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ON THE STRUCTURE OF 4 FOLDS WITH A HYPERPLANE SECTION WHICH IS A P^1 BUNDLE OVER A SURFACE THAT FIBRES OVER A CURVE

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In this article we want to analyze the structure of a 4 dimensional projective variety X which has a smooth ample divisor A that is a $\boldsymbol{P}^1$ bundle $\pi: A \to S$ over a smooth surface S.

In [Fa+So], as a consequence of a more general result, the first and third authors determined the structure of X in the case the base S of the $\boldsymbol{P}^1$ bundle A has a cover $\tilde{S}$ with $h^{2,0}(\tilde{S}) \neq 0$. Here we look at the remaining cases except for those surfaces which are the projectivization of a stable rank two vector bundle over a curve (the result is obviously true for S rational).

The key point is to extend the morphism $\pi: A \to S$ to a morphism $\bar{\pi}: X \to S$. If the surface S has a morphism $\Psi: S \to C$ onto a smooth curve C, then the morphism $\Psi \circ \pi: A \to S$ extends to a morphism $\varphi: X \to C$ (see [So1], Proposition V). Moreover the general fibre X_c of φ turns out to be a $\boldsymbol{P}^2$ bundle over a curve contained in S. We now construct $\bar{\pi}: X \to S$ geometrically. The idea is to take a general fibre P of the general $\boldsymbol{P}^1$ bundle X_c and look at all the deformations of P in X. Using the "universal" family of such deformations we will get our desired map.

The main result is the following

THEOREM. *Let X be a 4-dimensional projective variety which is a local complete interesection. Let A be an ample divisor on X which is a $\boldsymbol{P}^1$ bundle. $\pi: A \to S$ over a smooth surface S. Assume that there is a surjective holomorphic map $\Psi: S \to C$ with connected fibres, where C is a smooth curve. Then π can be extended to a holomorphic* map $\bar{\pi}: X \to S$ *unless $S = \boldsymbol{P}_c(V)$ with V a stable rank two vector bundle on C. Moreover $\bar{\pi}: X \to S$ is a $\boldsymbol{P}^2$*

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bundle.

The paper is organized as follows.

In Section 0 we recall some background material.

In Section 1 we study the structure of X in the case the surface S, base of the $\boldsymbol{P}^1$ bundle A has a surjective morphism $\Psi : S \to C$ onto a curve.

In Section 2 we completely determine the structure of X in the case $S=\boldsymbol{P}^2$. Also, for completeness, we determine the structure of those X with an ample divisor A which is a $\boldsymbol{P}^1$ bundle over $\boldsymbol{P}^n$, with $n \geqq 3$.

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§ 0. Background material

(0.1) Throughout this article the varieties considered will be projective and defined over $\boldsymbol{C}$. Given a variety X we denote its structure sheaf by $\mathcal{O}_X$. We do not distinguish between a holomorphic vector bundle E on a variety X and its sheaf of germs of holomorphic sections. We denote the tautological line bundle of E by ζ_E or $\mathcal{O}_{\boldsymbol{P}(E)}(1)$, where $\boldsymbol{P}(E)= E^V-\{\text{zero section}\}/\boldsymbol{C}^*$ and E^V is the dual bundle of E. If Y is a subvariety of X we denote by $E|_Y$ the restriction of E to Y. For more details on vector bundles see [Ok+Sc+Sp].

(0.2) Let $p : X \to Y$ be a map of projective varieties. We will use interchangeably the word morphism and holomorphic map, as well as rational map and meromorphic map.

(0.3) Let X be a projective variety. Let D be an effective Cartier divisor on X. We denote by $[D]$ or $\mathcal{O}_X(D)$ the line bundle defined by D. If L is a line bundle on X, let $|L|$ denote the linear system of all Cartier divisors associated to L.

(0.4) By F_r with $r \geqq 0$ we denote the rth Hirzebruch surface. F_r is the unique $\boldsymbol{P}^1$ bundle $\pi : F_r \to \boldsymbol{P}^1$ over $\boldsymbol{P}^1$ with a section E satisfying $E \cdot E = -r$. By $\tilde{F}_r$ with $r \geqq 1$ we denote the surface obtained from F_r by blowing down E.

The next result will be used often. We will state it for the convenience of the reader and refer to [So2], (0.6.1) for a proof.

(0.5) Lemma. *Let X be a normal irreducible compact surface. Let L be an ample line bundle on X, with a smooth $C \in |L|$ being a rational curve* *and* $C \subseteq X_{\text{reg}}$. *Then* L *is very ample and either*

a) X *is* F_r *and* $L=[E]\otimes[f]^k$ *with* $k \geqq r+1$, *or*

b) X *is* $\tilde{F}_r$ *and* $p^*L=[E]\otimes[f]^r$ *where* $p: F_r \to \tilde{F}_r$ *is the map that blows down* E. (*Here* f *denotes a fibre of* $\pi: F_r \to \boldsymbol{P}^1$).

