$, $1 \leq s \leq p$.*

*Proof.* i) Since $f_1^*, \cdots, f_h^*$ is a regular sequence of elements of degree $t$ we have $\dim \mathfrak{M}^p/\mathfrak{M}^{p+1} = 1$ and $\mathfrak{M}^{p+1} = 0$. But $\mathfrak{M}^p/\mathfrak{M}^{p+1} = \mathfrak{N}^p/\mathfrak{N}^{p+1} + J \cap \mathfrak{N}^p = \mathfrak{N}^p/\mathfrak{N}^{p+1} + J\mathfrak{N}^{p-t}$, thus for some element $x \in \mathfrak{N}^p$ we have $\mathfrak{N}^p = \mathfrak{N}^{p+1} + J\mathfrak{N}^{p-t} + xA$. The conclusion follows by Nakayama lemma.

ii) This is Lemma 4.15 in [H-R-V].

iii) If for some $n$ we have $a\mathfrak{M} \subset \mathfrak{M}^n$ and $a$ has order $r < n-1$, then $a^* \in 0\colon \mathfrak{M}^* = (0\colon \mathfrak{M})^*$, hence $a \in (0\colon \mathfrak{M}) \cap \mathfrak{M}^r + \mathfrak{M}^{r+1}$. Repeating this argument we get the conclusion.

iv) If $s = 1$ the conclusion follows by iii). Using induction on $s$, if $a\mathfrak{M}^s \subset \mathfrak{M}^p$ we get $a\mathfrak{M} \subset \mathfrak{M}^p\colon \mathfrak{M}^{s-1} = (0\colon \mathfrak{M}) + \mathfrak{M}^{p-s+1}$. Since $p - s + 1 \leq p$, this implies $a \in \mathfrak{M}^{p-s+1}\colon \mathfrak{M} = (0\colon \mathfrak{M}) + \mathfrak{M}^{p-s}$ as wanted.

v) We have $J\colon \mathfrak{N}^{p-s} = (J\colon \mathfrak{N})\colon \mathfrak{N}^{p-s-1} = (J, x)\colon \mathfrak{N}^{p-s-1} = (J, \mathfrak{N}^p)\colon \mathfrak{N}^{p-s-1} = (J\colon \mathfrak{N}) + \mathfrak{N}^{s+1} = (J, \mathfrak{N}^p) + \mathfrak{N}^{s+1} = (J, \mathfrak{N}^{s+1})$.

PROPOSITION 5.3. *Let $I$ be a perfect codimension $h$ ideal of $A$. Let us assume that $I$ is t-isomultiple and $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay. If one of the following conditions holds*

i) $h = t = 2$

ii) $h = 2$, $t = 3$ *and* $v(I) > 2$

iii) $h = 3$, $t = 2$ *and* $v(I) > 3$

*then* $e(R) = \binom{h+t}{h} - v(I)$.

*Proof.* If $h = 2 = t$ and $v(I) = 2$, then $e(R) = 4 = 6 - v(I)$; if $v(I) = 3$, then $e(R) = \binom{3}{2} = 6 - v(I)$ by the result of Elias (see Example 4.3). As for the other cases, we have $\mathrm{gr}_{\mathfrak{M}}(R) = k[X_1, \cdots, X_{d+h}]/I^*$ is Cohen-

Macaulay, hence we may assume $X_{h+1}, \cdots, X_{h+d}$ is a regular sequence modulo $I^*$: thus we have to compute the multiplicity of the graded ring $k[X_1, \cdots, X_h]/\mathfrak{A}$ where $\mathfrak{A}$ is generated by forms $F_1, \cdots, F_r$ of degree $t$ and $r = v(I)$. For this we may assume that $F_1, \cdots, F_h$ is a regular sequence and let $J$ be the ideal they generate. If $B = k[X_1, \cdots, X_h]$, we have by the above lemma $B_{h(t-1)} = Fk + J_t B_{h(t-1)-t}$. Since every non zero ideal in a Gorenstein ring contains the socle, and we have $\mathfrak{A} \supsetneqq J$, we get $B_{h(t-1)} = \mathfrak{A}_{h(t-1)}$. Now it is clear that if $h(t-1) = t+1$, then

$$e(B/\mathfrak{A}) = \sum_{i=1}^{t-1} \binom{h+i-1}{i} + \binom{h+t-1}{t} - v(I) = \binom{h+t}{h} - v(I).$$

Thus the conclusion follows.

