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SPHERICAL SUBMANIFOLDS WHICH ARE OF 2-TYPE VIA THE SECOND STANDARD IMMERSION OF THE SPHERE

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§1. Introduction

Let $S^m(r)$ be an m-sphere of constant sectional curvature $1/r^2$ and M an n-dimensional compact minimal submanifold of $S^m(r)$. If $S^m(r)$ is imbedded in E^{m+1} by its first standard imbedding, then, by a well-known result of Takahashi [11], the Euclidean coordinate functions restricted to M are eigenfunctions of Δ on M with the same eigenvalue n/r^2. Moreover, the center of mass of M in E^{m+1} coincides with the center of the hypersphere $S^m(r)$ in E^{m+1}. Thus, M is mass-symmetric in $S^m(r) \subset E^{m+1}$. Consequently, we see that if one wants to study the spectral geometry of a submanifold of $S^m(r)$, it is natural to immerse $S^m(r)$ by its k-th standard immersion, in particular, by its second standard immersion.

In [9], A. Ros has used this idea to study compact minimal submanifolds of $S^m(r)$ via the second standard immersion. In [9], he obtained a formal characterization of a compact minimal submanifold M, fully in S^m, such that the Euclidean coordinate functions restricted to M via the second standard immersion f of S^m are described by means of two different eigenvalues of Δ, i.e., M is of 2-type via f. He showed that such submanifolds are Einstein and mass-symmetric via f. However, he did not obtain any classification result for such submanifolds.

In this paper, we study compact submanifolds of a sphere which are mass-symmetric and of 2-type via the second standard immersion of the sphere. In Section 3, we obtain a generalization of Ros' characterization (Lemma 1). Some primary classifications are obtained in this section (Theorems 1 and 2). In Section 4, hypersurfaces of a sphere which are mass-symmetric and of 2-type via f are completely classified (Theorem 3).

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In Section 5, submanifolds of S^m with "maximal possible" codimension are studied. In the last section, results in previous sections are applied to obtain a classification theorem of compact surfaces of S^m which have the desired properties via f.

§ 2. Basics

Let $x: M \to E^m$ be an isometric immersion of a compact, connected, n-dimensional, Riemannian manifold M into a Euclidean m-space. Denote by $\mathrm{Spec}(M) = \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots \uparrow \infty\}$ the spectrum of Δ acting on differentiable functions in $C^\infty(M)$. If we extend the Laplace-Beltrami operator Δ to E^m-valued functions on M in a natural fashion, then, we have the following spectral decomposition of x (in L^2-sense) (cf. [1, 3, 5, 6, 9]):

$$x = x_0 + \sum_{t=1}^{\infty} x_t, \ \Delta x_t = \lambda_t x_t, \ x_t : M \longrightarrow E^m, \tag{2.1}$$

where x_0 is the center of mass of M in E^m. The submanifold M is said to be of *finite type* if the spectral decomposition of x consists of only finitely many nonzero terms. More precisely, M is said to be of *k-type* if there are exactly k nonzero x_t's ($t \geqq 1$) in the decomposition of x ([5, 6]).

From the Takahashi Theorem [11] we know that M is of 1-type if and only if M is a minimal submanifold of a hypersphere $S^{m-1}(r)$ of E^m. In this case, M is mass-symmetric in $S^{m-1}(r) \subset E^m$, i.e., the center of mass of M in E^m coincides with the center of $S^{m-1}(r)$ in E^m (cf. [6]).

Let $x: M \to E^m$ be a 2-type submanifold with mean curvature vector H. Then we have

$$x = x_0 + x_p + x_q, \quad \Delta x_p = \lambda_p x_p, \quad \Delta x_q = \lambda_q x_q \tag{2.2}$$

for some integers p, q ($q > p \geqq 1$). Since $\Delta x = -nH$, (2.2) implies

$$\Delta H = bH + e(x - x_0), \tag{2.3}$$

where $b = \lambda_p + \lambda_q$ and $e = \lambda_p \lambda_q / n$.

On E^m we consider an inner product $\langle\ ,\ \rangle$ given by $\langle u, v\rangle = u \cdot v^t$ for any $u, v \in E^m$, where each vector in E^m is regarded as a row matrix and v^t is the transpose of v. Let $r > 0$. Then the sphere $S^{m-1}(r) = \{x \in E^m | \langle x, x\rangle = r^2\}$ with the induced metric has constant sectional curvature $1/r^2$. Let $SM(m) = \{P \in gl(m; \boldsymbol{R}) | P^t = P\}$ be the space of symmetric m by m matrices over $\boldsymbol{R}$ endowed with the metric $g(P, Q) = (1/2r^2)\mathrm{tr}(PQ)$ for