§ 1. Proof of the main theorem

(1.0) THEOREM. *Let* X *be a four dimensional projective variety which is a local complete intersection. Let* A *be an ample divisor on* X *which is a* $\boldsymbol{P}^1$ *bundle,* $\pi: A \to S$ *over a smooth surface* S. *Assume that there is a surjective holomorphic map* $\Psi: S \to C$ *with connected fibres, where* C *is a smooth curve. Then* π *can be extended to a holomorphic map* $\bar{\pi}: X \to S$ *unless* $S=\boldsymbol{P}_c(V)$ *with* V *a stable rank two vector bundle on* C (see Remark (1.0.1)). *Moreover* $\bar{\pi}: X \to S$ *is a* $\boldsymbol{P}^2$ *bundle.*

(1.0.1) *Remark.* We do not need to assume that $\Psi: S \to C$ has connected fibres and that C is smooth. In fact if otherwise we can Remmert-Stein factorize $\Psi = s \circ r$ where $r: X \to C'$ is a holomorphic map onto a smooth curve C' and $s: C' \to C$ is a finite to one holomorphic map. Then the theorem is true unless $S=\boldsymbol{P}_{C'}(V)$ where V is a stable rank two vector bundle on C'.

Proof of the theorem. We notice that $\dim \operatorname{Sing}(X) \leqq 0$ since the ample divisor A on X is smooth. The holomorphic map $\Psi \circ \pi$ extends to a holomorphic map $\varphi: X \to C$, see [So1] Proposition V or [Fu]. Let X_c and A_c denote the general fibre of φ and $\Psi \circ \pi$ respectively. Note that A_c is a geometrically ruled surface over $\Psi^{-1}(c)$ and moreover A_c is an ample divisor on X_c. We claim that either

α) X_c is a $\boldsymbol{P}^2$ bundle over $\Psi^{-1}(c)$ and $[A_c]$ is the tautological line bundle on the $\boldsymbol{P}^2$ bundle X_c, or

β) $(\Psi \circ \pi)^{-1}(c) \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and X_c is a $\boldsymbol{P}^2$ bundle over $\boldsymbol{P}^1$ with $[A_c]$ the tautological line bundle on the $\boldsymbol{P}^2$ bundle X_c where the canonical projection is not an extension of $\pi: A_c \to \Psi^{-1}(c)$ $(\simeq \boldsymbol{P}^1)$. Note that the line bundle $[A_c]_{|\boldsymbol{P}^1 \times \boldsymbol{P}^1} = \mathcal{O}(1, t)$ with $t > 1$.

Proof of the claim. The general fibre of Ψ is a smooth curve of genus $g \geqq 0$. If $g>0$ or if $g=0$ and $A_c \simeq F_r$ with $r>0$, where F_r is as in (0.4), then using ([Ba2], [Ba3]), we conclude that X_c is a $\boldsymbol{P}^2$ bundle over $\Psi^{-1}(c)$ and A_c is the tautological line bundle on X_c. If $g=0$ and $A_0 \simeq F_0 \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ then we will show that

$$(*) \qquad \mathrm{Pic}(X_c) \simeq \mathrm{Pic}(A_c) \simeq \boldsymbol{Z} \otimes \boldsymbol{Z}\,.$$

Therefore the result will follow from [Ba1] once we know (*).

Proof of (*). From the following diagram

$$\begin{array}{ccc} H_2(A, \boldsymbol{Q}) & \longrightarrow & H_2(X, \boldsymbol{Q}) \\ \uparrow & & \uparrow \\ H_2(A_c, \boldsymbol{Q}) & \longrightarrow & H_2(X_c, \boldsymbol{Q}) \end{array}$$

we see that $\dim H_2(X_c, \boldsymbol{Q})=1$ is possible only if the two rulings of A_c ($\simeq F_0$) get identified in X. But the two rulings were in different homology classes in A therefore they cannot go in the same homology class in X. Using Kroncecker duality and the first Lefschetz theorem we conclude that $\mathrm{Pic}(X_p) \simeq \mathrm{Pic}(A_c)$. □

The proof of the theorem will be split up in two parts. We will treat case α) first and then the case β).

Case α) Fix a general $\boldsymbol{P}^2$ which is a fibre of $X_c \to \Psi^{-1}(c)$ and denote it by P. Using the fact that $P \subseteq X_c \subseteq X$ and the exact sequence of normal bundles

$$0 \longrightarrow N_{P/X_c} \longrightarrow N_{P/X} \longrightarrow N_{X_c/X|P} \longrightarrow 0$$

it is straightforward to see that $N_{P/X} = \mathcal{O}_P \oplus \mathcal{O}_P$, where $N_{P/X}$ is the normal bundle of P in X, and that $H^1(P, N_{P/X}) = 0$. Under the above assumption, using a basic result on Hilbert schemes, it follows that there exist irreducible projective varieties $\mathscr{W}$ and $\mathscr{Z}$ with the following properties:

1) $\mathscr{W} \subseteq \mathscr{Z} \times X$ and the map $p : \mathscr{W} \to \mathscr{Z}$ induced by the product projection is a flat surjection,

2) there is a smooth point $a \in \mathscr{Z}$ with p of maximal rank in a neighborhood of $p^{-1}(a)$ and $p^{-1}(a)$ is identified with $P \simeq \boldsymbol{P}^2$ via q, where $q : \mathscr{W} \to X$ is the map induced by the product projection.

(1.0.2) LEMMA. *There exists a Zariski open neighborhood U of a, where a is as in 2), such that for every $z \in U$*

i) $p^{-1}(z) = \mathscr{W}_z$ *is isomorphic to $\boldsymbol{P}^2$ and it is a fibre of $X_c \to \Psi^{-1}(c)$ for some $c \in C$,*

ii) $\mathscr{W}_z \cap A = f$ ($\simeq \boldsymbol{P}^1$), *where f is a fibre of π.*

Proof. From 2) above there exists a smooth neighborhood U of a in $\mathscr{Z}$ such that $p^{-1}(U) \to U$ and $q^{-1}(A) \cap p^{-1}(U) \to U$ are smooth morphisms.

