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MINIMAL RATIONAL THREEFOLDS II

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The Enriques-Fano classification ([E.F], [F]) of the maximal connected algebraic subgroups of the three variable Cremona group, despite of its group theoretic feature, seems to be the most significant result on the rational threefolds so far known. In this paper as in [MU] we interpret the Enriques-Fano classification from a geometric view point, namely the geometry of minimal rational threefolds. We explained in [MU] the link between the two objects; the maximal algebraic subgroups and the minimal rational threefolds. Let (G, X) be a maximal algebraic subgroup of three variable Cremona group. We denote by $\mathscr{C}(G, X)$ the set of all the algebraic operations (G, Y) such that Y is non-singular and projective and such that (G, Y) is isomorphic to (G, X) as law chunks of algebraic operation: namely (G, Y) is birationally equivalent to (G, X). Then we define an order in $\mathscr{C}(G, X)$: for (G, Z), $(G, W) \in \mathscr{C}(G, X)$, $(G, Z) > (G, W)$ if there exists an G-equivariant birational morphism of Z onto W.

Using the classification of [U4], we can state our result.

If (G, X) *is one of the maximal algebraic subgroups except for* (J9) *and* (J11) *listed in Theorem* (2.1), [U4], *then there exists the unique minimal element in the ordered set* $\mathscr{C}(G, X)$ *and any other element of* $\mathscr{C}(G, X)$ *is an equivariant blow-up of the minimal element. For the operations* (J9) *and* (J11), *we can describe the relatively minimal elements in* $\mathscr{C}(G, X)$; *there are countable many relatively minimal elements and they are explicitly constructed and are related each other by the equivariant elementary transformation. In these cases too, any other element is an equivariant blow-up of a relatively minimal elements.*

Since our result thus reveals a new fascinating corner where the simplicity dominates, we have not tried to relate our result with the recent attempts of constructing minimal models for threefolds allowing

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terminai singularities. Therefore there remains a very interesting problem of studying minimal elements in wider categories allowing some reasonable singularities.

As in our preceding papers, we work over the complex number field $k=\mathbf{C}$.

I should like to express my appreciation to the referee who pointed out an error in the argument of Section 8 and revealed the Euclidean model in a particular case.

§ 1. Preliminaries

We start with recalling the convention and some definitions.

(1.1.1) A Borel subgroup of SL_2 is denoted by B. We denote by D_∞ the 1-dimensional dihedral subgroup of SL_2; $D_\infty = \left\langle \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \right\rangle$, $t \in k^*$.

(1.1.2) An integral divisor on a variety is an irreducible reduced closed subscheme of codimension 1.

The rational ruled surfaces appear very often in the discussion. We recall some of their properties.

(1.2) The ruled surface F_n $(n \geq 0)$ is by definition $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-n))$ which has a natural $\mathbf{P}^1$-bundle structure f: $\mathrm{F}_n \to \mathbf{P}^1$. We denote $f^*\mathcal{O}_{\mathbf{P}^1}(m)$ by $\mathcal{O}_{\mathrm{F}_n}(m)$. The projection $\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-n) \to \mathcal{O}_{\mathbf{P}^1}(-n)$ gives a section C_∞ of $\mathrm{F}_n \to \mathbf{P}^1$. The self-intersection number $C_\infty^2 = -n$ and C_∞ is characterized by this property if $n \geq 1$ (cf. [Mar1]). $\mathcal{O}(C_\infty)$ is the tautological line bundle on $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-n))$ so that $f_*(jC_\infty) \simeq S^j(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-n))$ for $j \geq 0$, where $S^j(E)$ denotes the j-th symmetric tensor of E. We have a spectral sequence for f.

$$(*) \quad E_2^{p,q} = H^p(\mathbf{P}^1, R^q f_*(\mathcal{O}_{\mathrm{F}_n}(m) \otimes \mathcal{O}(jC_\infty))) \Longrightarrow H^*(\mathrm{F}_n, \mathcal{O}_{\mathrm{F}_n}(m) \otimes \mathcal{O}_{\mathrm{F}_n}(jC_\infty)).$$

For $j \geq 0$,
$$\begin{aligned} R^q f_*(\mathcal{O}_{\mathrm{F}_n}(m) \otimes \mathcal{O}(jC_\infty)) &\simeq \mathcal{O}_{\mathbf{P}^1}(m) \otimes R^q f_* \mathcal{O}(jC_\infty) \\ &\simeq \mathcal{O}_{\mathbf{P}^1}(m) \otimes R^q f_* \mathcal{O}_{\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(-n))}(j) \\ &\simeq \begin{cases} \mathcal{O}_{\mathbf{P}^1}(m) \otimes S^j(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-n)) \simeq \bigoplus_{s=0}^{j} \mathcal{O}(m-sn) & \text{if } q=0 \\ 0 & \text{if } q>0, \end{cases} \end{aligned}$$

by the projection formula. Therefore the spectral sequence (*) degenerates giving

$$(1.2.1) \quad \bigoplus_{s=0}^{j} H^p(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m-sn)) \simeq H^p(\mathrm{F}_n, \mathcal{O}_{\mathrm{F}_n}(m) \otimes \mathcal{O}_{\mathrm{F}_n}(jC_\infty)) \quad \text{for } j \geq 0,$$

(1.3.0) F'_n is by definition $\mathrm{Spec}(S(\mathcal{O}_{P^1}(-n))$, where $S(E) = \bigoplus_{j\geq 0} S^j(E)$ denotes the symmetric algebra on E. F'_n has a natural structure of $\mathbf{A}^1$-bundle f': $F'_n \to \mathbf{P}^1$. We denote $f'^*\mathcal{O}_{P^1}(m)$ by $\mathcal{O}_{F'_n}(m)$. $\mathbf{A}^1$-bundle $F'_n \to \mathbf{P}^1$ is the total space of line bundle $\mathcal{O}_{P^1}(n)$ and hence we can regard as an open subvariety of F_n: namely $F_n - C_\infty \simeq F'_n$.

F'_n has the 0-section C_0 of the line bundle $\mathcal{O}_{P^1}(n)$ which is defined in F_n by a surjective morphism $\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(-n) \to \mathcal{O}_{P^1}$. We have $C_0^2 = n$ and C_0 is disjoint from C_∞. And conversely if we have a section C' on F_n with $C'^2 = n$, $C' \cap C_\infty = \varnothing$ or equivalently if we have a surjective morphism $\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(-n) \to \mathcal{O}_{P^1}$, then there is a $\mathbf{P}^1$-automorphism of $F'_n/\mathbf{P}^1$ (or a $\mathbf{P}^1$-automorphism of $F_n/\mathbf{P}^1$ fixing C_∞) which transforms C to C_0. Since $\mathrm{Aut}^0_{P^1}F'_n$ is identified by Lemma (4.4), [U4] with the semi-direct product $H^0(\mathbf{P}^1, \mathcal{O}(n)) \rtimes \mathbf{G}_m$, the 0-section of $F'_n/\mathbf{P}^1$ is not characteristic to F'_n: or they are determined only up to $\mathrm{Aut}_{P^1}F'_n = \mathrm{Aut}_{P^1}F_n$.

It follows from the definition $f'_*\mathcal{O}_{F'_n} \simeq S(\mathcal{O}(-n))$. For a coherent sheaf M on $\mathcal{O}_{F'_n}$ the spectral sequence for f' degenerates since f is affine. If we write the isomorphism deduced from the degeneracy of the spectral sequence for $M = \mathcal{O}_{F'_n}(m)$, we get

$$(1.3.1)\quad \begin{aligned} H^p(F'_n, \mathcal{O}_{F'_n}(m)) &\simeq H^p(\mathbf{P}^1, \mathcal{O}_{P^1}(m) \otimes f'_*\mathcal{O}_{F'_n}) \\ &\simeq H^p(\mathbf{P}^1, \mathcal{O}_{P^1}(m) \otimes (\bigoplus_{j\geq 0} S^j(\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(-n)))) \\ &\simeq \bigoplus_{j\geq 0} H^p(\mathbf{P}^1, \mathcal{O}_{P^1}(m - jn)). \end{aligned}$$

(1.3.2) Let $S \to \mathbf{P}^1$ be a $\mathbf{P}^1$-bundle. As the Brauer group for a curve vanishes, there exists a line bundle E on $\mathbf{P}^1$ such that S is $\mathbf{P}^1$-isomorphic to $\mathbf{P}(E)$. If there exists sections D_0, D_∞ of S such that D_0 is disjoint from D_∞ with either $D_0^2 = n$ or $D_\infty^2 = -n$, then we have both $D_0^2 = n$ and $D_\infty^2 = -n$ and there exists a $\mathbf{P}^1$-isomorphism $S \simeq F_n$ mapping D_0 (resp. D_∞) to C_0 (resp. C_∞) (cf. Atiyah [A]).

(1.3.3) Let $D \subset F_n$ $(n > 0)$ be a section with $D^2 = -n$. Hence as we mentioned above $D = C_\infty$ (cf. Maruyama [Mar1] and [U3]). A non-trivial operation of SL_2 on F_n gives a semi-simple part of $\mathrm{Aut}^0\, F_n$ and leaves invariant C_∞ and another section D_0 disjoint from C_∞ hence $D_0^2 = n$ by (1.3.2): the SL_2-orbit decomposition of F_n is (open orbit)$\cup D_0 \cup D_\infty$ [U3]. In particular D_∞ (resp. D_0) is characterised as the SL_2-invariant curve on F_n whose self-intersection number is negative (resp. positive). Conversely if we give a section D_0 with $D_0^2 = n$ disjoint of C_∞, there exist a unique semi-simple part of $\mathrm{Aut}^0 \mathrm{F}_n$ or equivalently non-trivial operation of SL_2 on F_n leaving D_0 and C_∞ invariant (cf. (1.3.0) and [U3]).

We consider the ordered set $\mathscr{C}(G, X)$ of equivariant completions which are birationally equivalent to a fixed operation.

DEFINITION (1.4). For a law chunk of algebraic operation (G, X), we define $\mathscr{C}(G, X) = \{(G, Y) \mid (G, Y)$ is an algebraic operation isomorphic to (G, X) as law chunks of algebraic operation: in the usual language (G, Y) is birationally equivalent to (G, X). Y is non-singular and projective (cf. [U1], [MU]).

$\mathscr{C}(G, X)$ is non-empty by [Su] if G is linear, in particular if X is rational (see Theorem (3.2), [U2]) (see also [U5]). We define an order $>$ in $\mathscr{C}(G, X)$.

DEFINITION (1.5). For (G, X_1), $(G, X_2) \in \mathscr{C}(G, X)$, $(G, X_1) > (G, X_2)$ if there exists a G-equivariant birational morphism of X_1 onto X_2.

Our result is a fruitful application of the theory of extremal rays due to Mori, which is a generalization of the classical theory of both ruled surfaces and of exceptional divisors of the first kind over a surface. Referring the reader to [Mo] for the detail, we recall briefly the framework of his theory and indicate how we can apply it. Let $N(X)$ be the $\mathbf{R}$-vector space of all numerical equivalence classes of 1-cycles over a non-singular projective threefold X with coefficients in $\mathbf{R}$ and $NE(X)$ be the smallest convex cone in $N(X)$ containing all effective 1-cycles closed under the multiplication by $\mathbf{R}_+$. We denote by $\overline{NE}(X)$ the closure of $NE(X)$ in the $\mathbf{R}$-vector space $N(X)$. A half line $R = \mathbf{R}_+[Z]$ in $\overline{NE}(X)$ is called an extremal ray if (i) $(Z, c_1(X)) > 0$ and if (ii) for $Z_1, Z_2 \in \overline{NE}(X)$, $Z_1 + Z_2 \in R$ implies $Z_1, Z_2 \in R$.

THEOREM (Mori [Mo]). *Let X be a non-singular projective threefold and $R \subset \overline{NE}(X)$ be an extremal ray. Then there exists a morphism ϕ: $X \to Y$ to a projective variety Y such that* (i) $\phi_* \mathcal{O}_X = \mathcal{O}_Y$ *and* (ii) *for any irreducible curve C in X, $[C] \in R$ if and only if* $\dim \phi(C) = 0$.

The structure of the morphism ϕ is analysed in Theorem (3.3), Corollary (3.4) and Theorem (3.5), [Mo]. Roughly speaking we have one of the following: (i) ϕ is a blow-down, (ii) ϕ makes X into a fibration over a surface or a curve, (iii) X is a Fano threefold with $\rho(X) = 1$.

Since the morphism ϕ is functorial, if an algebraic group G operates on X, then G operates also on Y such that ϕ is G-equivariant. We apply this for an operation (G, X) of a linear algebraic group on a non-singular projective threefold X such that (G, X) gives a maximal connected algebraic subgroup of the three variable Cremona group. In most of our applications, Y is automatically smooth and we get one of the following:

(1) ϕ is an equivariant blow-up of non-singular Y,

(2) ϕ makes X into an equivariant $\mathbf{P}^1$-bundle over a rational ruled surface,

(3) ϕ makes X into a del Pezzo bundle over $\mathbf{P}^1$,

(4) X is a Fano threefold with $\rho(X) = 1$.

In the case (1), we apply the Mori theory again to Y and continue to look for a minimal model. In the cases (2) and (3), we can determine the structure of X completely. The case (4) rarely happens and in fact never when we limit ourselves to the de Jonquières type operations (G, X): recall that (G, X) is of de Jonquières type if there exist an operation (G, Y), $\dim Y = 1$ or 2 and a dominant G-equivariant rational map $X \dashrightarrow Y$.

(1.6) We mean by a blow-up $X \to Y$ a blow-up of a non-singular variety Y at a non-singular irreducible center. But by abuses of language, we often call a sequence of blow-ups f_i also a blow-up: $f = f_1 \circ f_2 \circ \cdots \circ f_n$: $X_n \to X_0$, where f_{i+1}: $X_{i+1} \to X_i$ is a blow-up of X_i at a non-singular irreducible center $(0 \leq i \leq n-1)$. We shall distinguish them clearly to avoid the confusion.

The following theorem is useful when we analyse the elements of $\mathscr{C}(G, X)$.

THEOREM (Danilov [Da]) (1.7). *Let f: $X \to Y$ be a birational morphism of non-singular projective varieties. If $\dim f^{-1}(y) \leq 1$ for any point $y \in Y$, then up to an automorphism of X, f can be decomposed into a sequence of blow-ups with smooth centers of codimension 2.*

When $\dim X = \dim Y = 2$, the Theorem is classical and well-known. We use the Theorem for threefolds.

We need the following Lemma which is finer than Lemma (1.21), [U3].

LEMMA (1.8). *Let f: $X \to Y$ be a projective, flat morphism of algebraic varieties. Let ψ: $G \times X \to X$ be an operation of a reductive algebraic group G on X such that $f \circ \psi = f \circ p_2$, where p_2: $G \times X \to X$ is the projec-*

tion. Then the following conditions are equivalent.

(1) *(G, X) is effective (resp. almost effective).*

(2) *There exists a point $y_0 \in Y$ such that the induced operation $(G, f^{-1}(y_0))$ is effective (resp. almost effective).*

(3) *For any point $y \in Y$, the induced operation $(G, f^{-1}(y))$ is effective (resp. almost effective).*

Proof. The implications $(3) \Rightarrow (2) \Rightarrow (1)$ are evident. Thus we have to prove $(1) \Rightarrow (3)$. This follows from Exposé IX, S.G.A.D. but we prove this directly. First we prove $(1) \Rightarrow (3)$ for the almost effective case. We show that if (3) does not hold for (G, X), then (1) does not hold. If there exists a point $y_0 \in Y$ such that $(G, f^{-1}(y_0))$ is not almost effective, then there exists a positive dimensional normal subgroup of G which operates trivially on $f^{-1}(y)$. Since any positive dimensional normal subgroup of the reductive group G contains a non-trivial torus, we can thus find a torus $T \neq 1 \subset G$ which operates trivially on $f^{-1}(y_0)$. It is sufficient to show that (T, X) is trivial. Namely to prove $(1) \Rightarrow (3)$, it is sufficient to prove it for the following special case: if G is a torus T and if there exists a point $y_0 \in Y$ such that T operates on $f^{-1}(y)$ trivially, then (T, X) is trivial. Let L be a relatively ample line bundle for $f: X \to Y$ such that f_*L is locally free of finite rank and $R^i f_* L = 0$ for all $i \geq 1$: such a bundle L exists replacing L by $L^{\otimes n}$, $n \gg 0$ if necessary. ψ defines a morphism $\tilde{\psi}: G \times Y = T \times Y \to \mathrm{Aut}_Y \mathbf{P}(f_*L)$ of group schemes over Y. For a point $y_0 \in Y$, we can find an affine neighbourhood Y' of y_0 such that f_*L is a free $\mathcal{O}_{Y'}$-Module. Therefore we may assume that Y is affine. Then $\tilde{\psi}$ gives over $Y = \operatorname{Spec} A$ a morphism $\tilde{\psi}': T \times Y \to \mathrm{PGL}_n \times Y$. Using embedding $\mathrm{PGL}_n \hookrightarrow \mathrm{GL}_N$, we finally get $\phi: T \times Y \to \mathrm{GL}_N \times Y$ and have to prove that ϕ is trivial. Putting by $T_{\ell^n} = \{t \in T \mid t^{\ell^n} = 1\}$ the finite subgroup of elements of order ℓ^n, ϕ defines a T_{ℓ^n}-module $M = A^{\oplus N}$, where ℓ is a prime number (different from p if $\operatorname{ch} k = p > 0$, which is not the case). The character χ: $T_{\ell^n} \to A$ is a function of an element t of T_{ℓ^n} and of a point y of Y: $\chi(t; y)$. If we fix $t \in T_{\ell^n}$, $y \mapsto \chi(t; y)$ is regular function on Y. Since T_{ℓ^n} is finite, there are only finitely many representations of T_{ℓ^n} of rank N over k and hence the function $y \mapsto \chi(t; y)$ is constant. Namely for all $y \in Y$, the reduction of T_{ℓ^n}-module M at y is isomorphic each other. Since the reduction at y_0 is the trivial T_{ℓ^n}-module, T_{ℓ^n} operates trivial on each fibre. Taking all $n > 0$, $\cup T_{\ell^n}$ is dense in T_{ℓ^n}, we conclude T operates trivial on M. Hence ϕ, $\tilde{\psi}'$ are trivial. The proof (1) $\Rightarrow$ (3) for effectiveness follows from the last argument for finite groups.

Lemma (1.8) is the rigidity of operation of a reductive group.

We prove 2 lemmas for the automorphism group of $\mathbf{A}^1$-bundle. As we explained above $\mathbf{A}^1$-bundles appear very often in the sequel.

LEMMA (1.9). *Let L be a line bundle over a projective algebraic variety X and D an effective Cartier divisor on X.* (a) *Then we denoting by L the total space* $\mathrm{Spec}(\bigoplus_{i\geq 0} L^{\otimes -i})$ *of the line bundle L,* $\mathrm{Aut}_X L$ *is representable in the category of reduced schemes over k and* $\mathrm{Aut}_X^0 L$ *is isomorphic to the semi-direct product* $H^0(X, L) \rtimes \mathbf{G}_m$. (b) *If the inclusion* $H^0(X, L(-D)) \to H^0(X, L)$ *is bijective, then* $H^0(X, L)$ *fixes all the points of L lying over D.*

Proof. For any reduced scheme $T \to X$, $\alpha \in \mathrm{Aut}_T(T \times_X L)$ is locally an affine transformation and hence can be extended to an T-automorphism of the $\mathbf{P}^1$-bundle $T \times_X \mathbf{P}(\mathcal{O} \oplus L^{-1})$. Therefore $\mathrm{Aut}_T(T \times_X L) = \{\sigma \in \mathrm{Aut}_T(T \times_X \mathbf{P}(\mathcal{O} \oplus L^{-1}))$ leaving the ∞-section invariant$\}$, which is a closed subgroup of the automorphism group $\mathrm{Aut}\,\mathbf{P}(\mathcal{O} \oplus L^{-1})$ of the projective variety $\mathbf{P}(\mathcal{O} \oplus L^{-1})$. Let $\bigcup_{\alpha\in I} U_\alpha$ be a covering of X such that L is defined by a 1-cocycle $\{g_{\alpha\beta}\}_{\alpha,\beta\in I}$, $g_{\alpha,\beta} \in H^0(U_\alpha \cap U_\beta, \mathcal{O}^*)$. Let $s \in H^0(X, L)$ which is locally given by $s_\alpha \in H^0(U_\alpha, 0)$ such that $g_{\alpha\beta}s_\alpha = s_\beta$. L is an $\mathbf{A}^1$-bundle gluing $U_\alpha \times \mathbf{A}^1$ ($\alpha \in I$) by $g_{\alpha\beta}$: $(x, u_\alpha) \in U_\alpha \times \mathbf{A}^1$ and $(y, u_\beta) \in U_\beta \times \mathbf{A}^1$ are identified if $x = y$ and $g_{\alpha\beta}u_\alpha = u_\beta$. Thus the automorphism $(x, u_\alpha) \mapsto (x_1 u_\alpha + s_\alpha)$ of $U_\alpha \times \mathbf{A}^1$ defines an action of the vector group $H^0(X, L)$ and $\mathbf{G}_m$ operates on each fibre by the scalor multiplication. If we notice that the $\mathrm{Lie}(\mathrm{Aut}_X L) \cong \{f \in \mathrm{End}(\mathcal{O} \oplus L) \mid f(\mathcal{O} \oplus 0) \subset \mathcal{O} \otimes 0\}$/scalor multiplications, (a) follows from Lemma (1.8). If $H^0(X, L(-D)) \cong H^0(X, L)$, all the section of $H^0(X, L)$ vanish on D and (b) is proved.

A similar argument gives

LEMMA (1.10). *Let L be a line bundle over a projective variety X and D an effective Cartier divisor on X.* (a) *We denoting by Z an $\mathbf{A}^1$-bundle over X defined by a non-trivial extension* $0 \to \mathcal{O} \to \tilde{\mathscr{E}} \to L \to 0$, $\mathrm{Aut}_X Y$ *is representable in the category of reduced schemes and* $\mathrm{Aut}_X^0 Y$ *is isomorphic to the vector group* $H^0(X, L)$. (b) *If the inclusion* $H^0(X, L(-D)) \to H^0(X, L)$ *is bijective, then* $\mathrm{Aut}_X^0 Z$ *fixes all the points of Z lying over D.*

Proof. This is proved by the same method as Lemma (1.9). See also

Lemma (4.22) [U4].

Certain SL_2-actions on a rational threefold do not have a fixed point as we see in [MU] and shall see later. In general, an SL_2-action on a threefold does not have too many fixed points.

LEMMA (1.11). *Let* (SL_2, X) *be an operation of* SL_2 *on a non-singular threefold* X. *Then the dimension of the subvariety* S *of* SL_2*-fixed points is at most* 1 *unless* $X = S$.

Proof. Assume that SL_2-fixes all the point of a surface S on X and that (SL_2, X) is non-trivial i.e. $X \neq S$. Let $x \in S$ be a smooth point of S. SL_2 operates on the Zariski tangent space m_x/m_x^2 but this operation fixes the S-direction so that m_x/m_x^2 contains a trivial representation $k \oplus k$ of degree 2. Hence by the complete reducibility of SL_2, m_x/m_x^2 is a trivial SL_2-module. As we have $S^j(m_x/m_x^2) \simeq m_x^j/m_x^{j+1}$, m_x^j/m_x^{j+1} is a trivial SL_2-module. Therefore by the complete irreducibility. $\mathcal{O}_x/m_x^j$ is a trivial SL_2-module for $j \geq 1$. Consequently $\hat{\mathcal{O}}_x = \varprojlim_j \mathcal{O}_x/m_x^j$ is a trivial SL_2-module and SL_2 operates on $\mathcal{O}_x$ trivially and hence on its quotient field, which is the function field of X. This is absurd as we assumed that (SL_2, X) is non-trivial.

The following Lemma gives an obstruction for blowing-down a divisor to a smooth point.

LEMMA (1.12). *Let* $\varphi\colon X \to Y$ *be a birational morphism of smooth projective threefolds and* $D \subset X$ *be an irreducible subvariety of* X. *If the image* $\varphi(D)$ *is a point, and if* φ *induces an isomorphism* $X - D \simeq Y - \varphi(D)$, *then* $\mathcal{O}(D) \otimes \mathcal{O}_D$ *is an ample line bundle on* D.

Proof. Let A be an ample divisor on X. $\varphi^*\varphi_*A$ is linearly equivalent to $A + nD$ with $n > 0$. Since for any divisor B on Y, $\mathcal{O}(\varphi^*B) \otimes \mathcal{O}_D$ is trivial on D, $\varphi^*\varphi_*A = A + nD$ is trivial when restricted on D. Thus $\mathcal{O}(A) \otimes \mathcal{O}(nD) \otimes \mathcal{O}_D \simeq \mathcal{O}_D$ hence $\mathcal{O}(A) \otimes \mathcal{O}_D \simeq \mathcal{O}(-nD) \otimes \mathcal{O}_D$ and $\mathcal{O}(-D) \otimes \mathcal{O}_D$ is ample.

The following Lemma would be well-known among the specialists. For the definition of a conic boudle, see [Be].

LEMMA (1.13). *Let* $(\varphi, f)\colon (G, X) \to (G, Y)$ *be a morphism of algebraic operations such that* $f\colon X \to Y$ *is a conic bundle over a non-singular surface* Y *and* $\rho(X) = \rho(Y) + 1$. *If any reduced G-invariant curve on* Y *is isomorphic to the disjoint union of some* $\mathbf{P}^1$*'s, then* $f\colon X \to Y$ *is a* $\mathbf{P}^1$*-bundle.*

Proof. Let $C \subset Y$ be an irreducible component of discriminant locus of the conic curve. We prove $C = \emptyset$. For otherwise it follows from the assumption that $\rho(X) = \rho(Y) + 1$ and from p. 87, Lemma, [Mi] that $\varphi^{-1}(C)$ is irreducible. Therefore by Proposition 1.5, [Be], there exists an étale 2-covering $C' \to C$ with an irreducible C'. This impossible since by hypotheses $C \simeq \mathbf{P}^1$.

§ 2. Compact case

Let (G, X) denote one of the following operations of Theorem (2.1), [U4]: (P1) $(\mathrm{PGL}_4, \mathbf{P}^3)$, (P2) $(\mathrm{PSO}_5$, quadric $\subset \mathbf{P}^4)$, (J1) $(\mathrm{PGL}_2 \times \mathrm{PGL}_3, \mathbf{P}^1 \times \mathbf{P}^2)$, (J2) $(\mathrm{PGL}_2 \times \mathrm{PGL}_2 \times \mathrm{PGL}_2, \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1)$ and (J4) $(\mathrm{PGL}_3, \mathrm{PGL}_3/B)$. These are the maximal subgroups where X is a projective homogeneous space. Let $(G, Y) \in \mathscr{C}(G, X)$. Then by definition, there exists a G-equivariant birational map $f\colon X \dashrightarrow Y$ which should be biregular as X is a complete homogeneous space. Therefore we have proved.

THEOREM (2.1). *Let* (G, X) *be one of the operations* (P1), (P2), (J1), (J2) *and* (J4) *of Theorem* (2.1), [U4]. *Then the set* $\mathscr{C}(G, X)$ *consists of a single element.*

§ 3. Equivariant completions of J3

(J3) is the operation $(\mathrm{PGL}_2 \times \mathrm{Aut}^0 \mathrm{F}'_m, \mathbf{P}^1 \times \mathrm{F}'_m) = (G, X)$ $(m \geq 2)$ which has a projective non-singular compactification $(G, Y) = (\mathrm{PGL}_2 \times \mathrm{Aut}^0 \mathrm{F}_m, \mathbf{P}^1 \times \mathrm{F}_m)$. The orbit decomposition is $\mathbf{P}^1 \times \mathrm{F}'_m \cup (\mathbf{P}^1 \times \mathrm{F}_m - \mathbf{P}^1 \times \mathrm{F}'_m)$. The latter is a divisor on $\mathbf{P}^1 \times \mathrm{F}_m$, which we denote by D. Let $(G, Z) \in \mathscr{C}(G, X)$. Then we have a G-equivariant birational map $f\colon Y \dashrightarrow Z$. By Hironaka's theorem of equivariant resolution, we can blow-up Y equivariantly to eliminate the indeterminacy of f. But as there is no orbit of codimension ≥ 2, f should be a birational morphism inducing an isomorphism between the open orbits. We show that f is biregular. Assume that $f(D)$ is a curve. Then it follows from Theorem (1.7), f is a blow-up but this is impossible since $D \simeq \mathbf{P}^1 \times \mathbf{P}^1$ and $\mathcal{O}(D) \otimes \mathcal{O}_D$ is $\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(0, -m)$, $m \geq 2$. It follows from Lemma (1.12) that $f(D)$ is not a point since $\mathcal{O}(D) \otimes \mathcal{O}_D \simeq \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(0, -m)$, $m \geq 2$. Therefore it follows from Zariski's main theorem that f is biregular. Hence we have proved.