$P, Q \in SM(m)$. Consider the mapping $f: S^m(r) \to SM(m+1)$ defined by $f(u) = u^t \cdot u$. Then f is an isometric immersion which is in fact the second standard immersion of $S^m(r)$. The image $f(S^m(r))$ is a real projective space which lies fully in an $(m + m(m+1)/2)$-dimensional linear space of $SM(m+1)$. We call $f(S^m(r))$ a *Veronese submanifold.*

For each point $u \in S^m(r)$, the normal space of $S^m(r)$ in $SM(m+1)$ at u (or more precisely at $f(u)$) is given by

$$T_u^\perp(S^m(r)) = \{P \in SM(m+1) \,|\, uP = \mu u \text{ for some } \mu \in \boldsymbol{R}\}. \tag{2.4}$$

In particular, we have $f(u) \in T_u^\perp(S^m(r))$.

If $\bar{\sigma}$ is the second fundamental form of f, we have

$$\bar{\sigma}(X, Y) = X^t \cdot Y + Y^t \cdot X - (2/r^2)\langle X, Y\rangle f(u) \tag{2.5}$$

for X, Y in $T_u(S^m(r))$. It is known that $\bar{\sigma}$ is parallel and it satisfies

$$g(\bar{\sigma}(X, Y), \bar{\sigma}(V, W)) = (1/r^2)\{2\langle X, Y\rangle\langle V, W\rangle + \langle X, V\rangle\langle Y, W\rangle + \langle X, W\rangle\langle Y, V\rangle\}, \tag{2.6}$$

$$g(\bar{\sigma}(X, Y), f(u)) = -\langle X, Y\rangle, \quad g(\bar{\sigma}(X, Y), I) = 0, \tag{2.7}$$

$$\bar{A}_{\bar{\sigma}(X,Y)} V = (1/r^2)\{2\langle X, Y\rangle V + \langle X, V\rangle Y + \langle Y, V\rangle X\}, \tag{2.8}$$

where $\bar{A}$ is the Weingarten map of f, $X, Y, V, W \in T_u(S^m(r))$, and I the identity matrix.

It is known that $S^m(r)$ is immersed by the second standard immersion f as a minimal submanifold of a hypersphere of $SM(m+1)$ centered at $r^2 I/(m+1)$ and with radius $(r^2 m/2(m+1))^{1/2}$. For more details, see [6, 9, 10].

In the following, we simply denote $S^m(1)$ by S^m.

§ 3. Submanifolds of S^m which are of 2-type via f

Let $\psi: M \to S^m$ be an isometric immersion of M into S^m. We denote by σ', H' and A the second fundamental form, the mean curvature vector and the Weingarten map of ψ, respectively. Denote by ∇ and $\bar{\nabla}$ the Levi-Civita connections on M and S^m, respectively, and by D the normal connection of ψ.

We consider the isometric immersion $x: M \to SM(m+1)$ defined by

$$x = f \circ \psi : M \xrightarrow{\psi} S^m \xrightarrow{f} SM(m+1).$$

Then the mean curvature vector H of x satisfies

$$H = H' + \frac{1}{n}\sum_{i=1}^{n}\bar{\sigma}(E_i, E_i)\,, \tag{3.1}$$

where H' is identified with the image f_*H' of H' under f_* and $E_1, \cdots, E_n$ is an orthonormal frame tangent to M.

Let u be an arbitrary point in M. We may assume that $\nabla_{E_j}E_i = 0$ at u. We compute $\Delta H'$ at u.

$$\begin{aligned}(\Delta H')(u) &= -\sum_{i=1}^{n} E_iE_iH' \\ &= \sum_{i=1}^{n}\{\bar{\nabla}_{E_i}A_{H'}E_i + \bar{\sigma}(E_i, A_{H'}E_i) - \bar{\nabla}_{E_i}D_{E_i}H' \\ &\quad - \bar{\sigma}(E_i, D_{E_i}H') + \bar{A}_{\bar{\sigma}(E_i, H')}E_i - \bar{D}_{E_i}\bar{\sigma}(E_i, H')\}\,,\end{aligned}$$

where $\bar{D}$ denotes the normal connection of f. By applying (2.8) and the fact that $\bar{\sigma}$ is parallel, we find

$$\begin{aligned}(\Delta H')(u) = \Delta^D H' &+ \mathrm{tr}(\bar{\nabla}A_{H'}) + \sum\sigma'(E_i, A_{H'}E_i) + 2\sum\bar{\sigma}(E_i, A_{H'}E_i) \\ &- 2\sum\bar{\sigma}(E_i, D_{E_i}H') + nH' - n\bar{\sigma}(H', H')\end{aligned} \tag{3.2}$$

where Δ^D is the Laplacian with respect to the normal connection D and

$$\mathrm{tr}(\bar{\nabla}A_{H'}) = \sum(\nabla_{E_i}A_{H'})E_i + \sum A_{D_{E_i}H'}E_i\,. \tag{3.3}$$