The following examples show that we cannot improve the above proposition

EXAMPLE 5.4. Let $A = k[\![X, Y]\!]$ and $I = (X^4, X^3Y, Y^4)$. Then $I$ is 4-isomultiple, but $e(R) = 13 \neq \binom{2+4}{2} - 3 = 12$.

EXAMPLE 5.5. Let $A = k[\![X, Y, Z, W]\!]$ and $I = (X^2, Y^2, Z^2, W^2, XY)$. Then $I$ is 2-isomultiple but $e(R) = 12 \neq \binom{2+4}{4} - 5 = 10$.

EXAMPLE 5.6. Let $A = k[\![X, Y, Z]\!]$ and $I = (X^3, Y^3, Z^3, X^2Y)$. Then $I$ is 3-isomultiple but $e(R) = 21 \neq \binom{3+3}{3} - 4 = 16$.

*Remark* 5.7. It is clear that if $I$ is a codimension three 2-isomultiple ideal then $v(I) \leq 6$. For each values of $v(I) = 4, 5, 6$ we want to exhibit a typical example. Let $R = k[\![t^6, t^7, t^8, t^9]\!]$; then $v(I) = 4$ and $e(R) = 6$. Let $I$ be the ideal generated by the pfaffians of a $5 \times 5$ skew-symmetric generic matrix, then $v(I) = e(R) = 5$. Let $I$ be the defining ideal of the Veronese surface in $\boldsymbol{P}^5$ or of the Segre embedding $\boldsymbol{P}^1 \times \boldsymbol{P}^3 \to \boldsymbol{P}^7$; then $v(I) = 6$ and $e(R) = 4$.

As we have seen in Remark 6 of the last section and Examples 5.5 and 5.6 show, it is difficult to control the multiplicity of the ring $R$ when $I$ is a $t$-isomultiple ideal. As soon as $h$ or $t$ increase, we cannot say what values for the multiplicity are allowed. Thus we restrict ourselves to an ideal I which is 2-isomultiple and try to give an upper bound for the multiplicity of the ring $R = A/I$, when $I$ is not a complete intersection.

The following result is the key point for our further investigations.

PROPOSITION 5.8. *Let $I$ be an homogeneous ideal of $A = k[X_1, \cdots, X_h]$ such that $R = A/I$ is Gorenstein, Artinian and with socle degree $j(R) = h - 2$. If $P(R, z) = \sum_{i=0}^{h-2} a_i z^i$ and $I$ contains a maximal regular sequence of forms of degree two, then we have*:

i) $a_1 \geq h - 2$

ii) *If $a_1 = h - 2$, then $I$ is a complete intersection*

iii) *If $h \leq 6$ then $e(R) \geq 2^{h-2}$*

*Proof.* i) Let $a_1 = h - p$ with $0 \leq p \leq h$; then $\dim I_1 = p$ and we may assume $X_1, \cdots, X_p \in I$. By the assumption made on $I$, we can find elements $F_{p+1}, \cdots, F_h$ in $I$ of degree two such that $X_1, \cdots, X_p, F_{p+1}, \cdots, F_h$ is a regular sequence in $I$. Of course we may assume $F_{p+1}, \cdots, F_h \in B = k[X_{p+1}, \cdots, X_h]$ and denote by $J$ the ideal they generate in $B$. Then we have $a_{h-p+1} = \dim(A/I)_{h-p+1} \leq \dim(B/J)_{h-p+1} = 0$; hence $h - p + 1 > h - 2$ which implies $a_1 = h - p \geq h - 2$.