Note that $A \cap \mathscr{W}_a = \boldsymbol{P}^1$. Moreover using the fact that small deformations of $\boldsymbol{P}^2$ and $\boldsymbol{P}^1$ are $\boldsymbol{P}^2$ and $\boldsymbol{P}^1$ respectively we conclude that the fibres of the maps $p_{|p^{-1}(U)}$ and $q_{|q^{-1}(A)\cap p^{-1}(U)}$ are $\boldsymbol{P}^2$ and $\boldsymbol{P}^1$ respectively. On the other hand a morphism φ from $\boldsymbol{P}^2 \subseteq X$ to C is constant. Hence any fibre of $p_{|p^{-1}(U)}$ is contained in a fibre of φ. Therefore the rest of (1.0.2) is obvious

(1.0.3) LEMMA. *The intersection number* $A\cdot A\cdot \mathscr{W}_z = 1$ *for every* $z \in \mathscr{Z}$. *And if* $\mathscr{W}_z = \overline{\mathscr{W}}_z \cup \{$*embedded part*$\}$ *then* $\overline{\mathscr{W}}_z$ *is reduced and irreducible.*

Proof. By α) we have that $\mathcal{O}_X(A)_{|\boldsymbol{P}^2} = \mathcal{O}_{\boldsymbol{P}^2}(1)$. Hence $(A\cdot A\cdot \boldsymbol{P}^2)_X = (\mathcal{O}_X(A)_{|\boldsymbol{P}^2}\cdot \mathcal{O}_X(A)_{|\boldsymbol{P}^2})_{\boldsymbol{P}^2} = 1$, which implies that $A\cdot A\cdot \mathscr{W}_z = 1$ since the intersection number is preserved by flat maps. Clearly $\overline{\mathscr{W}}_z$ is reduced and irreducible (since $A\cdot A\cdot \mathscr{W}_z = 1$).

Note that the general fibre of the morphism $\Psi : S \to C$ is either isomorphic to $\boldsymbol{P}^1$ or to a curve of positive genus.

(1.0.4) LEMMA. *For every* $z \in \mathscr{Z}$, $\mathscr{W}_z \not\subseteq A$.

Proof. Let $z \in \mathscr{Z}$ and let $\{z_n\}$ be a sequence of points in $\mathscr{Z}$ such that $\lim_{n\to\infty} z_n = z$ and $\mathscr{W}_{z_n} \simeq \boldsymbol{P}^2$ for every n. The above is possible by (1.0.2) Now use the fact that $\varphi(\mathscr{W}_{z_n})$ is one point for every n, to conclude that $\varphi(\mathscr{W}_z)$ is also one point. Assume that $\mathscr{W}_z \subseteq A$.

Since $\pi : A \to S$ is a $\boldsymbol{P}^1$ bundle and since $(\Psi\circ\pi)(\mathscr{W}_z) = c$, with c a point in C, we get that $\Phi = \pi_{|\overline{\mathscr{W}}_z} : \overline{\mathscr{W}}_z \to \pi(\overline{\mathscr{W}}_z)$ is a $\boldsymbol{P}^1$ bundle, where $\overline{\mathscr{W}}_z$ denotes the non-embedded part of $\mathscr{W}_z$. Note that $\pi(\mathscr{W}_z) \subseteq \psi^{-1}(c)$. To continue the proof of the lemma we distinguish two cases:

Case 1. The general fibre of Ψ is isomorphic to $\boldsymbol{P}^1$. If $\Psi^{-1}(c)$ with c as above is isomorphic to $\boldsymbol{P}^1$ then $\mathscr{W}_z$ is a $\boldsymbol{P}^1$ bundle over $\boldsymbol{P}^1$. Moreover there exists an ample line bundle $([A]_{|\mathscr{W}_z})$ on $\mathscr{W}_z$ whose selfintersection is 1. This last fact is impossible.

If $\Psi^{-1}(c)$ is singular then $\Psi^{-1}(c) = \sum n_i C_i$ with $C_i \simeq \boldsymbol{P}^1$. Also $\pi(\overline{\mathscr{W}}_z) = C_i$ for some i otherwise we would get a contradiction with the fact that $\overline{\mathscr{W}}_z$ is irreducible. Hence $\mathscr{W}_z$ is a $\boldsymbol{P}^1$ bundle over $\boldsymbol{P}^1$ which is impossible as noticed earlier.

Case 2. The general fibre of Ψ is isomorphic to a curve of positive genus. Take a general fibre of $\mathscr{W} \to \mathscr{Z}$ and consider all the lines on such fibre. Let T denote the irreducible component of the Hilbert scheme of X parametrizing such lines. Denote by M the universal family. Thus

$M \subseteq T \times X$. Note that the non embedded part of every fibre of M is irreducible and reduced (since $L \cdot M_t = L \cdot \boldsymbol{P}^1 = 1$, where M_t is a fibre of M over T).

