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ON AN ESTIMATE FOR SOLUTIONS OF NONLINEAR ELLIPTIC VARIATIONAL INEQUALITIES[1]

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Introduction

Let Ω be a bounded domain in R^n with the boundary $\partial\Omega$ of class $C^{0,1}$ and E be a compact subset (resp. a compact subset on an $(n-1)$-dimensional hypersurface of class $C^{0,1}$) in Ω. We assume that the usual function spaces $C^k(\bar{\Omega})$, $C_0^k(\Omega)$, $L^p(\Omega)$, $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ are known.

The first constraint condition is given by the following set:

$$K_1 = \{v \in W_0^{1,p}(\Omega);\ v(x) \in \boldsymbol{K}(x) \text{ a.e. (resp. p.p) } x \in E\}, \tag{1}$$

where $\boldsymbol{K}(x)$ is a closed convex set in R^1 depending on x.

Next let $\partial_1\Omega$ and $\partial_2\Omega$ be two disjoint open subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\partial_1\Omega} \cup \overline{\partial_2\Omega}$ and $\partial_1\Omega \neq \emptyset$. We set

$$C^1_{(0)}(\bar{\Omega}) = \{v \in C^1(\bar{\Omega});\ v = 0 \text{ in a neighborhood of } \overline{\partial_1\Omega}\}.$$

The completion of $C^1_{(0)}(\bar{\Omega})$ with respect to the norm $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$[2] is denoted by $W^{1,p}_{(0)}(\Omega)$. The following set K_2 defines the second constraint condition:

$$K_2 = \{v \in W^{1,p}_{(0)}(\Omega);\ v(x) \in \boldsymbol{k}(x) \text{ p.p. } x \in \overline{\partial_2\Omega}\}, \tag{2}$$

where $\boldsymbol{k}(x)$ is also a closed convex set in R^1 depending on x.

The aim of this paper is to establish an estimate for the solution $u \in K_i$ of the following variational inequality:

$$\sum_{j=1}^{n} \langle a_j(x, \nabla u),\ \partial x_j(u-v)\rangle + (a_0(x, u), u-v) \leqq (f, u-v) \tag{3}$$
$$\text{for any } v \in K_i,$$

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[2] For the sake of simplicity we write $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_p$ and $(\partial_1 u, \partial_{x_2} u, \cdots, \partial_{x_n} u) = \nabla u$.

where we describe the pairing between $W_0^{1,p}(\Omega)(W_{(0)}^{1,p}(\Omega))$ and its dual by $\langle\ ,\ \rangle$ and the inner product of $L^2(\Omega)$ by $(\ ,\ )$.

At first we note that the norm $\|u\|_{1,p}$ is equivalent to the norm $\|\nabla u\|_p$ in $W_0^{1,p}(\Omega)$ or $W_{(0)}^{1,p}(\Omega)$ by Poincaré's inequality.

In the case when $K = K_1$ (resp. $K = K_2$) and $E = \Omega$ (resp. $\partial_1\Omega = \emptyset$), many authors obtained a number of results with respect to the properties of solutions of (3) for operators a_j and $K(x)$ (resp. $k(x)$) of various types. Here we do not refer explicitly to such cases.

Before stating our theorem, we will refer to results related to our problems. D. Kinderlehrer ([23]) proved the existence of a Lipschitz continuous solution in R^2 under the assumption that $a_j = a_j(\eta)$[3] satisfies $\sum_{j=1}^2 (a_j(\eta) - a_j(\eta'))(\eta_j - \eta'_j) \geqq \nu|\eta - \eta'|^2$ $(\nu>0)$ and E is a segment. In [13] J. Frehse discussed the continuity of the first order derivatives of solutions when a_j satisfies $\sum_{i,j=1}^n (\partial a_j/\partial\eta_i)(\eta)\xi_i\xi_j \geqq \nu|\xi|^2\,(\nu>0)$ and E is an $(n-1)$-dimensional manifolds. Also G. H. Williams ([38]) proved the existence of a Lipschitz continuous solution for the case that a_j satisfies the inequality $\sum_{j=1}^n (a_j(x, z, \eta) - a_j(x, z', \eta'))(\eta_j - \eta'_j) + (a_0(x, z, \eta) - a_0(x, z', \eta'))(z - z') \geqq C|\eta - \eta'|^2\,(C > 0)$ and E is a subset in Ω. In the above three cases $K(x)$ is only of the type $K(x) = \{u(x) \geqq \phi(x)\}$ for a given function $\phi(x)$. Additionally when $n = 2$ and $\partial_2\Omega = (a, b) \times \{x_2 = 0\}$ particularly, D. Kinderlehrer ([24]) showed that solutions belong to class $C^{1,\alpha}(\Omega \cup \partial_2\Omega)$ under the same condition on a_j as in [23] and with the assumption that $k(x) = \{u(x) \geqq \psi(x)\}$ for some function $\psi(x)$.