For each point u in M, we choose an orthonormal basis $\{\xi_{n+1}, \cdots, \xi_m\}$ of the normal space of M is S^m at u such that ξ_{n+1} is parallel to H' at u (if $H' = 0$ at u, any orthonormal frame satisfies this condition). Simply denote A_{ξ_r} $(r = n+1, \cdots, m)$ by A_r. We have

$$\sum_{i=1}^{n}\sigma'(E_i, A_{H'}E_i) = |A_{n+1}|^2H' + \mathfrak{A}'(H') \tag{3.4}$$

where $\mathfrak{A}'(H') = \sum_{r=n+2}^{m}\mathrm{tr}(A_{H'}A_r)\xi_r$ is the so-called allied mean curvature vector of M in S^m. It is clear that if $H' = 0$ at u, then $\mathfrak{A}'(H') = |A_{n+1}|^2H' = 0$ at u. It is easy to see that $\mathfrak{A}'(H')$ and both sides of (3.4) are independent of the choice of $\xi_{n+1}, \cdots, \xi_m$ such that ξ_{n+1} is parallel to H'. By combining (3.2) and (3.4) we obtain

$$\begin{aligned}(\Delta H')(u) = \Delta^D H' &+ \mathrm{tr}(\bar{\nabla}A_{H'}) + (|A_{n+1}|^2 + n)H' + \mathfrak{A}'(H') \\ &+ 2\sum\bar{\sigma}(E_i, A_{H'}E_i) - 2\sum\bar{\sigma}(E_i, D_{E_i}H') - n\bar{\sigma}(H', H')\,.\end{aligned} \tag{3.5}$$

On the other hand, from (2.6), (2.7) and parallelism of $\bar{\sigma}$, we have

$$\begin{aligned}(3.6)\qquad \frac{1}{n}\Delta\Big(\sum_{i=1}^{n}\bar{\sigma}(E_i, E_i)\Big)(u) &= 2(n+2)H' + \frac{2}{n}(n+1)\sum_j \bar{\sigma}(E_j, E_j)\\ &+ \frac{2}{n}\sum_{i,j}\bar{\sigma}(A_{\sigma'(E_i,E_j)}E_j, E_i) - \frac{2}{n}\sum_{i,j}\bar{\sigma}(\sigma'(E_i, E_j), \sigma'(E_i, E_j))\\ &- \frac{2}{n}\sum_{i,j}\bar{\sigma}((\tilde{\nabla}\sigma')(E_i, E_j, E_j), E_i)\,,\end{aligned}$$

where $\tilde{\nabla}\sigma'$ denotes the covariant derivative of σ'. From Codazzi's equation, we have

$$(3.7)\qquad \sum(\tilde{\nabla}\sigma')(E_i, E_j, E_j) = nD_{E_i}H'\,.$$

Thus, we obtain, from (3.1), (3.5), (3.6) and (3.7),

$$\begin{aligned}(8.8)\qquad (\Delta H)(u) &= \Delta^D H' + \mathrm{tr}(\bar{\nabla}A_{H'}) + \mathfrak{A}'(H') + (\|A_{n+1}\|^2 + 3n + 4)H'\\ &+ \frac{2(n+1)}{n}\sum_j \bar{\sigma}(E_j, E_j) + 2\sum_i \bar{\sigma}(E_i, A_{H'}E_i)\\ &+ \frac{2}{n}\sum_{i,j}\bar{\sigma}(A_{\sigma'(E_i,E_j)}E_i, E_j) - 4\sum_i \bar{\sigma}(E_i, D_{E_i}H')\\ &- n\bar{\sigma}(H', H') - \frac{2}{n}\sum_{i,j}\bar{\sigma}(\sigma'(E_i, E_j), \sigma'(E_i, E_j))\,.\end{aligned}$$

As we mentioned in Section 2, $f: S^m \to SM(m+1)$ is of 1-type and S^m is isometrically immersed in a hypersphere, say W, of $SM(m+1)$ centered at $I/(m+1)$ as a minimal submanifold.

The general assumptions we made in this paper are

(1) $x = f\circ\psi: M \to S^m \to SM(m+1)$ *is of 2-type* and

(2) $x = f\circ\psi$ *is mass-symmetric*, i.e., the center of mass of M in $SM(m+1)$ is the center of the hypersphere W in $SM(m+1)$, which means that $x_0 = I/(m+1)$; and

(3) *the immersion* $\psi: M \to S^m$ *is full*, i.e., $\psi(M)$ is not contained in any great hypersphere of S^m.