THEOREM (3.2). *The set* $\mathscr{C}(\mathrm{PGL}_2 \times \mathrm{Aut}^0 \mathrm{F}'_m, \mathbf{P}^1 \times \mathrm{F}'_m)$ $(m \geq 2)$ *consists of a single element.*

§ 4. Equivariant completions of J5

(J5) is the operation (PGL_2, PGL_2/D_{2n}), where D_{2n} is the dihedral subgroup of order $2n$ with $n \geqq 4$. We denote by $\tilde{D}_{2n}$ the binary dihedral subgroup of order $4n$ of SL_2: $\tilde{D}_{2n}$ is the subgroup of SL_2 generated by $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ and $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$, $\zeta^{2n} = 1$. Since $(G, W) = (\mathrm{SL}_2, \mathrm{SL}_2/\tilde{D}_{2n})$ almost effectively realizes $(\mathrm{PSL}_2, \mathrm{SL}_2/\tilde{D}_{2n}) = (\mathrm{PGL}_2, \mathrm{PGL}_2/D_{2n})$, we study $\mathscr{C}(G, W)$. Sometimes $\mathscr{C}(G, W)$ is denoted by $\mathscr{C}(\mathrm{J5}\colon n)$.

The following Lemma was proved in [MU] but we give a proof because later we need a more general assertion which one can prove easily once one review the following

LEMMA (4.1). *Let* (SL_2, X) *be an operation of* SL_2 *on a projective nonsingular threefold* X. *If* SL_2 *has an open orbit on* X, *then there is no* SL_2*-fixed point on* X. *In particular if* $(G, Y) \in \mathscr{C}(\mathrm{J5}; n)$, *then there is no* G*-fixed point on* Y.

Proof. Let $P \in Y$ and assume that P is a fixed point of G. Since Y is projective and $G = \mathrm{SL}_2$ is simple, there exists an SL_2-invariant affine neighbourhood Spec A of P. We denote by I the ideal of A consisting of the elements of A vanishing at P. Putting $m = IA_I$, we get a surjective SL_2-linear map $\varphi\colon I^2 \to m^2/m^3$. We can choose an SL_2-invariant finite dimensional subspace V of I^2 such that $\varphi(V) = m^2/m^3$ by (1.9) Proposition, [Bo]. The Zariski tangent space m/m^2 is an SL_2-module and $S^2(m/m^2) \simeq m^2/m^3$. Thus m^2/m^3 contains a non-zero SL_2-invariant $\bar{f}$. For, there certainly exists $\bar{f}$ if m/m^2 is reducible (which implies that m/m^2 contains a trivial representation since m/m^2 is 3-dimensional) and if m/m^2 is irreducible, m/m^2 is isomorphic to the vector space of homogeneous polynomials of degree 2 in x, y where SL_2 operates on x, y in usual way and the discriminant which is a polynomial of degree 2 in the coefficients of a given homogeneous polynomial, is the SL_2-invariant. Since the image $\varphi(V) = m^2/m^3$ contains a trivial representation $k\bar{f}$, by the complete reducibility of SL_2, we can find a non-zero lifting $f \in V$ of $\bar{f}$ which is SL_2-invariant hence constant as SL_2 has an open orbit. This is absurd since f vanishes at P.

To apply the main Theorem of [Mo], we need a more general assertion than Lemma (4.1).

LEMMA (4.2). *Let (G, X) be an (algebraic) operation of a reductive algebraic group G on an affine variety $X =$ Spec A. Let $P \in X$ be a fixed point, n a non-negative integer and let W be a G-invariant subspace of I^n/I^{n+1}, where I is the ideal of A consisting of regular functions on X vanishing at P. Then there exists a G-invariant subspace W' of I^n such that the canonical map $W' \to W' + I^{n+1}/I^{n+1} \to I^n/I^{n+1}$ induces a G-isomorphism of W' and W.*

Proof. The lemma is proved by the same method as in the proof of Lemma (4.1).

COROLLARY (4.3). *Let (G, X), P and n be as in Lemma (4.2). If there exists a G-invariant element $\bar{f}$ in I^n/I^{n+1}, then there is a G-invariant lifting f of $\bar{f}$ to I^n.*

Proof. This is a particular case of Lemma (4.2) since G is completely reducible.

LEMMA (4.4). *Let $(G, X) \in \mathscr{C}(\mathrm{J5}; n)$. Then the 4 cases (3.3.2) $\cdots$ (3.3.5) for X in Theorem (3.2), [Mo] never occur.*

Proof. We excluded the case (3.3.2) by Lemma (4.1) since SL_2-operates on Y. The case (3.3.4) does not occur neither. For otherwise, the divisor D to be mapped to a singular point is SL_2-invariant and the single singular point of the cone D is left fixed by SL_2, which contradicts Lemma (4.1). Assume that the case (3.3.3) happens. Then $D \simeq \mathbf{P}^1 \times \mathbf{P}^1$ and $\mathcal{O}_D(D)$ is of bidegree $(-1, -1)$. Let $\phi: X \to Y$ be the morphism arising from the Mori theory and $\phi(D) = Q$. SL_2-operates on Y and ϕ is equivariant. It follows from Lemma (3.3.2), [Mo] that $I_Q/I_Q^2 \simeq H^0\ (\mathcal{O}_D(-D)) \simeq H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(1, 1))$. Thus the Zariski tangent space I_Q/I_Q^2 at Q, as an SL_2-module, is isomorphic to the tensor product $M_1 \otimes M_2$. Thus we have either (1) M_1 and M_2 are irreducible or (2) M_1 or M_2 is trivial. The first case never occurs. For otherwise since Y is projective and SL_2 is simple, we can find an SL_2-invariant affine neighbourhood Spec A of Q. Since there is a non-zero SL_2-invariant in I_Q/I_Q^2 in case (1), it follows from Corollary (4.3) that we can find a non-zero SL_2-invariant meromorphic function regular on Spec A vanishing at Q. This is absurd since SL_2 has an open orbit on Y. In case (2) we can find as before an SL_2-invariant affine neighbourhood Spec A of Q. Since there is a 2-dimensional SL_2-invariant subspace in I_Q/I_Q^2 in case (2), it follows that we can find a 2-dimensional SL_2-invariant subspace of A by Lemma (4.2) hence an SL_2-invariant linear system of dimension 1. Namely we get an SL_2 equivariant dominant rational map $X \dashrightarrow \mathbf{P}^1$, hence in particular a morphism $(\mathrm{SL}_2, \mathrm{SL}_2/\tilde{D}_{2n}) \to (\mathrm{SL}_2, \mathbf{P}^1)$ of algebraic operations from the open orbit. This is impossible as $\tilde{D}_{2n}$ can not be contained in a Borel subgroup. Now assume that the case (3.3.5) occurs. We argue similarly, using the notation in Mori [Mo], p. 146–147 as above. It follows from Lemma (3.32) that $I_Q/I_Q^2 \simeq H^0(D, \mathcal{O}_D(-D)) \simeq H^0(\mathbf{P}^2, \mathcal{O}(2))$. The latter is isomorphic to $S^2(H^0(\mathbf{P}^2, \mathcal{O}(1))$ hence contains a non-zero SL_2-invariant element $\bar{f}$ as we see in the Proof of Lemma (4.1). Arguing as in the preceding cases, $\bar{f}$ yields a non-zero SL_2-invariant (hence constant) rational function h vanishing at Q.

Let $(\mathrm{SL}_2, \mathbf{P}^1 \times \mathbf{P}^1)$ be the diagonal operation and E denote the irreducible SL-module of dimension 3. Then SL_2 operates on $\mathbf{P}(E)$ giving an operation $(\mathrm{SL}_2, \mathbf{P}^2)$. SL_2 has an open orbit on $\mathbf{P}^2$ and the orbit decomposition of $\mathbf{P}^2$ is $\mathrm{SL}_2/D_\infty \cup \mathrm{SL}_2/B$, where B is a Borel subgroup and D_∞ is a subgroup generated by $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, $a \in k^*$ and $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. The latter orbit is a quadric in $\mathbf{P}^2$. Let us define $\varphi\colon \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^2$ by $\varphi((x, y), (u, v)) = (xu, yu + xv, yv) \in \mathbf{P}^2$ for $((x, y), (u, v)) \in \mathbf{P}^1 \times \mathbf{P}^1$. Then $\varphi\colon \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^2$ is an SL_2-equivariant 2-covering whose branch locus is the diagonal of $\mathbf{P}^1 \times \mathbf{P}^1$ and whose ramification locus is the quadric $X_1^2 - 4X_0X_2 = 0$ in $\mathbf{P}^2$. Let $\mathcal{O}(i, j)$ denote the line bundle over $\mathbf{P}^1 \times \mathbf{P}^1$ of bidegree i, j. The direct image sheaf $\varphi_*\mathcal{O}(i, j) = E(i, j)$ is the rank 2 vector bundle studied by Schwartzenberger [Sc]. SL_2 operates on $E(i, j)$ hence on $\mathbf{P}(E(i, j))$. Since $\varphi^*\mathcal{O}_{\mathbf{P}^2}(1)$ is isomorphic to $\mathcal{O}(1, 1)$, $E(i, j) \simeq E(i - j, 0) \otimes \varphi^*\mathcal{O}_{\mathbf{P}^2}(j)$ hence $\mathbf{P}(E(i, j)) \simeq \mathbf{P}(E(i - j, 0))$. Thus we denote by E_i the vector bundle $E(i, 0)$.

LEMMA (4.5). *SL_2 operates on $\mathbf{P}(E_n)$ and has an open orbit isomorphic to $\mathrm{SL}_2/\tilde{D}_{2n}$ for $n \geq 1$.*

Proof. The stabilizer H at $(0, 1, 0) \in \mathbf{P}^2$ is the one dimensional subgroup D_∞ of SL_2 generated by $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in \mathrm{SL}_2$, $t \in k^*$ and $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. $\varphi^{-1}(0, 1, 0) = ((0, 1), (1, 0)) \cup ((1, 0), (0, 1))$ and the fibre of E_n at $(0, 1, 0)$ is identified with the direct sum of the fibre of $\mathcal{O}(n, 0)$ at $((0, 1), (1, 0))$ and $((1, 0), (0, 1))$. The operations of $T = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \middle| t \in k^* \right\}$ on the fibres of $\mathcal{O}(n, 0)$ at $((0, 1), (1, 0))$ and $((1, 0), (0, 1))$ are respectively by t^n and t^{-n}. The operation of $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ on $\mathbf{P}^1 \times \mathbf{P}^1$ interchanges the points $((0, 1), (1, 0))$ and $((1, 0), (0, 1))$

hence the fibres of $\mathcal{O}(n, 0)$. Thus the stabilizer at a suitable point of $\mathbf{P}$ (the fibre of E_n at (0, 1, 0)) is $\tilde{D}_{2n}$.

LEMMA (4.6). *Let* $(G, X) \in \mathscr{C}(\mathrm{J5}; n)$, (G, Y) *be an algebraic operation and* $f: X \to Y$ *be a G-equivariant dominant rational map. If Y is non-singular and complete and if* $\dim Y \leqq 2$, *then* (G, Y) *is isomorphic to* $(\mathrm{SL}_2, \mathbf{P}^2)$ *where* SL_2 *operates on* $\mathbf{P}^2$ *through the irreducible representation of degree* 3.

Proof. Assume that Y is a curve. As X is rational, Y is isomorphic to $\mathbf{P}^1$ by the Lüroth theorem and (SL_2, Y) is the usual operation of SL_2 on $\mathbf{P}^1$ since SL_2 must have an open orbit on Y. Thus f defines a surjective morphism $\mathrm{SL}_2/\tilde{D}_{2n} \to \mathrm{SL}_2/B$. Namely up to conjugacy $\tilde{D}_{2n}$ is contained in the Borel subgroup B of SL_2. This is absurd because $\tilde{D}_{2n} \subset \mathrm{SL}_2$ is an irreducible representation of degree 2 of $\tilde{D}_{2n}$. Hence Y is a surface. G has an open orbit also on Y since (G, X) has the open orbit and f is dominant. Let the open orbit on Y be isomorphic to SL_2/H. Then there is a surjective morphism $\mathrm{SL}_2/\tilde{D}_{2n} \to \mathrm{SL}_2/H$ for the open orbit and hence we may assume $\tilde{D}_{2n} \subset H$. It follows now that $H = D_\infty$. Thus (G, Y) is isomorphic to $(\mathrm{SL}_2, \mathbf{P}^2)$ as law chunks of algebraic operation, where SL_2 operates on $\mathbf{P}^2$ through the irreducible representation of degree 3 (cf. [U3]). Let $g: \mathbf{P}^2 \dashrightarrow Y$ be an SL_2-equivariant birational map. By Hironaka, there exists an equivariant blow-ups $g_1: Z \to \mathbf{P}^2$, $g_2: Z \to Y$ such that $g = g_2 \circ g_1^{-1}$. But the orbit decomposition of $\mathbf{P}^2$ is $\mathrm{SL}_2/D^\infty \cup \mathrm{SL}_2/B$ and hence $Z = \mathbf{P}^2$ and g is a birational morphism hence an isomorphism since $\mathbf{P}^2$ is relatively minimal and Y is non-singular.

THEOREM (4.7). *For* $n \geqq 4$, *the ordered set* $\mathscr{C}(\mathrm{J5}; n)$ *contains the unique minimal element which is given by* $(\mathrm{SL}_2, \mathbf{P}(E_n))$ *and any other element of* $\mathscr{C}(\mathrm{J5}; n)$ *is obtained by an equivariant blow-up of the minimal element along curves isomorphic to* $\mathbf{P}^1$.

Proof. Let $(G, X) \in \mathscr{C}(\mathrm{J5}; n)$. We show that if $\rho(X) \geqq 3$, then we can equivariantly blow down X to a non-singular projective Y. In fact, as X is rational, by Mori [Mo] there exists an extremal ray and a morphism $\phi: X \to Y$ in Theorem (3.1), [Mo]. The 4 cases (3.3.2) $\cdots$ (3.3.5) are already excluded by Lemma (4.4). We have to exclude all the cases in Theorem (3.5) in [Mo].

The case (3.5.1) in [Mo] never happens. Otherwise there would be a morphism $(G, X) \to (G, Y)$ with Y non-singular projective surface. It follows from Lemma (4.11) that $Y \cong \mathbf{P}^2$. Now $\rho(X) = \rho(Y) + 1 = \rho(\mathbf{P}^2) + 1 = 2$, which is a contradiction.

The case (3.5.2) in [Mo] is impossible by Lemma (4.6). The case (3.5.3) is excluded in Section 7, [MU].

We conclude therefore that if $\rho(X) \geqq 3$, then only the case (3.3.1) in [Mo] occurs.

Now we must show that if $(G, X) \in \mathscr{C}(\mathrm{J5}; n)$ and $\rho(X) = 2$, then (G, X) is isomorphic to $(\mathrm{SL}_2, \mathbf{P}(E_n))$. Then as in the $\rho(X) \geqq 3$ case, the extremal ray exists. Since as we have seen in Section 7 [MU] $\rho(Y) \geqq 2$ for $(G, Y) \in \mathscr{C}(\mathrm{J5}; n)$, the case (3.3.1), [Mo] never happens. The cases (3.3.2) $\cdots$ (3.3.5), [Mo] are excluded by Lemma (4.4). The case (3.5.2), [Mo] is excluded by Lemma (4.11) and we need not consider the case (3.5.3), [Mo] since $\rho(X) = 2$. The only possible case is the (3.5.1), [Mo] and it follows from Lemma (4.4) and Lemma (1.13) that φ: $X \to Y \cong \mathbf{P}^2$ is an SL_2-equivariant $\mathbf{P}^1$-bundle over $\mathbf{P}^2$ and hence X is isomorphic to $\mathbf{P}(E)$ for a suitable vector bundle E over $\mathbf{P}^2$ of rank 2 since the Brauer group of $\mathbf{P}^2$ vanishes. Let U denote the open SL_2-orbit on $\mathbf{P}^2$; $U = \mathbf{P}^2$-(the invariant quadric). We show that the orbit decomposition of $\mathbf{P}(E|U)$ is $\mathrm{SL}_2/\tilde{\boldsymbol{D}}_{2n}$ and $\mathrm{SL}_2/\mathbf{G}_m$ and the morphism φ makes the latter orbit an étale 2-covering of U. In fact as we saw in the proof of Lemma (4.5), the stabilizer at $(0, 1, 0) \in \mathbf{P}^2$ is $\tilde{\boldsymbol{D}}_\infty$. $\tilde{\boldsymbol{D}}_\infty$ operates on the fibre $\mathbf{P}(E)|_{(0,1,0)}$ which contains $\tilde{\boldsymbol{D}}_\infty/\tilde{\boldsymbol{D}}_{2n} \cong \mathbf{P}^1 - 2$ points $= \{(a, b) \in \mathbf{P}^1 \,|\, a, b \neq 0\} = Q$. The operation of $\tilde{\boldsymbol{D}}_\infty$ on Q is $(a, b) \mapsto (t^{2n}a, b)$ for $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ and $(a, b) \mapsto (b, a)$ for $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. Thus $\tilde{\boldsymbol{D}}_\infty$ operates on $\mathbf{P}(E)|_{(0,1,0)}$ as on Q. Let us put $D' = \mathbf{P}(E|U) - \mathrm{SL}_2/\tilde{\boldsymbol{D}}_{2n}$. Then $\varphi^{-1}(0, 1, 0) - Q$ consists of 2 distinct points and these 2 points are interchanged by $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ hence in the same SL_2-orbit. Since the morphism $D' \to U$ is SL_2-equivariant and surjective, D' is an SL_2-orbit. Consequently D' is isomorphic to $\mathrm{SL}_2/\mathbf{G}_m$ and $\mathrm{SL}_2/\mathbf{G}_m \to \mathrm{SL}_2/\tilde{\boldsymbol{D}}_\infty$ is of course étale. Now let D be the closure of D' in $\mathbf{P}(E) = X$. There exists a line bundle L over $\mathbf{P}^2$ such that $\mathcal{O}(D) \cong \mathcal{O}_{\mathbf{P}(E)}(2) \otimes \phi^*L$. Hence we have an exact sequence

$$\begin{array}{c} 0 \longrightarrow \mathcal{O}(-D) \longrightarrow \mathcal{O}_{\mathbf{P}(E)} \longrightarrow \mathcal{O}_D \longrightarrow 0\,. \\ \wr\Vert \\ \mathcal{O}_{\mathbf{P}(E)}(-2) \otimes \phi^*L^{-1} \end{array}$$

Tensoring $\mathcal{O}_{\mathbf{P}(E)}(1)$ we get

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}(E)}(-1) \otimes \phi^* L^{-1} \longrightarrow \mathcal{O}_{\mathbf{P}(E)}(1) \longrightarrow \mathcal{O}_D(1) \longrightarrow 0 .$$

If we take the direct image ϕ_*,

$$\begin{array}{ccccccc} 0 \longrightarrow \phi_*(\mathcal{O}_{\mathbf{P}(E)}(-1)) \otimes L \longrightarrow & \phi_*\mathcal{O}_{\mathbf{P}(E)}(1) & \longrightarrow \phi_*\mathcal{O}_D(1) \longrightarrow & R^1\mathcal{O}_{\mathbf{P}(E)}(-1) \otimes L . \\ & \| \wr & & \| \\ & E & & 0 \end{array}$$

Consequently $E \cong \phi_* \mathcal{O}_D(1)$. We shall show below that $D \cong \mathbf{P}^1 \times \mathbf{P}^1$. Admitting this for a moment, we conclude that $X = \mathbf{P}(E)$ is isomorphic to $\mathbf{P}(E_i)$ for a some integer $i \geqq 0$. SL_2 has an open orbit on $\mathbf{P}(E_i)$ isomorphic to $\mathrm{SL}_2/\tilde{\boldsymbol{D}}_{2i}$. Therefore $i = n$ by Theorem (2.1), [U4].

LEMMA (4.8). *D is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$.*

Proof. It is sufficient to prove that D is non-singular. In fact if we know that D is non-singular, D is a projective non-singular equivariant completion of $\mathrm{SL}_2/\mathbf{G}_m$. A particular projective, non-singular equivariant completion of $\mathrm{SL}_2/\mathbf{G}_m$ is given by $(\mathrm{SL}_2, \mathbf{P}^1 \times \mathbf{P}^1)$, where SL_2-operates diagonally. Therefore $\mathbf{P}^1 \times \mathbf{P}^1$ and D are connected by equivariant blowing-ups and downs. But as SL_2-has no fixed point on $\mathbf{P}^1 \times \mathbf{P}^1$ and $\mathbf{P}^1 \times \mathbf{P}^1$ is relatively minimal, $\mathbf{P}^1 \times \mathbf{P}^1$ is isomorphic to D.

Let us now prove that D is non-singular. Let us denote by $f\colon \tilde{D} \to D$ the normalization of D and by $\tilde{L}$ the inverse image $f^*(\mathcal{O}_{\mathbf{P}(E)}(1) \otimes \mathcal{O}_D)$. SL_2 operates on $\tilde{D}$ and has no fixed point since SL_2 has no fixed point on D. Therefore $\tilde{D}$ is a non-singular equivariant completion of $(\mathrm{SL}_2, \mathrm{SL}_2/\mathbf{G}_m)$ and it follows from the argument above that $\tilde{D}$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ with the diagonal action of SL_2 and if we put $\varphi' = \varphi | D$, $\varphi' \circ f\colon \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^2$ is the map studied above. Since f is an isomorphism outside the diagonal, $(\varphi' \circ f)_* \tilde{L}$ is isomorphic to E when restricted on $\mathbf{P}^2 -$ (the ramification quadric curve). Therefore SL_2 has an open orbit isomorphic to $\mathrm{SL}_2/\tilde{\boldsymbol{D}}_{2n}$ on $\mathbf{P}((\varphi' \circ f)_* \tilde{L})$ and consequently by Proposition (7) [Sc], and Theorem (2.1), [U4], $\tilde{L} \cong \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(a + n, a)$ (or $\mathcal{O}(a, a + n)$). This is a contradiction once the following Lemma is proved.

LEMMA (4.9). *Let M be a line bundle on D. If D is singular, then $f^*M \cong \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(a, a)$.*

Proof. The following sublemma is well-known (see [Sc]). To explain the assertion we need

NOTATION (4.10). Let Y be a variety, L a line bundle and φ a global section of L^2. Let L be defined by a cocycle g_{ij} for an open covering $(U_i)_{i\in I}$ and assume that φ is locally given by φ_i so that $g_{ij}^2\varphi_i = \varphi_j$. Let $X_i = \{(z_i, x_i) \in \mathbf{A}^1 \times U_i \mid z_i^2 = \varphi_i(x_i)\}$. Identifying $(z_i, x_i) \in \mathbf{A}^1 \times U_i$ and $(z_j, x_j) \in \mathbf{A}^1 \times U_j$ when $(z_j, x_j) = (g_{ij}z_i, x_i)$, we get the $\mathbf{A}^1$-bundle $\operatorname{Spec}(\bigoplus_{j\geq 0} L^{\otimes -j})$. Then X_i's are glued together to give $X = \cup X_i$. We say that the scheme X is the 2-covering defined by the section φ of L. The ramification locus of $X \to Y$ is the 0-locus of φ.

SUBLEMMA (4.11). *Let $X \to Y$ be a finite morphism of degree 2, Y being non-singular. If X is Cohen-Macauley, then there exist a line bundle L on Y and a section φ of L^2 such that X is Y-isomorphic to the 2-covering of Y defined by φ.*

By Sublemma (4.11), D is defined by a section s of $\mathcal{O}_{\mathbf{P}^2}(2\ell)$ whose reduced 0-locus is the SL_2-invariant quadric. Namely let $h(x_0, x_1, x_2)$ be the quadric defining the ramification quadric. Then $s = h^\ell$. Therefore, as $f\circ\varphi'$: $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^2$ is nothing but the canonical map associated with $\mathbf{P}^1 \times \mathbf{P}^1 \to$ (the symmetric product of 2 copies of $\mathbf{P}^1$) $\cong \mathbf{P}^2$, $f\circ\varphi'$ and φ' are locally written as follows:

$$\begin{array}{ccccc} \tilde{D} = \mathbf{P}^1 \times \mathbf{P}^1 & \xrightarrow{\quad f \quad} & D & \xrightarrow{\quad \varphi' \quad} & \mathbf{P}^2 \\ \cup & & \cup & & \cup \\ \operatorname{Spec} k[x, y] & \longrightarrow & \operatorname{Spec} k[xy, x+y, (x-y)^\ell] & \longrightarrow & \operatorname{Spec} k[x+y, xy] \end{array}$$

ℓ is odd and $\geqq 3$ since D is an irreducible 2-covering of $\mathbf{P}^2$ and singular. Let $\Delta(2)$ denote the non-reduced subscheme of $\mathbf{P}^1 \times \mathbf{P}^1$ defined by I_Δ^2, where I_Δ is the defining ideal of the diagonal. The inclusion morphism L: $\Delta(2) \to \mathbf{P}^1 \times \mathbf{P}^1$ is locally written by $k[x, y]/(x-y)^2 \leftarrow k[x, y]$. Composing with f, we get ψ: $\Delta(2) \to D$ which is locally given by $k[x, y]/(x-y)^2 \leftarrow k[x+y, xy, (x-y)^\ell]$. Let now i be the automorphism of $\mathbf{P}^1 \times \mathbf{P}^1$ interchanging factors. It follows from the local expression that $f\circ i\circ L = \psi$. Therefore if $f^*M = \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(a, b)$, then $L^*\mathcal{O}(a, b) \cong L^*\mathcal{O}(b, a)$. Now the Lemma follows from

LEMMA (4.12). *If $\mathcal{O}_{\Delta(2)} \otimes \mathcal{O}(a, b) \cong \mathcal{O}_{\Delta(2)} \otimes \mathcal{O}(b, a)$, then $a = b$.*

Proof. By tensoring $\mathcal{O}(-b, -a)$, we have to show that $\mathcal{O}_{\Delta(2)} \otimes \mathcal{O}(j, -j)$ is not isomorphic to $\mathcal{O}_{\Delta(2)}$ if $j \geqq 1$. In fact we have an exact sequence:

$$0 \longrightarrow \mathcal{O}(-2, -2) \longrightarrow \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1} \longrightarrow \mathcal{O}_{\Delta(2)} \longrightarrow 0. \tag{4.13}$$

Assume that $\mathcal{O}_{\Delta(2)} \otimes \mathcal{O}(j, -j) \cong \mathcal{O}_{\Delta(2)}$ with $j \geqq 1$. Tensoring $\mathcal{O}(j, -j)$ with the exact sequence (4.13), we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(j-2, -j-2) & \longrightarrow & \mathcal{O}(j, -j) & \longrightarrow & \mathcal{O}_{\Delta(2)}(j, -j) & \longrightarrow & 0\,. \\ & & & & & & \| \wr & & \\ & & & & & & \mathcal{O}_{\Delta(2)} & & \end{array} \tag{4.14}$$

Since the morphisms in this exact sequence are SL_2-morphisms, we get SL_2-exact sequence

$$\begin{array}{ccccc} 0 & \longrightarrow & H^0(\mathcal{O}_{\Delta(2)}) & \longrightarrow & H^1(\mathcal{O}(j-2, -j-2))\,. \\ & & \| \wr & & \\ & & k & & \end{array} \tag{4.15}$$

The SL_2-module $H^1(\mathcal{O}(j-2, -j-2))$ contains the trivial module k. Let W_d denote the irreducible SL_2-module of degree $d+1$ if $d \geqq 0$ and $W_d = 0$ if $d < 0$. By the Künneth formula and the Serre duality, $H^1(\mathcal{O}(j-2, j-2)) \cong W_{j-2} \otimes W_j$ for $j \geqq 1$. By the Clebsch-Gordan formula (p. 126, [Hu]) $W_{j-2} \otimes W_j \cong W_{2j-2} \oplus W_{2j-4} \oplus \cdots \oplus W_2$, $W_{j-2} \otimes W_j$ contains no trivial SL_2-module k which is absurd.

§ 5. Equivariant completions of J6

We have to consider the operation $(G, W_{m,n}) = (\mathbf{G}_m \times SL_2 \times SL_2, \mathbf{G}_m \times SL_2 \times SL_2/H_{m,n})$, where

$$H_{m,n} = \left\{\left(t_1^m t_2^n, \begin{pmatrix} t_1 & x \\ 0 & t_1^{-1} \end{pmatrix}, \begin{pmatrix} t_2 & y \\ 0 & t_2^{-1} \end{pmatrix}\right) \in \mathbf{G}_m \times SL_2 \times SL_2 \,\middle|\, t_1, t_2 \in k^*, x, y \in k\right\}$$

and m, n are integers with $m \geqq 2$, $-2 \geqq n$. We denote $\mathscr{C}(G, W_{m,n})$ by $\mathscr{C}(J6; m, n)$. $W_{m,n}$ is the principal $\mathbf{G}_m$-bundle of bidegree (m, n) over $SL_2/B \times SL_2/B \cong \mathbf{P}^1 \times \mathbf{P}^1$. Hence $(G, W_{m,n})$ has a natural equivariant completion $L_{m,n} = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(-m, -n))$ and we know that we have a G-equivariant morphism (φ, f): $(G, \mathbf{P}(\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(m, n)) \to (SL_2 \times SL_2, \mathbf{P}^1 \times \mathbf{P}^1)$, φ being the projection ([U4]). We can regard $\mathrm{Spec}\,(\bigoplus_{\ell\geqq 0} \mathcal{O}(-\ell m, -\ell n))$ as a Zariski open set in $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(-m, -n))$. We set

$$D_\infty = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(-m, -n)) - \mathrm{Spec} \bigoplus_{\ell \geq 0} \mathcal{O}(-\ell m, \ell n))$$

and $D_0 = \mathrm{Spec}\,(\bigoplus_{\ell\geq 0}(-\ell m, \ell n)) - W_{m,n}$ (= the zero section of the line bundle $\mathrm{Spec}\,(\bigoplus_{\ell\geqq 0} \mathcal{O}(-\ell m, \ell n))$. Namely adding the 0-section D_0 to the $\mathbf{G}_m$-bundle $W_{m,n}$, we get the line bundle $\mathrm{Spec}\,\mathcal{O}(\bigoplus_{\ell\geqq 0}(-\ell m, -\ell n))$ and furthermore by adding the ∞-section D_∞ to the line bundle Spec$(\bigoplus_{\ell\geqq 0}\mathcal{O}(-\ell m, -\ell n))$, we get the $\mathbf{P}^1$-bundle $\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(-m, -n))$ over $\mathbf{P}^1\times\mathbf{P}^1$. By the equivariant morphism f, the orbit decomposition of $\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(-m, -n))$ is $W_{m,n}\cup D_0\cup D_\infty$.