Secondly we mention results for the cases when a_j degenerate. To begin with we impose the next conditions on a_j $(j = 0, 1, \cdots, n)$:

$$(4)\qquad \begin{cases} \sum_{j=1}^n a_j(x, z, \eta)\eta_j \geqq a|\eta|^p - b(x)|z|^p - l(x), \\ |a_j(x, z, \eta)| \leqq a'|\eta|^{p-1} + g(x)|z|^{p-1} + h(x), \qquad j = 1, \cdots, n \\ |a_0(x, z, \eta)| \leqq d(x)|\eta|^{p-1} + e(x)|z|^{p-1} + m(x). \end{cases}$$

Assuming that a_j $(j = 1, \cdots, n)$ are independent of z and that b, g, h, and l equal to zero and that furthermore $a_0 = |z|^{p-2}z$, H. Beirão da Veiga ([35]) obtained the $L^\infty(\Omega)$ estimate for solutions of variational inequalities under the constraint condition that $u(x) \geqq \phi_i(x)$ on E_i $(i = 1, \cdots, m)$. In [35] the boundary condition is of either Dirichlet, Neumann or Signorini's

[3] In place of the operator $a_j(x, u, \nabla u)$ we consider the function $a_j(x, z, \eta)$ defined on $\Omega \times R^1 \times R^n$.

type, furthermore the mixed type of these conditions is also treated. Moreover in [36] he established the $L^\infty(\Omega)$ estimate and the $C^\alpha(\bar{\Omega})$ estimate under the assumption (4) for the constraint condition $K = \{v \in W^{1,p}(\Omega);\ v(x) \geqq \psi(x)$ on $\partial\Omega$, $v(x) = \psi(x)$ on $\Gamma\}$. H. Beirão da Veiga-F. Conti ([37]) proved that solutions belong to $C^\alpha(\bar{\Omega})$ when a_j $(j = 1, \cdots, n)$ are independent of z and $a_0 = 0$ in (4), in addition a_j satisfies the condition $\sum_{j=1}^n (a_j(x, \eta) - a_j(x, \eta'))(\eta_j - \eta'_j) > 0$. Here the constraint condition is of type: $u(x) \geqq \phi(x)$ on E. When $b = e = g = 0$ in (4), A. Domarkas ([9], [10]) showed that solutions belong to $C^\alpha(\bar{\Omega})$ for the constraint condition K: $K = \{v \in W_0^{1,p}(\Omega);\ v(x) \geqq \phi_1(x)$ on E_1 and $v(x) \leqq \phi_2(x)$ on $E_2\}$ or $K = \{v \in W^{1,p}(\Omega);\ v(x) \geqq \psi_1(x)$ on Γ_1 and $v(x) \leqq \psi_2(x)$ on $\Gamma_2\}$. In the above works E and E_i are subsets of Ω and Γ, Γ_i are subsets of $\partial\Omega$, additionally $\phi_i(x)$, $\psi(x)$ and $\psi_i(x)$ are some given functions.

Throughout this paper let $p \geqq 2$ and let us assume that $\alpha \geqq 0$ if $p \geqq n$, $0 \leqq \alpha < n(p-1)/(n-p) - 1$ if $p < n$.

Now we impose the following assumptions on a_j $(j = 0, 1, \cdots, n)$:

ASSUMPTION A.

(I) (i) $a_j = a_j(x, \eta) \in C^0(\Omega \times R^n) \cap C^1(\Omega \times (R^n - \{\mathbf{0}\}))$, $a_j(x, \mathbf{0}) = 0$, $j = 1, \cdots, n$.