CLAIM *Every fibre of $M \to T$ has $\boldsymbol{P}^1$ as normalization.*

Proof of the claim. Consider a curve B in T through a point t'. Also choose B of positive genus. Let M_B denote the inverse image of B under the natural projection $M \to T$. Note that most fibres of $M_B \to B$ are linear $\boldsymbol{P}^1$'s since B is chosen of positive genus. If we take a minimal model of a desingularization of $\tilde{M}_B$, where $\tilde{M}_B$ denotes the normalization of M_B, we get a ruled surface over the normalization of B. This last conclusion follows from the fact that M_B has infinitely many $\boldsymbol{P}^1$'s and from the fact that the genus of B is positive. Thus since going from $M_B \to$ normalization $\to$ desingularization $\to$ minimal model does not destroy a positive genus curve and the normalization of M_t, goes in a fibre of a $\boldsymbol{P}^1$ bundle we conclude that every fibre of $M \to T$ has $\boldsymbol{P}^1$ as a normalization. □

Now choose 2 points $(a, b) \subseteq \mathscr{W}_z$ with $\Phi(a) \neq \Phi(b)$. Let $(x_n, y_n) \subseteq \mathscr{W}_{z_n}$ be a sequence of pairs of points such that $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} y_n = b$. Let M_{t_n} be a sequence of lines containing (x_n, y_n). The limit of M_{t_n} is (maybe after passing to a subsequence) an irreducible curve M_t containing the (a, b) plus possibly some embedded points. As shown in our previous claim, M_t is birational to $\boldsymbol{P}^1$ and therefore $\Phi(M_t)$ is birational to $\boldsymbol{P}^1$. Thus the normalization, $\mathscr{D}$, of $\mathscr{W}_z$ is a $\boldsymbol{P}^1$ bundle over $\boldsymbol{P}^1$ under the map induced by Φ. But the pullback of $[A]$ to $\mathscr{D}$ is an ample bundle, $\mathscr{L}$, which satisfies $\mathscr{L} \cdot \mathscr{L} = 1$ by (1.0.3). This is impossible for an ample line bundle on a $\boldsymbol{P}^1$ bundle over $\boldsymbol{P}^1$. □

(1.0.5). LEMMA. *$\mathscr{W}_z \cap A = f$, where f is fibre of π. (The equality here is only up to embedded points).*

Proof. By (1.0.2) we can take a sequence of points $\{z_n\}$ in $\mathscr{Z}$ with $\lim_{n\to\infty} z_n = z$, such that $\lim_{n\to\infty} \mathscr{W}_{z_n} = \mathscr{W}_z$, $\mathscr{W}_{z_n} \simeq \boldsymbol{P}^2$ for all n and $\mathscr{W}_{z_n} \cap A$ = fibre of π. Hence $\mathscr{W}_z \cap A = f + C$, where f is a fibre of π and C is a possibly empty effective 1-cycle. From (1.0.3) and the fact that A is ample it follows that $C = \emptyset$. □

Therefore we get a map $v : \mathscr{Z} \to S$ which is a continuous and meromorphic and whose fibres are connected. Let $\mathscr{W}'$ denote $v \times i_X(\mathscr{W})$, where i_X is the identity map on X.

(1.0.6) LEMMA. *$\mathscr{W}' \subseteq S \times X$ is a family with $\mathscr{W}'_s$ for every $s \in S$ equal to $\mathscr{W}_z$ for some $z \in \mathscr{Z}$.*

Proof. Assume otherwise. Then there is a curve $Y = v^{-1}(s) \subseteq \mathscr{Z}$ such that for every $y \in Y$, $\mathscr{W}_y \supseteq f$. Note that $(\bigcup_{y \in Y} \mathscr{W}_y) \cap A = f$ by (1.0.5). On the other hand $\bigcup_{y \in Y} \mathscr{W}_y$ is a divisor on X. Thus $\dim((\bigcup_{y \in Y} \mathscr{W}_y) \cap A) \geqq 2$. This contradiction proves our lemma. □

From (1.0.5) it follows that $\mathscr{W}' \xrightarrow{q'} X$ is one to one, where q' is the map induced by the product projection. Moreover X is normal. Therefore $q' : \mathscr{W}' \to X$ is a biholomorphism. Hence $\bar{\pi} = p' \circ (q')^{-1} : X \to S$ is holomorphic.

Before passing to the caes β) we will show that the above $\bar{\pi}$ gives to X the structure of a $\boldsymbol{P}^2$ bundle over S.

By construction the general fibre of $\bar{\pi}$ is $\boldsymbol{P}^2$. Also $\mathcal{O}_X(A)_{|\boldsymbol{P}^2} = \mathcal{O}_{\boldsymbol{P}^2}(1)$. As for the possible singular fibre F of $\bar{\pi}$, we notice that F is reduced and irreducible since $L \cdot L \cdot F = 1$. Since $\boldsymbol{P}^1$ is an hyperplane section of F it is well known, see (0.5) that F is either F_r with $r \geqq 0$ or $\tilde{F}_r$ with $r \geqq 1$, where F_r and $\tilde{F}_r$ are as in (0.4). There are no F_r with an ample line bundle of degree 1. Among the $\tilde{F}_r$ the only one with an ample line bundle of degree 1 is $\tilde{F}_1 \simeq \boldsymbol{P}^2$. Now we use a theorem of Hironaka ([Hi], Theorem 1.8) to conclude that $\bar{\pi} : X \to S$ is a $\boldsymbol{P}^2$ bundle.

Let us now consider the case β).