THEOREM (5.1). *$\mathscr{C}$(J6; m, n) ($m\geqq 2$, $-2\geqq n$) consists of a single element ($G, L_{m,n}$).*

Proof. Let $(G, X)\in\mathscr{C}(\mathrm{J6}; m, n)$. By Hironaka's Theorem, there exists an equivariant blow-up $f_1\colon Y\to\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(-m, -n))$ and a birational morphism $f_2\colon Y\to X$. It follows from orbit decomposition that f_1 is an isomorphism. Thus we get a birational morphism $f_2\colon \mathbf{P}(\mathcal{O}\oplus\mathcal{O}(-m, -n))\to X$. We have $\mathcal{O}(-D_0)\otimes\mathcal{O}_{D_0}\cong\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(m, n)$ and $\mathcal{O}(-D_\infty)\otimes\mathcal{O}_{D_\infty}\cong\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(-m, -n)$ if we identify D_0 and D_∞ with $\mathbf{P}^1\times\mathbf{P}^1$ by the equivariant map f. It follows from Lemma (1.12) that neither $f_2(D_0)$ nor $f_2(D_\infty)$ is reduced to a point. Since $m\geqq 2$, $-2\geqq n$, then we can blow down neither D_0 nor D_∞ to a curve (cf. Theorem (1.7)). Hence by Zariski's Main Theorem f_2 is an isomorphism.

§ 6. Equivariant completions of J7

J'_m is by definition Spec$(S(\mathcal{O}_{\mathbf{P}^2}(-m))$; the total space of the line bundle of degree m over $\mathbf{P}^2$. We study the operation (J7) ($\mathrm{Aut}^0\,\mathrm{J}'_m$, J'_m) ($m\geqq 2$). We know by Proposition (4.8), [U4] $\mathrm{Aut}\,^0\mathrm{J}'_m$ respects the fibration $\mathrm{J}_m\to\mathbf{P}^2$. Therefore the operation (J7) ($\mathrm{Aut}^0\,\mathrm{J}'_m$, J'_m) has a natural equivariant completion ($\mathrm{Aut}^0\,\mathrm{J}_m$, J_m) by Corollary (4.10), [U4] where J_m denotes $\mathbf{P}(\mathcal{O}_{\mathbf{P}^2}\oplus\mathcal{O}_{\mathbf{P}^2}(-m))$. We denote $\mathscr{C}(\mathrm{Aut}^0\,\mathrm{J}'_m, \mathrm{J}_m)$ by $\mathscr{C}(\mathrm{J7}; m)$.

THEOREM (6.1). *$\mathscr{C}$(J7; m) ($m\geqq 2$) consists of a single element* ($\mathrm{Aut}^0\,J_m$, J_m).

Proof. For the same reason as in the case (J6), it is sufficient to notice that we can not collapse the orbit $D_\infty=\mathrm{J}_m-\mathrm{J}'_m\cong\mathbf{P}^2$ to a smooth point. In fact, let $f\colon \mathrm{J}_m\to X$ be a birational morphism such that X is non-singular and projective, f is biregular outside D_∞ and such that $f(D_\infty)$ is a point. Let $H=\pi^*\mathcal{O}_{\mathbf{P}^2}(1)$ so that $H\otimes\mathcal{O}_{D_\infty}\cong\mathcal{O}_{\mathbf{P}^2}(1)$, where $\pi\colon \mathrm{J}_m\to\mathbf{P}^2$ is the projection. Then $\mathcal{O}_{D_\infty}\cong f^*(f_*H)\otimes\mathcal{O}_{D_\infty}\cong(H\otimes\mathcal{O}(rD_\infty))\otimes\mathcal{O}_{D_\infty}\cong\mathcal{O}_{\mathbf{P}^2}(1)\otimes(\mathcal{O}(rD_\infty)\otimes\mathcal{O}_{D_\infty})$. Hence $r=1$ and $\mathcal{O}(D_\infty)\otimes\mathcal{O}_{D_\infty}\cong\mathcal{O}_{\mathbf{P}^2}(-1)$. But $\mathcal{O}(D_\infty)\otimes\mathcal{O}_{D_\infty}\cong\mathcal{O}_{\mathbf{P}^2}(-m)$, which is a contradiction.

§ 7. Equivariant completions of J8

We study (J8) ($\mathrm{Aut}^0 L'_{m,n}$, $L'_{m,n}$) ($m \geqq n \geqq 1$): $L'_{m,n}$ is the total space $\mathrm{Spec}(S(\mathcal{O}(-m)\otimes\mathcal{O}(-n)))$ of the line bundle $\mathcal{O}(m, n)$ of bidegree (m, n) over $\mathbf{P}^1\times\mathbf{P}^1$. The operation (J8) ($\mathrm{Aut}^0 L'_{m,n}$, $L'_{m,n}$) respects the fibration $L'_{m,n}\to \mathbf{P}^1\times\mathbf{P}^1$ by Proposition (4.11) and hence by Corollary (4.12) the operation (J8) ($\mathrm{Aut}^0 L'_{m,n}$, $L'_{m,n}$) has a natural equivariant completion ($\mathrm{Aut}^0 L_{m,n}$, $L_{m,n}$), where $L_{m,n}$ denotes the $\mathbf{P}^1$-bundle $\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(-m, -n))$ over $\mathbf{P}^1\times\mathbf{P}^1$. If $n = 1$, by the composite morphism $L_{m,1}\xrightarrow{\pi}\mathbf{P}^1\times\mathbf{P}^1\xrightarrow{p_1}\mathbf{P}^1$, we can regard $L_{m,1}$ equivariantly as F_1-bundle over $\mathbf{P}^1$. Replacing each fibre F_1 by $\mathbf{P}^2$, we get an equivariant completion ($\mathrm{Aut}^0 L'_{m,1}$, X_m). The $\mathbf{P}^2$-bundle X_m over $\mathbf{P}^1$ is isomorphic to $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}\oplus\mathcal{O}_{\mathbf{P}^1}\oplus\mathcal{O}_{\mathbf{P}^1}(-m))$ by its construction. We denote $\mathscr{C}(\mathrm{Aut}^0 L'_{m,n}, L'_{m,n})$ by (J8; m, n).

THEOREM (7.1). *If* $m\geqq n\geqq 2$, $\mathscr{C}$(J8; m, n) *consists of one element* ($\mathrm{Aut}^0 L_{m,n}$, $L_{m,n}$). *The ordered set* $\mathscr{C}$(J8; m, 1) *consists of* 2 *elements*; ($\mathrm{Aut}^0 L'_{m,1}$, $L_{m,1}$) $>$ ($\mathrm{Aut}^0 L'_{m,1}$, X_m) ($m\geqq 1$).

Theorem is proved by the same method as Theorem (6.1). Hence we omit the proof.

§ 8. Equivariant completions of J9

Let $C_\infty = \mathrm{F}_n - \mathrm{F}'_n$; namely C_∞ is the section of $\mathbf{P}^1$-bundle f_n: $\mathrm{F}_n\to\mathbf{P}^1$ with $C_\infty^2 = -n$. The section with this property is uniquely determined and called the infinity section of F_n (see (1.2)). We denote by $\mathcal{O}_{\mathrm{F}_n}(t)$ the line bundle $f_n^*\mathcal{O}_{\mathbf{P}^1}(t)$ and by $\mathcal{O}_{\mathrm{F}'_n}(t)$ its restriction on F'_n.

The variety $\mathrm{F}'_{m,n}$ ($m\geqq n\geqq 1$) is the total space of the vector bundle $\mathcal{O}_{\mathbf{P}^1}(m)\oplus\mathcal{O}_{\mathbf{P}^1}(n)$ over $\mathbf{P}^1$: $\mathrm{F}'_{m,n} = \mathrm{Spec}(S(\mathcal{O}_{\mathbf{P}^1}(-m)\oplus\mathcal{O}_{\mathbf{P}^1}(-n)))$, where $S(E)$ denotes the symmetric algebra on E. We can regard $\mathrm{F}'_{m\,n}$ also as the total space of a line bundle $\mathcal{O}_{\mathrm{F}'_n}(m)$ over F'_n. Namely $\mathrm{F}'_{m,n} = \mathrm{Spec}(S(\mathcal{O}_{\mathrm{F}'_n}(-m)))$ and we have a morphism $\mathrm{F}'_{m,n}\to\mathrm{F}'_n$ giving the bundle structure over F'_n. (J9) is the operation ($\mathrm{Aut}^0\mathrm{F}'_{m,n}$, $\mathrm{F}'_{m,n}$) ($m > n\geqq 2$). Since ($\mathrm{Aut}^0\mathrm{F}'_{m,n}$, $\mathrm{F}'_{m,n}$) respects the fibration by Proposition (4.13), [U4], to complete equivariantly the $\mathbf{A}^1$-bundle $\mathrm{F}'_{m,n}$ over F'_n, first we want to equivariantly extend it to an $\mathbf{A}^1$-bundle over F_n. For this purpose, in view of Lemma (1.9) in section 1 of this paper, Lemma (4.4), Proposition (4.13) and Corollary (4.17) all in [U4], it is sufficient to find a line bundle M over F_n such that (1) M is $\mathrm{Aut}^0\mathrm{F}_n$-equivariant, M is isomorphic to $\mathcal{O}_{\mathrm{F}'_n}(m)$

when restricted over F'_n and such that (2) the restriction $H^0(F_n, M) \to H^0(F'_n, M) \cong H^0(F'_n, \mathcal{O}_{F'_n}(m))$ induces an isomorphism. In fact, then $\mathrm{Spec}(S(M^{-1})) \supset \mathrm{Spec}(S(\mathcal{O}_{F'_n}(-m))$ is a desired extension to F_n and $\mathbf{P}(\mathcal{O}_{F_n} \oplus M^{-1})$, which is an equivariant completion of $\mathrm{Spec}(S(M^{-1}))$, is an equivariant completion of $(\mathrm{Aut}^0 F'_{m,n}, F'_{m,n})$.

We choose an integer ℓ such that $m = \ell n + r$, $0 \leqq r < n$ (cf. Corollary (4.17), [U4]).

LEMMA (8.1). *$(\mathrm{Aut}^0 F'_{m,n}, F'_{m,n})$ has an equivariant completion $(\mathrm{Aut}^0 F^k_{m,n}, F^k_{m,n})$ for any integer $k \geqq \ell$, where $F^k_{m,n}$ denotes the $\mathbf{P}^1$-bundle $f^k_{m,n}$: $\mathbf{P}(\mathcal{O}_{F_n} \oplus \mathcal{O}_{F_n}(-m) \otimes \mathcal{O}_{F_n}(-kC_\infty)) \to F_n$.*

Proof. By the argument preceding the Lemma, we look for a condition for the restriction $0 \to H^0(F_n, \mathcal{O}_{F_n}(m) \otimes \mathcal{O}_{F_n}(kC_\infty)) \to H^0(F'_n, \mathcal{O}_{F'_n}(m))$ to be isomorphism. It follows from the spectral sequences in (1.2), (1.2.1) and (1.3.1),

$$\begin{array}{ccc}
0 \to H^0(F_n, \mathcal{O}_{F_n}(m) \otimes \mathcal{O}_{F_n}(kC_\infty)) & \longrightarrow & H^0(F'_n, \mathcal{O}_{F'_n}(m)) \\
\wr\| & & \wr\| \\
0 \to H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m) \otimes S^k(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-n)) & \longrightarrow & H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m) \otimes S(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-n)) \\
\wr\| & & \wr\| \\
0 \to \bigoplus_{s=0}^{k} H^0(\mathbf{P}^1, \mathcal{O}(m - sn)) & \longrightarrow & \bigoplus_{t \geqq 0} H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m - tn)) .
\end{array}$$

Therefore $k \geqq \ell$ is the necessary and sufficient condition.

We denote by D^k_2 the ∞-section of the $\mathbf{P}^1$-bundle $F^k_{m,n}$ over F_n so that $D^k_2 = F^k_{m,n} - \mathrm{Spec}(S(\mathcal{O}_{F_n}(-m)) \otimes \mathcal{O}_{F_n}(-kC_\infty))$ and by D^k_1 the inverse image $(f^k_{m,n})^{-1}C_\infty$. We have a morphism

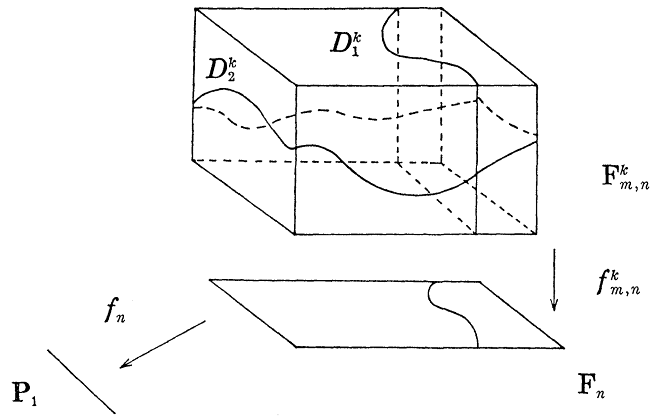
$(\mathrm{Id}, f^k_{m,n})$: $(\mathrm{Aut}^0 F^k_{m,n}, F^k_{m,n}) \longrightarrow (\mathrm{Aut}^0 F^k_{m,n}, F_n)$ of algebraic operations.

The projection $f^k_{m,n}$ induces an isomorphism $D^k_2 \cong F_n$ and the divisor D^k_1 is isomorphic to $\mathbf{P}((\mathcal{O} \oplus \mathcal{O}(-m) \otimes \mathcal{O}(-kC_\infty)) \otimes \mathcal{O}_{C_\infty})$ hence to $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-m + kn)) \cong F_{|-m+kn|}$. The intersection $D^k_2 \cap D^k_1$ is the ∞-section of the ruled surface $D^k_1 \cong F_{-m+kn}$ (see Fig (8.2)).

We use the orbit decomposition of $F^k_{m,n}$ in a substantial way.

LEMMA (8.3). *The orbit decomposition of $(\mathrm{Aut}^0 F'_{m,n}, F^\ell_{m,n})$ is $F'_{m,n} \cup (D^\ell_1 - D^\ell_1 \cap D^\ell_2) \cup (D^\ell_2 - D^\ell_1 \cap D^\ell_2) \cup (D^\ell_1 \cap D^\ell_2)$. For any integer $k > \ell$, the orbit decomposition of $(\mathrm{Aut}^0 F'_{m,n}, F^k_{m,n})$ is $F'_m \cup (D^k_1 - D^k_1 \cap D^k_2 - C) \cup (D^k_2 - D^k_1 \cap D^k_2) \cup (D^k_1 \cap D^k_2) \cup C$, where C is a 0-section of the ruled surface $D^k_1 \cong F_{-m+nk}$.*

Fig. (8.2)



Proof. It follows from Corollary (4.17) and Lemma (4.4), [U4] that $F'_{m,n}$, D_1^j and D_2^j, $D_1^j \cap D_2^j$ are $\operatorname{Aut} F'_{m,n}$-invariant subvarieties of $F^j_{m,n}$ for $j \geqq \ell$. By the morphism $f^j_{m,n}$, the invariant divisor D_2^j decomposes into the union $(D_2^j - D_1^j \cap D_2^j) \cup (D_1^j \cap D_2^j)$ of 2 invariant subvarieties. It follows from Corollary (4.17), [U4] that the unipotent part of the $\operatorname{Ker}(\operatorname{Aut}^0 F'_{m,n} \to \operatorname{Aut}^0 F'_n)$ is $H^0(F'_n, \mathcal{O}_{F'_n}(m))$. In the Proof of Lemma (8.1) we choose $j \geqq \ell$ so that the restriction

$$(*)\quad H^0(F_m, \mathcal{O}_{F_n}(m) \otimes \mathcal{O}_{F_n}(jC_\infty)) \longrightarrow H^0(F'_n, \mathcal{O}_{F_n}(m) \otimes \mathcal{O}_{F_n}(jC_\infty)) = H^0(F'_n, \mathcal{O}_{F'_n}(m))$$

induces an isomorphism between the cohomology groups and ℓ is the smallest among such integers. Now the injection $\mathcal{O}_{F_n}(m) \otimes \mathcal{O}_{F_n}(jC_\infty) \to \mathcal{O}_{F_n}(m) \otimes \mathcal{O}_{F_n}((j+1)C_\infty)$ induces an injection $0 \to H^0(F_n, \mathcal{O}_{F_n}(m) \otimes \mathcal{O}_{F_n}(jC_\infty)) \to H^0(F_n, \mathcal{O}_{F_n}(m) \otimes \mathcal{O}_{F_n}((j+1)C_\infty))$, which is thus an isomorphism since their dimensions coincide by (*). Therefore all the sections of $H^0(F_n, \mathcal{O}_{F_n}(m) \otimes \mathcal{O}_{F_n}(j+1)C_\infty)$ vanishes on C_∞. It follows from Lemma (1.9) that $H^0(F_n, \mathcal{O}_{F_n}(m) \otimes \mathcal{O}_{F_n}(jC_\infty))$ operates on D_1^j trivially if $j > \ell$ and non-trivially on D_1^ℓ. The Lemma now follows from Lemma (1.8) and from [U3].

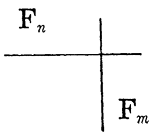
DEFINITION (8.4). Let X be a projective non-singular threefold and S a finite set of irreducible divisors D_i on X $(1 \leqq i \leqq n)$ such that (1) each D_i is isomorphic to a rational ruled surface F_{m_i}, $m_i \geqq 0$ $(1 \leqq i \leqq n)$, (2) if $D_i \cap D_j \neq \emptyset$, then D_i and D_j intersects transversely along $D_i \cap D_j = C_{ij} (\cong P^1)$ which is a section of the rational ruled surfaces F_{m_i} and F_{m_j} with $|(C_{ij} \cdot C_{ij})_{D_i}| = m_i$ and $|(C_{ij} \cdot C_{ij})_{D_j}| = m_j$ and such that (3) for any three distinct divisors $D_a, D_b, D_c \in S$, $D_a \cap D_b \cap D_c = \emptyset$. We associate to

S a diagram D. D consists of n segments $\ell_1, \ell_2, \cdots, \ell_n$ on the plane satisfying the following conditions.

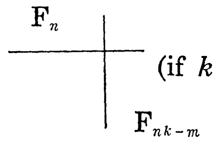
(a) $\ell_i \cap \ell_j$ is either empty or a point for any $1 \leqq i < j \leqq n$.

(b) $\ell_i \cap \ell_j \neq \emptyset$ if and only if $D_i \cap D_j \neq \emptyset$.

EXAMPLES (8.5.1). We taking for S the set of G-invariant divisors on $\mathrm{F}^k_{m,n}$, it follows from Lemma (8.3) that S is described by

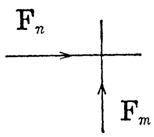


and



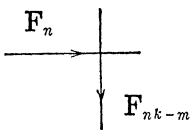
(if $k > \ell$).

Assuming $m - n\ell > 0$, to indicate that $C' = \mathrm{F}_n \cap \mathrm{F}_{m-n\ell}$ has the intersection numbers; $(C'^2)_{\mathrm{F}_{m-n\ell}} = n\ell - m < 0$, $(C'^2)_{\mathrm{F}_n} = -n < 0$, we complete the diagram by making the segments into arrows:



.

For k with $nk - m > 0$, $C' = \mathrm{F}_n \cap \mathrm{F}_{nk-m}$ has the intersection number $(C'^2)_{\mathrm{F}_n} = -n < 0$, $(C'^2)_{\mathrm{F}_{nk-m}} = nk - m > 0$. Therefore our diagram is



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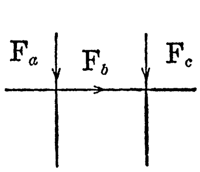
(8.5.2). The diagram



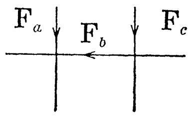
signifies

F_a and F_b (resp. F_b and F_c) intersect transversely along sections of the ruled surfaces and we have the intersection numbers

$$(\mathrm{F}_a \cap \mathrm{F}_a)^2_{\mathrm{F}_a} = -a,\ (\mathrm{F}_a \cap \mathrm{F}_b)^2_{\mathrm{F}_b} = b,\ (\mathrm{F}_a \cap \mathrm{F}_c)^2_{\mathrm{F}_b} = -b$$

and $(\mathrm{F}_b \cap \mathrm{F}_c)^2_{\mathrm{F}_c} = -c$.

(8.5.3). The diagram



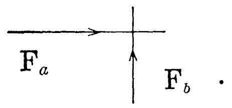
signifies among other things

$(F_a \cap F_b)^2_{F_a} = -a$, $(F_a \cap F_b)^2_{F_b} = -b$, $(F_b \cap F_c)^2_{F_b} = b$ and $(F_b \cap F_c)^2_{F_c} = -c$.

The following Lemma shows the convenience of the diagram.

LEMMA (8.5.4). *Let X be a non-singular projective threefold.*

(i) *Assume that we have two divisors on X isomorphic respectively to F_a and F_b such that they are expressed in the diagram*



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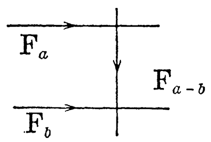
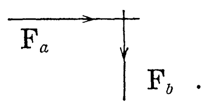
Let p: $Y \to X$ be a blow-up of X along $F_a \cap F_b$. If $a > b \geq 1$, then the divisor $p^{-1}(F_a \cup F_b)$ on Y is expressed in terms of the diagram



where F_a and F_b are the proper transforms and F_{a-b} is the exceptional divisor.

(ii) *Assume that we have two divisors on X isomorphic to F_a and F_b such that they are expressed in the diagram*



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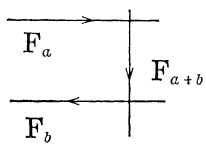
Let p: $Y \to X$ be a blow up of X along $F_n \cap F_b$. If a, $b \geq 1$, then the divisor $p^{-1}(F_a \cup F_b)$ on Y is expressed in terms of the diagram

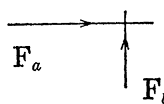


where F_a *and* F_b *are the proper transforms and* F_{a+b} *is the exceptional divisor for p.*

Proof. Since F_a and F_b intersect transversely, the exceptional divisor E is a rational ruled surface and $E \cap F_a$, $E \cap F_b$ are disjoint sections of the ruled surfaces. To distinguish F_a, F_b from their proper transforms, the proper transforms of F_a and F_b are denoted by $\tilde{F}_a$ and $\tilde{F}_b$. Let us first prove (i). $(\tilde{F}_a \cap E)^2_E = ((\tilde{F}_a \cap E)\cdot(\tilde{F}_a \cap E))_E = (\tilde{F}_a \cdot \tilde{F}_a \cdot E)_Y = (\tilde{F}_a + E - E \cdot \tilde{F}_a \cdot E) = (p^*F_a - E \cdot \tilde{F}_a \cdot E) = (p^*F_a \cdot \tilde{F}_a \cdot E)_Y - (E \cdot \tilde{F}_a \cdot E) = (F_a \cdot p_*(\tilde{F}_a \cdot E))_X - (-a) = -b + a$.

Similarly we get $(F_b \cap E)^2_E = a - b$. Therefore E is isomorphic to F_{a-b}. Since $a - b > 0$, the arrow on $E \simeq F_{a-b}$ points from F_a to F_b. The second assertion is proved by the same method.

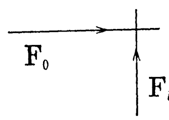
Remark (8.5.5). In the diagram it is convenient to extend the arrow to the ruled surface $F_0 = \mathbf{P}^1 \times \mathbf{P}^1$. To be precise, for example by definition the diagram



$(a, b \geqslant 1)$

signifies that the divisors intersect transversely, $(F_a \cap F_b)^2_{F_a} = -a$, $(F_a \cap F_b)^2_{F_a} = -b$ and $F_a \cap F_b$ is a section of the ruled surfaces F_a, F_b. Let us allow here $a = 0$ or $b = 0$ and then an arrow means one intersection number is greater than or equal to the other. The diagram

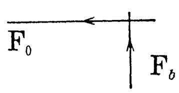
(8.5.5)



$(b > 0)$

shows that $F_0 \cap F_b$ is a section of the ruled surfaces F_0, F_b and $(F_0 \cap F_b)^2_{F_b} = 0$ and $(F_0 \cap F_b)^2_{F_b} = -b$.

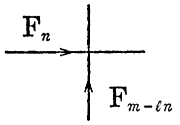
Therefore the diagram (8.5.5) is equivalent to



This remark being done, Lemma (8.5.4) (i) holds for $a \geq b \geq 0$ and (ii) is correct for a, $b \geq 0$.

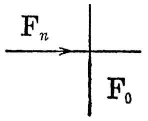
It follows from Lemma (8.3),

(8.6) the set of the G-invariant divisors on $\mathrm{F}^{\ell}_{m,n}$ are described by,



if $m - \ell n > 0$

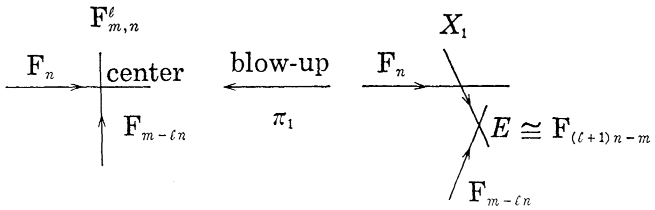
or by,



if $m - \ell n = 0$.

Let us treat only the case $m - \ell n > 0$ because the case $m - \ell n = 0$ is treated without any essential modification. There is only one choice of the center to blow up $\mathrm{F}^{\ell}_{m,n}$ equivariantly: the intersection $\mathrm{F}_n \cap \mathrm{F}_{m-\ell n}$. Let us blow-up $\mathrm{F}^{\ell}_{m,n}$ at $\mathrm{F}_n \cap \mathrm{F}_{m-\ell n}$: π_1: $X_1 \to \mathrm{F}^{\ell}_{m,n}$, we get a new diagram of G-invariant divisors

(8.7)



by Lemma (8.5.4).

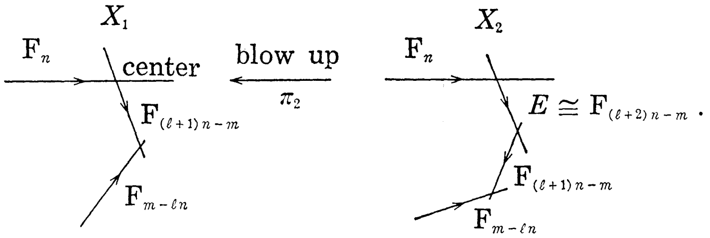
We can now show that we can blow down $\mathrm{F}_{m-\ell n}$ in X_1 to $\mathbf{P}^1$ and we get $X^{\ell+1}_{m,n}$. This procedure is called the elementary transformation applied for the $\mathbf{P}^1$-bundle $f^{\ell}_{m,n}$: $\mathrm{F}^{\ell}_{m,n} \to \mathrm{F}_n$ with center $D^{\ell}_1 \cap D^{\ell}_2$ (see Maruyama [Mar 2]). Similarly the $\mathbf{P}^1$-bundle $f^{k}_{m,n}$: $\mathrm{F}^{k}_{m,n} \to \mathrm{F}_n$ and the $\mathbf{P}^1$ -bundle $f^{k+1}_{m,n}$: $\mathrm{F}^{k+1}_{m,n} \to \mathrm{F}_n$ are related one another by equivariant elementary transformations. Since $\mathrm{F}^{\ell}_{m,n}$ is a $\mathbf{P}^1$-bundle over F_n obtained from the line bundle $\mathrm{Spec}(S(\mathcal{O}_{\mathrm{F}_n}(-m)) \oplus \mathcal{O}_{\mathrm{F}_n}(-\ell C_\infty))$ by adding the ∞-section, we have $\mathcal{O}_{D^{\ell}_2}(D^{\ell}_2) \cong \mathcal{O}_{\mathrm{F}_n}(-m) \otimes \mathcal{O}_{\mathrm{F}_n}(-\ell C_\infty)$ hence, denoting by f the fibre of the ruled surface $D^{\ell}_2 \cong \mathrm{F}_n$, $(f \cdot D^{\ell}_2)_{\mathrm{F}^{\ell}_{m,n}} = -\ell$.

(8.8) If $\ell = 1$, we can equivariantly blow down $D^{\ell}_2 \cong \mathrm{F}_n$ to get $\mathrm{F}^1_{m\,n} \to \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-m) \oplus \mathcal{O}_{\mathbf{P}^1}(-n))$.