Under these assumptions we have

$$(3.9)\qquad \Delta H = bH' + \frac{b}{n}\sum_{i=1}^{n}\bar{\sigma}(E_i, E_i) + e\Big(x - \frac{1}{m+1}I\Big)\,,$$

where $b = \lambda_p + \lambda_q$ and $e = \lambda_p\lambda_q/n$. We put

$$(3.10)\qquad L = \sum \bar{\sigma}(E_i, D_{E_i}H')\,.$$

Then, by using (2.6) and (3.8), we obtain

$$g(\Delta H, L) = -4g(L, L) = -4\sum_{i,j}\langle E_i, E_j\rangle\langle D_{E_i}H', D_{E_j}H'\rangle = -4|DH'|^2. \tag{3.11}$$

On the other hand, (2.6), (2.7) and (3.9) imply

$$g(\Delta H, L) = eg(x, L) = -e\sum\langle E_i, D_{E_i}H'\rangle = 0. \tag{3.12}$$

Therefore, from (3.11) and (3.12), we see that $\psi : M \to S^m$ has parallel mean curvature vector, i.e., $DH' = 0$. Thus, we have $\Delta^D H' = \mathrm{tr}(\bar{\nabla} A_{H'}) = 0$.

For the immersion $x : M \to S^m$ we may regard the Weingarten map A as a linear map from the normal bundle $T^\perp M$ into the space of self-adjoint endomorphisms $S_n(TM)$ of the tangent bundle TM:

$$A : T^\perp M \to S_n(TM)$$

which carries $\xi \in T^\perp M$ onto A_ξ. On $S_n(TM)$ there is a canonical inner product defined by $\langle\langle B, C\rangle\rangle = (1/n)\,\mathrm{tr}(BC)$ for $B, C \in S_n(TM)$. We say that the Weingarten map A is *homothetic* if there exists a positive number ρ such that $\langle\langle A_\xi, A_\eta\rangle\rangle = \rho\langle\xi, \eta\rangle$ for $\xi, \eta \in T^\perp M$. Submanifolds with conformal or homothetic Weingarten map were investigated in [2].

LEMMA 1. *Let $\psi : M \to S^m$ be a full isometric immersion. If $x = f \circ \psi$ is mass-symmetric and of 2-type, then*

(1) *the mean curvature vector of ψ is parallel, i.e., $DH' = 0$,*

(2) *$\mathfrak{A}'(H') = 0$, i.e., $\sum \sigma'(E_i, A_{H'}E_i)$ is parallel to H',*

(3) *$\|A_{H'}\|$ is constant,*

(4) *the Weingarten map A of ψ is homothetic on $\langle H'\rangle^\perp$, where $\langle H'\rangle^\perp$ is the orthogonal complement of $\langle H'\rangle = \mathrm{Span}\,\{H'\}$, and*

(5) *the Ricci tensor S of M satisfies*

$$S(X, Y) = 2n\langle A_{H'}X, Y\rangle + k\langle X, Y\rangle$$

for some constant k. (k depends only on λ_p and λ_q).

Proof. Since $x = f \circ \psi : M \to SM(m+1)$ is assumed to be mass-symmetric and of 2-type, H' is parallel in the normal bundle of M in S^m. In particular, the length of H' is constant. Since $\Delta^D H' = \mathrm{tr}(\bar{\nabla} A_{H'}) = 0$, (3.8) and (3.9) imply $\mathfrak{A}'(H') = 0$ and $\|A_{n+1}\|^2 + 3n + 4 = b$. This proves (2) and (3).

From (2.6) and (3.8) we have

$$\begin{aligned} g(\Delta H, \bar{\sigma}(\xi, \eta)) &= [4(n+1) + 2n\|H'\|^2]\langle\xi, \eta\rangle \\ &\quad - 2n\langle H', \xi\rangle\langle H', \eta\rangle - \frac{4}{n}\mathrm{tr}(A_\xi A_\eta) \end{aligned} \tag{3.13}$$

for any normal vector fields ξ, η of M in S^m.

On the other hand, (2.7) and (3.9) give

$$g(\Delta H, \bar{\sigma}(\xi, \eta)) = (2b - e)\langle \xi, \eta \rangle . \tag{3.14}$$

From (3.13) and (3.14) we find

$$\begin{aligned} \langle\langle A_\xi, A_\eta \rangle\rangle = &\frac{1}{4}[4(n+1) + 2n\|H'\|^2 + e - 2b]\langle \xi, \eta \rangle \\ &- \frac{n}{2}\langle H', \xi \rangle \langle H', \eta \rangle \end{aligned} \tag{3.15}$$

which proves the homotheticy of A on $\langle H' \rangle^\perp$.