Case β). Let $c \in C$ be a general point. We take a general rational curve ℓ in $A_c = (\Psi \circ \pi)^{-1}(c) \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ such that $\ell \cdot \ell = 0$ and ℓ is not a fibre of π. From now on we denote by ℓ the ruling of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ which is not a fibre of π. It is straightforward to see that

$$N_{\ell/A} = \mathcal{O}_\ell \oplus \mathcal{O}_\ell \quad \text{and} \quad H^1(\ell, N_\ell, N_{\ell/A}) = 0\,.$$

Denote by S' the irreduible component of the Hilbert scheme of A parametrizing flat deformations of ℓ in A and by $\mathscr{Y}$ the universal family. Thus $\mathscr{Y} \subseteq S' \times A$. Denote by $p : \mathscr{Y} \to S'$ and $q : \mathscr{Y} \to A$ the maps induced by the product projections. Note that such deformations fill up the whole space A, i.e., $q(\mathscr{Y}) = A$.

CLAIM 1. *$\Psi : S \to C$ is a geometrically ruled surface.*

Proof of claim 1. Assume that there exists a point $c_0 \in C$ such that $\Psi^{-1}(c_0)$ is a singular fibre. Then the number of irreducible components of $\Psi^{-1}(c_0)$ is at least 2. Let $\{c_n\}$ be a sequence of points in C approaching the point c_0. Let $\{\ell_n\}$ be the corresponding sequence of lines in $\mathscr{Y}$. Thus $\lim_{n\to\infty} \pi(\ell_n) = \Psi^{-1}(c_0)$, where the equality is only setwise (Here we have identified ℓ_n with $q(\ell_n)$). But the above equality is impossible since by β) $A\cdot\ell_n = 1$ for all n, while the number of irreducible components of $\Psi^{-1}(c)$ is at least 2 and A is an ample divisor. □

We note that for every $c\in C$, $(\Psi\circ\pi)^{-1}(c)\simeq \boldsymbol{P}^1\times\boldsymbol{P}^1$. In fact since S is geometrically ruled it follows that for every $c\in C$, $(\Psi\circ\pi)^{-1}(c)\simeq F_r$ with $r\geqq 0$. Assume that there exists a $c_0\in C$ such that $(\Psi\circ\pi)^{-1}(c)\simeq F_r$ with $r>0$.

By a slight variation of the argument used in the proof of the above claim it follows that for each $x\in(\psi\circ\pi)^{-1}(c)$ there exists an irreducible curve $\ell\subseteq(\psi\circ\pi)^{-1}(c)$ such that:

1) $A\cdot\ell=1$,

2) the image of ℓ under π is $\boldsymbol{P}^1$.

A simple direct check shows that this is not possible on F_r unless $r=0$.

Let S' and $\mathscr{Y}$ be as before. We denote by ℓ_s the fibre of $\mathscr{Y}$ over $s\in S'$. Clearly the smooth fibres of the flat family $\mathscr{Y}$ are isomorphic to $\boldsymbol{P}^1$. Recall that $A\cdot\ell_s=1$. Hence the Hilbert polynomial $\chi(\mathcal{O}_{\ell_s}(A_{|\ell_s})^{\otimes n})$ of ℓ_s is equal to $n+1$. Let $s\in S'$ be such that ℓ_s is singular. Denote by $\bar{\ell}_s$ the one dimensional closed subscheme of ℓ_s defined by removing the embedded points of ℓ_s.

CLAIM 2. *$\ell_s=\bar{\ell}_s$ and S' is smooth.*

Proof of Claim 2. Note that since $\bar{\ell}_s$ is contained in a fibre of $\Psi\circ\pi$ which is isomorphic to $\boldsymbol{P}^1\times\boldsymbol{P}^1$ and since $A\cdot\bar{\ell}_s=1$ it follows that $\bar{\ell}_s$ is a fibre of $\boldsymbol{P}^1\times\boldsymbol{P}^1$, so $\bar{\ell}_s=\boldsymbol{P}^1$. In order to see that $\ell_s=\bar{\ell}_s$ we consider the following exact sequence

$$0\longrightarrow T\longrightarrow\mathcal{O}_{\ell_s}\longrightarrow\mathcal{O}_{\bar{\ell}_s}\longrightarrow 0 \tag{1.0.7}$$

where the sheaf T is the torsion part of $\mathcal{O}_{\ell_s}$. Tensoring (1.0.7) with $\mathcal{O}(A_{|\ell_s})^{\otimes n}$ and using the fact that the Euler characteristic is additive on a short exact sequence it follows that

$$\chi(\mathcal{O}_{|\ell_s}(A_{|\ell_s})^{\otimes n})=\chi(T\otimes\mathcal{O}_{\ell_s}(A_{|\ell_s})^{\otimes n})+\chi(\mathcal{O}_{\bar{\ell}_s}(A_{|\bar{\ell}_s})^{\otimes n}).$$

Note that the Hilbert polynomial of ℓ_s and of $\bar{\ell}_s$ are equal. Thus T is the 0-sheaf. To see that S' is smooth note that $N_{\ell_s/A}=\mathcal{O}_\ell\oplus\mathcal{O}_\ell$. Therefore it follows that S' is smooth at s.

(1.0.8) *Remark.* i) $\mathscr{Y}$ is isomorphic to A
ii) A is a $\boldsymbol{P}^1$ bundle $\sigma : A \longrightarrow S'$ over S'.