(8.9) Let us blow up X_1 at $\mathrm{F}_{(\ell+1)n-m} \cap \mathrm{F}_n$: π_2: $X_2 \to X_1$, where $\mathrm{F}_{(\ell+1)n-m}$

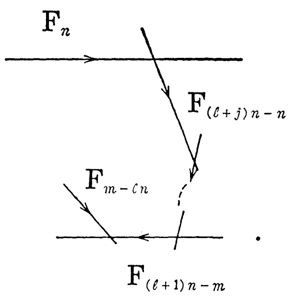
is the exceptional divisor for π_1 and F_n is the proper transform $\pi_1^{-1}[D_2]$. It follows from Lemma (8.5.4)

(8.9.1)



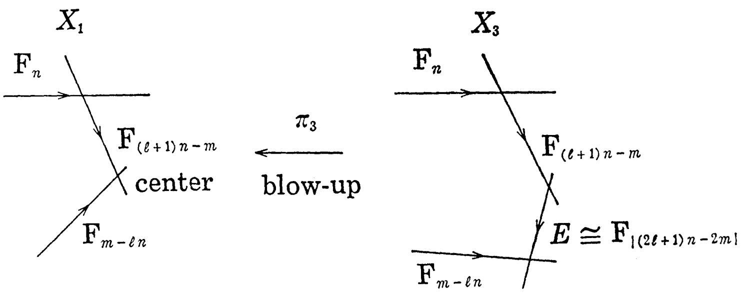
If we continue blowing up j-times on the invariant curve ($\cong \mathbf{P}^1$) on F_n, we get

(8.9.2)



(8.10) Let us blow up X_1 at $\mathrm{F}_{(\ell+1)n-m} \cap \mathrm{F}_{m-\ell n}$: π_3: $X_3 \to X_1$. We get by Lemma (8.5.4) a diagram

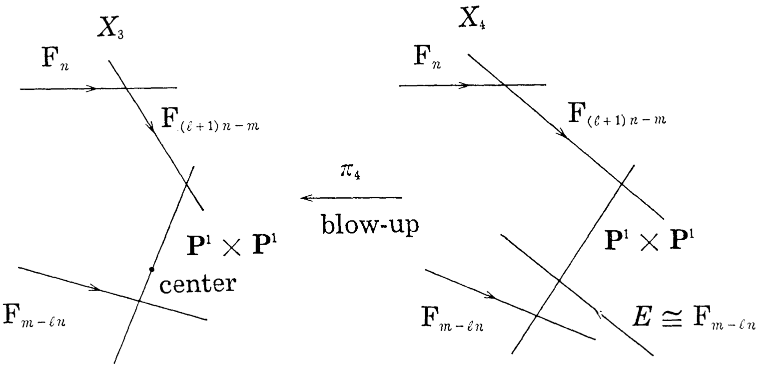
(8.10.1)



if $(2\ell + 1)n - 2m > 0$, otherwise the arrow on E should be reversed. Assume now that $(2\ell + 1)n - 2m = 0$ so that $E \cong \mathbf{P}^1 \times \mathbf{P}^1 = \mathrm{F}_0$ and we can choose a G-invariant section L of $\mathbf{P}^1 \times \mathbf{P}^1$ disjoint from $F_{(\ell+1)n-m} \cap E$ and $F_{m-\ell n} \cap E$.

(8.11) Let us blow up X_3 at L: π_4: $X_4 \to X_3$. Denoting by E the exceptional divisor for π_4, we can calculate the intersection number $(E.\mathbf{P}^1 \times \mathbf{P}^1.\mathbf{P}^1 \times \mathbf{P}^1) = \ell n - m$. In fact let

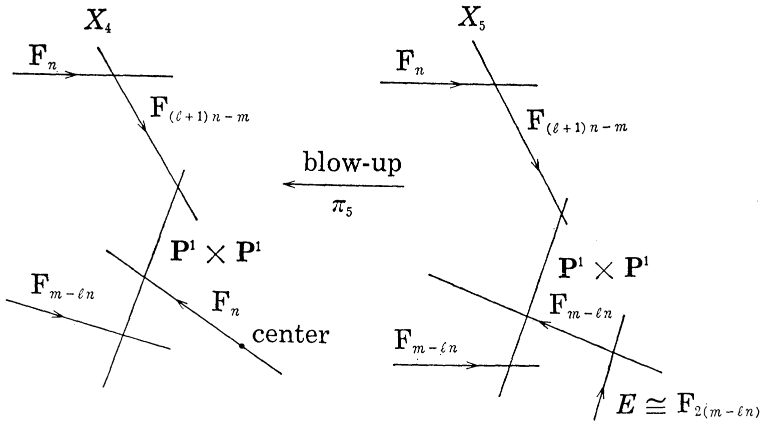
(8.11.1)



be the diagrams for X_3 and X_4. Then we have $(E. F_0. F_0)_{X_4} = (E. F_0. F_0 + E - E)_{X_4} = (E. F_0. (\pi_0^* F_0) - E)_{X_4} = (E. F_0. \pi_4^* F_0)_{X_4} - (E. F_0. E)_{X_4} = (E. F_0. \pi_4^* F_0)_{X_4} = (\pi_{4*} E \cap F_0. F_0)_{X_3} = (F_{m-\ell n} \cap F_0. F_0) = \ell n - m$. Since by Lemma (1.11) SL_2 operates non-trivially on E leaving invariant the intersection $E \cap \mathbf{P}^1 \times \mathbf{P}^1$, then it follows from (1.3.3) $E - \mathrm{F}_{m-\ell n}$ and $E \cap \mathbf{P}^1 \times \mathbf{P}^1$ is the ∞-section of E.

(8.12) Let us blow up X_4 at another section of F_n disjoint from $E \cap \mathbf{P}^1 \times \mathbf{P}^1$; π_5: $X_5 \to X_4$. As SL_2 operates on E non-trivially leaving the intersection $\mathrm{F}_n \cap E$ by Lemma (1.11), for the same reason as in (8.11) we get

(8.12.1)



LEMMA (8.12.2). *Let X be a (successive) equivariant blow-up of $\mathrm{F}^{\ell}_{m,n}$. Then a G-invariant irreducible divisor on X is isomorphic to F_a, $a \geq 0$ with $a \in \mathbf{Z}m + \mathbf{Z}n$. There is at most only one G-invariant divisor on X isomorphic to F_0 (see also Lemma (8.14)).*

Proof. This is a cosequence of Lemma (8.5.4), Remark (8.5.5) and the argument of (8.11.1) and (8.12.1).

LEMMA (8.13). *Let* $(G, X) \in C(\mathrm{Aut}^0 \mathrm{F}'_{m,n}, \mathrm{F}'_{m,n}) = \mathscr{C}(\mathrm{J9}; m, n)$. *We assume* $(m, n) \geq 2$. *Then* (a) *there is no G-invariant divisor on X which is isomorphic either to* $\mathbf{P}^2$ *or* F_1. (b) *There is no G-fixed point on X.*

Proof. We have an equivariant birational map $\varphi: X \dashrightarrow \mathrm{F}^{\ell}_{m,n}$. We can eliminate the indeterminacy of φ by equivariantly blowing up X;

$$\begin{array}{ccc} & \tilde{X} & \\ {}^{\varphi_2}\swarrow & & \searrow^{\varphi_1} \\ \mathrm{F}^{\ell}_{m,n} & \underset{\varphi}{\longleftarrow\cdots} & X \end{array} .$$

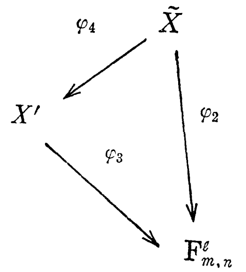
Since there is no fixed point on $\mathrm{F}^{\ell}_{m,n}$ by Lemma (8.3), φ_2 is also an equivariant blow-up by Theorem (1.7). If there were a G-invariant divisor isomorphic to $\mathbf{P}^2$ or F_1 on X, then it proper transform would be a G-invariant divisor on X isomorphic to $\mathbf{P}^2$ or to F_1. This contradicts Lemma (8.12.2) since $(m, n) \geq 2$. The assertion (a) is proved. If we blow up X at a G-fixed point, (b) follows from (a).

LEMMA (8.14). *Let* $(G, \tilde{X}) \in \mathscr{C}(\mathrm{Aut}^0 \mathrm{F}'_{m,n}, \mathrm{F}'_{m,n}) = \mathscr{C}(\mathrm{J9}; m, n)$ *and* (Id, φ_2): $(G, \tilde{X}) \to (G, \mathrm{F}^{\ell}_{m,n})$ *be an equivariant blow-up, where we choose the integer* ℓ *as before*: $m = n\ell + r$, $0 \leqq r < n$. *We denote by S the set of all the irreducible reduced effective G-invariant divisors on* $\tilde{X}$. *Then* (a) *S is a finite set.* (b) *The subvariety* $\bigcup_{D \in S} D \subset \tilde{X}$ *is connected.* (c) *S satisfies the condition* (1), (2), (3) *of Definition* (8.4). (d) *If a divisor* $D \in S$ *is exceptional for* φ_2 *and if there is only one divisor* $D' \in S$, $D \neq D'$) *with* $D \cap D' \neq \emptyset$, *then* $(D \cap D')^2_D > 0$. (e) *For any G-invariant irreducible reduced curve C, there exists a G-invariant irreducible reduced effective divisor D such that C lies on D such that C is a section of the rational ruled surface* $D \cong \mathrm{F}_t$ *and* $(C^2)_D = \pm t$.

Remark (8.14.1). Later we shall not use (d). But we need (a), (b), (c), (d) to prove (a), (b), (c) inductively.

Proof. First of all, we notice that the center of each step of φ_2 is a G-orbit by Lemma (8.13). Therefore the morphism φ_2: $\tilde{X} \to \mathrm{F}^{\ell}_{m,n}$ maps the centers onto the unique G-invariant curve $D^{\ell}_1 \cap D^{\ell}_2$ on $\mathrm{F}^{\ell}_{m,n}$. This proves

(b) by induction. Now let $SL_2 \to G$ be a non-trivial morphism and hence give a semi-simple part of G by Corollary (4.17), [U4]. Then by Lemma (1.11) for any $D \in S$, SL_2 operates non-trivially on D. We prove the Lemma by induction on the length of the sequence φ_2 of blow-ups at smooth irreducible centers or on the Picard number $\rho(\tilde{X})$ of $\tilde{X}$. The Lemma holds for $\mathbf{F}^{\ell}_{m,n}$ by Lemma (8.3). The assertion (a) is evidently proved by induction. The remaining assertions (c), (d), (e) are not obvious. Let us factor $\varphi_2 = \varphi_3 \circ \varphi_4$:



where φ_3 is a sequence of G-equivariant blow-ups for which the Lemma holds and φ_4 is a G-equivariant blow-up at an irreducible G-invariant center, which is a curve C isomorphic to $\mathbf{P}^1$ by Lemma (8.3). We assume that the Lemma holds for X' and prove it for X. There are three possibilities:

(i) there exists 2 divisors $D_1, D_2 \in S'$ such that $C = D_1 \cap D_2$, where S' denotes the set of G-invariant irreducible reduced effective divisors on X' (cf. Fig. (8.14.2. (i)), Examples (8.7), (8.9.1), (8.9.2) and (8.10)).

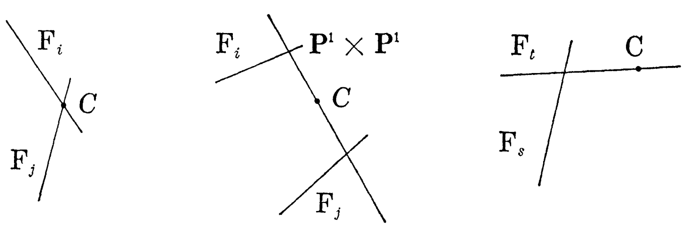
(ii) C lies on the unique G-invariant divisor D on X'.

We may assume in the case of (ii) that D is exceptional for φ_3 by Lemma (8.3). For otherwise C is in the case (i) by Lemma (8.3).

The case (ii) is divided into 2 subcases:

(ii a) C lies on the unique G-invariant divisor D on X' and D is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ and exceptional for φ_3 (cf. Fig. (8.14.2. (ii a)).

Fig (8.14.2)

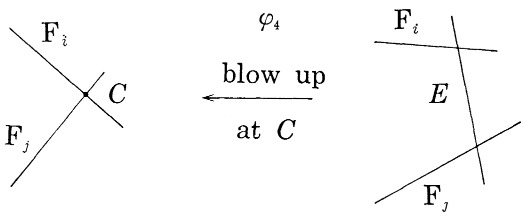


(i) (ii a) (ii b)

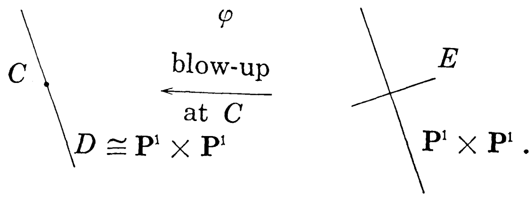
Example (8.11)),

(ii b) C lies on the unique G-invariant divisor D on X' and D is isomorphic to F_t $(t > 0)$ and exceptional for φ_3 (cf. Fig. (8.14.2. (ii b)), Example (8.12)).

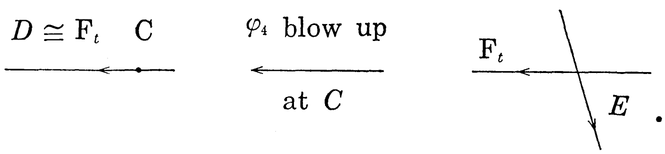
In the case (i), it is evident that Lemma (8.14) holds for $\tilde{X}$ by (1.3.2) and Lemma (1.11):



Let us study the case (ii a). In this case, it follows from (e) for C on X' that $(C, C)_D = 0$. By the assertions (c), (d), (e) for X', we have:



Namely the exceptional divisor E for φ_4 intersects on $\tilde{X}$ with the unique G-invariant divisor which is the proper transform of D. Let us calculate the intersection number: by the projection formula $-n = (C_\infty . C_\infty)_{\mathrm{F}_n}$ $= (f^{\ell}_{m,n} C_\infty . (D^{\ell}_1 \cap D^{\ell}_2))_{\mathrm{F}^{\ell}_m,}$ $= ((f^{\ell}_{m,n} \circ \varphi_2)^* C_\infty . E . \varphi_4^{-1}[D])_{\tilde{X}}$ $= (\{aE + a\varphi_4^{-1}[D] +$ (a divisor disjoint from $E \cap \varphi_4^{-1}[D])\} . E . \varphi_4^{-1}[D])_{\tilde{X}}$ with an integer $a > 0$, $= a(E . E . \varphi_4^{-1}[D])_{\tilde{X}} + a(\varphi_4^{-1}[D] . E . \varphi_4^{-1}[D])_{\tilde{X}} = a(\varphi_4^{-1}[D] . E . \varphi_4^{-1}[D])_{\tilde{X}}$ by the assumption (e). This proves that for the new born G-invariant divisor E the assertions (c), (d), (e) hold by (1.3.3) since SL_2-operates non-trivially on any G-invariant divisor on $\tilde{X}$ by Lemma (1.11). The Lemma is proved for $\tilde{X}$ in this case. Now we study the last case (ii b). By (1.3.3), Lemma (1.11) and by the assertion (c), (d), (e) for X' we have:



Namely the exceptional divisor E for φ_4 intersects with the unique G-invariant divisor which is the proper transform of D. We calculate the intersection number as above: $-n = (C_\infty . C_\infty)_{F_n} = ((f^{\ell}_{m,n} \circ \varphi_2)^* C_\infty . E . \varphi_4^{-1}[D])_{\tilde{X}} = (\{aE + a\varphi_4^{-1}[D] + (\text{a divisor disjoint from } E \cap \varphi_4^{-1}[D]\} . E . \varphi_4^{-1}[D])_{\tilde{X}}$, with an integer $a > 0$, $= a(E . E . \varphi_4^{-1}[D])_{\tilde{X}} + a(\varphi_4^{-1}[D] . E . \varphi_4^{-1}[D])_{\tilde{X}}$. Hence $(E . E . \varphi_4^{-1}[D])_{\tilde{X}} < 0$ since $(\varphi_4^{-1}[D] . E . \varphi_4^{-1}[D])_{\tilde{X}} = (C, C)_D > 0$. This shows that the new born divisor E satisfies the assertions (c), (d), (e) of the Lemma for the same reason as in the case (ii a). This proves the Lemma for $\tilde{X}$ in the case (ii b) and hence the Lemma.

COROLLARY (8.15). *We assume* $(m, n) \geq 2$. *Let* $(G, X) \in \mathscr{C}(\mathrm{Aut}^0 F'_{m,n}\ F'_{m,n}) = \mathscr{C}(\mathrm{J}9;\ m\ n)$, $(G, \tilde{X}) \in \mathscr{C}(\mathrm{J}9; m, n)$, (Id, φ_1): $(G, \tilde{X}) \to (G, F^{\ell}_{m,n})$ *and* (Id, φ_1): $(G, \tilde{X}) \to (G, X)$ *equivariant blow-ups with smooth centers. Then there is no G-invariant divisor D isomorphic to* $\mathbf{P}^1 \times \mathbf{P}^1$ *blown-down to* $\mathbf{P}^1$ *along different projections by* φ_1 *and* φ_2: *the restriction of* φ_i *on* $D \cong \mathbf{P}^1 \times \mathbf{P}^1$ *coincides with the projection* p_i: $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ *onto the i-th factor* $(i = 1, 2)$.

Proof. Assume that the existence of such a divisor D. It follows from Lemma (8.14) that G leaves invariant $a \times \mathbf{P}^1 \subset \mathbf{P}^1 \times \mathbf{P}^1$ for a suitable point $a \in \mathbf{P}^1$. Therefore $\varphi_1(a \times \mathbf{P}^1)$ is a G-fixed point on X, which contradicts Lemma (8.13).

Remark (8.15.1). We can prove Corollary (8.15) directly. It follows from Lemma (8.12.2) that in any (successive) blow-up of $F^{\ell}_{m,n}$, there is at most 1 invariant divisor E is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. Let ℓ be the line $\mathbf{P}^1$ to be contracted to a new direction. Then there is an invariant divisor D such that ℓ is numerically equivariant to $E \cap D$. The divisor D is isomorphic to F_a for a suitable $a \in \mathbf{Z}m + \mathbf{Z}n$. Since E is contracted, $-1 = (E . \ell) = (E . E . D) = \pm a$. This is a contradiction as we assume $(m, n) \geq 2$.

LEMMA (8.16). *Under the hypothesis* $(m, n) \geq 2$, *none of the cases* (3.3.2), (3.3.3), (3.3.4) *and* (3.3.5) *of Theorem* (3.3), [Mo] *occurs.*

Proof. The cases (3.3.2), (3.3.5) are avoided by (a), Lemma (8.13). The case (3.3.4) is excluded by (b), Lemma (8.13) since the singular point of the G-invariant divisor is a G-fixed point.

Assume now that (3.3.3) is the case. Let $F^{\ell}_{m,n} \leftarrow\cdots X_1$ be an equivariant birational map, where we choose ℓ as before. Since there appears no fixed point by Lemma (8.13), there exist equivariant blow-ups φ_2: $\tilde{X} \to F^{\ell}_{m,n}$

and φ_1: $\tilde{X} \to X$ such that $\varphi_2 \circ \varphi_1^{-1} = \varphi$ by Theorem (1.7). The proper transform $\varphi_1^{-1}[D]$ of the divisor $D \cong \mathbf{P}^1 \times \mathbf{P}^1$ of case (3.3.3) in [Mo] is G-invariant $\mathbf{P}^1 \times \mathbf{P}^1$ on $\tilde{X}$. Therefore $\varphi_2(\mathbf{P}^1 \times \mathbf{P}^1)$ is a G-invariant divisor and hence by Lemma (8.3) contained in D_1^ℓ: in fact $\varphi_2(\mathbf{P}^1 \times \mathbf{P}^1) = D_1^\ell \cap D_2^\ell$ unless $m = \ell n$. Since SL_2 operates on $\mathrm{F}'_{m,n}$, it operates also on $D \cong \mathbf{P}^1 \times \mathbf{P}^1$. There are 3 possibilities: (1) SL_2 fixes all the point of $\mathbf{P}^1 \times \mathbf{P}^1$, (2) SL_2 operates through one of the factors of $\mathbf{P}^1 \times \mathbf{P}^1$, (3) SL_2 operates diagonally on $\mathbf{P}^1 \times \mathbf{P}^1$. The case (1) never occurs by Lemma (1.11). We now exclude the case where SL_2 operates on D through one of the factors of $\mathbf{P}^1 \times \mathbf{P}^1$, say the second. It follows from the assumption (cf. Theorem (3.3) [Mo]) that $s \times \mathbf{P}^1$ and $\mathbf{P}^1 \times t$ are numerically equivalent. Taking s general, we may assume that $s \times \mathbf{P}^1$ is disjoint from the center of the blow-up φ_1 since φ_1 is equivariant. Then $\varphi_1^*\ s \times \mathbf{P}^1$ is numerically equivalent to $s \times \mathbf{P}^1$ in the proper transform $\varphi^{-1}[\mathbf{P}^1 \times \mathbf{P}^1]$ and hence $\varphi_{2*}\varphi_1^* s \times \mathbf{P}^1$ is numerically equivalent to either $D_1^\ell \cap D_2^\ell$ or to a 0-section C_0 of the rational ruled surface D_1^ℓ since φ_2 is SL_2-equivariant and $\varphi_{2*}\varphi_1^* s \times \mathbf{P}^1$ is SL_2-invariant curve on D_1^ℓ. It follows from Corollary (8.15) that $\varphi_{2*}\varphi_1^*(\mathbf{P}^1 \times t)$ is numerically equivalent to $aC_1 + bC_2$ for some integers $a, b \geqq 0$, where C_1 (resp. C_2) is a fibre of the ruled surface D_1^ℓ (resp. D_2^ℓ). π: $\mathrm{F}^\ell_{m,n} \to \mathrm{F}_n \to \mathbf{P}^1$ denote the fibration which is respected by G and we put $\pi^*\mathcal{O}_{\mathbf{P}^1}(1) = H$ (see Proposition (4.13), [U4]). Then $(H. C_0) = (H. D_1 \cap D_2) = 1$ but $(H. C_1) = (H. C_2) = 0$. Therefore $(\varphi_1^*(s \times \mathbf{P}^1). \varphi_2^* H) = (\varphi_{2*}\varphi_1^*(s \times \mathbf{P}^1). H)$ which is equal to $(C_0. H)$ or $((D_1^\ell \cap D_2^\ell). H)$ hence to 1 and $(\varphi_1^*(\mathbf{P}^1 \times t). \varphi_{2*} H) = (\varphi_{2*}\varphi_1^*(\mathbf{P}^1 \times t). H)$ $((aC_1 + bC_2). H) = 0$. This contradicts the numerical equivalence of $\mathbf{P}^1 \times t$ and $s \times \mathbf{P}^1$ on X. Now we have to exclude the last possibility; the diagonal operation of SL_2 on $D \cong \mathbf{P}^1 \times \mathbf{P}^1$. Assume to the contrary. Then $s \times \mathbf{P}^1 - \mathbf{P}^1 \times t \sim 0$ and $0 = \varphi_1^*(s \times \mathbf{P}^1 - \mathbf{P}^1 \times t) = (s \times \mathbf{P}^1$ in $\varphi_1^{-1}[D])$ $-$ $(\mathbf{P}^1 \times t$ in $\varphi_1^{-1}[D])$ hence taking φ_{2*}, $\varphi_{2*}(s \times \mathbf{P}^1$ in $\varphi_1^{-1}[D])$ is numerically equivalent to $\varphi_{2*}(\mathbf{P}^1 \times t$ in $\varphi_1^{-1}[D])$ for any $s, t \in \mathbf{P}^1$. Let us put $C_1 = \varphi_2(s \times \mathbf{P}^1$ in $\varphi_1^{-1}[D])$ and $C_2 = \varphi_2(\mathbf{P}^2 \times t$ in $\varphi_1^{-1}[D])$. We have one of the following: (1) φ_2 blows down $\varphi_1^{-1}[D] = \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$. (2) φ_2 is biregular at $\mathbf{P}^1 \times \mathbf{P}^1$. In the first case one of C_i $(i = 1, 2)$ is reduced to a point and the other coincides with the unique G-invariant curve $D_1^\ell \cap D_2^\ell$. Thus $D_1^\ell \cap D_2^\ell$ is numerically equivalent to 0. This is absurd and the case (1) never happens. In the second case, $m = \ell n$ and $D_1 \cong \mathbf{P}^1 \times \mathbf{P}^1$, $s \times \mathbf{P}^1$ and $t \times \mathbf{P}^1 \subset D_1 = \mathbf{P}^1 \times \mathbf{P}^1$ are numerically equivalent, which does not happen (compare the intersection numbers $(D_1. C_i)$, $i = 1, 2$).

LEMMA (8.17). *Let* $(G, X) \in \mathscr{C}(\mathrm{J9};\, m, n)$. *Let* $\phi\colon X \to Y$ *be the morphism of* (3.5.1), *Theorem* (3.5), [Mo]. *Then the surface* Y *is isomorphic to* F_n *and* X *is a* $\mathbf{P}^1$*-bundle over* F_n.

Proof. By Castelnuovo's Theorem [Z] Y is rational and G has an open orbit on Y. ϕ induces a morphism $\psi\colon G \to \mathrm{Aut}^0 Y$. A semi-simple part of G is SL_2 and its image under ψ does not reduce to 1. For otherwise $\mathrm{Ker}\,\psi$ would be isogeneous to SL_2 since the general fibre of ϕ is isomorphic to $\mathbf{P}^1$. Hence the semi-simple part of G would be normal, which contradicts Corollary (4.17), [U4]. Therefore SL_2 operates non-trivially on Y. Since the unipotent radical of G is not abelian by Corollary (4.17) [U4], it follows from Lemma (4.4), [U4] and Lemma (1.8) that the unipotent radical of G operates non-trivially on Y, Y is a ruled surface and G operates on Y with orbit decomposition: (open orbit)$\cup \mathbf{P}^1$. By Lemma (1.13), X is a $\mathbf{P}^1$-bundle over Y. We show that Y is isomorphic to F_n. First assume that $Y \cong \mathrm{F}_n$. We denote by W_j the irreducible SL_2-module of rank $j+1$. It follows from Corollary (4.17), [U4] that the irreducible SL_2-module W_{m-n} $(m-n \geqq 1)$ operates on $\mathrm{F}'_{m,n}$; $x \mapsto x$, $y \mapsto y$, $z \mapsto z + y\varphi_{m-n}(x)$, where $\varphi_{m-n}(x) \in k[x]$ with $\deg \varphi_{m-n}(x) \leqq m-n$. Hence the map $\mathrm{F}^k_{m,n} \to \mathrm{F}_n$ is generically the quotient by W_{m-n}. Since W_{m-n} operates trivially on F_m, the $\mathbf{P}^1$-bundle is generically the quotient of X by W_{m-n} and hence Y is G-equivariantly isomorphic to F'_n. Consequently Y is $\mathrm{Aut}^0\,\mathrm{F}_n$-equivariantly isomorphic to F_n by Corollary (4.17), [U4]. Hence $\mathrm{F}_n \cong Y$, which contradicts the assumption. Let us now assume that Y is not isomorphic to F_m. The using W_m; $x \mapsto x$, $y \mapsto y$, $z \mapsto z + \varphi_m(x)$ in place of W_{m-n}, by the same argument we conclude $\mathrm{F}_n \cong Y$.

LEMMA (8.18). *Let* $(G, X) \in \mathscr{C}(\mathrm{J9};\, m, n)$. *The case* (3.5.2) *of Theorem* (3.5) *in* [Mo] *occurs if and only if* $2n > m > n$ *and then* X *is isomorphic to* $\mathbf{P}(\mathscr{O}_{\mathbf{P}^1}(-m) \oplus \mathscr{O}_{\mathbf{P}^1}(-n) \oplus \mathscr{O}_{\mathbf{P}^1})$.

Proof. Assume that the case (3.5.2), [Mo] happens and let $\phi\colon X \to Y$ be the resulted morphism. G operates on Y and ϕ is equivariant. By Lüroth's theorem, Y is isomorphic to $\mathbf{P}^1$. Hence induces a morphism $\psi\colon G \to \mathrm{PGL}_2$. A semi-simple part of G which is isogeneous to SL_2, is mapped surjectively onto PGL_2. For otherwise there would be a subgroup of $\mathrm{Ker}\,\psi$ isogeneous to the semi-direct product $(U_{m+1} \otimes U_{n+1})\mathrm{SL}_2$ by Corollary (4.17), [U4], where U_{m+1} and U_{n+1} denote respectively the irreducible SL_2-modules of degree $m+1$ and $n+1$, since PGL_2 contains no unipotent group bigger than $\mathbf{G}_a$. Therefore $(U_{m+1} \oplus U_{n+1})\mathrm{SL}_2$ would operate on the fibres of ϕ. We taking $y \in Y$ general, it follows that $(U_{m+1} \oplus U_{n+1})\mathrm{SL}_2$ would operate almost effectively on the fibre X_y, which contradicts Theorem (2.25), [U3]. Thus ψ: $G \to \mathrm{PGL}_2$ is surjective and gives the semi-simple part of G. Hence by letting $G/H \cong \mathrm{F}'_{m,n}$ and B a Borel subgroup of G containing H, both φ: $X \to Y$ and π: $\mathrm{F}^k_{m,n} \to \mathbf{P}^1$ are birationally equivalent to $G/H \to G/X$. In particular φ and π are birationally equivalent. It follows that Y is a homogeneous space under G and all the fibres of φ is a non-singular del Pezzo surface S. The surface S is an equivariant completion of $\mathbf{A}^2$ under the action: $(y, z) \mapsto (y + a, z + f(y))$ $f(y) \in k[y]$ with $\deg f(y) \leqq \ell$ by Corollary (4.17), [U4], where ℓ is an integer with $m = n\ell + r$, $0 \leqq r < n$. Therefore by Theorem 24.3 (ii) Manin [Man] S is either $\mathbf{P}^2$ or F_1 and $\ell = 1$. But the F_1-bundles are excluded in the Mori theory, Theorem (3.5), [Mo]. Thus S is $\mathbf{P}^2$. Therefore there exists a vector bundle $E \cong \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}$, $a \leqq b \leqq 0$ such that $\mathbf{P}(E) \cong X$. Looking at End E, it follows from Corollary (4.17), [U4] that this is possible if and only if $a = -m$, $b = -n$. The if part follows from Corollary (4.17), [U4].