ii) If $a_1 = h - 2$ then, as before, we can find elements $F_3, \cdots, F_h \in I$ of degree two such that $X_1, X_2, F_3, \cdots, F_h$ is a regular sequence in $I$. Also we may assume $F_3, \cdots, F_h \in B = k[X_3, \cdots, X_h]$ and we denote by $J$ the ideal they generate in $B$ and by $\bar{I}$ the ideal $I/(X_1, X_2)$ in $B$. Then we have $h - 2 = a_1 = a_{h-3} = \dim(B/\bar{I})_{h-3} = \dim(B/J)_{h-3}$, hence $\bar{I}_{h-3} = J_{h-3}$. Now if $F \in \bar{I}_d$ with $d \leq h - 3$, then $FB_{h-3-d} \in J$, hence $F \in J$: $B_{h-3-d} = (J, B_{d+2})$ where the last equality follows by Lemma 5.2 iv). This implies $F \in J$, hence $J = \bar{I}_d$ and $I$ is a complete intersection.

iii) If $h \leq 5$ we get the conclusion using i). If $h = 6$, using the numerical characterization of Hilbert functions due to Macaulay (see for example [St] Theorem 2.2) we can prove that if $a_1 = a_3 = 5$ then $a_2 \geq 4$, and if $a_1 = a_3 = 6$ then $a_2 \geq 5$. Since, if $a_1 = a_3 = 4$ then $a_2 = 6$ by part ii), this gives the conclusion.

THEOREM 5.9. *Let $I$ be a 2-isomultiple codimension $h$ ideal of the $n$-dimensional regular local ring $(A, \mathfrak{N})$. If $v(I) > h$ and $h \leq 6$, then $e(R) \leq 3 \cdot 2^{h-2}$.*

*Proof.* We have $\mathrm{gr}_{\mathfrak{M}}(R) = k[X_1, \cdots, X_n]/I^*$ where $I^*$ is generated by forms of degree two, $v(I^*) = v(I)$ and $h = \mathrm{height}(I^*)$. Thus we may assume $A = k[X_1, \cdots, X_n]$ and $I$ is an homogeneous ideal generated by forms of degree two. Let $F_1, \cdots, F_h, F$ be elements in a minimal base of

$I$ such that $F_1, \cdots, F_h$ is a regular sequence in $I$ and let $\mathfrak{A} = (F_1, \cdots, F_h, F)$. Then $e(A/I) = e[(A/\mathfrak{A})/(I/\mathfrak{A})]$; since $\dim A/\mathfrak{A} = \dim A/I$ we have, by Proposition 1.5, $e(A/I) \leq e(A/\mathfrak{A})$. Hence we may assume $v(I) = h + 1$, and let $I = (F_1, \cdots, F_h, F)$ where $F_1, \cdots, F_h$ is a regular sequence in $A$. If $n = h$ and we put $J = (F_1, \cdots, F_h)$, then $A/J \colon F$ is an artinian graded Gorenstein ring (see [K] Proposition 3.1). Also, by Lemma 5.2 v), $F \in J \colon A_{h-1}$, hence $A_{h-1} \subset J \colon F$, while if $A_{h-2} \subset J \colon F$ then $F \in A_3$, a contradiction. Thus $A/J \colon F$ has socle degree $h - 2$ and since $J \subset J \colon F$ we may apply Proposition 5.8. to get $e(A/J \colon F) \geq 2^{h-2}$.

But we have an exact sequence $0 \to A/J \colon F \to A/J \to A/I \to 0$; hence $e(A/I) = e(A/J) - e(A/J \colon F) \leq 2^h - 2^{h-2} = 3 \cdot 2^{h-2}$.