From (2.6) and (3.8) we find

$$\begin{aligned} g(\Delta H, \bar{\sigma}(E_k, E_l)) = &\left[4(n+1) + \frac{4(n+1)}{n} + 2n\|H'\|^2\right]\langle E_k, E_l \rangle \\ &+ 4\langle \sigma'(E_k, E_l), H' \rangle + \frac{4}{n}\sum_i \langle \sigma'(E_k, E_i), \sigma'(E_l, E_i) \rangle . \end{aligned} \tag{3.16}$$

From (2.6), (2.7) and (3.9) we get

$$g(\Delta H, \bar{\sigma}(E_k, E_l)) = \left(2b + \frac{2b}{n} - e\right)\langle E_k, E_l \rangle . \tag{3.17}$$

Since the Ricci tensor S of M satisfies

$$\begin{aligned} S(E_k, E_l) = (n-1)\langle E_k, E_l \rangle - \sum_i \langle \sigma'(E_k, E_i), \sigma'(E_l, E_i) \rangle \\ + n\langle \sigma'(E_k, E_l), H' \rangle , \end{aligned} \tag{3.18}$$

(3.16), (3.17) and (3.18) imply

$$\begin{aligned} S(E_i, E_j) = &\, 2n\langle A_{H'}E_i, E_j \rangle \\ &+ \left[n(n+3) + \frac{n^2}{2}\|H'\|^2 + \frac{ne}{4} - \frac{b(n+1)}{2}\right]\langle E_i, E_j \rangle . \end{aligned}$$

This proves (5). (Q.E.D.)

Remark 1. (i) It is not difficult to verify that if a submanifold M of S^m satisfies conditions (1)–(5) of Lemma 1, then $x = f \circ \psi$ is mass-symmetric and it is of 1 or 2-type.

(ii) Lemma 1 was obtained in [9] in the special case when M is a minimal submanifold of S^m. So Lemma 1 is a generalization of Ros' characterization theorem.

By applying Lemma 1, we have the following,

THEOREM 1. *Let* $\psi: M \to S^m$ *be an isometric immersion of a compact Riemannian manifold such that the immersion is full. If* $x = f \circ \psi$ *is mass-symmetric and of* 2*-type in* $SM(m+1)$, *then either*

(a) M *is of* 1*-type in* E^{m+1} *and so* M *is minimal in a hypersphere of* E^{m+1} *or*

(b) M *is of* 2*-type in* E^{m+1} *and mass-symmetric in* $S^m \subset E^{m+1}$.

Proof. Under the hypothesis, Lemma 1 implies $DH' = 0$ $\mathfrak{A}'(H') = 0$ and $\|A_{H'}\|$ being constant. Therefore, by applying Theorem 4.4 of [6, p. 278], we conclude that either M is of 1-type in E^{m+1} or M is mass-symmetric and of 2-type in $S^m \subset E^{m+1}$. (Q.E.D.)

If M is Einsteinian, then case (b) of Theorem 1 cannot occur. In fact, we have

THEOREM 2. *Let* $\psi: M \to S^m$ *be an isometric immersion of a compact Einstein manifold* M *into* S^m *such that the immersion is full. If* $x = f \circ \psi$ *is mass-symmetric and of* 2*-type, then either* M *is minimal in* S^m *or* M *is minimal in a small hypersphere of* S^m. *In both cases,* M *is of* 1*-type in* E^{m+1}.

Proof. Under the hypothesis, statement (5) of Lemma 1 implies that M is pseudo-umbilical in S^m. Moreover, from statement (1) of Lemma 1, M has parallel mean curvature vector H' in S^m. Thus, by applying Proposition 4.2 of [6, p. 133], we obtain the theorem. (Q.E.D.)

We give the following lemma for later use.

LEMMA 2. *Let* $M = S^n(r)$ *be a small hypersphere of radius* r $(r<1)$ *of* S^{n+1}. *Then* M *is of* 2*-type in* $SM(n+2)$ *via* $f: S^{n+1} \to SM(n+2)$. *Moreover,* M *is mass-symmetric and of* 2*-type in* $SM(n+2)$ *if and only if* $r^2 = (n+1)/(n+2)$.

Proof. Let V_i be the eigenspace of Δ on M with eigenvalue λ_i. Then we have $V_1 V_1 \subset V_0 + V_1 + V_2$. Without loss of generality we may assume that M is given by the intersection of $S^{n+1} \subset E^{n+2}$ and the hyperplane P of E^{n+1} whose last coordinate is given by $\sqrt{1-r^2}$. Thus, $M = \{(y, \sqrt{1-r^2}) \in E^{n+2} | y \cdot y^t = r^2\}$. Since the immersion $f: S^{n+1} \to SM(n+2)$ is defined by $f(u) = u^t \cdot u$ for $u \in S^{n+1}$, it is clear that M is of 2-type in $SM(n+2)$ via f. Since M is immersed in $SM(n+2)$ by $(y, \sqrt{1-r^2})^t \cdot (y, \sqrt{1-r^2})$, we see that the center of mass x_0 of M in $SM(n+2)$ is proportional to the identity matrix I of $SM(n+2)$ if and only if $r^2=(n+1)/(n+2)$. Moreover, in this case, we have $x_0=(1/(n+2))I$ which is exactly the center of the hypersphere W which S^{n+1} lies via f. (Q.E.D.)