LEMMA (8.19). *Let* $(G, X) \in \mathscr{C}(\mathrm{J}; m, n)$, *and* $(m, n) \geq 2$. (1) *If* $m \geq 2n$, *then the Picard number* $\rho(X) \geqq 3$. (2) *If* $2n > m$ $(>n)$, *then* $\rho(X) \geqq 2$. (3) *Moreover* $\rho(X) = 2$ *if and only if* $2n > m$ $(>n)$ *and* X *is isomorphic to* $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(-m) \oplus \mathcal{O}_{\mathbf{P}^1}(-n) \otimes \mathcal{O}_{\mathbf{P}_1})$.

Proof. Let us first show that $\rho(X) \geqq 2$ in general. Assume $\rho(X) = 1$. There exists an equivariant birational map χ: $\mathrm{F}^k_{m,n} \dashrightarrow X$, where we fix a large integer k. Then we can eliminate the indeterminacy of χ by equivariantly blowing up $\mathrm{F}^k_{m,n}$:

$$\begin{array}{ccc} & \tilde{X} & \\ {}^{p}\swarrow & & \searrow^{q} \\ \mathrm{F}^k_{m,n} & \overset{\chi}{\dashrightarrow} & X \end{array}$$

Since there is no fixed point on X, q is also an equivariant blow-up by Throrem (1.7). $\rho(\mathrm{F}^k_{m,n}) = 3$ and there are exactly 2 G-invariant integral divisors D^k_1 and D^k_2 on $\mathrm{F}^k_{m,n}$ and they form with $\pi^*\mathcal{O}_{\mathbf{P}^1}(1)$ a basis of $\mathrm{Pic}\,\mathrm{F}^k_{m,n} \otimes \mathbf{Q}$. Since $\rho(X) = 1$, q should blow-down all the components of $p^{-1}(D^k_1 \cup D^k_2)$.

SUBLEMMA (8.19.1). (a) *$qp^{-1}(D_1^k \cup D_2^k)$ is a G-invariant curve C.* (b) *$q^{-1}(y)$ is a 1-cycle on $\tilde{X}$ for any point $y \in C$.* (c) *Putting $q^{-1}(y) = f$, we have the intersection number $(E.f) = 0$ for all but one irreducible components E of $p^{-1}(D_1^k \cup D_2^k)$ and $(E.f) = -1$ for one particular irreducible component E'.*

Proof. We have already observed above (a). (b) follows from Lemma (8.13). Let us write q as a product of the blow-ups $q_1\colon X' \to X$ with irreducible center C and $q'\colon \tilde{X} \xrightarrow{q'} X'$. Let us denote $q_1^{-1}(C)$ by E'.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\;q'\;} & X' \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle q_1} & \\ & X & \end{array}$$

E' is the only one G-invariant divisor on X'. For an irreducible component E of $p^{-1}(D_1^k \cup D_2^k)$, we have $(E.f) = (E.q'^*(q_1^{-1}(y)) = (q_*E.\, q_1^{-1}(y)) = 0$ or -1 according as $q'_*E = 0$ or $q'_*E = E'$ (notice that q' is a blow-down).
q.e.d.

Let us continue the proof of the Lemma. Since as in the proof of Corollary (8.15) q blows-down the exceptional divisors to the same directions as p, $(p^*D_2^k.f) = (D_2^k.p_*f) = (D_2^k.f_1 + f_2)$, where f_i is the fibre of the ruled surface D_i^k. It follows from Sublemma $(p^*D_2^k.f) = 0$ or -1. But $(D_2^k.f) = 1$, $(D_2^k.f) = -k$ hence $(D_2^k, f_1 + f_2) = 1 - k$. This is absurd. Now Lemma (8.19) follows from Lemmas (8.16), (8.17) and (8.18).

THEOREM (8.20). *We assume $(m, n) \geq 2$. Any element of $\mathscr{C}(\mathrm{J9}; m, n)$ $(m > n \geqq 2)$ can be equivariantly blown-down to a relatively minimal element of the ordered set $\mathscr{C}(\mathrm{J9}; m, n)$: an element x of an ordered set Z is said to be relatively minimal if $x > y$ for $y \in Z$ implies $x = y$. The centers of the blow-down are curves isomorphic to $\mathbf{P}^1$.*

(1) *If $m \geqq 2n$, the relatively minimal elements in $\mathscr{C}(\mathrm{J9}; m, n)$ are the* $(\mathrm{Aut}^0 \mathrm{F}_{m,n}^k, \mathrm{F}_{m,n}^k)$*'s $(k \geqq \ell)$.*

(2) *If $2n > m$, the relatively minimal elements in $\mathscr{C}(\mathrm{J9}; m, n)$ are the* $(\mathrm{Aut}^0 \mathrm{F}_{m,n}^k, \mathrm{F}_{m,n}^k)$*'s $(k > \ell)$ and* $(\mathrm{Aut}^0 \mathbf{P}(\mathscr{O}_{\mathbf{P}^1}(-m) \oplus \mathscr{O}_{\mathbf{P}^1}(-n) \oplus \mathscr{O}_{\mathbf{P}^1}), \mathbf{P}(\mathscr{O}_{\mathbf{P}^1}(-m) \oplus \mathscr{O}_{\mathbf{P}^1}(-n) \oplus \mathscr{O}_{\mathbf{P}^1}))$.

Proof. It follows from Lemma (8.16), Lemma (8.17), Lemma (8.18), Lemma (8.19) and Mori [Mo] that if $(G, X) \in \mathscr{C}(\mathrm{J9}; m, n)$ with $\rho(X) \geqq 4$, then we can find an equivariant blow-down $(G, Y) \in \mathscr{C}(\mathrm{J9}; m, n)$ with $\rho(Y) = \rho(X) - 1$. By construction the $(\mathrm{Aut}^0 \mathrm{F}_{m,n}^k, \mathrm{F}_{m,n}^k)$'s are relatively minimal under the condition on m, n, ℓ, k of (1) and (2) (cf. (8.8)). Therefore in view of Lemma (8.17) and Lemma (8.19), it remains to show: Let $(G, X) \in \mathscr{C}(\mathrm{J9}; m, n)$ such that by an equivariant morphism $(G, X) \to (G, \mathrm{F}_n)$ compactifying $(\mathrm{Aut}^0 \mathrm{F}'_{m,n}, \mathrm{F}'_{m,n}) \to (\mathrm{Aut}^0 \mathrm{F}'_{m,n}, \mathrm{F}'_n)$, X is a $\mathbf{P}^1$-bundle over F_n, then (G, X) is isomorphic to $(\mathrm{Aut}^0 \mathrm{F}^k_{m,n}, \mathrm{F}^k_{m,n})$, $k \geqq \ell$. Since the Brauer group of F_n vanishes, there exists a vector bundle E of rank 2 such that X is F_n-isomorphic to $\mathbf{P}(E)$. The restriction of E over $\mathrm{F}'_n = \mathrm{F}_n - C_\infty$ is denoted by E'. Then it follows from the argument of the Proof of Lemma (8.17) that $\mathbf{P}(E')$ is the natural relative completion of $\mathbf{A}^1$-bundle $\mathrm{F}'_{m,n} \to \mathrm{F}'_n$ over F'_n hence we may assume $E' \cong \mathcal{O}_{\mathrm{F}'_n} \oplus \mathcal{O}_{\mathrm{F}'_n}(-m)$. The $\mathbf{P}^1$-bundle $\mathbf{P}(E')/\mathrm{F}'_n$ has an equivariant section s': $\mathrm{F}'_n \to \mathbf{P}(E')$ corresponding to the projection $\mathcal{O}_{\mathrm{F}'_n} \oplus \mathcal{O}_{\mathrm{F}'_n}(-m) \to \mathcal{O}_{\mathrm{F}'_n}(-m)$. In other words s' is an equivariant rational section s: $\mathrm{F}_n \to \mathbf{P}(E)$. Since a rational map to a complete variety is regular in codimension 1, s is G-equivariant and $C_\infty = \mathrm{F}_n - \mathrm{F}'_n$ is a G-orbit, s is an equivariant section of $\mathbf{P}(E)/\mathrm{F}_n$ extending s'. Therefore we can find a line bundles L, M on F_n such that (1) $M|\mathrm{F}'_n \cong \mathcal{O}_{\mathrm{F}'_n}(m)$ and (2) E is an extension: (*) $0 \to L \to E \to M \to 0$. Since by definition of s $L|\mathrm{F}'_n \cong \mathcal{O}_{\mathrm{F}'_n}$, we may assume $L \cong \mathcal{O}_{\mathrm{F}_n}$ by tensoring L^{-1} if necessary. The first condition implies $M \cong \mathcal{O}_{\mathrm{F}_n}(-m) \otimes \mathcal{O}_{\mathrm{F}_n}(-jC_\infty)$. It follows from the proof of Lemma (8.1) $j \geqq \ell$. Let us show that the extension is trivial; $E \cong L \oplus M$. The extension (*) is parametrized by $H^1(\mathrm{F}_n, M^{-1})$ and by the restriction map (**) $H^1(\mathrm{F}_n, M^{-1}) \to H^{-1}(\mathrm{F}'_n, M^{-1}) \cong H^1(\mathrm{F}'_n, \mathcal{O}_{\mathrm{F}'_n}(m))$, the class $[E]$ map to 0. Now the triviality of the class $[E]$ follows from the following diagram that shows the injectivity of the restriction map (***).

$$
\begin{array}{ccc}
H^1(\mathrm{F}_n, M) & \longrightarrow & H^1(\mathrm{F}'_n. M) \\
\wr\| & & \wr\| \\
H^1(\mathrm{F}_n, \mathcal{O}_{\mathrm{F}_n}(m) \otimes \mathcal{O}_{\mathrm{F}_n}(jC_\infty)) & & H^1(\mathrm{F}'_n, \mathcal{O}_{\mathrm{F}'_n}(m)) \\
\wr\| & & \wr\| \\
H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m) \otimes f_{n*}\mathcal{O}(jC_\infty)) & & \bigoplus_{i \in \mathbf{Z}} H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m + in)) \\
\wr\| & \nearrow & \\
H^1(\mathbf{P}^1, \mathcal{O}(m) \otimes S^j(\mathcal{O} \oplus \mathcal{O}(-n)) & & \\
\wr\| & & \\
\bigoplus_{s=0}^{j} H^1(\mathbf{P}^1; \mathcal{O}(m - sn)) & &
\end{array}
$$

(cf. (1.2.1) and (1.3.1)).

Remark (8.20.1). We can prove Theorem (8.20) without using [Mo]. To this end we associate to a (successive) equivariant blow up of $F^{\ell}_{m,n}$ a diagram. We define inductively the diagram whose set of vertices consists of the irreducible invariant divisors. The diagram is composed of vertices, an edge and arrows joining vertices. Each vertex is painted white or black. Namely, we associate to $F^{\ell}_{m,n}$ a diagram consisting of 2-white vertices representing the invariant divisors D^{ℓ}_1, D^{ℓ}_2 and connect them by an edge.

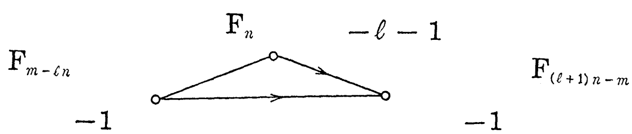
$$F_n \circ\!\!-\!\!\!-\!\!\!-\!\!\circ F_{m-\ell n}$$

We add to this diagram the intersection number of the divisor with its fibre. For example, on the vertex F_n, we associate the intersection number $(F_n . f) = -\ell$, where f denotes a fibre of the $\mathbf{P}^1$-bundle $F_n \to \mathbf{P}^1$. Therefore the number $(F_{m-\ell n} . f) = 0$ is associated to $F_{m-\ell n}$. Our diagram for $F^{\ell}_{m,n}$ is

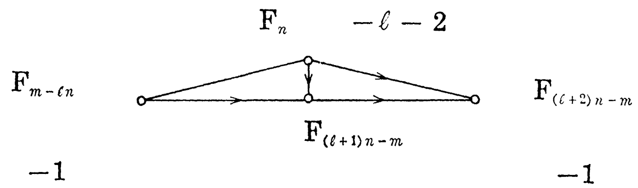
$$\begin{array}{ccc} F_n & \circ\!\!-\!\!\!-\!\!\!-\!\!\circ & F_{m-\ell n} \\ -\ell & & 0 \ . \end{array}$$

Now we assume that a diagram is associated with a (successive) blow-up Y of $F^{\ell}_{m,n}$ and we define a diagram for an equivariant blow-up $\pi: X \to Y$. The diagram for X is defined to be the union of the diagram for Y and a vertex which represents the exceptional divisor E for π so that set of the vertices of the diagram for X is considered as the set of the irreducible invariant divisors on X. The colour of the new vertex corresponding to the exceptional divisor E is white if the center of the blow-up π lies on the proper transform F_n of D^{ℓ}_2 on $F^{\ell}_{m,n}$ and otherwise black. We join a vertex corresponding to a divisor D on Y with the vertex E by an arrow pointing to E if the center of the equivariant blow up is on D. We write at each vertex the intersection number of the corresponding divisor with its fibre. This is easily done by the following rule: (1) Write -1 for the new exceptional divisor E; (2) For vertices to be connected with the vertex E, diminish by 1 from the number for Y; (3) Write the same number for other vertices coming from Y.

Here are some examples. For X_1 in (8.7) we have



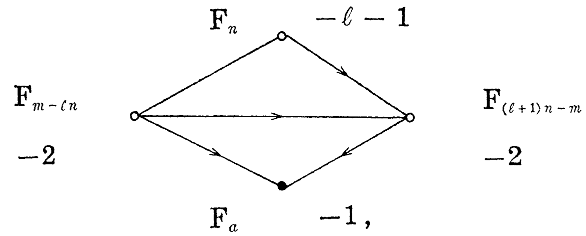
If we blow up X_1 along $F_n \cap F_{(\ell+1)n-m}$, the diagram is



here $F_{(\ell+2)n-m}$ is the esceptional divisor (c.f (8.7)).

If we blow up X_1 along $F_{m-\ell n} \cap F_{(\ell+1)n-m}$, then the diagram is

(8.20.2)

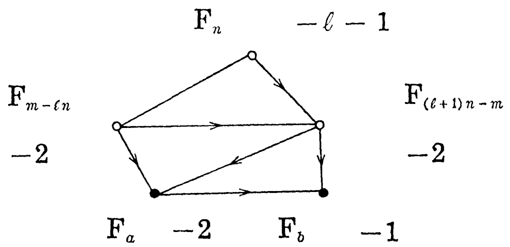


where F_a is the exceptional divisor. We know

$$a = \begin{cases} (2\ell+1)n - 2m & \text{if } (2\ell+1)n - 2m \geq 0, \\ 2m - (2\ell+1)n & \text{if } (2\ell+1)n - 2m < 0, \end{cases}$$

by Lemma (8.5.4).

In (8.20.2) if we blow up along $F_{(\ell+1)n-m} \cap F_a$, we get



We can determine b by Lemma (8.5.4).

The set of vertices or the invariant divisors is an ordered set by the arrows. Namely let A, B be vertices for X. Then we define $A > B$ if there is an arrow from A to B. We say that an element x of an ordered set s is relatively minimal if there is no element $y \in S$ such that $x > y$. We conclude by Corollary (8.15).

PROPOSITION (8.20.3). *We assume $(m, n) \geq 2$. Let Y be an equivariant blow-up of $F^{\ell}_{m,n}$ and D be an irreducible invariant divisor on Y. Then*

(A) *the diagram for* Y *is connected and* (B) *the following conditions are equivalent.*

(1) *The divisor* D *can be equivariantly blown down giving a non-singular projective threefold.*

(2) *In the diagram of* Y *the number associated with the divisor* D *is* -1.

If the divisor D *corresponds to a black vertex, then the conditions* (1), (2) *are equivalent to the following condition.*

(3) *The vertex corresponding to* D *is relatively minimal.*

(C) *Let* x *be a black vertex. Then any vertex* y *with* $x > y$ *is black.*

Proof. The assertion follows from Corollary (8.15) by calculating the intersection number. Notice that Y is a successive equivariant blow-up with centers isomorphic to $\mathbf{P}^1$.

Let $\tilde{X}$ be a (successive) equivariant blow-up of $F^{\ell}_{m,n}$. Let us study a successive equivariant blow-down of $\tilde{X}$; $\tilde{X} \to Y_1 \to Y_2 \to \cdots Y_s$, where we assume that the Y_i's are projective and non-singular. Let us first study p_1: $\tilde{X} \to Y$. Thus the exceptional divisor for p_1 is a vertex of the diagram for $\tilde{X}$ with intersection number -1. We associate to Y_1 a diagram, whose vertices are the invariant divisors on Y_1. We write for each vertex the intersection number similarly as in $\tilde{X}$. We explain how we join the vertices. We construct the diagram for Y_1 from that of $\tilde{X}$ as follows. We eliminate from the diagram for $\tilde{X}$ the vertex of the exceptional divisor for p_1, arrows and edge related with this vertex. The number for a remaining vertex is unchanged if the corresponding divisor on $\tilde{X}$ is disjoint from the exceptional divisor for p_1 and otherwise the number is increased by 1. For the remaining vertices, we keep arrows and colours unchanged. The diagram for Y_1 has the following properties.

(8.20.4) (1) The diagram is connected.

(2) The vertices consist of the invariant divisors.

(3) For an invariant divisor, the following conditions are equivalent.

(i) The divisor is exceptional and can be equivariantly blown down on a projective non-singular threefold.

(ii) The number associated with the vertex is -1.

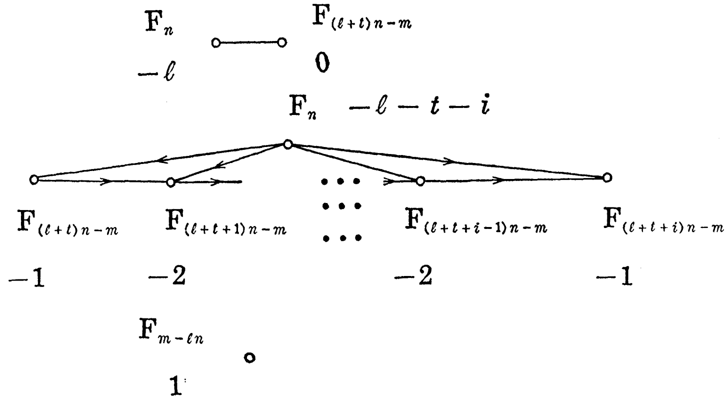
If the vertex is black, these conditions are equivalent to the following.

(iii) The vertex is relatively minimal.

(4) Let x be a black vertex. Then any vertex y with $x > y$ is black.

Now for p_2: $Y_1 \to Y_2$, we can argue similarly and define a diagram for Y_2. The diagram for Y_2 has the properties (8.20.8). Inductively we define a diagram for Y_s and the diagram for Y_s has the properties (8.20.4) too.

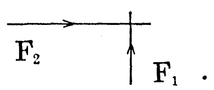
If the diagram for Y_s has a black vertex, it follows from the properties (4) and (3.iii) of (8.20.4) that we can find a black-vertex which can be blown down. We can find an equivariant blown down $Y_s \to Y$ such that the diagram for Y consists only of white vertices and then the diagram for Y is one of the following:



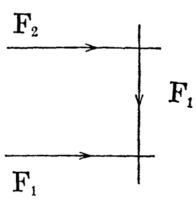
In other words Y and hence Y_s is an equivariant blow-up of $F^{\ell+t}_{m,n}$ in case of the first and second diagram. The case of the third diagram occurs when $\ell = 1$ and in this case Y_s is an equivariant blow up of $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(-m) \oplus \mathcal{O}_{\mathbf{P}^1}(-n) \oplus \mathcal{O}_{\mathbf{P}^1})$.

Now let us treat the case $(m, n) = 1$. First we give an example to illustrate the situation.

EXAMPLE (8.21). $m = 3$, $n = 2$ and $\ell = 1$, $r = 1$. On $F_{3,2}$ we have isomorphisms $D^1_2 \simeq F_2$, $D^1_1 \simeq F_1$. In terms of the graph,

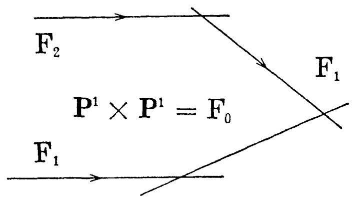


If we blow up $F^1_{3,2}$ along $F_1 \cap F_2$, we get π_1: $X_1 \to F^1_{3,2}$ with



as we have seen in (8.7).

We blow up X_1 along $F_1 \cap F_1$ to get $\pi_2\colon X_2 \to X_1$. On X_2 we have



We show that we can collaspe the divisor F_0 to another direction to get a projective non-singular threefold $U_{3,2}$ which we call the Euclidean model of the operation $(\mathrm{Aut}^0 F'_{3,2}, F'_{3,2})$. Namely there exists a projective non-singular threefold $U_{3,2}$ which is an equivariant completion of $(\mathrm{Aut}^0 F'_{3,2}, F'_{3,2})$ and an equivariant morphism $\pi\colon X_2 \to U_{3,2}$. The morphism π is an equivariant blow-up morphism $\pi\colon X_2 \to U_{3,2}$. The morphism π is an equivariant blow-up with center $\mathbf{P}^1$ and the divisor $F_0 = \mathbf{P}^1 \times \mathbf{P}^1$ on X_2 is the exceptional divisor for π. The restrictions $\pi_2 | F_0$ and $\pi | F_0$ are equivalent to different projections; $a \times \mathbf{P}^1 \subset \mathbf{P}^1 \times \mathbf{P}^1$ is collapsed by one of the morphisms π_2 and π, and is mapped isomorphically by the other for any $a \in P$. By calculating the intersection number we know that we can contract analytically the divisor F_0 for a different direction. However the result of Mori [Mo] tells us that we can do it algebraically. In fact for this purpose we show that the cone $NE(X)$ is polyhedral. Precisely the cone $NE(X_2)$ is spanned by a finite number of elements of $NE(X_2)$. We determine even a linearly independent generators of the cone $NE(X_2)$. The argument works not only for X_2 but also for any equivariant blow up of $F^1_{3,2}$. To begin with, we show that the cone $NE(F^1_{3,2})$ is spanned by the fibres of $D^1_2 \simeq F_2$ and $D^1_1 \simeq F_1$ and by the curve $F_1 \cap F_2$. In fact let $\mathbf{G}_a \subset \mathrm{SL}_2$ be a subgroup of SL_2 isomorphic to $\mathbf{G}_a$. since a semi-simple part of $\mathrm{Aut}^0 F'_{3,2}$ is SL_2, we can consider $\mathbf{G}_a \subset \mathrm{SL}_2$ as a one dimensional subgroup of $\mathrm{Aut}^0 F'_{3,2}$. We have a $\mathbf{G}_a$-equivariant morphism

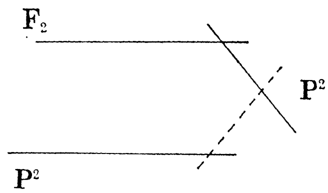
$$f_2 \circ f^1_{3,2}\colon F_{3,2} \longrightarrow \mathbf{P}^1 .$$

The subgroup $\mathbf{G}_a$ has an open orbit $\mathbf{A}^1 \subset \mathbf{P}^1$. $F'_{3,2}$ is trivial over $\mathbf{A}^1$; $F'_{3,2} \cap (f_2 \circ f'_{3,2})^{-1}\mathbf{A}^1 \simeq \mathbf{A}^3$ making the following diagram commutative

$$\begin{array}{ccc} F'_{3,2} \cap (f_2 \circ f^1_{3,2})^{-1}\mathbf{A}^1 & \xrightarrow{\sim} & \mathbf{A}^3 \\ {\scriptstyle f_2 \circ f^1_{32}} \downarrow & & \downarrow {\scriptstyle p_1} \\ \mathbf{A}^1 & = & \mathbf{A}^1 \end{array}$$

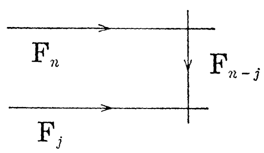
We may assume that the operation of $\mathbf{G}_a$ on $\mathbf{A}^3$ is $(x, y, z) \to (x + a, y, z)$ (cf. [U4] p. 396). We show that C is rationally equivalent to an effective curve on $F_1 \cup F_2$ and hence the cone $NE(\mathrm{F}^1_{3,2})$ is spanned by the fibres of $D^1_2 \simeq \mathrm{F}_2$ and $D^1_1 \simeq \mathrm{F}_1$ and by $D^1_1 \cap D^1_2$ since in general an effective curve on the ruled surface F_a $(a \geq 0)$ is linearly equivariant to a non-negative linear combination $mf + nS$ where f is the fibre of F_a and S is the section of the ruled surface $\mathrm{F}_a \to \mathbf{P}^1$ with $S^2 = -a$.

If the curve C is not contained in $\mathrm{F}^1_{3,2} - \mathbf{A}^3$ and if $C \cap \mathbf{A}^3$ is not $\mathbf{G}_a$-invariant (or equivalently $C \cap \mathbf{A}^3$ is not of the form $\{(x, a, b) \in \mathbf{A}^3 \mid x \in \mathbf{A}^1\}$ for a certain point $(a, b) \in \mathbf{A}^2$), then using the operation of $\mathbf{G}_a$, the curve C is rationally equivalent to an effective curve over $\mathrm{F}_{3,2} - \mathbf{A}^3$. For any curve on $\mathrm{F}^1_{3,2} - \mathbf{A}^3$ or for any curve on $\mathbf{A}^3$ excluded above, using operations of the additive group $\mathbf{G}_a$ along the fibres of $\mathrm{F}^1_{3,2} \to P^1$, we conclude that they are rationally equivalent to effective curves on $D^1_1 \cup D^1_2$. We proved that the cone $NE(\mathrm{F}^1_{3,2})$ is spanned by the fibres of $D^1_2 \simeq \mathrm{F}_2$ and $D^1_1 \simeq \mathrm{F}_1$ and by $D^1_1 \cap D^1_2$. Since the Picard number of $\mathrm{F}^1_{3,2}$ is 3, $\dim N(\mathrm{F}^1_{3,2}) = 3$ and hence the 1-cycles the fibres of D^1_1 and D^1_2 and $D^1_1 \cap D^1_2$ are linearly independent and therefore they are edges of $NE(\mathrm{F}^1_{3,2})$. The same argument works for any equivariant blow-up of $\mathrm{F}^1_{3,2}$. For example for X_1 any effective 1-cycle on X_1 is rationally equivalent to an effective 1-cycle on $\mathrm{F}_2 \cup \mathrm{F}_1 \cup \mathrm{F}_1$. Therefore $NE(X_1)$ is spanned by $\mathrm{F}_2 \cap \mathrm{F}_1$, $\mathrm{F}_1 \cap \mathrm{F}_1$ and by the fibres of F_2 and the F_1's. As we noticed above, any effective divisor on F_a is linearly equivalent to a non-negative linear combination $mf + nS$ and hence the 1-cycle $\mathrm{F}_2 \cap \mathrm{F}_1$ is numerically equivalent to a non-negative linear combination of 1-cycles $\mathrm{F}_1 \cap \mathrm{F}_1$ and a fibre of the exceptional divisor F_1. Therefore the cone $NE(X_1)$ is spanned by $\mathrm{F}_1 \cap \mathrm{F}_1$ and fibres of F_2 and the F_1's. They are linearly independent and they are the edges of $NE(X_1)$. For X_2, we conclude that $NE(X_2)$ is spanned by $\mathrm{F}_1 \cap \mathrm{F}_0$ and fibres of F_2, F_1, F_0 and F_1, here in the intersection $\mathrm{F}_1 \cap \mathrm{F}_0$ we may take any one of the F_1's. By the adjunction formula the canonical bundle of F_0 is $(\mathrm{F}_0 + K_{X_2}) \mid \mathrm{F}_0$. Using again the adjunction formula for the curve $\mathrm{F}_0 \cap \mathrm{F}_1 = \ell$ we get $(((\mathrm{F}_0 + K_{X_2}) \mid \mathrm{F}_0) + \ell . \ell)_{\mathrm{F}_0} = -2$. On the other hand $(((\mathrm{F}_0 + K_{X_2}) \mid \mathrm{F}_0) + \ell . \ell)_{\mathrm{F}_0} = (((\mathrm{F}_0 + K_{X_2} + \mathrm{F}_1)\mathrm{F}_0 . \mathrm{F}_1 . \mathrm{F}_0)_{\mathrm{F}_0} = (\mathrm{F}_0 + K_{X_2} + \mathrm{F}_1 . \mathrm{F}_1 . \mathrm{F}_0) = (\mathrm{F}_0 . \mathrm{F}_1 . \mathrm{F}_0) + (K_{X_2} . \mathrm{F}_1 . \mathrm{F}_0) + (\mathrm{F}_1 . \mathrm{F}_1 . \mathrm{F}_0) = -1 + (K_{X_2} . \mathrm{F}_1 . \mathrm{F}_0)$. Therefore $(K_{X_2} . \ell) = -1$ and $\ell = \mathrm{F}_1 \cap \mathrm{F}_0$ is an extremal rational curve. It follows from Lemmas (8.16) and (8.17) and from [Mo] that we can contract F_0 to another direction; π; $X_2 \to U_{3,2}$ we have

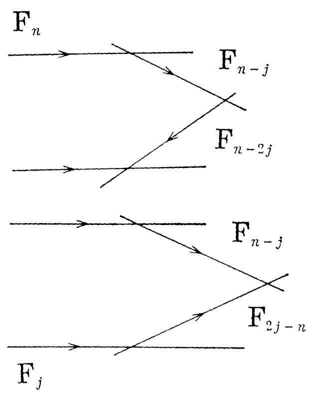


where the doted line represents $\mathbf{P}^1$ consisting of the fixed points. Since $(F_2.F_0.F_0) = -2$, we get $\mathcal{O}(\mathbf{P}^2)\otimes\mathcal{O}_{\mathbf{P}^2} \simeq \mathcal{O}_{\mathbf{P}^2}(-2)$ for $\mathbf{P}^2$ intersecting F_2. For another $\mathbf{P}^2$, we have over $F^1_{3,2}$, $\mathcal{O}(F_1)\otimes\mathcal{O}_{F_1} = p^*\mathcal{O}_{\mathbf{P}^1}(-2)$ since $F_1 = p^{-1}C_2$ and $C^2 = -2$. There is a line on this $\mathbf{P}^2$ which is isomorphically mapped to a zero section of F_1 on $F^1_{3,2}$ and hence we have $\mathcal{O}(\mathbf{P}^2)\otimes\mathcal{O}_{\mathbf{P}^2} \simeq \mathcal{O}_{\mathbf{P}^2}(-2)$ too. Therefore we can not equivariantly contract none of the divisors on $U_{3,2}$ onto a non-singular variety. Namely $(\mathrm{Aut}^0 F'_{3,2}.\ U_{3,2})$ is a relatively minimal element of $\mathscr{C}(\mathrm{J}; 3, 2)$. It follows from Lemma (8.5.4) and from the argument of (8.12) that for an equivariant blow-up of X_2 there is only one invariant divisor isomorphic to $\mathbf{P}^1\times\mathbf{P}^1$ and there is no invariant $\mathbf{P}^2$.