(ii)
$$\sum_{i,j=1}^n \frac{\partial a_j}{\partial \eta_i}(x, \eta)\xi_i\xi_j \geqq \gamma(\kappa_0 + |\eta|^{p-2})|\xi|^2,$$
$$\left|\frac{\partial a_j}{\partial \eta_i}(x, \eta)\right| \leqq \Lambda(\kappa_0 + |\eta|^{p-2}), \qquad i = 1, \cdots, n,$$
$$\left|\frac{\partial a_j}{\partial x_k}(x, \eta)\right| \leqq \Lambda(\kappa_0|\eta| + |\eta|^{p-1}), \quad k = 1, \cdots, n.$$

(II) (i) $a_0 = a_0(x, z) \in C^0(\Omega \times R^1) \cap C^1(\Omega \times (R^1 - \{0\}))$, $a_0(x, 0) = 0$.

(ii)
$$\frac{\partial a_0}{\partial z}(x, z) \geqq 0, \qquad \left|\frac{\partial a_0}{\partial z}(x, z)\right| \leqq \Lambda(|z|^\alpha + 1),$$
$$\left|\frac{\partial a_0}{\partial x_k}(x, z)\right| \leqq \Lambda(|z|^{\alpha+1} + 1), \qquad k = 1, \cdots, n.$$

in the above $x \in \Omega$, $z \in R^1 - \{0\}$, $\eta \in R^n - \{\mathbf{0}\}$ and $\xi \in R^n$. And κ_0 is a non-negative constant, γ and Λ are some positive constants.

Hereafter we write by the same C all constants independent of u and f, which appear in (3). We define the following function $\Phi(x)$ in R^n:

$$\Phi(x) = \begin{cases} \mathrm{dis}\,(x, E \cup \partial\Omega) & \text{for the case of } K = K_1 \\ \mathrm{dis}\,(x, \partial\Omega) & \text{for the case of } K = K_2 \end{cases}$$

where $\mathrm{dis}(A, B)$ is the distance between A and B.

Our theorem is as follows.

THEOREM. *Under the assumption A if $f \in W^{1,p^*}(\Omega)$ (resp. $L^2(\Omega)$) in the case of $\kappa_0 = 0$* (resp. $\kappa_0 \neq 0$), *the following assertions hold for the solution u of the nonlinear variational inequality* (3).

(I) *In any case $\Phi(x)a_j(x, \nabla u) \in W^{1,p^*}(\Omega)$, $\Phi(x)|\nabla u|^{p/2} \in W^{1,2}(\Omega)$, $j = 1, \cdots, n$.*

(II) *If $\kappa_0 = 0$,*

$$\|\Phi(\cdot)a_j(\cdot, \nabla u)\|_{1,p^*}^{p^*} + \|\Phi(\cdot)|\nabla u|^{p/2}\|_{1,2}^2 \leqq C(1 + \|f\|_{1,p^*}^{p^*} + \|f\|_{p^*}^{(\alpha+2)/(p-1)}).$$

(III) *If $\kappa_0 \neq 0$, $\Phi(x)\nabla u \in W^{1,2}(\Omega)$ and*

$$\|\Phi(\cdot)\nabla u\|_{1,2}^2 + \|\Phi(\cdot)a_j(\cdot, \nabla u)\|_{1,p^*}^{p^*} + \|\Phi(\cdot)|\nabla u|^{p/2}\|_{1,2}^2 \leqq C(1 + \|f\|_2^{2\beta/p}),$$

where $\beta = \max(p, \alpha + 2)$ and p^* is the dual number of p, i.e., $1/p + 1/p^* = 1$.

In connection to the estimate in our theorem the following results were obtained by G. H. Yakovlev. He gave first the estimate of $\|\partial_{x_j}(|\partial_{x_i}u|^{p/2})\|_{L^2(\Omega_\delta)}$ for solutions of the variational problem which is derived from the functional of the form $I(v) = \int_\Omega \left\{\sum_{j=1}^n |\partial_{x_j}v|^p + a(x)v\right\}dx$ with the Dirichlet boundary condition ([40]). Here $\Omega_\delta = \{x \in \Omega;\ \mathrm{dis}(x, \partial\Omega) > \delta\}$.