In the following three sections, we shall apply previous results to obtain some classifications results.

§ 4. Hypersurface of S^m which are of 2-type via f

The main purpose of this section is to classify hypersurfaces of S^m which are mass-symmetric and of 2-type via f.

Let $M=S^p(r_1)\times S^{n-p}(r_2)$ be the Riemannian product of two spheres with radii r_1 and r_2, respectively. Let M be a hypersurface of $S^{n+1}=S^{n+1}(1)$. Then we have $r_1^2+r_2^2=1$. We recall that

$$\mathrm{Spec}(S^p(r_1))=\{\bar\lambda_k=k(p+k-1)/r_1^2|k\geq 0\}\quad\text{and}$$
$$\mathrm{Spec}(S^{n-p}(r_2))=\{\lambda'_k=k(n-p+k-1)/r_2^2|k\geq 0\}.$$

Moreover, the coordinate functions of x_i of $S^p(r_1)$ in E^{p+1} are eigenfunctions with eigenvalue $\bar\lambda_1$ and the coordinate functions y_t of $S^{n-p}(r_2)$ in E^{n-p+1} are eigenfunctions with eigenvalue λ'_1. Therefore, the coordinate functions of $M=S^p(r_1)\times S^{n-p}(r_2)$ in $SM(n+2)$ via f are given by the following matrix

$$\left[\begin{array}{c:c} x_ix_j & x_iy_t \\ \hdashline x_iy_t & y_ty_s \end{array}\right]_{\substack{1\leq i,j\leq p+1,\\ 1\leq s,t\leq n+1-p}}. \tag{4.1}$$

So the coordinate functions of M in $SM(n+2)$ are eigenfunctions on M associated with at most three eigenvalues of Δ on M given by $\bar\lambda_2$, λ'_2 and $\lambda'_1+\bar\lambda_1$.

LEMMA 3. *$M=S^p(r_1)\times S^{n-p}(r_2)$ $(r_1^2+r_2^2=1)$ is of 2-type in $SM(n+2)$ via f if and only if either*

(1) *$r_1^2=(p+1)/(n+2)$ and $r_2^2=(n-p+1)/(n+2)$ or*

(2) *$r_1^2=(p+2)/(n+2)$ and $r_2^2=(n-p)/(n+2)$, or*

(3) *$r_1^2=p/(n+2)$ and $r_2^2=(n-p+2)/(n+2)$.*

Proof. M is of 2-type via f if and only if two of $\bar\lambda_2$, λ'_2 and $\lambda'_1+\bar\lambda_1$ are equal. This implies the Lemma.

LEMMA 4. *$M=S^p(r_1)\times S^{n-p}(r_2)$ $(r_1^2+r_2^2=1)$ is mass-symmetric in*

$SM(n+2)$ via f if and only if $r_1^2=(p+1)/(n+2)$ and $r_2^2=(n-p+1)/(n+2)$.

Proof. First we regard $M=S^p(r_1)\times S^{n-p}(r_2)$ as a submanifold in $E^{n+2}=E^{p+1}\oplus E^{n-p+1}$ in a natural way. It is easy to see that the center of mass of M in $SM(n+2)$ via f is given by

$$\begin{bmatrix} \dfrac{r_1^2}{p+1}I_{p+1} & 0 \\ 0 & \dfrac{r_2^2}{n-p+1}I_{n-p+1} \end{bmatrix}.$$

Thus, M is mass-symmetric if and only if $(n-p+1)r_1^2=(p+1)r_2^2$. Because $r_1^2+r_2^2=1$, we obtain the Lemma.

Now, we give the following main result of this section.

THEOREM 3. *Let $\psi: M\to S^{n+1}$ be an isometric immersion of a compact n-dimensional Riemannian manifold M into S^{n+1}. Then $x=f\circ\psi$ is mass-symmetric and of 2-type if and only if either*

(1) *M is a small hypersphere of S^{n+1} with radius $r=[(n+1)/(n+2)]^{1/2}$, or*

(2) *$M=S^p(r_1)\times S^{n-p}(r_2)$ with $r_1^2=(p+1)/(n+2)$ and $r_2^2=(n-p+1)/(n+2)$.*

The immersions of M into S^{n+1} in (1) and (2) are given in natural way.