We can generalize Example (8.21) for $\mathscr{C}(\mathrm{J9}; m, n)$, $m > n \geq 2$ with $(m, n) = 1$. In fact, we blow up $F^{\ell}_{m,n}$ along $D^{\ell}_1 \cap D^{\ell}_2$, we get $\pi_1\colon X_1 \to F^{\ell}_{m,n}$ as in (8.7);



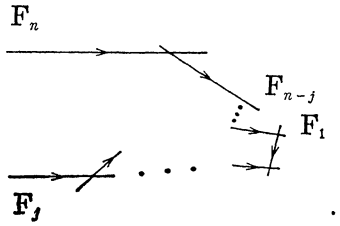
where we put $j = m - \ell n$ so that $(n, j) = 1$. We blow up X_1 at $F_{(\ell+1)n-m} \cap F_{m-\ell n}$ to get $\pi_2\colon X_2 \to X_1$. We have on X_2



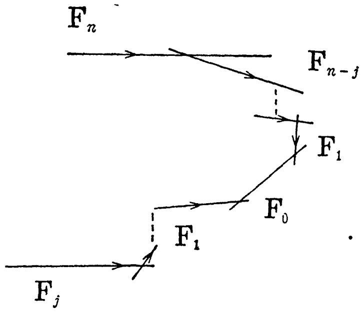
if $n - 2j \geqslant 0$

if $2j - n > 0$

Continuing blowing up along the unique curve where the two arrows gather, we arrive by the Euclidean algorithm at the exceptional divisor F_1 on X_r for some r. We have on X_r



Now we still continue blowing up X_r to get finally X_s



The argument in Example (8.21) shows that (i) an effective 1-cycle on X_s is rationally equivalent to an effective 1-cycle on $F_n \cup F_{n-j} \cup \cdots \cup F_1 \cup F_0 \cup F_1 \cup \cdots \cup F_j$, (ii) $NE(X_s)$ is spanned by the fibres of F_n, F_{n-j}, $\cdots$, F_1, F_0, F_1, $\cdots$ and F_j and by the 1-cycle $F_0 \cap F_1$, (ii) these 1-cycles are linearly independent and that (iv) they are the edges of the cone $NE(X_s)$. The same calculation as in Example (8.21) shows $(K_{X_s}. F_0. F_1) = -1$ so that $F_0 \cap F_1$ is an extremal rational curve. As in Example (8.21) we can contract equivariantly F_0 to another direction to get a projective non-singular threefold $U_{m\ n}$ on which the algebraic group $\mathrm{Aut}^0 F'_{m,n}$ acts. We call $U_{m,n}$ the Euclidean model of $(\mathrm{Aut}^0 F'_{m,n}, F'_{m,n})$. For any (successive) equivariant blow-up of X_s, there is only one invariant F_0 by Lemma (8.5.4) and there is no invariant $\mathbf{P}^2$. As in Example (8.21), we can show that $(\mathrm{Aut}^0 U_{m,n}, U_{m,n})$ is relatively minimal in $\mathscr{C}(\mathrm{J}9; m, n)$.

We can state in a form of Theorem.

THEOREM (8.22). *We assume* $(m, n) = 1$. *Any element of* $\mathscr{C}(\mathrm{J}9;\ m,\ n)$ $(m > n \geq 2)$ *can be equivariantly blown down to a relatively minimal element of the ordered set* $\mathscr{C}(\mathrm{J}9;\ m,\ n)$.

(1) *If* $m \geq 2n$, $(m, n) = 1$, *then the relatively minimal elements in* $\mathscr{C}(\mathrm{J9}; m, n)$ *are the* $(\mathrm{Aut}^0 \mathrm{F}^k_{m,} \mathrm{F}^k_{m,n})$'s $(k \geq \ell)$ *and the Euclidean model* $(\mathrm{Aut}^0 U_{m,n}, U_{m,n})$.

(2) *If* $2n > m$, $(m, n) = 1$, *then the relatively minimal elements in* $\mathscr{C}(\mathrm{J9}; m, n)$ *are the* $(\mathrm{Aut}^0 \mathrm{F}^k_{m,n}, \mathrm{F}^k_{m,n})$*'s* $(k \geq \ell)$, $(\mathrm{Aut}^0 \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(-m) \oplus \mathcal{O}_{\mathbf{P}^1}(-n) \oplus \mathcal{O}_{\mathbf{P}^1}))$, $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(-m) \oplus \mathcal{O}_{\mathbf{P}^1}(-n) \oplus \mathcal{O}_{\mathbf{P}^1}))$ *and the Euclidean model* $(\mathrm{Aut}^0 U_{m,n}, U_{m,n})$.

Proof is done by the same method as in remark (8.20.1). Since there may exist a divisor isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ which can be contracted for 2 different directions, in the diagram of invariant divisors we have to associate 2 numbers for $\mathbf{P}^1 \times \mathbf{P}^1$. We do not give here a proof since this is done quite similarly and the explication is messy.

Theorem (8.20) and (8.22) determine the relatively minimal elements of the ordered set $\mathscr{C}(\mathrm{J9}; m, n)$ and we constructed these relatively minimal elements in an explicit way.

Remark (8.23). In the model $U_{3,2}$, we do have the case (3.3.5) of Theorem (3.3) in [Mo].

§ 9. Equivariant completions of J10

It follows from Lemma (4.20), [U4] that $(\mathrm{Aut}^0 \mathrm{F}'_{m,m}, \mathrm{F}'_{m,m})$ has an equivariant completion $(\mathrm{Aut}^0 \mathrm{F}_{m,m}, \mathrm{F}_{m,m})$, where $\mathrm{F}_{m,m}$ denotes $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(-m) \oplus \mathcal{O}_{\mathbf{P}^1}(-m) \oplus \mathcal{O})$ $(m \geq 2)$. We put $\mathscr{C}(\mathrm{J10}; m) = \mathscr{C}(\mathrm{Aut}^0 \mathrm{F}'_{m,m}, \mathrm{F}'_m)$ $(m \geq 2)$. As in the other preceding cases we want to know the orbit decomposition.

LEMMA (9.1). *The orbit decomposition of* $(\mathrm{Aut}^0 \mathrm{F}_{m,m}, \mathrm{F}_{m,m})$ *is* $\mathrm{F}'_{m,m} \cup (\mathrm{F}_{m,m} - \mathrm{F}'_{m,m})$. *The latter orbit is a divisor* D *isomorphic to* $\mathbf{P}^1 \times \mathbf{P}^1$. *Furthermore* $\mathcal{O}_D(D) \simeq \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(-m, 1)$.

Proof. The first assertion follows from Lemma (4.20), [U4]. Let $U_0 = \{(x_0, x_1) \in \mathbf{P}^1 \mid x_0 \neq 0\}$ and $U_1 = \{(x_0, x_1) \in \mathbf{P}^1 \mid x_1 \neq 0\}$. Then it follows from the definition that $\mathrm{F}_{m,m}$ is defined by gluing together $U_0 \times \mathbf{P}^2$ and $U_1 \times \mathbf{P}^2$: $(s; (u_0, u_1, u_2)) \in U_0 \times \mathbf{P}^2$ and $(t; (v_0, v_1, v_2)) \in U_1 \times \mathbf{P}^2$ are identified if $t = 1/s$, $v_0 = s^{-m} u_0$, $v_1 = s^{-m} u_1$ and $u_2 = v_2$. $D = \mathrm{F}_{m,m} - \mathrm{F}'_{m,m}$ is

$$\{(s; (u_0, u_1, 0)) \in U_0 \times \mathbf{P}^2\} \cup \{(t; (v_0, v_1, 0)) \in U_1 \times \mathbf{P}^2\}$$

hence isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. Let f be a fibre of the projection $D \to \mathbf{P}^1$ induced by the map $\mathrm{F}_{m,m} \to \mathbf{P}^1$. Since f is a line in the fibre isomorphic to $\mathbf{P}^2$ and D is defined by $u_2 = v_2 = 0$ inducing the hyperplane section on the fibre $\mathbf{P}^2$, $(D.f) = 1$. To determine the degree for another projection of D, we put $\ell = \{(s;\ (1, 0, 0)) \in U_0 \times \mathbf{P}^2\} \cup \{(t;\ (1, 0, 0)) \in U_1 \times \mathbf{P}^2\}$ and calculate the intersection number $(D.\ \ell)$. On $W_0 = \{(s;\ (u_0, u_1, u_2)) \in U_0 \times \mathbf{P}^2 \mid u_0 \neq 0\}$ D is defined by a regular function $u_2/u_2 = 0$ and on $W_1 = \{(t;\ (v_0, v_1, v_2) \in U_1 \times \mathbf{P}^2 \mid v_0 \neq 0\}$ D is defined by a regular function $v_2/v_0 = 0$. Therefore on $W_0 \cup W_1$, D defines a line bundle whose transition function is $(v_2/v_0)/(u_2/u_0) = u_0/v_0 = t^m$. Thus $\mathcal{O}(D) \otimes \mathcal{O}_\ell \simeq \mathcal{O}_{\mathbf{P}^1}(-m)$.

THEOREM (9.2). *The set* $\mathscr{C}(\mathrm{J10};\ m)$ $(m \geq 2)$ *consists of one element* $(\mathrm{Aut}^0\, \mathrm{F}_{m,m},\ \mathrm{F}_{m,m})$.

Proof. Since $\mathcal{O}(D) \otimes \mathcal{O}_D \simeq \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(-m, 1)$, we can not collapse D neither to a smooth point by Lemma (1.12) nor to a curve. The theorem now follows by the argument of Theorem (3.2).

§ 10. Equivariant completions of J11

Let us first construct some small equivariant completions. Let us recall the definition of E'^{ℓ}_m: using the notations of the section 1 E'^{ℓ}_m is the $\mathbf{A}^1$-bundle over F'_m defined by the unique non-trivial extension

$$(*) \qquad 0 \longrightarrow \mathcal{O}_{\mathrm{F}'_m} \longrightarrow \mathscr{E} \longrightarrow \mathcal{O}_{\mathrm{F}'_m}(2 - \ell m) \longrightarrow 0$$

which is homogeneous under the operation of SL_2 on F'_m. We know by [U4] that J11 is the operation $(\mathrm{Aut}^0\, \mathrm{E}'^{\ell}_m,\ \mathrm{E}'^{\ell}_m)$ $(m \geq 2,\ \ell \geq 2$ or $m = 1,\ \ell \geq 3)$ which respects consequently the natural fibrations $\mathrm{E}'^{\ell}_m \to \mathrm{F}'_m \to \mathbf{P}^1$. The extension $0 \to \mathcal{O}_{\mathrm{F}'_m} \to \mathscr{F} \to \mathcal{O}_{m}'(2 - \ell m) \to 0$ is parametrized by the cohomology group $H^1(\mathrm{F}'_m,\ \mathcal{O}_{\mathrm{F}'_m}(\ell m - 2))$ which is SL_2-isomorphic to $\bigoplus_{k\geq 0} H^1(\mathbf{P}^1,\ \mathcal{O}(\ell m - 2 - km))$ by (1.3.1) and E'_m corresponds to the unique SL_2-invariant subspace $H^1(\mathbf{P}^1, 0(-2))$ of $\bigoplus_{k\geq 0} H^1(\mathbf{P}^1,\ \mathcal{O}(\ell m - 2 - km))$ (see (3.8), [U4]). If we take $j \geq \ell$, then by the degeneracy of the spectral sequence (1.2) of $f\colon \mathrm{F}_n \to \mathbf{P}^1$, the extension $\mathscr{E}$ on F'_m can be extended over F_m:

$$(**) \qquad 0 \longrightarrow \mathcal{O}_{\mathrm{F}_m} \longrightarrow \tilde{\mathscr{E}} \longrightarrow \mathcal{O}_{\mathrm{F}_m}(2 - \ell m) \otimes \mathcal{O}_{\mathrm{F}_m}(-jC_\infty) \longrightarrow 0\,,$$

where $\mathrm{C}_\infty = \mathrm{F}_m - \mathrm{F}'_m$ (see (1.30) and the section 8). The hypotheses $j \geq \ell$ implies $(\ell - j)m - 2 \leq 0$ and hence the morphism $H^0(\mathrm{F}_m,\ \mathcal{O}_{\mathrm{F}_m}(\ell m - 2) \otimes \mathcal{O}_{\mathrm{F}_m}(jC_\infty)) \to H^0(\mathrm{F}'_m,\ \mathcal{O}_{\mathrm{F}'_m}(\ell m - 2))$ induced by the restriction is an isomorphism. In fact by the degeneracy of spectral sequences (1.2) and (1.3.1), we have a commutative diagram:

$$\begin{array}{ccc} H^0(\mathrm{F}_m, \mathcal{O}_{\mathrm{F}_m}(\ell m-2)\otimes\mathcal{O}_{\mathrm{F}_m}(jC_\infty)) & \longrightarrow & H^0(\mathrm{F}'_m, \mathcal{O}_{\mathrm{F}'_m}(\ell m-2)) \\ \wr\| & & \wr\| \\ \bigoplus_{i=0}^{j} H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(\ell m-2)\otimes\mathcal{O}(-im)) & \simeq & \bigoplus_{i=0}^{\infty} H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(\ell m-2)\otimes\mathcal{O}(-im)). \end{array}$$

Hence the $\mathbf{P}^1$-bundle $f^\ell_m(j)\colon \mathrm{E}^\ell_m(j) = \mathbf{P}(\tilde{\mathscr{E}}) \to \mathrm{F}_m$ is equivariant completion of $(\mathrm{Aut}^0 \mathrm{E}'^\ell_m, \mathrm{E}'^\ell_m)$ by Lemma (4.22), [U4].

NOTATION (10.1). We denote by $D^\ell_1(j)$ a divisor $f^\ell_m(j)^{-1}(C_\infty)$ and $D^\ell_2(j)$ the ∞-section of the $\mathbf{P}^1$-bundle $f^\ell_m(j)\colon \mathrm{E}^\ell_m(j) \to \mathbf{P}^1$ arising from the $\mathbf{A}^1$-bundle or the extension (**) so that $\mathrm{E}^\ell_m(j) = \mathrm{E}'^\ell_m \cup D^\ell_1(j) \cup D^\ell_2(j)$.

LEMMA (10.2). (a) *The divisors $D^\ell_1(j)$ and $D^\ell_2(j)$ are* $\mathrm{Aut}^0 \mathrm{E}^\ell_m(j)$*-invariant.* (b) $\mathrm{E}^\ell_m(j) - (D^\ell_1(j) \cup (D^\ell_1(j))$ *is an open orbit.* (c) $D^\ell_1(\ell)$ *is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ and the* $\mathrm{Aut}^0 \mathrm{E}^\ell_m(\ell)$*-orbit decomposition of $D^\ell_1(\ell) \simeq \mathbf{P}^1 \times \mathbf{P}^1$ is $(\mathbf{P}^1 \times \mathbf{P}^1$ − diagonal$)$ ∪ diagonal.* (c′) *If $j > \ell$, then $D^\ell_1(j)$ is isomorphic to $\mathrm{F}_{(j-\ell)m+2}$ and the* $\mathrm{Aut}^0 \mathrm{E}^\ell_m(j)$*-orbit decomposition of $D^\ell_1(j) \simeq \mathrm{F}_{(j-\ell)m+2}$ consists of an open orbit and two $\mathbf{P}^1$'s which are disjoint sections of the ruled surface $\mathrm{F}_{(j-\ell)m+2} \to \mathbf{P}^1$: they are the ∞-section and a 0-section of $\mathrm{F}_{(j-\ell)m+2}$.* (d) $L^\ell_2(j)$ *is isomorphic to F_m with orbit decomposition $(\mathrm{F}_m - C_\infty) \cup C_\infty$.*

Proof. The assertions (a), (b) and (d) follow from the construction (cf. the section 8). The restriction to $C_\infty \simeq \mathbf{P}^1$ of the non-trivial extension $0 \to \mathcal{O}_{\mathrm{F}_m} \to \mathscr{E} \to \mathcal{O}_{\mathrm{F}_m}(2-\ell m)\otimes\mathcal{O}_{\mathrm{F}_m}(-\ell C_\infty) \to 0$ is an extension:

$$(*) \qquad 0 \longrightarrow \mathcal{O}_{\mathbf{P}^1} \longrightarrow \mathscr{E}\,|\,\mathbf{P}^1 \longrightarrow \mathcal{O}_{\mathbf{P}^1}(2-\ell m+\ell m) \longrightarrow 0$$

since $(C_\infty)^2 = -m$. We show that the extension (*) is non-trivial. In fact putting $M = \mathcal{O}_{\mathrm{F}_m}(-2+\ell m)\otimes\mathcal{O}_{\mathrm{F}_m}(\ell C_\infty)$, consider the exact sequence $0 \to \mathcal{O}_{\mathrm{F}_m}(-C_\infty) \to \mathcal{O}_{\mathrm{F}_m} \to \mathcal{O}_{C_\infty} \to 0$ defining the curve C_∞ on F_n. Tensoring M with (*), we get $0 \to \mathcal{O}_{\mathrm{F}_m}(-2+\ell m)\otimes\mathcal{O}_{\mathrm{F}_m}((\ell-1)C_\infty) \to M \to M\otimes\mathcal{O}_{C_\infty}(\simeq\mathcal{O}_{\mathbf{P}^1}(-2))$ $\to 0$. Hence combining with the spectral sequence for $\mathrm{F}_m \to \mathbf{P}^1$ we finally get an SL_2-exact sequence:

$$\begin{array}{ccc} H^1(\mathcal{O}_{\mathrm{F}_m}(-2+\ell m)\otimes\mathcal{O}_{\mathrm{F}_m}((\ell-1)C_\infty)) & \longrightarrow H^1(M) \longrightarrow & H^1(\mathbf{P}^1, \mathcal{O}(-2))) \\ \|\wr & \|\wr & \\ H^1(\mathbf{P}^1, \mathcal{O}(-2+\ell m)\otimes S^{\ell-1}(\mathcal{O}\oplus\mathcal{O}(-m))) & H^1(\mathbf{P}^1, \mathcal{O}(-2+\ell m)\otimes S^\ell(\mathcal{O}\otimes\mathcal{O}(-m))) & \\ \|\wr & \|\wr & \\ \bigoplus_{i=0}^{\ell-1} H^1(\mathbf{P}^1, \mathcal{O}(-2+\ell m-im)) & \longrightarrow \bigoplus_{j=0}^{\ell} H^1(\mathbf{P}^1, (-2+\ell m-jm)). & \end{array}$$

Hence the map $H^1(M) \to H^1(C_\infty, M\otimes\mathcal{O}_{C_\infty})$ induces an isomorphism between

$H^1(\mathbf{P}^1, \mathcal{O}(-2))$-factor in $H^1(M)$ and $H^1(M \otimes \mathcal{O}_{C_\infty})$, since $H^1(\mathbf{P}^1, \mathcal{O}(j))$ is the irreducible SL_2-module of degree $-1-j$ if $H^1(\mathbf{P}^1, \mathcal{O}(j)) \neq 0$. Therefore the exact sequence (*) is SL_2-isomorphic to the exact sequence:

$$(**) \qquad 0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1) \oplus \mathcal{O}(2) \longrightarrow \mathcal{O}(2) \longrightarrow 0\,.$$

Since it follows from the spectral sequence (1.2.1) that the inclusion map

$$(10.3) \qquad \begin{aligned} &H^0(\mathrm{F}_m, \mathcal{O}_{\mathrm{F}_m}(-2+\ell m) \otimes \mathcal{O}_{\mathrm{F}_m}((\ell-1)C_\infty)) \\ &\qquad \longrightarrow H^0(\mathrm{F}_m, \mathcal{O}_{\mathrm{F}_m}(-2+\ell m) \otimes \mathcal{O}_{\mathrm{F}}(jC_\infty)) \end{aligned}$$

is bijective for any $j \geq \ell$ (cf. the section 8), we conclude from Lmma (1.10) that $\mathrm{Ker}\,\varphi = H^0(\mathrm{F}_m, \mathcal{O}_{\mathrm{F}_m}(-2+\ell m))$ fixes all the points lying over C_∞, where φ denotes the morphism $\mathrm{Aut}^0\,\mathrm{E}^\ell_m(j) \to \mathrm{Aut}^0\,\mathrm{F}_m$ induced by the equivariant morphism π. In particular when $\ell = j$, the action of $\mathrm{Aut}^0\,\mathrm{E}^\ell_m(\ell)$ on $D_1 \simeq \mathbf{P}^1 \times \mathbf{P}^1$ is an operation of SL_2 on $\mathbf{P}^1 \times \mathbf{P}^1$ respecting the exact sequence (**) and hence the diagonal action of SL_2 on $\mathbf{P}^1 \times \mathbf{P}^1$ by Lemma (1.11). This proves (c). Since for $j > \ell$, there is no SL_2-invariant in $H^1(\mathbf{P}^1, \mathcal{O}(-2-(j-\ell)m))$, the restriction of the SL_2-exact sequence $0 \to \mathcal{O}_{\mathrm{F}_m} \to \mathcal{E} \to \mathcal{O}_{\mathrm{F}_m}(2-\ell m) \otimes \mathcal{O}_{\mathrm{F}}(jC_\infty) \to 0$ onto $C_\infty \simeq \mathbf{P}^1$; $0 \to \mathcal{O}_{\mathbf{P}^1} \to \mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^1} \to \mathcal{O}_{\mathbf{P}^1}(2+(j-\ell)m) \to 0$ splits and hence $D^\ell_1(j) \simeq \mathrm{F}_{2+(j-\ell)m}$. (c′) now follows from isomorphism (10.3), Lemma (1.10), Lemma (1.11) and [U3].

LEMMA (10.4). *Let* $(G, X) \in \mathscr{C}(\mathrm{Aut}^0\,\mathrm{E}'^\ell_m, \mathrm{E}'^\ell_m) = \mathscr{C}(\mathrm{J}11; m, \ell)$ $(m \geq 2,\ \ell \geq 2$ *or* $m = 1,\ \ell \geq 3)$. *Then there is no G-fixed point on X and none of the cases* (3.3.2), (3.3.3), (3.3.4) *and* (3.3.5) *of Theorem* (3.3) *Mori* [Mo] *occurs.*

Proof. A simple part of G is SL_2 and it has an open orbit by Corollary (4.23), [U4] and hence there is no SL_2-fixed point by Lemma (1.2.2), [MU] and consequently there is no G-fixed point on X. We show that there is no invariant divisor isomorphic to $\mathbf{P}^2$ on X. In fact let $\chi: X \cdots\!\to \mathrm{E}^\ell_m(\ell)$ be an equivariant birational map. Let the diagram

$$\begin{array}{ccc} & \tilde{X} & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & \overset{\chi}{\cdots\!\longrightarrow} & \mathrm{E}^\ell_m(\ell) \end{array}$$

be an equivariant elimination of indeterminancy, where p is an equivariant blow-up. Since there is no fixed point on $\mathrm{E}^\ell_m(\ell)$, it follows from Theorem (1.7) q is an equivariant blow-up too. Assume that there is an invariant $\mathbf{P}^2$ on X. Since there is no fixed point, the proper transform $p^{-1}(\mathbf{P}^2)$ is isomorphic to $\mathbf{P}^2$. This is a contradiction since there is neither invariant $\mathbf{P}^2$ nor fixed point on $\mathrm{E}_m^\ell(\ell)$.

LEMMA (10.5). *Let* $(G, X) \in \mathscr{C}(\mathrm{J}11; m, \ell)$ $(m \geq 2, \ell \geq 2$ *or* $m = 1, \ell \geq 3)$. *Let* ϕ: $X \to Y$ *be the morphism of* (3.5.1) *Theorem* (3.5) *of* [Mo]. *Then the surface is isomorphic to* F_m *and* X *is a* $\mathbf{P}^1$*-bundle over* F_m.

Proof. Since $\mathrm{Aut}^0(\mathrm{E}_m'^\ell; \mathrm{F}_m) = \mathrm{Aut}^0 \mathrm{E}_m'^\ell$ by Corollary (2.3), [U4], the lemma is proved by the same argument as in the proof of Lemma (8.17).

LEMMA (10.6). *Let* $(G, X) \in \mathscr{C}(\mathrm{J}11; m, \ell)$. *Case* (3.5.2) *of Theorem* (3.5), [Mo] *never occurs.*

Proof. It follows from Corollary (4.23), [U4] that a semi-simple part SL_2 has an open orbit on X. Therefore if Φ: $X \to Y$ is a morphism of case (3.5.2) of Theorem (3.5), in [Mo], SL_2 has an open orbit and we have a non-trivial morphism (ψ, ϕ): $(G, X) \to (\mathrm{PGL}_2, \mathbf{P}^1)$. Hence all the fibres of ϕ are isomorphic to one non-singular del Pezzo surface S and S is an equivariant completion of a homogeneous surface Y' under the unipotent radical U of G since U has an open orbit on each fibre (see [U4]). The homogeneous surface (U, Y') contains $(\mathbf{G}_a^{\oplus 2}, \mathbf{G}_a^{\oplus 2})$ since any homogeneous surface under unipotent group contains such operation. It follows now from Theorem (3.5), [Mo] and from Theorem 2.4.4, [Ma] that S is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ or $\mathbf{P}^2$. In the last case, there exists a vector bundle $\mathscr{E}$ of rank 3 on $\mathbf{P}^2$ such that $X \simeq \mathbf{P}(\mathscr{E})$. By a theorem of Grothendieck we can find 2 line bundles L, M on $\mathbf{P}^1$ such that $X \simeq \mathbf{P}(L \oplus M \oplus 0)$. Therefore $\mathbf{G}_m^3$ operates on X and $\mathbf{G}_m^3 \subset G$. This is absurd. We notice that the rank of G is equal to 2 (see [U4]). Now we exclude the first case $S \simeq \mathbf{P}^1 \times \mathbf{P}^1$. In fact assume that this is the case. We know that a semi-simple part of G, which is isogeneous to SL_2, has an open orbit (cf. Corollary (4.23), [U4]). Let B be a Borel subgroup of SL_2. B has an open orbit on $\mathbf{P}^1$. Let $y \in \mathbf{P}^1$ such that $\overline{BY} = \mathbf{P}^1$ and hence $T = \{g \in G \mid gy = y\}$ is isomorphic to $\mathbf{G}_m$. The operation of $\mathbf{G}_m \simeq T$ on the fibre $\phi^{-1}(y) \simeq \mathbf{P}^1 \times \mathbf{P}^1$ is, up to an automorphism of $\mathbf{P}^1 \times \mathbf{P}^1$, given by $(u, v) \to (t^a u, t^b v)$ for $t \in \mathbf{G}_m$, $(u, v) \in \mathbf{P}^1 \times \mathbf{P}^1$, where a, b are integers. $(a, b) \neq 0$ since SL_2 has an open orbit on X lying over $\mathbf{P}^1$. Therefore there exists a $T \simeq \mathbf{G}_m$-orbit on $\mathbf{P}^1 \times \mathbf{P}^1 \simeq \varphi^{-1}(y)$ whose closure is $\mathbf{P}^1 \times v_0$ or $u_0 \times \mathbf{P}^1$. We may assume that the closure is $\mathbf{P}^1 \times v_0$. Then if for general point $u \in \mathbf{P}^1$, we put $D' = B(u, v_0)$

$\subset X$ and $D = \bar{D}'$, then D is a divisor since the stabilizer at (u, v_0) of B is finite. It follows from the construction $D \cap \phi^{-1}(y) = D \cap (\mathbf{P}^1 \times \mathbf{P}^1) = \mathbf{P}^1 \times v_0$. This is impossible since by Theorem (3.1), [Mo], $\mathbf{P}^1 \times v_0$, $u_0 \times \mathbf{P}^1$ belong to the same extremal ray but $(D. \mathbf{P}^1 \times a) = 0$, $(D. b \times \mathbf{P}^1) = 1$.