To see i) note that $\mathscr{Y}$ is birational to A. Moreover $\mathscr{Y}$ is in one to one correspondence with A, since for every $a \in A$ there exists a unique $\ell \subseteq A_c$ containing a, where $c = (\Psi \circ \pi)(a)$. Hence $\mathscr{Y}$ is isomorphic to A. From i) it follows that there is a morphism $\sigma = q \circ p^{-1}$ from A onto S' whose fibres are isomorphic to $\boldsymbol{P}^1$. Moreover $\mathcal{O}_A(A)_{|\boldsymbol{P}^1} = \mathcal{O}_{\boldsymbol{P}^1}(1)$. Thus ii) is clear.

CLAIM 3. *S' is geometrically ruled over C.*

Proof of Claim 3. Let $c \in C$ and let $\sigma_c : A_c \to \boldsymbol{P}^1$ be the restriction of the map σ to A_c. Let f_c denote a fibre of the map π restricted to A_c ($\cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$). By the universality of the Hilbert scheme σ_c embeds f_c into S'; we denote the smooth rational curve $\sigma(f_c)$ in S' by f'_c. To show that there exists a morphism from S' onto C we will distinguish the case $g(C) > 0$ and $g(C) = 0$ where $g(C)$ denotes the genus of C. In the case $g(C) > 0$ it follows that $H^1(S', \mathcal{O}_{S'}) \neq 0$. We get the following diagram

$$
*)\qquad
\begin{array}{ccccc}
A & \xrightarrow{\Psi \circ \pi} & C & \longrightarrow & \mathscr{J}(C) \simeq \mathrm{Alb}(A) \\
{\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle j} & & \wr\| \\
S' & \xrightarrow[\alpha]{} & \alpha(S') & \hookrightarrow & \mathrm{Alb}(S')
\end{array}
$$

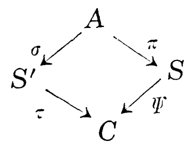
where α is the Albanese map. In the above diagram we have used the fact that $\mathrm{Alb}(A) \simeq \mathrm{Alb}(S) \simeq \mathscr{J}(C)$. Note that $\dim \alpha(S') = 1$. We claim that $j : C \to \alpha(S')$ is an isomorphism. Using the Riemann-Hurwitz formula the above claim is clear for $g(C) > 1$. For $g(C) = 1$ we get that the morphism j is a covering map. But this is impossible by the commutativity of the first square diagram in *). Therefore we get a morphism $\tau : S' \to C$, with $\tau = j^{-1} \circ \alpha$. Also f'_c (the closed subscheme induced in S' by f_c) are fibres of τ. Therefore S' is generically ruled over C. To see that S' is geometrically ruled we assume otherwise. Then there exists a fibre $F = \sum_i n_i C_i$. Let $c = \tau(F)$. Note that $\sigma^{-1}(F) = \sum n_i F_i$, where each F_i is a $\boldsymbol{P}^1$ bundle over C_i. By the commutativity of the first square diagram in *) we see that $\sum n_i F_i = \sigma^{-1}(F) = \varphi^{-1}(c) = \boldsymbol{P}^1 \times \boldsymbol{P}^1$ which is impossible. If $g(C) = 0$ then $H^1(S', \mathcal{O}_{S'}) = 0$. Thus there exists a line bundle L on S' such that the linear system $|L|$ contains infinitely many f'_c where f'_c is the closed subscheme induced in S' by f_c. It follows immediately from

$$0 \longrightarrow \mathcal{O}_{S'} \longrightarrow L \longrightarrow L_{|f'_c} \longrightarrow 0$$

that $\dim|L| = 1$. Also it can be easily seen that the linear system $|L|$ is base point free. Hence it defines a morphism onto $\boldsymbol{P}^1$. The general fibre of such morphism is isomorphic to $\boldsymbol{P}^1$. Therefore by Noether's lemma S' is rational. The same argument as in the case $g(C) > 0$, shows that S' is geometrically ruled.

From the above proof it also follows that the elements of $|L|$ are exactly $\{f'_c\}_{c \in C}$. □

Thus we have the following commutative diagram



We will now show that the case β) cannot occur unless $S = \boldsymbol{P}_c(V)$ where V is a stable rank two vector bundle on C. (Obviously does not occur if S is rational ruled).

By the universality of the fibre product of S and S' over C we get a morphism $A \to S \times_C S'$ which is an isomorphism by Zariski's Main Theorem. The surfaces S and S' are geometrically ruled over C and therefore there exist rank two vector bundles V and V' over C such that $S = \boldsymbol{P}(V)$ and $S' = \boldsymbol{P}(V')$. For the quadruple

$$X, A, S', \quad \text{and} \quad \pi' : A \longrightarrow S'$$

the hypotheses of (1.0) are satisfied. If we were in case β) with respect to X, A, S, and π, then we must be in case α) with respect to X, A, S', π'. To see this note that being in case β) with respect to X, A, S, and π, then

$$[A_c]|_{P^1 \times P^1} = \mathcal{O}(1, t) \text{ with } t > 1\,,$$

i.e. $[A]$ restricted to a fibre of π is of degree $t > 1$. π' restricted to A_c gives the ruling different from the ruling corresponding to π restricted to A_c. Therefore, with respect to X, A, S', π', it follows that $[A]$ restricted to a fibre of π' is of degree 1. Since this degree would have to be greater than 1 if we were in case β) with respect to X, A, S', π' it follows that we are in case α) with respect to X, A, S', π' Hence we conclude that the morphism $\sigma : A \to S'$ extends to a morphism $\tilde{\sigma} : X \to S'$ and that $\tilde{\sigma} : X$ S' is a $\boldsymbol{P}^2$ bundle. Therefore we have the following exact sequence of vector bundles on S'