Now we can use induction on $n - h$. Let $n > h$; then we can find an element $x \in A_1$ which is a non zero divisor modulo $J$. Since $x$ is a parameter modulo $I$ we get height $(I + x/x) = h$. If $v(I + x/x) = h + 1$, then we have by Proposition 1.5 and the inductive assumption: $e(A/I) = e(A/I + x) = e[(A/x)/(I + x)/(x)] \leq 3 \cdot 2^{h-2}$. If $v(I + x/x) = h$, then $F \in (J, x)$ and $I = (J, xy)$ for some $y \in A_1$. Now $y$ is zero divisor modulo $J$, hence we have $\dim(A/(J, y)) = \dim A/J = n - h$; also $I \colon y \supset (J, x)$ hence $\dim A/I \colon y \leq \dim(A/(J, x)) = n - h - 1$. We can apply Proposition 1.5 i) to infert $e(A/I) = e(A/(I, y)) = e(A/(J, y))$. But height $((J, y)/y) =$ height $(J, y) - 1 = h - 1$, hence we get $e(A/I) = e(A/(J, y)) \leq 2^{h-1} < 3 \cdot 2^{h-2}$, as required.

*Remark* 5.10. In the above theorem, unlike the rest of this paper, we assume neither $I$ nor $I^*$ to be perfect.

*Remark* 5.11. Let $A = k[\![X_1, \cdots, X_h]\!]$ and $I = (X_1^t, \cdots, X_h^t, X_1^{t-1}X_2)$. Then $(X_1^t, \cdots, X_h^t) \colon X_1^{t-1}X_2 = (X_1, X_2^{t-1}, X_3^t, \cdots, X_h^t)$, hence $e(A/I) = t^h - t^{h-2}(t - 1) = t^{h-2}(t^2 - t + 1)$.

The above theorem and this example suggest the following question: if $I$ is a $t$-isomultiple codimension $h$ ideal, is it true that $e(R) \leq t^{h-2}(t^2 - t + 1)$? For example, let $h = 2$; as in the proof of Theorem 5.9 we need to prove that if $A = k[X, Y]$ and $I = (F_1, F_2, F)$, where $F_1, F_2, F$ are forms of degree $t$ such that $F_1, F_2$ is a regular sequence, then $e(A/I) \leq t^2 - t + 1$. Now if $A_{t-2} \subset J \colon F$ then, by Lemma 5.2, $F \in J$; hence $e(A/J \colon F) \geq t - 1$ which implies $e(A/I) = t^2 - e(A/(J \colon F)) \leq t^2 - t + 1$.

*Remark* 5.12. Let $k$ be an algebraically closed field, $A = k[X_0, \cdots, X_h]$ and $I = (F_1, \cdots, F_{h+1})$ a codimension $h$ ideal with $\deg F_i = 2$. Let $C$ be the curve in $\boldsymbol{P}^h$ which is defined by $F_1, \cdots, F_{h-1}$ and which we assume to be non singular. Let $D_0 = C \cap F_h$ and $D_1 = C \cap F_{h+1}$; $D_0$ and $D_1$ are divisors on $C$ and we have $\deg(D_0) = 2^h$.

Further if $\sum_i P_i$ is a divisor defined by $I$, then $D_0 - \sum_i P_i \equiv D_1 - \sum_i P_i$, but $D_0 - \sum_i P_1 \neq D_1 - \sum_i P_i$ as Cartier divisors. Let $d$ be the minimum degree for a divisor $D$ on $C$ such that $\dim|D| \geq 1$; then it is clear that $e(A/I) \leq 2^h - d$. Now we can compute $d$ by using Riemann-Roch theorem which says

$$h^0(\mathcal{O}(D)) - h^1(\mathcal{O}(D)) = d + 1 - g(C).$$

Since $g(C) = 2^{h-2}(h-3) + 1$, we get $d = 2^{h-2}(h-3) + 2 - h^1(\mathcal{O}(D))$, hence $e(A/I) \leq 2^{h-2}(7-h) - 2 + h^1(\mathcal{O}(D))$. Thus we are led to compute $h^1(\mathcal{O}(D))$ which probably is not easy; but the above formula could justify the assumption $h \leq 6$ in our Theorem 5.9.