LEMMA (10.7). *If* $(G, X) \in \mathscr{C}(\mathrm{J}11; m, \ell)$, *then* $\rho(X) \geq 3$ $(m \geq 2, \ell \geq 2$ *or* $m = 1, \ell \geq 3)$.

Proof. By Lemma (10.4), Lemma (10.5), Lemma (10.6) and [Mo], it is sufficient to prove that $\rho(X) \neq 1$ and this is done by the same argument as in the Proof of Lemma (8.19).

LEMMA (10.8). *If* $(G, X) \in \mathscr{C}(\mathrm{J}11; m, \ell)$ *and* $\rho(X) = 3$, *then X is a* $\mathbf{P}^1$-*bundle over* F_m *and* F_m-*isomorphic to* $\mathrm{E}_m^\ell(k)$ *for a suitable integer* $k \geq \ell$.

Proof. It follows from Lemma (10.4), Lemma (10.5), Lemma (10.6) and [Mo] that X is a $\mathbf{P}^1$-bundle over F_m and $\pi: X \to \mathrm{F}_m$ is G-equivariant. Since the Brauer of F_m vanish, there exists a vector bundle $\mathscr{E}$ of rank 2 over F_m such that $X \simeq \mathbf{P}(\mathscr{E})$. The restriction $\mathbf{P}(\mathscr{E}) | \mathrm{F}'_m$ contains the $\mathbf{A}^1$-bundle E'^ℓ_m and $D' = (\mathbf{P}(\mathscr{E}) | \mathrm{F}'_m) - E'^\ell_m$ is isomorphic to F'_m, which is a 2-dimensional G-orbit on $\mathbf{P}(\mathscr{E}) | \mathrm{F}'_m$ hence on $\mathbf{P}(\mathscr{E})$ (cf. Proof of Lemma (8.17) and Theorem (8.20)). The closure D of D' in $\mathbf{P}(\mathscr{E})$ is an irreducible divisor. D contains no fibre of $\mathbf{P}(\mathscr{E})$ over F_m. In fact D contains no fibre of $\mathbf{P}(\mathscr{E})$ lying over F'_m and if it contained a fibre of $\mathbf{P}(\mathscr{E})$ lying over $\mathrm{F}_m - \mathrm{F}'_m$, since $\mathrm{F}_m - \mathrm{F}'_m = \mathbf{P}^1$ is a G-orbit, D would contain the surface $\pi^{-1}(\mathrm{F}_m - \mathrm{F}'_m)$, which contradicts the irreducibility of D. Thus D intersects properly with any fibre of π. The intersection number $(D. f) = 1$ for any fibre lying over F'_m hence over F_m. This shows that D is non-singular. Since D is an equivariant completion of $D' \simeq \mathrm{F}'_m$, D is isomorphic to F_m and $\pi | D: D \to \mathrm{F}_m$ is an isomorphism. Therefore we get an exact sequence:

$$0 \longrightarrow L \longrightarrow \mathscr{E} \longrightarrow M \longrightarrow 0,$$

where L and M are line bundles over F_m. Thensoring L^{-1} we may assume $L = \mathscr{O}$:

$$(*) \qquad 0 \longrightarrow \mathscr{O} \longrightarrow \mathscr{E} \longrightarrow M \longrightarrow 0.$$

The restriction of this extension over F'_m should be

$$(**) \qquad 0 \longrightarrow \mathscr{O}_{\mathrm{F}'_m} \longrightarrow \mathscr{E} | \mathrm{F}'_m \longrightarrow \mathscr{O}_{\mathrm{F}'_m}(2 - \ell m) \longrightarrow 0.$$

Thus $M \simeq \mathscr{O}_{\mathrm{F}_m}(2 - \ell m) \otimes \mathscr{O}_{\mathrm{F}_m}(-k C_\infty)$, where $C_\infty = \mathrm{F}_m - \mathrm{F}'_m$. It follows from the argument at the beginning of this section that $(\ell - k)m - 2 \leq 0$. The extension (*) is parametrized by

$$\begin{aligned} H^1(\mathrm{F}_m, M^{-1}) &= H^1(\mathrm{F}_m, \mathcal{O}_{\mathrm{F}_m}(\ell m - 2) \otimes \mathcal{O}_{\mathrm{F}_m}(-kC_\infty)) \\ &\simeq H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(\ell m - 2) \otimes S^k(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-m)) \\ &\simeq \bigoplus_{j=0}^{k} H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}_1}(\ell m - 2 - jm)) \end{aligned}$$

and the extension (**) by

$$H^1(\mathrm{F}'_m, \mathcal{O}_{\mathrm{F}'_m}(\ell m - 2)) \simeq \bigoplus_{j \geq 0} H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(\ell m - 2 - jm))$$

by (1.2.1) and (1.3.1). Hence the restriction $H^1(\mathrm{F}_m, M^{-1}) \to H^1(\mathrm{F}'_m, M^{-1})$ is injective and the extension defining $\mathcal{E}$ is the cone used to define $\mathrm{E}^\ell_m(k)$.

We have thus proved

THEOREM (10.9). *Any element of* $\mathscr{C}(\mathrm{Aut}\,\mathrm{E}'^\ell_m, \mathrm{E}'^\ell_m) = \mathscr{C}(\mathrm{J11}; m, \ell)$ $(m \geq 2, \ell \geq 2$ *or* $m = 1, \ell \geq 3)$ *can be equivariantly blown-down to a relatively minimal element of the ordered set* $\mathscr{C}(\mathrm{J11}; m, \ell)$ *and the centers of the blow-down are curves isomorphic to* $\mathbf{P}^1$. *The relatively minimal elements in* $\mathscr{C}(\mathrm{J11}; m, \ell)$ *are* $(\mathrm{Aut}^0\,\mathrm{E}^\ell_m(j), \mathrm{E}^\ell_m(j))$ $(j \geq \ell)$.

Remark (10.10). As in the section 8, we can show that the $(\mathrm{Aut}^0\,\mathrm{E}^\ell_m(j), \mathrm{E}^\ell_m(j))$ $(j \geq \ell)$ are related each other by equivariant elementary transformations.

§ 11. Equivariant completions of J12

Let us recall the definition of the operation (J12). Let $\pi\colon C_1 \to C_2$ be an étale 2-covering of a non-singular open rational curve C_2 by an irreducible curve C_1 of genus $g \geq 1$ so that C_1 is an elliptic or hyperelliptic curve of genus g. Let ι be the involution of C_1 giving π so that $C_1/\langle\iota\rangle \simeq C_2$ and we denote by i an involution of $C_1 \times \mathbf{P}^1 \times \mathbf{P}^1$:

$$i(t, x, y) = (\iota t, y, x) \qquad \text{for } t \in C_1,\ x, y \in \mathbf{P}^1.$$

We let operate SL_2 on $C_1 \times \mathbf{P}^1 \times \mathbf{P}^1$: $h(t, x, y) = (t, hx, hy)$ for $(t, x, y) \in C_1 \times \mathbf{P}^1 \times \mathbf{P}^1$, where for $h \in \mathrm{SL}_2$ $z \in \mathbf{P}^1$, hz denote the usual action of SL_2 on $\mathbf{P}^1$. The operation of SL_2 on $C_1 \times (\mathbf{P}^1 \times \mathbf{P}^1 - \text{diagonal})$ commutes with the involution i and thus defines an operation (SL_2, X_π), where $X_\pi = (\mathbf{P}^1 \times \mathbf{P}^1\text{-diagonal})/\langle i\rangle$. See (3.9), [U4].

Let us construdt a model. Let g be a non-negative integer and $n = g + 1$. Let V be an irreducible SL_2-module of dimension 3 so that we identify V with the vector space of the homogeneous polynomials of degree 2 in x, y. The operation of SL_2 on the vector space $kx \oplus ky$ is in the usual way. By letting SL_2 operator on $\mathcal{O}_{\mathbf{P}^1}(n)$ trivially, SL_2 operates on a vector bundle $\mathcal{E} = (\mathcal{O}_{\mathbf{P}^1} \otimes_k V) \oplus \mathcal{O}_{\mathbf{P}^1}(n)$ over $\mathbf{P}^1$. We use the basis a, b, c on V: $g(x, y) = ax^2 + bxy + cy^2 \in V$. Denoting by z the fibre coordinate of $\mathcal{O}_{\mathbf{P}^1}(n)$ locally, if $f(t) \in H^0(\mathbf{P}^1, \mathcal{O}(2u))$, $b^2 - 4ac + f(t)z^2$ is an SL_2-invariant section of $H^0(\mathbf{P}^1, S^2(\mathcal{E}))$. Therefore we get a quadric bundle $p: X_f = \{b^2 - 4ac + f(t)z^2 = 0 \mid (ax^2 + bxy + cy^2, z) \in \mathbf{P}(\mathcal{E})\} \to \mathbf{P}^1$ on which SL_2-operates. From now on we assume that the homogeneous polynomial $f(t)$ of degree $2n$ has only simple roots so that $y^2 = f(t)$ defines a hyperelliptic (or elliptic) curve.

LEMMA (11.1). *If $f(t) = 0$ has only simple roots, then X_f is non-singular and projective.*

Proof. In fact, locally X_f is defined by $\{(t; a, b, c, z) \in \mathbf{A}^1 \times \mathbf{P}^3 \mid b^2 - 4ac + f(t)z^2 = 0\}$. The smoothness of X_f follows from this local expression. The projectivity of X_f is obvious.

LEMMA (11.2). *Let $\pi: C \to \mathbf{P}^1$ be the hyperelliptic curve defined by $y^2 = f(t)$. Then (SL_2, X_π) and (SL_2, X_π) are mutually isomorphic as law chunks of algebraic operation.*

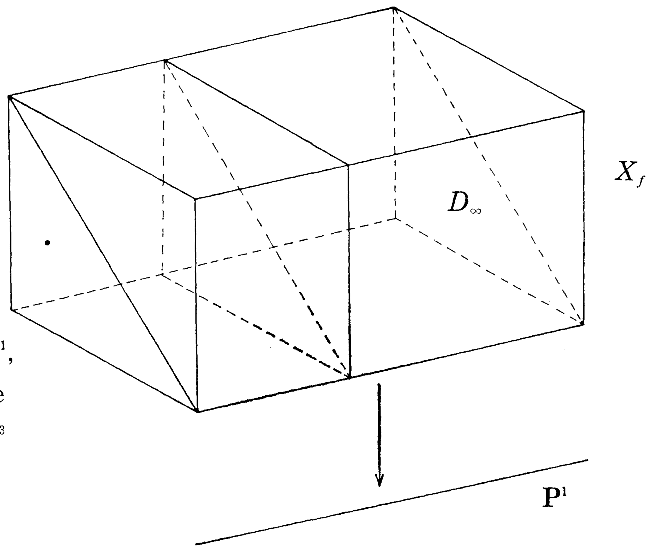
Proof. If we add $y = \sqrt{f(t)}$, then over $k(t, y)$ the quadratic form $b^2 - 4ac + f(t)z^2$ is isomorphic to; $b^2 - 4ac - w^2$. Namely $X_f \otimes_{\mathbf{P}^1} C$ is birationally equivalent to the product $C \times$ (quadric surface). Now the descent datum on $C \times$ (quadric surface) defining X_f is $(a, b, c, w) \to (a, b, c, -w)$ if written in terms of the coordinate system on $C \times \mathbf{P}^3$. Let us see how the quadric is identified with $\mathbf{P}^1 \times \mathbf{P}^1$ embedded in $\mathbf{P}^3$ by $\mathbf{P}^1 \times \mathbf{P}^1 \ni ((x_0, x_1); (u_0, u_1)) \to (x_0u_0, x_0u_1, x_1u_0, x_1u_1) \in \mathbf{P}^3$. If SL_2 operates diagonally on $\mathbf{P}^1 \times \mathbf{P}^1$, SL_2-module $H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(1, 1))$ is decomposed into the direct sum $U_3 \oplus U_1$, where U_3 is the irreducible SL_2-module of degree i. The explicit decomposition is $H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(1, 1)) \simeq H^0(\mathbf{P}^1, \mathcal{O}(1)) \otimes H^0(\mathbf{P}^1, (1)) = (kx_0u_0 + k(x_0u_1 + x_1u_0) + kx_1u_1) \oplus k(x_0u_1 - x_1u_0)$. Thus to connect X_f with X_π we had better consider the embedding $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ given by $((x_0, x_1); (u_0, u_1)) \to (x_0u_0, x_0u_1 + x_1u_0, x_1u_1, x_0u_1 - x_1u_0)$ so that the defining equation of the image is $X_1^2 - 4X_0X_2 - X_3^2 = 0$. Therefore the involution (a, b, c, w) $(a, b, c, -w)$ on the quadric just corresponds to the automorphism of $\mathbf{P}^1 \times \mathbf{P}^1$ interchanging the factors. Now Lemma follows from the definition of X_π.

Remark (11.3). We may expect to argue as in the preceding cases trying to exclude the cases (3.3.3), (3.3.4) and (3.3.5) in [Mo]. But the case (3.3.4) does occur as we see below (see (11.18)). Therefore we examine equivariant completions studying closely the operation (SL_2, X_f) in Lemma (11.1) and its equivariant blow-ups.

LEMMA (11.4). *The* SL_2*-invariant reduced irreducible effective divisors on* X_f *are*; (1) *the fibre* $p^{-1}(t)$, $t \in \mathbf{P}^1$ *which is isomorphic to* $(\mathrm{SL}_2, \mathbf{P}^1 \times \mathbf{P}^1)$ *with diagonal* SL_2*-action if* $f(t) \neq 0$ *and to the singular quadric* $\{u^2 + v^2 + w^2 = 0\} \subset \mathbf{P}^3$ *if* $f(t) = 0$, (2) $D_\infty = \{b^2 - 4ab = 0 \mid (ax^2 + bxy + cy^2, 0) \in X_f \subset \mathbf{P}(\mathscr{E})\}$ *which is isomorphic to* $(\mathrm{SL}_2, \mathbf{P}^1 \times \mathbf{P}^1)$ *with* SL_2*-action through the first factor.*

Remark (11.5). Since SL_2 is a semi-simple part of $\mathrm{Aut}^0 Z \simeq \mathrm{Aut}^0 \mathrm{F}_2$ by [U3], the operation of SL_2 on the singular quadric Z is essentially unique and given as follows: Let U_3 denote the irreducible SL_2-module of dimension 3 which we identify with the vector space of the homogeneous polynomials of degree 2 in 2 variables x, y. Then U_3 has an SL_2-invariant, the discriminant D of the degree 2 polynomial, $D(ax^2 + bxy + cy^2) = b^2 - 4ac$. k being the trivial SL_2-module, SL_2 acts on the direct sum $U_3 \oplus k$ hence on $\mathbf{P}^3 = \mathbf{P}(U_3 \oplus k)$ leaving the singular quadric $b^2 - 4av = 0$ in $\mathbf{P}^3$ invariant.

Proof. Lemma follows from the construction of X_f.



general fibre $\simeq \mathbf{P}^1 \times \mathbf{P}^1$, singular fibre $\simeq$ the singular quadric in $\mathbf{P}^3$

LEMMA (11.6). *The* SL_2*-fixed points on* X_f *are the singular points of the singular fibres. If we blow up* X_f *at one of the* SL_2*-fixed points,* SL_2 *operates on the exceptional divisor* $\mathbf{P}^2$ *through irreducible* SL_2*-module of dimension* 3.

Proof. The first assertion of the Lemma is a consequence of the construction of X_f. X_f is locally defined by $\{(t; (a. b, c, z)) \in \mathbf{A}^1 \times \mathbf{P}^3 \mid b^2 - 4ac + f(t)z^2 = 0\}$. We may assume that the fixed point is in the fibre $p^{-1}(0)$, namely $t = 0$ is a simple zero of $f(t) = 0$. Hence locally for the usual topology at the singular point y of the singular fibre $p^{-1}(0)$, we may assume that X_f is defined by $\{(t, a, b, c) \in \mathbf{A}^4 \mid b^2 - 4ac + t = 0\}$ and therefore X_f is isomorphic to, around $(0; (0, 0, 0))$, $\mathbf{A}^3$ with irreducible SL_2-action on $\mathbf{A}^3$.

NOTATION (11.7). We denote by W_f the blow-up of X_f at all the SL_2-fixed points of X_f.

LEMMA (11.8). *The* SL_2*-invariant irreducible reduced effective divisors on* W_f *are*; (1) *the proper transforms of* SL_2*-invariant divisors on* X_f, (2) *the exceptional divisors isomorphic to* $\mathbf{P}^2$. *The proper transforms of the singular fibres are isomorphic to* F_2. *There is no* SL_2*-fixed point on* W_f.

Proof. Since SL_2 operates on the exceptional divisor $\mathbf{P}^2$ through the irreducible representation, SL_2-orbit decomposition of $\mathbf{P}^2$ is $SL_2/\mathbf{G}_m$, which is an open orbit, and $SL_2/B \simeq \mathbf{P}^1$, which is a conic in $\mathbf{P}^2$.

Here we give a table of intersections of 2 divisors on W_f. We denote by q: $W_f \to X_f$ the blow-up and by $q^{-1}[D]$ the proper transform of D.

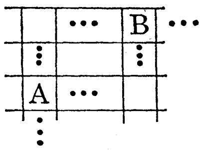
The statement at (A, B) in the table  describes the curve $A \cap B$ as a curve on A.

Table (11.9)

	$q^{-1}(D_\infty)\simeq \mathbf{P}^1\times\mathbf{P}^1$	q^{-1}(smooth fibre) $\simeq\mathbf{P}^1\times\mathbf{P}^1$	q^{-1}[singular fibre] $\simeq\mathbf{F}_2$	exceptional divisor $\simeq\mathbf{P}^2$
$q^{-1}(D_\infty)\simeq\mathbf{P}^1\times\mathbf{P}^1$		$\mathbf{P}^1\times a$ in $\mathbf{P}^1\times\mathbf{P}^1$	$\mathbf{P}^1\times a$ in $\mathbf{P}^1\times\mathbf{P}^1$	empty
q^{-1}(smooth fibre) $\simeq\mathbf{P}^1\times\mathbf{P}^1$	diagonal $\mathbf{P}^1\subset\mathbf{P}^1\times\mathbf{P}^1$		empty	empty
q^{-1}[singular fibre] $\simeq\mathbf{F}_2$	0-section in $\mathbf{F}_2$	empty		∞-section in $\mathbf{F}_2$
exceptional divisor $\simeq\mathbf{P}^2$	empty	empty	non-singular conic in $\mathbf{P}^2$	

where the ∞-section of F_2 is unique section of the ruled surface F_2 with self-intersection number -2 and a 0-section is a section of the ruled surface disjoint from the ∞-section, whose self-intersection number is necessarily equal to 2.

COROLLARY (11.10). *Let C be an irreducible reduced SL_2-invariant curve on W_f.* (a) *C is an SL_2-orbit hence isomorphic to $\mathbf{P}^1$.*

(b) *C is contained in a fibre $q^{-1}\circ p^{-1}(t)$ for a some point $t \in \mathbf{P}^1$.*

(c) *C is an intersection of 2 irreducible reduced effective divisors.*

(d) *If 2 irreducible reduced effective divisors on W_f intersect, they intersect transversely along $\mathbf{P}^1$.*

(e) *Any 3 distinct irreducible reduced effective divisors have no point in common.*

Proof. This is a consequence of Lemma (11.8) and the construction of X_f.

Now we describe the SL_2-invariant divisors on an equivariant blow-up of W_f.

LEMMA (11.11). *Let f_2: $\tilde{Y} \to W_f$ be a sequence of SL_2-equivariant blow-ups: $\chi_{i,i+1}$: $X_i \to X_{i+1}$ $(0 \leq i \leq m+1)$ is an SL_2-equivariant blow-up at an irreducible center, $X_0 \doteq \tilde{Y}$, $X_m = X_f$ and $f_2 = \chi_{m-1\,m} \circ \chi_{m-2\,m-1} \circ \chi_{01}$. For $1 \leq i \leq m$, we have the following.*

(a) *There is no SL_2-fixed point on X_i.*

(b) *Any SL_2-invariant irreducible reduced effective divisor on X_i is smooth.*

(c) *Any SL_2-invariant (integral) curve C on X_i has the following properties:*

(c1) *C is an SL_2-orbit hence isomorphic to $\mathbf{P}^1$,*

(c2) *C is contained in a fibre $\chi_{im}^{-1}\circ p^{-1}(t)$ for a some point $t \in \mathbf{P}^1$,*

(c3) *C is an intersection of 2 SL_2-invariant irreducible reduced effective divisors A, B such that $C = A \cap B$, $(C^2)_A \neq 0$.*

(d) *If 2 irreducible reduced effective divisors on X_i intersect, they intersect transversely along $\mathbf{P}^1$.*

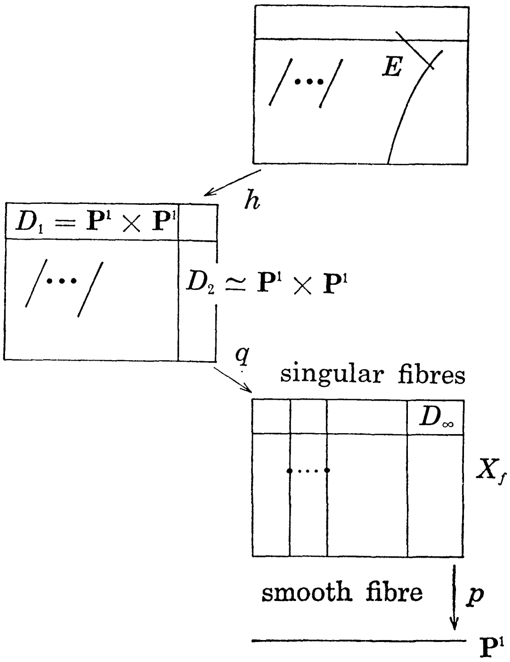
(e) *Any 3 distinct irreducible reduced effective divisors on X_i have no point in common.*

(f) *The exceptional divisor $E_i \subset X_i$ $(0 \leq i \leq m-1)$ for $X_{i,i+1}$ is isomorphic to the ruled surface F_{ℓ_i} for suitable integer $\ell_i \geq 1$ with non-trivial SL_2-action.*

Proof. (a) follows from Lemma (11.8) since the image of a fixed point is SL_2-fixed. We prove Lemma (11.11) by descending induction on i. If $i = m$, Lemma (11.11) is equivalent to Corollary (11.10). Now we have to prove the assertions for i assuming the lemma for $i+1$. Since there is no fixed point on X_{i+1}, $\chi_{i,i+1}$: $X_i \to X_{i+1}$ is the blow-up of X_{i+1} at an irreducible curve hence by (b) at $\mathbf{P}^1$. Therefore the exceptional divisor E_i is a ruled surface F_{ℓ_i}. Since if SL_2 operates on F_ℓ non-trivially and if $\ell \geq 1$, SL_2-orbit decomposition of F_ℓ is $F_\ell =$ open orbit (0-section) $\cup$ (∞-section), by Lemma (1.11) and (1.3.3) the other assertions follow once we prove (c3) for SL_2-invariant curves on $E_i \subset X_i$.

To illustrate our argument, let us first prove the lemma for $i = m - 1$, namely χ_{m-1m}: $X_{m-1} \to X_m = W_f$ is a blow-up at an irreducible non-singular SL_2-invariant curve. We give a formal induction later. We verify $\ell_{m-1} \geq 1$ case by case. Let h: $Y \to W_f$ be the blow-up of W_f at the intersection $q^{-1}(D_\infty) \cap q^{-1}$ (smooth fibre) (see Fig. (11.11.1)) and E be the exceptional divisor for h.

Fig. (11.11.1)



Since $D_1 = q^{-1}(D_\infty)$ and $D_2 = q^{-1}$ (smooth fibre) intersect transversely along the curve isomorphic to $\mathbf{P}^1$, E is isomorphic to F_ℓ for some integer $\ell \geq 0$.

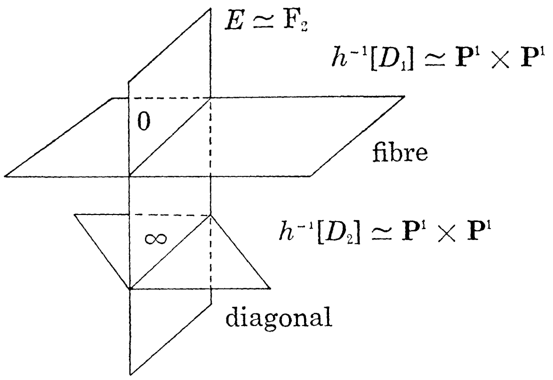
Moreover $h^{-1}[D_1] \cap E$ and $h^{-1}[D_2] \cap E$ are disjoint sections of ruled surface F_ℓ since D_1 and D_2 intersect transversely along the center of the blow-up. We calculate the intersection numbers $(h^{-1}[D_1]^2 . E)$ and $(h^{-1}[D_2]^2 . E)$. Since D_2 is a fibre of $p \circ q$, $h^* D_2 . E = 0$ hence

$$0 = (h^{-1}[D_2] . h^* D_2 . E) = (h^{-1}[D_2] . h^{-1}[D_2] + E . E)$$
$$= (h^{-1}[D_2] . h^{-1}[D_2] . E) + (h^{-1}[D_2] . E . E) = (h^{-1}[D_2] . h^{-1}[D_2] . E) + 2$$

by the table (11.9). Therefore

$$(h^{-1}[D_2] . h^{-1}[D_2] . E) = -2 \quad \text{and} \quad (h^{-1}[D_1], h^{-1}[D_1] . E) = 2$$

since $h^{-1}[D_2] \cap E$ is a section of F_ℓ disjoint from $h^{-1}[D_2] \cap E$. We have thus proved $\ell = 2$ and E is isomorphic to F_2:

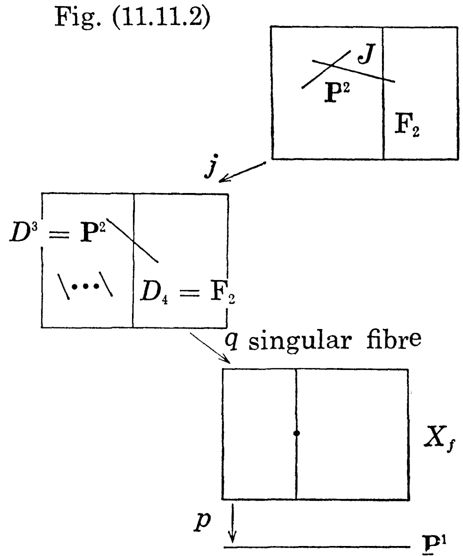


Let now $j \colon Z \to W_f$ be the blow up of W_f at the intersection of an exceptional divisor D_3 for q and q^{-1}[a singular fibre passing through $q(D_3)$] which we denote by D_4 (see Fig. (11.11.2)).

For the same reason as in case of h, the exceptional divisor J for j is isomorphic to F_ℓ for some integer $\ell \geq 0$ and

$$0 = (j^{-1}[D_3] . (\text{fibre of } p \circ q \circ j) . J) = (j^{-1}[D_3] . j^*(2D_3 + D_4) . J)$$
$$= (j^{-1}[D_3] . j^{-1}[2D_3] + j^{-1}[D_4] + 3J . J)$$
$$= (j^{-1}[D_3] . j^{-1}[2D_3] . J) + (j^{-1}[D_3] . j^{-1}[D_4] . J) + (j^{-1}[D_3] . 3J . J)$$
$$= 2(j^{-1}[D_3] . j^{-1}[D_3] . J) + 12$$

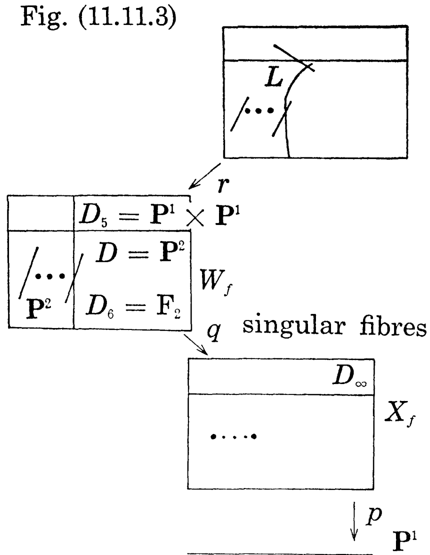
by Table (11.9). Hence $(j^{-1}[D_3] . j^{-1}[D_3] . J) = -6$. $j^{-1}[D_4] \cap J \subset J$ is a section disjoint from the section $j^{-1}[D_3] \cap J \subset J$ hence $(j^{-1}[D_4] . j^{-1}[D_4] . J) = 6$. Let us check this by another calculation. For the same reason as in preceding cases, Fig. (11.11.2)



$$
\begin{aligned}
0 &= (j^{-1}[D_4].\text{ fibre of } p\circ q\circ j.\,J) = (j^{-1}[D_4].\,j^{-1}[2D_3] + j^{-1}[D_4] + 3J.\,J)\\
&= 2(j^{-1}[D_4], j^{-1}[D_3].\,J) + (j^{-1}[D_4].\,j^{-1}[D_4].\,J) + 3(j^{-1}[D_4].\,J.\,J)\\
&= (j^{-1}[D_4].\,j^{-1}[D_4].\,J) - 6.
\end{aligned}
$$

Let r: $V \to W_f$ be the blow-up of W_f at the intersection of $D_5 = q^{-1}(D_\infty)$ and $D_6 = q^{-1}$ [singular fibre](see Fig. (11.11.3)). We denote by D the irreducible exceptional divisor for q intersecting with D_6.