Proof. If M is mass-symmetric and of 2-type in $SM(n+2)$ via f, then Lemma 1 implies that $DH'=0$, $\|A_{H'}\|$ is constant and the Ricci tensor S of M satisfies

$$S(X, Y)=2n\langle A_{H'}X, Y\rangle+k\langle X, Y\rangle, \tag{4.2}$$

where k is a constant. On the other hand, from Gauss' equation, we have

$$S(X, Y)=(n-1)\langle X, Y\rangle+n\alpha'\langle AX, Y\rangle-\langle A^2X, Y\rangle \tag{4.3}$$

where A is the Weingarten map of M in S^{n+1} Combining (4.2) and (4.3) we find $A^2+n\alpha'A+(k+1-n)I=0$. This shows that M has at most two distinct principal curvatures and the principal curvatures are constant. If M has only one principal curvature, M is a small hypersurface of S^{n+1}. In this case, Theorem 3 follows from Lemma 2. If M has two distinct principal curvatures, then M is the product of two spheres. In this case, Theorem 3 follows from Lemma 3 and Lemma 4. (Q.E.D.)

Remark. Let W be the hypersphere of $SM(n+2)$ in which S^{n+1} is immersed as a minimal submanifold via f. Examples (2) and (3) of Lemma 3 give the first known examples of 2-type submanifolds in W which are not mass-symmetric.

§ 5. Submanifolds with maximal codimension

Let M be an n-dimensional submanifold of S^m. Consider the associated Weingarten map $A : T^\perp M \to S_n(TM)$ from the normal space of M in S^m into the vector bundle of self-adjoint endomorphisms of TM. In the vector bundle $S_n(TM)$ we consider the subbundle $M_n = \{B \in S_n(TM) | \text{trace } B = 0\}$. Then we have

$$S_n(TM) = M_n \oplus \boldsymbol{R}I_n \,. \tag{5.1}$$

With respect to the usual inner product $\langle\langle\ ,\ \rangle\rangle$ on $S_n(TM)$, the subbundles M_n and $\boldsymbol{R}I_n$ are orthogonal. It is easy to see that the fibres of $S_n(TM)$ are of $\frac{1}{2}n(n+1)$-dimensional.

LEMMA 5. *Let $\psi : M \to S^m$ be an isometric immersion of a compact n-dimensional Riemannian manifold M into S^m such that the immersion is full. If $x = f \circ \psi$ is mass-symmetric and of 2-type, then we have $m \leq n(n+3)/2$. In particular, if $m = n(n+3)/2$, then M is immersed as a minimal submanifold in a small hypersphere of S^m via ψ.*

Proof. Under the hypothesis, Lemma 1 implies that M has parallel mean curvature vector in S^m. Thus, M has constant mean curvature. If M is minimal in S^m, then $A(T^\perp M) \subset M_n$. Since ψ is full, statement (4) of Lemma 1 implies $m - n \leq n(n+1)/2 - 1$ which gives $m \leq n(n+3)/2 - 1$. Therefore, we may assume that M has nonzero constant mean curvature in S^m. In this case, we obtain $m \leq n(n+3)/2$. If $m = n(n+3)/2$, then we see that $A : T^\perp M \to S_n(TM) = M_n \oplus \boldsymbol{R}I$ is surjective. Since A maps $\nu = \langle H' \rangle^\perp$ onto M_n, we have $A(H') \in \boldsymbol{R}I_n$. This shows that M is pseudo-umbilical in S^m. Because M has parallel mean curvature vector H' in S^m, we conclude that M lies in a hypersphere $S^{m-1}(r)$ of S^m as a minimal submanifold. Since M is not minimal in S^m, we have $r < 1$. (Q.E.D.)

By applying Lemma 5 we may obtain the following.

THEOREM 4. *Let $\psi : M \to S^{n(n+3)/2}$ be an isometric immersion of a compact, n-dimensional, Riemannian manifold M into $S^{n(n+3)/2}$ such that the* *immersion is full. If $x = f \circ \psi$ is mass-symmetric and of 2-type, then M is a real-space-form which is immersed fully in a small hypersphere of $S^{n(n+3)/2}$ as a minimal, isotropic submanifold.*

Proof. Under the hypothesis, Lemma 5 implies that M is immersed as a minimal submanifold in a small hypersphere $S^{n(n+3)/2-1}(r) = S$ of $S^{n(n+3)/2}$. Moreover, from Lemma 1, we know that the Weingarten map A of M in S is homothetic. Thus, for any fixed point $p \in M$, the Weingarten map at p; $A(p) : T_p^{\perp}M \to M_n(p)$ is an isomorphism. Since $A(p)$ is homothetic, we have

$$\langle\langle A_{\xi}, A_{\eta}\rangle\rangle = c^2\langle \xi, \eta\rangle$$

for some constant c. Let v be a given unit vector in T_pM. We choose an orthonormal basis $B = \{e_1, \cdots, e_n\}$ such that $e_1 = v$. Since $A(p) : T_p^{\perp}M \to M_n(p)$ is an isomorphism, there exists an orthonormal basis $\xi_{n+1}, \cdots, \xi_{n(n+3)/2-1}$ in $T_p^{\perp}M$ such that, with respect to B, the associated Weingarten endmorphisms are given by