$$(1.0.9) \qquad 0 \longrightarrow \mathcal{O}_{S'} \longrightarrow E \xrightarrow{\gamma} F \longrightarrow 0$$

with $X = \boldsymbol{P}(E)$ and $A = \boldsymbol{P}(F)$ is embedded in X via the map γ. Since for every $c \in C$ $(\tau \circ \sigma)^{-1}(c) \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ we have that $F_{|\tau^{-1}(c)} = \mathcal{O}_{\boldsymbol{P}^1}(a)_c \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_c)$. It is an easy check to see that a_c is independent of c in C. Thus we can omit the subscript c. Consider the vector bundle $F \otimes \xi^{-a}$ where ξ is the tautological line bundle of V'. By the base change theorem $\tau_*(F \otimes \xi^{-a}) = \tilde{V}$ is a vector bundle on C of rank two. Thus (1.0.9) becomes

$$(1.0.10) \qquad 0 \longrightarrow \mathcal{O}_{S'} \longrightarrow E \longrightarrow \tau^* \tilde{V} \otimes \xi^a \longrightarrow 0$$

(1.0.11) LEMMA. $S = \boldsymbol{P}(\tilde{V})$.

Proof. Note that $A = \boldsymbol{P}(F) = \boldsymbol{P}(\tau^* V \otimes \xi^a) = \boldsymbol{P}(\tau^* V)$. Also $A = S \times_c S' = \boldsymbol{P}(V) \times_c S' = \boldsymbol{P}(\tau^* \tilde{V})$. Therefore there exists a line bundle $\mathcal{L}$ on S' such that $\tau^* \tilde{V} = \tau^* V \otimes \mathcal{L}$. Taking the 0-th direct image via τ on both sides of the equality we get that $\tilde{V} = V \otimes \tau_* \mathcal{L}$. Also $\tau^* \mathcal{L}$ is a line bundle since $\mathcal{L}_{|\tau^{-1}(c)}$ is trivial. Hence $\boldsymbol{P}(\tilde{V}) = \boldsymbol{P}(V \otimes \tau_* \mathcal{L}) = \boldsymbol{P}(V) = S$.

(1.0.12) LEMMA. *If $\tilde{V}$ is not a stable vector bundle on C then A not an ample divisor on X.*

Proof. It is enough to show that the sequence (1.0.10) splits. Since $\tilde{V}$ is a vector bundle of rank 2 on the curve C which is not stable, there exists an exact sequence

$$0 \longrightarrow M \longrightarrow \tilde{V} \longrightarrow N \longrightarrow 0$$

such that $\deg M \geqq \deg N$. If we pull back the above exact sequence via τ and we tensor it with ξ^a we get

$$(1.0.13) \qquad 0 \longrightarrow \tau^* M \otimes \xi^a \longrightarrow \tau^* \tilde{V} \otimes \xi^a \longrightarrow \tau^* N \otimes \xi^a \longrightarrow 0\,.$$

Note that $\tau^* N \otimes \xi^a$ is ample. Hence $\tau^* M \otimes \xi^a$ is ample since $\deg M \geqq \deg N$. Therefore using the cohomology sequence associated to the dual sequence of (1.0.13), the ampleness of $\tau^* N \otimes \xi^a$ and of $\tau^* M \otimes \xi^a$ and the fact that $a > 1$, we conclude that $H^1(S', (\tau^* \tilde{V} \otimes \xi^a)^v) = 0$.
(Note that $a = 1$ would imply that (1.0.10) splits). □

Thus we have shown that the case β) does not occur unless $S = \boldsymbol{P}_C(V)$ with V a stable rank 2 vector bundle on C.

§ 2. $\boldsymbol{P}^1$ bundles over $\boldsymbol{P}^n$ with $n \geqq 2$ as ample divisors

(2.0) THEOREM. *Let X be a projective local complete intersection. Let*

A be an ample divisior on X which is a $\boldsymbol{P}^1$ bundle $p: A \to \boldsymbol{P}^2$ over $\boldsymbol{P}^2$. Then X is a $\boldsymbol{P}^2$ bundle over $\boldsymbol{P}^2$ unless $A \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^2$.

Proof. We claim that the map $p: A \to \boldsymbol{P}^2$ extends to a map $\tilde{p}: X \to \boldsymbol{P}^2$ unless $A \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^2$. Think of p as the map associated to the linear system $|p^*\mathcal{O}_{\boldsymbol{P}^2}(1)|$. To show that the map p extends it is enough to check that the sections of $\Gamma(A, p^*\mathcal{O}_{\boldsymbol{P}^2}(1))$ can be extended to X as sections of $\mathcal{L}$ where $\mathcal{L}$ is the unique extension of $p^*\mathcal{O}_{\boldsymbol{P}^2}(1)$ to X, see [So1]. Now to show that the sections extend it is sufficient to prove that $H^1(X, \mathcal{L} \otimes [-A]) = 0$. This is implied by $H^1(A, (\mathcal{L} \otimes [-A]^t)_{|A}) = 0$ for all $t > 0$, see [So1] or [Fa + So]. Let $F \in |p^*\mathcal{O}_{\boldsymbol{P}^2}(1)$, i.e., $F = p^{-1}(\ell)$ where ℓ is a linear hyperplane of $\boldsymbol{P}^2$. Using the long cohomology sequence associated to the following exact sequence

$$0 \longrightarrow K_A \otimes [A]^t \otimes [F]^{-1} \longrightarrow K_A \otimes [A]^t \longrightarrow (K_A \otimes [A]^t)_{|F} \longrightarrow 0\,,$$

the Kodaira vanishing theorem and the fact that F is a $\boldsymbol{P}^1$ bundle over $\boldsymbol{P}^1$, we get that $H^1(A, \mathcal{L}_A \otimes [-A]^t{}_{|A}) = 0$ for all $t > 0$ unless $F = F_0$, with F_0 as in (0.4).