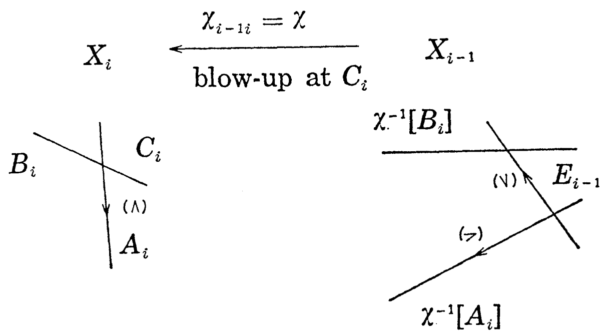
Fig. (11.11.3)



We denote by L the exceptional divisor for r. For the same reason as before $0 = (r^{-1}[D_6].\ (\text{fibre of } p \circ q \circ r).\ L) = (r^{-1}[D_6].\ r^{-1}[D_6] + L + 2r^*D.\ L) = (r^{-1}[D_6].\ r^{-1}[D_6].\ L) + (r^{-1}[D_6].\ L.\ L) = (r^{-1}[D_6].\ r^{-1}[D_6].\ L) + 2$. Hence L is isomorphic to F_2.

Let us now give a rigorous induction. As we have seen at the beginning of the proof, $\chi_{i-1,i}\colon X_{i-1} \to X_i$ is the blow-up at an SL_2-invariant curve C_i on X_i and it is sufficient to prove (c3) for SL_2-invariant curves on $E_{i-1} \subset X_{i-1}$ under the assumption that Lemma (11.11) holds for X_i. For the center C_i, we can find a smooth SL_2-invariant divisor A_i such that $C_i \subset A$, $p \circ q \circ \chi_{im}(A)$ is a point of $\mathbf{P}^1$ and such that $(C_i^2)_{A_i} \neq 0$. Let B_i be the G-invariant smooth irreducible divisor on X_i with $C_i = A_i \cap B_i$: B_i exists by (c3) for X_i.

Fig. (11.11.4)



$\chi = \chi_{i-1,i}\colon X_{i-1} \to X_i$ is the blow-up at $C_i = A_i \cap B_i$ as Fig. (11.11.4) shows. We have:

$$
\begin{aligned}
0 &= ((a \text{ fibre of } p \circ q \circ \chi_{im}).\ E_{i-1}.\ \chi^{-1}[A_i])_{X_{i-1}} \\
&= ((a\chi^{-1}[A_i] + bE_{i-1} + \text{other divisors disjoint from } E_{i-1})E_{i-1}.\ \chi^{-1}[A_i])_{X_{i-1}} \\
&\qquad\qquad (\text{here } a, b \text{ are positive integers}) \\
&= a(\chi^{-1}[A_i].\ E_{i-1}.\ \chi^{-1}[A_i])_{X_{i-1}} + b(E_{i-1}.\ E_{i-1}.\ \chi^{-1}[A_i])_{X_{i-1}}.
\end{aligned}
$$

Thus $(E_{i-1}.\ E_{i-1}.\ \chi^{-1}[A_i])_{X_{i-1}} \neq 0$ and $E_{i-1} \simeq F_{\ell_{i-1}}$ with $\ell_i \geq 1$ since $E_{i-1} \cap \chi^{-1}[A_i]$ and $E_{i-1} \cap \chi^{-1}[B_i] \subset E_{i-1}$ are disjoint sections (cf. (1.3.3)). This proves (c3) for $i-1$. For, since there is no SL_2-fixed point on X_{i-1}, new born SL_2-invariant curves on X_{i-1} are $\chi^{-1}[B_i] \cap E_{i-1}$ and $\chi^{-1}[A_i] \cap E_{i-1}$.

The fibration $p\colon X_f \to \mathbf{P}^1$ is a dell Pezzo fibration of Theorem (3.5) (3.5.2), [Mo]. In fact we show (see Example (11.18)).

LEMMA (11.12). *Pic* X_f *is generated by* D_∞ *and* p^* Pic $\mathbf{P}^1$. *In partic-*

ular the Picard number $\rho(X_f) = 2$.

Proof. As in the Proof of Lemma (11.2), let $C \to \mathbf{P}^1$ be the hyperelliptic curve defined by the extension $k(\sqrt{f}, t)/k(t)$. Let $B \subset C$ be the set of branch points and $R \subset \mathbf{P}^1$ be the set of ramification poits so that $\pi\colon C - B \to \mathbf{P}^1 - R$ is étale. Let us denote $C - B$ by C_1 and $\mathbf{P}^1 - R$ by C_2. As we have seen in the Proof of Lemma (11.2). The restriction $C_2 \times_{\mathbf{P}^1} X_f$ is the quotient of $C_1 \times \mathbf{P}^1 \times \mathbf{P}^1$ by the involution i: $i(t, x, y) = (\iota t, y, x)$ for $t \in C_1$, $x, y \in \mathbf{P}^1$ where ι is the involution arising from π. $\operatorname{Pic}(C_1 \times \mathbf{P}^1 \times \mathbf{P}^1)$ is isomorphic to $\operatorname{Pic} C_1 \oplus \operatorname{Pic} \mathbf{P}^1 \otimes \operatorname{Pic} \mathbf{P}^1$ hence to $\operatorname{Pic} C_1 \oplus \mathbf{Z}^{\oplus 2}$. By the descent theory,

$$\operatorname{Pic} C_2 \times_{\mathbf{P}^1} X_f \simeq (\operatorname{Pic} C_1 \oplus \mathbf{Z}^{\oplus 2})^i \simeq \mathbf{Z}\,. \tag{11.12.1}$$

Given a line bundle L over X_f. It follows from (11.12.1) that we can find an integer n such that $L(nD_\infty)$ is trivial when restricted over $C_2 \times_{\mathbf{P}^1} X_f$. Therefore $L(nD_\infty) \simeq \mathcal{O}_{X_f}(D)$ for some divisor D whose support lies in the singular fibres of $p\colon X_f \to \mathbf{P}^1$. We have thus proved $\operatorname{Pic} X_f = \mathbf{Z}D_\infty \otimes p^* \operatorname{Pic} \mathbf{P}^1$.

Let now $(\mathrm{SL}_2, Y) \in \mathscr{C}(\mathrm{J}12, \pi) = \mathscr{C}(\mathrm{SL}_2, X_\pi)$. We have an equivariant birational map $f\colon Y \dashrightarrow W_f$ by Lemma (11.1). By blowing up Y, we can eliminate the indeterminacy of f:

$$\begin{array}{ccccc} & & \tilde{Y} & & \\ & \swarrow^{f_1} & & \searrow^{f_2} & \\ Y & & \overset{f}{\dashrightarrow} & & W_f \overset{q}{\longrightarrow} X_f\,. \end{array}$$

After Hironaka, everything is done SL_2-equivariantly. Since there is no SL_2-fixed point by Lemma (11.8), the dimension of any fibre of $f_2 \leq 1$ and hence f_2 is also a blow-down by Theorem (1.7). We may take f_2 minimal so that there is no non-trivial factorization;

$$\begin{array}{ccccc} & & \tilde{Y} & & \\ & \swarrow^{f_1} & \downarrow g & \searrow^{f_2} & \\ & & \tilde{Y}' & & \\ & \swarrow^{f_1'} & & \searrow^{f_2'} & \\ Y & & & & W_f\,, \end{array}$$

where g, f_1' and f_2' are blow-ups. Let us decompose f_2 into a product of blow ups: $\chi_{i,i+1}$: $X_i \to X_{i+1}$ $0 \leq i \leq m-1$, $X_0 = \tilde{Y}$, $X_m = W_f$, $f_2 = \chi_{m-1,m} \circ \chi_{m-2,m-1} \circ \cdots \circ \chi_{01}$ and $\chi_{i,i+1}$ is an SL_2-equivariant blow-up of X_{i+1} at non-singular irreducible curve.

Similarly, let us decompose f_1 as a product of blow-ups with irreducible centers: $\psi_{j,j+1}$: $Y_j \to Y_{j+1}$ $(0 \leq j \leq s-1)$, $Y_0 = \tilde{Y}$, $Y_s = Y$ and $\psi_{j,j+1}$ is a blow-up with a non-singular irreducible center.

LEMMA (11.13). *Let D be a smooth fibre $p^{-1}(t)$, $t \in \mathbf{P}^1$. The proper transform $(q \circ f_2)^{-1}[D]$ is not an exceptional divisor for f_1.*

Proof. D_∞ and $p^*\mathcal{O}_{\mathbf{P}^1}(1)$ generate Pic X_f by Lemma (11.12). The smooth fibre D is a non-singular quadric in $\mathbf{P}^3$ hence isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. Let us set $\ell_1 = \mathbf{P}^1 \times a$ and $\ell_2 = a \times \mathbf{P}^1$. They are curves on D hence on X_f. It follows from the construction $(\ell_1 . D_\infty) = (\ell_2 . D_\infty) = 1$, $(\ell_1 . p^*\mathcal{O}_{\mathbf{P}^1}(1)) = (\ell_2 . p^*\mathcal{O}_{\mathbf{P}^1}(1)) = 0$. Therefore ℓ_1 is numerically equivalent to ℓ_2. Hence $f_2^*(\ell_1 - \ell_2) = f_2^{-1}[\ell_1] - f_2^{-1}[\ell_2]$ is numerically trivial since SL_2- operation on $\mathbf{P}^1 \times \mathbf{P}^1 = D$ is diagonal. Assume now that $(q \circ f)^{-1}[D]$ is exceptional for f_1. Then there exist a blow-down $\tilde{f}$: $\tilde{Y} \to Z$ collapsing $(q \circ f)^{-1}[D]$ to a curve $C \simeq \mathbf{P}^1$ on a non-singular projective variety. Putting $\tilde{D} = (q_2 \circ f)^{-1}[D]$, the restriction $\tilde{f}$: $\mathbf{P}^1 \times \mathbf{P}^1 \simeq \tilde{D} \to C \simeq \mathbf{P}^1$ is a projection, say onto the first factor. Then $\tilde{f}_* f_2^{-1}[\ell_1] = aC$, $a > 0$ and $\tilde{f}_* f_2^{-1}[\ell_2] = 0$. Hence C is numerically trivial. This is absurd since Z is non-singular projective.

LEMMA (11.14). $\mathcal{O}_{D_\infty}(D_\infty) \simeq \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(2, -n)$.

Proof. Notice that in our notation, the restriction $p \mid D_\infty \to \mathbf{P}^1$ is the projection from $\mathbf{P}^1 \times \mathbf{P}^1$ onto the second factor (cf. Table (11.9)). Let $\mathcal{O}_{D_\infty}(D_\infty) \simeq \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(a, b)$. Then it follows from the definition of X_f and that D_∞ cuts on each fibre a divisor coming from the hyperplane section bundle of $\mathbf{P}^3$ hence $a = 2$. Let us calculate b. For this purpose we recall the definition of X_f. We cover $\mathbf{P}^1$ by 2 $\mathbf{A}^1$'s U_0, U_1 in usual way: $\mathbf{P}^1 = U_0 \cup U_1$ and the points $t \in U_0$ and $1/t \in U_1$ are identified. Gluing $V_0 = U_0 \times \mathbf{P}^3$ and $V_1 = U_1 \times \mathbf{P}^3$ together by identifying $(t, (x_0, x_1, x_2, x_3)) \in V_0$ and $(s, (y_0, y_1, y_2, y_3)) \in V_1$ if $t = 1/s$, $x_0 = y_0$, $x_1 = y_1$, $y_2 = x_2$, $y_3 = t^n x_3$, we thus get $\mathbf{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(n))$. Let $f(t)$ be a polynomial of degree $2n$ with only simple roots. X_f is defined locally on V_0 by $x_0^2 + x_1^2 + x_2^2 + f(t)x_3^2 = 0$ and on V_1 by $y_0^2 + y_1^2 + y_2^2 + f(t)(1/2n)y_3^2 = 0$. Therefore D_∞ is defined by $x_3 = 0$ on $V_0 \cap X_f$ any $y_3 = 0$ on $V_1 \cap X_f$. Now we put $\ell = \{(t, (1, 0, 0, 0))\} \in V_0 \mid t \in k\}$

$\cup\{(s,(1,0,0,0))\in V_1 \mid s\in k\}$. ℓ is a section of $p|D_\infty : D_\infty \to \mathbf{P}^1$ and b is the degree of the line bundle $\mathcal{O}(D_\infty)\otimes\mathcal{O}_\ell$ on ℓ. On $V_0\cap\{x_0\neq 0\}$ D_∞ is defined by x_3/x_0 and on $V_1\cap\{y_0\neq 0\}$ by $y_3/y_0=0$. Thus $\mathcal{O}(D_\infty)\otimes\mathcal{O}_\ell$ is defined by the transition function $(y_3/y_0)/(x_3/x_0)=y_3/x_3=t^n$ and hence $\mathcal{O}(D_\infty)\otimes\mathcal{O}_\ell$ is isomorphic to $\mathcal{O}_{\mathbf{P}}(-n)$. Therefore $b=-n$.

LEMMA (11.15). *The exceptional divisor for* ψ_{01} *is exceptional for* $q\circ f_2$.

Proof. Lemma (11.13) shows that the proper transform of a smooth fibre of p is not exceptional for f_1. In view of Lemma (11.11) we have to show that the proper transform of one of the following divisors can not be exceptional for f_1: (1) singular fibres for p, (2) D_∞. In fact let D be a singular fibre of p. $q^*D=q^{-1}[D]+2E$ with $q^{-1}[D]\simeq \mathrm{F}_2$, $E\simeq\mathbf{P}^2$ by Lemma (11.8). Let ℓ be the fibre of the ruled surface $q^{-1}[D]\simeq\mathrm{F}_2$. Since D is a fibre, $0=(\ell.\,q^*(\text{fibre of } p))=(\ell.\,q^*D)=(\ell.\,q^{-1}[D]+2E)=(\ell.\,q^{-1}[D])+2(\ell.\,E)=(\ell.\,q^{-1}[D])+2$ by Lemma (11.8). Therefore $\mathcal{O}(q^{-1}[D])\otimes\mathcal{O}_\ell\simeq\mathcal{O}_{\mathbf{P}^1}(-2)$ and hence the restriction of $\mathcal{O}(q\circ f_2^{-1}[D])$ on a fibre of the ruled surface $(f_2\circ q)^{-1}[D]\simeq\mathrm{F}_2$ is $\mathcal{O}_{\mathbf{P}^1}(-k)$, $k\geq 2$ and can not be exceptional for ψ_{01}. By Lemma (11.14) $\mathcal{O}_{D_\infty}(D_\infty)=(2,-n)$. Therefore putting $D'=(f_2\circ q)^{-1}[D_\infty]$, $\mathcal{O}_{D'}(D')=(2,-b)$ with $b\geq n$ since SL_2 operates on $D_\infty\simeq\mathbf{P}^1\times\mathbf{P}^1$ hence on $(\chi_{im}\circ q)^{-1}[D_\infty]$ through the first factor. Thus D' can not be exceptional for ψ_{01}.

PROPOSITION (11.16). $q\circ f\colon Y \dashrightarrow X_f$ *is a blow-up morphism or* Y *is an* SL_2*-equivariant blow-up of* (SL_2, X_f).

Proof. It follows from Lemma (11.15) the exceptional divisor for ψ_{01} is an exceptional divisor for $f_1\circ q$ and no exceptional divisor is isomorphic to $\mathbf{P}^1\times\mathbf{P}^1$ by Lemma (11.11).

THEOREM (11.17). (SL_2, X_f) *is the unique minimal element of* $\mathscr{C}(\mathrm{J}12;\pi)$ *and any other element of* $\mathscr{C}(\mathrm{J}12;\pi)$ *is an equivariant blow-up of* (SL_2, X_f).

The following example shows that the case (3.3.4), [Mo] occurs in $\mathscr{C}(\mathrm{J}12;\pi)$.

EXAMPLE (11.18). Let $p^{-1}(t)=S\subset\mathbf{P}^3$, $t\in\mathbf{P}^1$ be a singular fibre of $\rho\colon X_f\to\mathbf{P}^1$. Let m be a line in $\mathbf{P}^3$ lying on the singular quadric S. Regarding ℓ as a 1-cycle on X_f, we have $(m.\,D_\infty)=1$, $(m.\,p^*\mathcal{O}_{\mathbf{P}^1}(1))=0$. Denoting by ℓ a fibre of $D_\infty\simeq\mathbf{P}^1\times\mathbf{P}^1\to\mathbf{P}^1$ with $(\ell.\,p^*\mathcal{O}_{\mathbf{P}^1}(1))=1$ as in Lemma (11.14). Let us denote by Y $\mathbf{P}(\mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}(n))$, by $\tilde{p}\colon Y\to\mathbf{P}^1$ the projection and by $\tilde{D}_\infty$ the divisor on Y defined as follows: using the notation of the Proof of Lemma (11.14), $\tilde{D}_\infty$ is defined on V_0 by $x_3 = 0$ and on V_1 by $y_3 = 0$ so that $\tilde{D}_\infty \cap X_f = D_\infty$. We need one more divisor $\tilde{D}$ on Y which is defined on V_0 by $x_0 = 0$ and on V_1 by $y_0 = 0$. By simple local calculations, we get the following.

(11.15.1) $$\tilde{p}_*\mathcal{O}_Y(\tilde{D}_\infty) \simeq \mathcal{O}_{\mathbf{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbf{P}^1}(-n)$$

(11.15.2) $$\tilde{p}_*\mathcal{O}_Y(\tilde{D}) \simeq \mathcal{O}_{\mathbf{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbf{P}^1}(n).$$

(11.15.3) $\mathcal{O}_Y(\tilde{D})$ is generated by its global sections.

Let us content ourselves with checking (11.15.2) and (11.15.3) since (11.15.1) is proved by the same method. Using the notation of the Proof of Lemma (11.14), if we calculate $\tilde{p}_*(\mathcal{O}_Y(D))$ locally for $U_0 \times \mathbf{P}^3 \to U_0$, we get $\tilde{p}_*(\mathcal{O}_Y(\tilde{D})) \mid U_0$ is a free $k[t]$-module

$$\left\{a_0(t) + a_1(t)\frac{x_1}{x_0} + a_2(t)\frac{x_2}{x_0} + a_3(t)\frac{x_3}{x_0} \,\middle|\, a_i(t) \in k[t],\ 1 \le i \le 3\right\}.$$

Similarly on U_1,

$$\tilde{p}_*\mathcal{O}_Y(\tilde{D}) \mid U_1 \simeq \left\{b_0(s) + b_1(s)\frac{y_1}{y_0} + b_2(s)\frac{y_2}{y_0} + b_3(s)\frac{y_3}{y_0} \,\middle|\, b_i(s) \in k[s],\ 0 \le i \le 3\right\}.$$

Since $t = 1/s$, $x_0 = y_0$, $x_1 = y_1$, $y_2 = x_2$, $y_3 = t^n x_3$, the identification of these 2 free modules on the intersection $U_0 \cap U_1$ gives $\mathcal{O}^{\oplus 3} \oplus \mathcal{O}(n)$. Thus the set of global sections

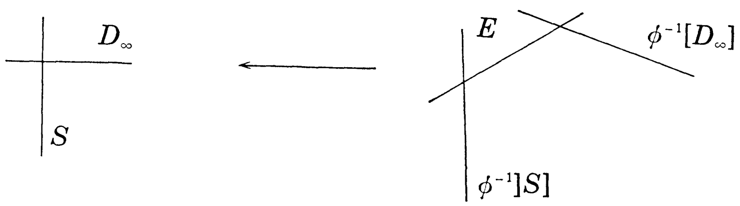
$$\begin{aligned} H^0(Y, \mathcal{O}_Y(\tilde{D})) &\simeq \Big\{a_0 + a_1\frac{x_1}{x_0} + a_2\frac{x_2}{x_0} + a_3(t)\frac{x_3}{x_0} \,\Big|\, a_i \in k \text{ for } 0 \le i \le 2, \\ &\qquad a_3(t) \in k[t] \text{ with } \deg a_3(t) \le n\Big\} \\ &= \Big\{b_0 + b_1\frac{y_1}{y_0} + b_2\frac{y_1}{y_0} + b_3(s)\frac{y_1}{y_0} \,\Big|\, b_i \in k \text{ for } 0 \le i \le 2, \\ &\qquad b_3(s) \in k[s] \text{ with } \deg b_3(s) \le n\Big\}. \end{aligned}$$

This shows (11.15.3).

It follows from (11.15.1), (11.15.2) and (11.15.3) that $\mathcal{O}_{\tilde{Y}}(\tilde{D}) \simeq \mathcal{O}_{\tilde{Y}}(\tilde{D}_\infty) \otimes \tilde{p}^*\mathcal{O}_{\mathbf{P}^1}(n)$ and $\mathcal{O}_{\tilde{Y}}(\tilde{D}_\infty) \otimes \tilde{p}^*\mathcal{O}_{\mathbf{P}^1}(n)$ is generated by its global sections. In particular the restriction $\mathcal{O}_{\tilde{Y}}(\tilde{D}_\infty) \otimes \tilde{p}^*\mathcal{O}_{\mathbf{P}^1}(n) \otimes \mathcal{O}_{X_f} \simeq \mathcal{O}_{\tilde{X}_f}(D_\infty) \otimes \tilde{p}^*\mathcal{O}_{\mathbf{P}^1}(n)$ onto X_f is generated by its global sections. Now we can show that $\mathbf{R}_+ m +$

$\mathbf{R}_+\ell$ coincides with the cone $NE(X_f)$ spanned by the effective 1-cycles on X_f (see the section 1). In fact by Lemma (11.14) $\mathbf{R}m + \mathbf{R}\ell = N(X_f)$ since we know $(m.p^*\mathcal{O}_{\mathbf{P}^1}(1)) = 0$, $(\ell.p^*\mathcal{O}_{\mathbf{P}^1}(1)) = 1$. It is sufficient to show that for $a, b \in \mathbf{R}$ $am + b\ell$ is numerically equivalent to an effective 1-cycle on X_f if and only if $a, b \geq 0$. As the if part is evident, we have to show "only if" part. Assume that $am + b\ell$ is numerically equivalent to an effective 1-cycle. Since $\mathcal{O}(D_\infty) \otimes p^*\mathcal{O}_{\mathbf{P}^1}(n)$ is generated by its global sections, we have $0 \leq (am + b\ell.\mathcal{O}(D_\infty) \otimes p^*\mathcal{O}_{\mathbf{P}^1}(n)) = a(m.D_\infty) + b(\ell.D_\infty) + b(\ell.p^*\mathcal{O}_{\mathbf{P}^1}(n)) = a - nb + nb = a$ by Lemma (11.14). As $p^*\mathcal{O}_{\mathbf{P}^1}(n)$ is also generated by its global sections, $0 \leq (am + b\ell.p^*\mathcal{O}_{\mathbf{P}^1}(1)) = b$ as wanted.

Since by the adjunction formula $\mathcal{O}(K_{X_f}) \simeq \mathcal{O}_Y(K_Y + X_f) \otimes \mathcal{O}_{X_f} \simeq \mathcal{O}_{X_f}(-2D_\infty) \otimes p^*M$ for some line bundle M over $\mathbf{P}^1$, $(\ell.K_{X_f}) = -2$. Hence $\mathbf{R}_+\ell$ is an extremal ray. Let us now blow up X_f at $C = S \cap D_\infty$: ϕ: $Z \to X_f$.



We denote by E the exceptional divisor. Then $K_Z = \phi^*K_{X_f} + E$ and

$$\begin{aligned}(\phi^{-1}[\ell].K_Z) &= (\phi^{-1}[\ell].\phi^*K_{X_f} + E) = (\phi^{-1}[\ell].\phi^*K_{X_f}) + (\phi^{-1}[\ell].E) \\ &= (\phi_*\phi^{-1}[\ell].K_{X_f}) + 1, \quad \text{by the projection formula,} \\ &= (\ell.K_{X_f}) + 1 = -2 + 1 = -1.\end{aligned}$$

Hence $\mathbf{R}_+\phi^{-1}[\ell]$ is an extremal ray. Moreover since

$$\begin{aligned}(\phi^{-1}[\ell].\phi^{-1}[S]) + 1 &= (\phi^{-1}[\ell], \phi^{-1}[S] + E) = (\phi^{-1}[\ell].\phi^*S) \\ &= (\phi_*\phi^{-1}[\ell].S) = (\ell.S) = 0, \quad (\phi^{-1}[\ell].\phi^{-1}[S]) = -1.\end{aligned}$$

Since S is the singular quadric in $\mathbf{P}^3$, this shows that on $(\mathrm{SL}_2, W) \in \mathscr{C}(\mathrm{J}12; \pi)$, $\mathbf{R}_+\phi^{-1}[\ell]$ is an example of an extremal ray giving rise to case (3.3.4).

BIBLIOGRAPHY

[A] Atiyah, M. F., Complex fibre bundles and ruled surfaces, Proc. London Math. Soc. (3), **5** (1955), 407–434.

[Be] Beauville, A., Variété de Prym et Jacobiennes intermédiaires, Ann. Sci. Ec. Norm. Sup., 4e série, t. **10** (1977), 304–392.

[Bo] Borel, A., Linear algebraic groups, Benjamin, New York, 1969.

[Da] Danilov, V. I., Decomposition of certain birational morphisms, Math. USSR Izv., **16** (1981), 419–429.

[De] Demazure, M., Sous-groupes algébriques de rang maximum du groupe de Cremona, Ann. Sci. Ec. Norm. Sup., 4e série, t. **3** (1970), 507–588.

[EF] Enriques, F. e Fano, G., Sui gruppi di transformazioni cremoniane dello spazio, Annali di Matematica pura ed applicata, s. 2^a to. **15** (1897), 59–98.

[F] Fano, G., I gruppi di Jonquières generalizzati, Atti della R. Acc. di Torino, **33** (1898), 221–271.

[Hi] Hironaka, H., Resolution of singularities of an algebraic variety over a field of characteristic zero, Ann. of Math., **79** (1964), 109–326.

[Hu] Humphreys, J. E., Introduction to Lie algebra and representation theory, GTM 9, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[I1] Iskovskih, V. A., Fano threefolds I, Math. USSR Izv., **11** (1977), 485–527.

[I2] ——, Fano threefolds II, Math. USSR Izv., **12** (1978), 469–506.

[Man] Manin, Yu. I., Cubic forms, North-Holland, Amsterdam, 1974.

[Mar1] Maruyama, M., On classification of ruled surfaces, Lect. in Math., Dept. Math. Kyoto Univ. 3, Kinokuniya, Tokyo 1970.

[Mar2] ——, On a family of algebraic vector bundles, Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki, Kinokuniya, Tokyo, 1973, 95–146.

[Mi] Miyanishi, M., Algebraic methods in the theory of algebraic threefolds, Advanced Studies in Pure Math., 1, 1983, Alg. Var. and An. Var., 69–99.

[Mo] Mori, S., Threefolds, whose canonical bundles are not numerically effective, Ann. of Math., **116** (1982), 133–176.

[MU] Mukai, S. and Umemura, H., Minimal rational threefolds, Algebraic geometry, Lecture Notes in Math., **1016**, Springer-Verlag, Berlin-Heidelberg-New York, 1983, 490–518.

[Re] Reid, M., Canonical 3-folds, Journée de géométrie algébrique d'Angers, ed. A. Beauville, Sijthoff and Noordhoff, Alphen 1980, 273–310.

[Ro] Rosenlicht, M., Some basic theorems on algebraic groups, Amer. J. Math., **78** (1956), 401–443.

[Sc] Schwarzenberger, R. L. E., Vector bundles on the projective plane, Proc. London Math. Soc., (3) **11** (1961), 623–640.

[Su] Sumihiro, H., Equivariant completion, J. Math. Kyoto Univ., **14-1** (1974), 1–28.

[U1] Umemura, H., Sur les sous-groupes algébriques primitifs de groupe de Cremona à trois variables, Nagoya Math. J., **79** (1980), 47–67.

[U2] ——, Maximal algebraic subgroups of the Cremona group of three variables, Nagoya Math. J., **87** (1982), 59–78.

[U3] ——, On the maximal connected algebraic subgroups of the Cremona group I, Nagoya Math. J., **88** (1982), 213–246.

[U4] ——, On the maximal connected algebraic subgroups of the Cremona group II, Algebraic groups and related topics, Advanced Studies in Pure Math., **6**, Kinokuniya-North-Holland, 1985, 349–436.

[U5] ——, Algebro-geometric problems arising from Painlevé's works, Algebraic and

Topological theories—to the memory of Dr. T. Miyata, Kinokuniya, Tokyo, 1985, 467–495.

[Z] Zariski, O., Introduction to the problem of minimal models in the theory of algebraic surfaces, Math. Soc. Japan, Tokyo, 1958.

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