$$A_{n+1} = c\left[\begin{array}{c|c} -(n-1)a_{n-1} & 0 \\ \hline 0 & a_{n-1}I_{n-1} \end{array}\right],$$

$$A_{n+2} = c\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -(n-2)a_{n-2} & \\ \hline \multicolumn{2}{c|}{0} & a_{n-2}I_{n-2} \end{array}\right],$$

$$\vdots$$

$$A_{2n-2} = c\left[\begin{array}{c|c|c} 0 & 0 & \\ \cline{1-2} 0 & -2a_2 & 0 \\ \hline \multicolumn{2}{c|}{0} & a_2I_2 \end{array}\right],$$

$$A_{2n-1} = c\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -a_1 & \\ \hline \multicolumn{2}{c|}{0} & a_1 \end{array}\right],$$

$$A_{n+[i,j]} = c\begin{array}{cc} \begin{array}{ccccc} & i & & j & \end{array} & \\ \left[\begin{array}{ccccc} & \vdots & & \vdots & \\ \cdots & 0 & \cdots & \sqrt{\frac{n}{2}} & \cdots \\ & \vdots & & \vdots & \\ \cdots & \sqrt{\frac{n}{2}} & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{array}\right] & \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} \end{array},$$

where $[i, j] = i + \frac{1}{2}(j - i)(2n + 1 - j + i) - 1$, $a_{n-k}^2 = n/(n - k)(n - k + 1)$; $1 \leq k \leq n - 1$ and $1 \leq i < j \leq n$. From these we see that the second fundamental form $\bar{\sigma}$ of M in S satisfies $\|\bar{\sigma}(v, v)\|^2 = (n - 1)c^2$ which shows the isotropy of M in S. The constancy of sectional curvature of M follows from the equation of Gauss. (Q.E.D.)

Remark. Isotropic isometric immersions from a real-space-form into another real-space-form have been studied by Itoh and Ogiue [8].

By a similar argument we have the following.

THEOREM 5. *Let $\psi : M \to S^m$ be an isometric minimal immersion of a compact, n-dimensional, Riemannian manifold such that the immersion is full. If $x = f \circ \psi$ is mass-symmetric and of 2-type, then $m \leq n(n + 3)/2 - 1$. In particular, if $m = n(n + 3)/2 - 1$, then M is a real-space-form which is immersed as an isotropic submanifold.*

Since this theorem can be proved in the same way as that of Theorem 4, so we omit it.

§ 6. Classification of 2-type surfaces

In this section we classify surfaces in S^m which are mass-symmetric and of 2-type via f.

THEOREM 6. *Let $\psi : M \to S^m$ be an isometric immersion of a compact surface M into S^m such that the immersion is full. If $x = f \circ \psi$ is mass-symmetric and of 2-type, then one of the following statements holds:*

(1) *$m = 3$ and M is immersed as a small hypersphere $S^2(r)$ with radius $r = \sqrt{3}/2$;*

(2) *$m = 3$ and M is immersed as a Clifford (minimal) torus $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$ in S^3;*

(3) *$m = 4$ and M is immersed as a Veronese (minimal) surface in S^4;*

(4) *$m = 5$ and M is immersed as a Veronese (minimal) surface in a small hypersphere $S^4(\sqrt{5/6})$ of S^5.*

The converse is also true.

Proof. Under the hypothesis, Lemma 1 implies that M has parallel mean curvature vector in S^m. Thus, by applying a result of Chen and Yau (cf. [4, p. 106]), we have $m > 3$ and either M is a minimal surface of S^m or M is a minimal surface of a small hypersphere $S^{m-1}(r)$ of S^m, or M lies in totally geodesic S^3 of S^m. If the later case holds, then $m = 3$ since

ψ is full. In this case, Theorem 3 implies that either case (1) or case (2) occurs.

If $m > 3$, then, by Lemma 5, $m = 4$ or $m = 5$. If $m = 4$, Theorem 5 and Theorem 2 of [8] imply that M is a Veronese surface in S^4. If $m = 5$, by using Theorem 4, we see that M is immersed in a small hypersphere $S^4(r)$ of S^5 as a Veronese surface. Without loss of generality, we may assume that $S^4(r)$ is given by $u_6 = \sqrt{1 - r^2}$, where $(u_1, \cdots, u_6)$ are the Euclidean coordinates of S^5 in E^6. From direct computation, we see that the center of mass of M in $SM(6)$ via f is given by

$$x_0 = \left(\begin{array}{c|c} \frac{r^2}{5} I_5 & 0 \\ \hline 0 & 1 - r^2 \end{array}\right).$$

Since M is mass-symmetric in $W \subset SM(6)$, we have $x_0 = I/6$. Thus, we see that M is mass-symmetric in $SM(6)$ if and only if $r^2 = 5/6$.

The converse follows from direct computation. (Q.E.D.)

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