Note that since A is a $\boldsymbol{P}^1$ bundle over $\boldsymbol{P}^2$ we have that $A = \boldsymbol{P}(V)$, where V is a rank 2 vector bundle on $\boldsymbol{P}^2$. In the case $F = F_0$ we have that for every line ℓ in $\boldsymbol{P}^2$, $V_{|\ell} = \mathcal{O}_\ell(a_\ell) \oplus \mathcal{O}_\ell(a_\ell)$. Also it is easy to see that a_ℓ is independent of ℓ. Therefore the vector bundle V is uniform and so $V = \mathcal{O}_{\boldsymbol{P}^2}(a) \oplus \mathcal{O}_{\boldsymbol{P}^2}(a)$. Therefore $A = \boldsymbol{P}(V) \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^2$. Thus the map p extends to a holomorphic map $\tilde{p}: X \to \boldsymbol{P}^2$ unless $A \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^2$. Now the same argument as in [Fa + So], (3.0) shows that X is a $\boldsymbol{P}^2$ bundle over $\boldsymbol{P}^2$. □

(2.1) THEOREM. *Let X be a projective local complete intersection. Let A be an ample divisor on X which is a $\boldsymbol{P}^1$ bundle $p: A \to \boldsymbol{P}^n$ over $\boldsymbol{P}^n$. If $n \geqq 3$ then $A \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^n$ and hence X is a $\boldsymbol{P}^{n+1}$ bundle over $\boldsymbol{P}^1$.*

Proof. Note that $A = \boldsymbol{P}(V)$ for some rank 2 vector bundle V on $\boldsymbol{P}^n$. We can assume, without loss of generality that V is normalized. We will prove the theorem for $n = 3$. The same proof yields the general case also. Let $F = p^{-1}(\boldsymbol{P}^2)$, where $\boldsymbol{P}^2$ is a hyperplane of $\boldsymbol{P}^3$. Let $\mathcal{L} \in \mathrm{Pic}(X)$ be such that $\mathcal{L}_A = [F]$. If $\Gamma(X, \mathcal{L}) \to \Gamma(A, \mathcal{L}_A) \to 0$ then the map p extends to X. And we will have the contradiction that $n \leqq 2$, see [So1], Proposition V. Thus we can assume that $H^1(X, \mathcal{L} \otimes [A]^{-1}) \neq 0$. This implies that $H(A, \mathcal{L}_A \otimes [A]^{-t}_{|A}) \neq 0$ for some $t > 0$. For such t we consider the following exact sequence

$$0 \longrightarrow K_A [A]^t \otimes [F]^{-1} \longrightarrow K_A \otimes [A]^t \longrightarrow K_F \otimes [A]^t_F \otimes [F]^{-1}_F \longrightarrow 0 .$$

From the long exact cohomology sequence associated to the above sequence, Kodaira vanishing theorem and the fact that $H^3(A, K_A \otimes [A]^t \otimes [F]^{-1}) \neq 0$ by hypothesis, it follows that $H^2(F, K_F \otimes [A]^t_F \otimes [F]^{-1}_F) \neq 0$.

Note that F is a $\boldsymbol{P}^1$ bundle $p_F : F \longrightarrow \boldsymbol{P}^2$ over $\boldsymbol{P}^2$. Let $\tilde{F} = p_F^{-1}(\boldsymbol{P}^1)$, where $\boldsymbol{P}^1$ is a hyperplane of $\boldsymbol{P}^2$. We consider the sequence

$$0 \longrightarrow K_F \otimes [A]^t_F \otimes [\tilde{F}]^{-1} \longrightarrow K_F \otimes [A]^t_F \longrightarrow K_{\tilde{F}} \otimes [A]^t_{\tilde{F}} \otimes [F]^{-1}_{\tilde{F}} \longrightarrow 0 .$$

And now, as above, we conclude that $H^1(\tilde{F}, K_{\tilde{F}} \otimes [A]^t_{\tilde{F}} \otimes [\tilde{F}]^{-1}_{\tilde{F}}) \neq 0$. This together with the fact $\tilde{F}$ is a $\boldsymbol{P}^1$ bundle over $\boldsymbol{P}^1$ implies that $F = F_0$, where F_0 is as in (0.4). Therefore we conclude that $V_{|\ell}$ is trivial for all lines $\ell \subseteq \boldsymbol{P}^3$, which implies that V is trivial. Thus $A \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^3$. But $A(\simeq \boldsymbol{P}^1 \times \boldsymbol{P}^3)$ is ample on X. Hence X is a $\boldsymbol{P}^{3+1}$ bundle, see [So1]. □

Note Added in Proof. The main theorem of this paper which is stated in the introduction leaves open what the structure of the fourfold X is when S is the projectivization of a stable rank 2 vector bundle. This last open case has been settled by the second author E. Sato and H. Spindler in "On the structure of 4-folds with hyperplane section which is a $\boldsymbol{P}^1$ bundle over a ruled surface", Springer Lecture Notes in Mathematics, **1194** (1986), 145–149.

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