On the other hand if we assume $C$ is a generic curve (in the sense of the moduli space), then by the theorem of Brill-Noether we have that there exists on $C$ a linear system of degree $d$ and dimension $r$ if and only if $d \geq [rg/(r+1)] + r$; since we may assume $r = 1$ we get $e(A/I) \leq 2^h - d \leq 2^h - g/2 - 1 = 2^h - 2^{h-3}(h-3) - 1/2 - 1 = 2^{h-3}(11-h) - 3/2$. If we compare this bound with the one given in Theorem 5.9 we see that they coincide until $h = 4$, but for $h = 5$ we get $3 \cdot 2^{h-2} = 24$ while $2^{h-3}(11-h) - 3/2 = 22 + 1/2$. We remark that for $h = 5$ the corresponding curve has genus 17 and ask weather our result could have some application to the study of the moduli space of curves with such a genus.

*Remark* 5.13. The converse of the above theorem does not holds. Let $A = k[\![X, Y, Z]\!]$ and $I = (X^3, Y^2, Z^2, XY, XZ)$; then $e(R) = 6 \leq 3 \cdot 2^{h-2}$, but $I$ is not 2-isomultiple.

Strangely enough, if we assume $I$ to be homogeneous and $v(I) = h + 1$, then we can prove that $I$ is an intersection of quadrics.

PROPOSITION 5.14. *Let $I$ be an homogeneous codimension $h$ ideal of $A = k[X_1, \cdots, X_h]$ such that $v(I) = h + 1$. If $h \leq 6$ and $e(R) \leq 3 \cdot 2^{h-2}$, then $I$ can be generated by forms of degree two.*

*Proof.* Let $r$, $j$ and $s$ be the number of generators in a minimal base of $I$ of degree 2, 3 and 4 respectively. Then we have $\dim I_2 = r$, $\dim I_3 \leq hr + j$ and

$$\dim I_4 \leq \binom{r+1}{2} + r\left[\binom{h+1}{2} - r\right] + hj + s = r\binom{h+1}{2} - \binom{r}{2} + hj + s;$$

hence we get:

$$e(R) \geq 1 + h + \binom{h+1}{2} - r + \binom{h+2}{3} - hr - j + \binom{h+3}{4} - r\binom{h+1}{2} + \binom{r}{2} - hj - s.$$

Since $r + j + s \leq h + 1$ we get

$$e(R) \geq \binom{h+2}{3} + \binom{h+3}{4} + \binom{r}{2} - (r+1)\binom{h+1}{2}.$$

If $r \leq h \leq 6$ it is easy to see that the term on the right is strictly bigger than $3 \cdot 2^{h-2}$, a contradiction. Hence $r \geq h$, which implies $r = h + 1$; the conclusion follows.

The last result of this paper deals with the problem of finding a lower bound for the multiplicity of $A/I$, where $I$ is a $t$-isomultiple codimension $h$ ideal. If we don't make any assumption on $v(I)$, the bound is $\binom{h+t-1}{h}$ as proved in section 3. But if we assume $I$ to be an almost complete intersection, then we can prove the following result, where given the integers $h$ and $t$, we define $s$ to be the integer part of $(h+1)(t-1)/2$ and we let $(\sum_{i=0}^{t-1} z^i)^{h+1} = \sum_i d_i z^i$.

THEOREM 5.15. *Let $I$ be a perfect codimension $h$ ideal of the regular local ring $(A, \mathfrak{N})$. If $v(I) = h + 1$, $I$ is $t$-isomultiple and $\mathrm{gr}_{\mathfrak{M}}(R)$ is Cohen-Macaulay, then $e(R) \geq d_s$, and the equality holds if and only if the socle degree of $R$ is $s$.*

*Proof.* We have $\mathrm{gr}_{\mathfrak{M}}(R) = k[X_1, \cdots, X_{d+h}]/I^*$; since this is Cohen-Macaulay, we may assume $X_{h+1}, \cdots, X_{d+h}$ is a regular sequence modulo $I^*$, hence $e(R) = e(\mathrm{gr}_{\mathfrak{M}}(R)) = e(k[X_1, \cdots, X_h]/\mathfrak{A})$ where $\mathfrak{A}$ is a codimension $h$ ideal generated by $h + 1$ forms of degree $t$. Let $\mathfrak{A} = (F_1, \cdots, F_h, F)$ where $F_1, \cdots, F_h$ is a regular sequence in $B = k[X_1, \cdots, X_h]$; further let $J$ be the ideal generated by $F_1, \cdots, F_h$. We have that $B/J$ and $B/J\colon F$ are graded artinian Gorenstein rings with socle degree $h(t-1)$ and $h(t-1) - t$ respectively (use Lemma 5.2). Then we have $P(B/J, z) = z^{h(t-1)}P(B/J, 1/z)$ and $P(B/J\colon F, z) = z^{h(t-1)-t}P(B/J\colon F, 1/z)$. Hence we get $P(B/\mathfrak{A}, z) = P(B/J, z) - z^t P(B/J\colon F, z) = P(B/J, z) - z^{h(t-1)}P(B/J\colon F, 1/z)$. But it is clear that we have $P(B/\mathfrak{A}, 1/z) = P(B/J, 1/z) - z^{-t}P(B/J\colon F, 1/z)$, hence $P(B/\mathfrak{A}, z) = P(B/J, z) - z^{h(t-1)+t}[P(B/J, 1/z) - P(B/\mathfrak{A}, 1/z)] = P(B/J, z) - z^t P(B/J, z) + z^{h(t-1)+t}P(B/\mathfrak{A}, 1/z) = (1 - z^t)P(B/J, z) + z^{h(t-1)+t}P(B/\mathfrak{A}, 1/z)$

$= (1 - z^t)(\sum_{i=0}^{t-1} z^i)^h + z^{h(t-1)+t}P(B/\mathfrak{A}, 1/z) = (1 - z)(\sum_{i=0}^{t-1} z^i)^{h+1} + z^{h(t-1)+t}$
$P(B/\mathfrak{A}, 1/z)$.

Now we remark that $B/\mathfrak{A}$ has socle degree less or equal to $h(t-1) - 1$ and we let $(\sum_{i=0}^{t-1} z^i)^{h+1} = \sum_i d_i z^i$ and $P(B/\mathfrak{A}, z) = \sum_i a_i z^i$. Then it is easy to prove that

$$e(R) = d_s + 2\sum_{i=s+1}^{h(t-1)-1} a_i \qquad \text{if } (h+1)(t-1) = 2s$$
$$e(R) = d_s + a_{s+1} + 2\sum_{i=s+2}^{h(t-1)-1} a_i \qquad \text{if } (h+1)(t-1) = 2s+1.$$

Of course, this gives the conclusions.

*Remark.* If $h = 2$, then by the above theorem, we get $e(R) \geq 3n^2$ if $t = 2n$ and $e(R) \geq 3n^2 + 3n + 1$ if $t = 2n + 1$.

In this case the bound is sharp as the following examples show.

Let $A = k[\![X, Y]\!]$, $I = (X^{2n}, Y^{2n}, X^nY^n)$ or $I = (X^{2n+1}, Y^{2n+1}, X^nY^{n+1})$. If $t = 2$ then we get $e(R) \geq \binom{h+1}{n}$ if $h = 2n$ and $e(R) \geq 2\binom{h}{n}$ if $h = 2n + 1$. Also in this case the bound is sharp: let $k$ be a field of characteristic zero and $A = k[\![X_1, \cdots, X_h]\!]$, $I = (X_1^2, \cdots, X_h^2, X_1X_2 + X_3X_4 + \cdots + X_{h-1}X_h)$ if $h = 2n$, $I = (X_1^2, \cdots, X_h^2, X_1X_2 + X_3X_4 + \cdots + X_{h-2}X_{h-1})$ if $h = 2n + 1$.

We can give examples which prove the bound is sharp for many other values of $h$ and $t$ and we think that this should be always possible. But, for the moment, we do not have a general proof of this fact.

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