In [43] also he obtained estimates of

$$\|a_j(\cdot, u, \nabla u)\|_{W^{1,p^*}(\Omega_\delta)} \quad \text{and} \quad \left\|\frac{\partial a_j}{\partial x_k}(\cdot, u, \nabla u)\right\|_{W^{1,p^*}(\Omega_\delta)}$$

for weak solutions of the nonlinear elliptic equation $\sum_{j=1}^n \partial_{x_j}(a_j(x, u, \nabla u)) = a_0(x, u, \nabla u)$ under weaker assumptions than ours. The method in [43] is to use the quotient of differentials with the usual parallel transformation, namely,

$$h^{-1}[u(x_1, x_2, \cdots, x_{i-1}, x_i + h, x_{i+1}, \cdots, x_n) - u(x)],$$

which implies that the estimate of the norm is obliged to be restricted in Ω_δ. In this paper we use a transformation with weight function $\Phi(x)$, so we can estimate the norm in the whole Ω. Our estimates are very similar to that of G. H. Yakovlev ([41]), where a nonlinear elliptic equation with the Dirichlet boundary condition was treated. He prepared the estimate for smooth solutions and applied the Galerkin's method, so his technique can not be applied to our variational inequality.

Finally we refer to the regularity of weak soultions of the nonlinear elliptic equation. Let us put $v = u \pm \phi$ in (3) for a solution u and any function ϕ in $C_0^\infty(\Omega \setminus E)$ (resp. $C_0^\infty(\Omega)$) in the case of $K = K_1$ (resp. $K = K_2$), then we see that u is a weak solution of the nonlinear elliptic equation

$$(5) \qquad -\sum_{j=1}^{n} \partial_{x_j}(a_j(x, \nabla u)) + a(x, u) = f$$

in $\Omega \setminus E$ (resp. Ω).

J. C. Evans ([11]) and J. L. Lewis ([26]) proved that weak solutions of (5) belong to $C^{1,\alpha}_{\mathrm{loc}}(\Omega \setminus E)$ (resp. $C^{1,\alpha}_{\mathrm{loc}}(\Omega)$) when $a_j = |\nabla u|^{p-2}\partial_{x_j} u$ and $a_0 = 0$. Besides E. DiBenedetto ([8]) and P. Tolksdorf ([33]) showed the same results under weaker assumptions on a_j and a_0 than those of [11] and [26].

1.

The existence of a unique solution for the variational inequality (3) is derived from Lemma 1.2 in this section.

First we will prepare the following lemma which will be frequently used in this paper:

LEMMA 1.1. *Under the assumption* A *the following assertions hold*:
(i) a_j $(j = 0, 1, \cdots, n)$ *are estimated in such a way that*

$$(1.1) \qquad |a_j(x, \eta)| \leqq C(\kappa_0 + |\eta|^{p-2})|\eta|, \qquad j = 1, \cdots, n,$$

$$(1.2) \qquad |a_0(x, z)| \leqq C(1 + |z|^{\alpha})|z|.$$

(ii) (P. Tolksdorf [33, p. 129] and P. Lindqvist [28, p. 310]) *There exists a positive number γ_0 depending only on γ and p such that*

$$(1.3) \qquad \sum_{j=1}^{n} (a_j(x, \eta) - a_j(x, \eta'))(\eta_j - \eta'_j) \geqq \gamma_0(\kappa_0 + |\eta|^{p-2} + |\eta'|^{p-2})|\eta - \eta'|^2,$$

$$(1.4) \qquad (a_0(x, z) - a_0(x, z'))(z - z') \geqq 0.$$

Here $x \in \Omega$, η, $\eta' \in R^n$ and z, $z' \in R^1$.

Proof.

(i) The estimates (1.1) and (1.2) are derived from (I)-(ii) and (II)-(ii) in the assumption A and the equalities

$$a_j(x, \eta) = \int_0^1 \sum_{i=1}^{n} \frac{\partial a_j}{\partial \eta_i}(x, t\eta)\eta_i dt, \qquad a_0(x, z) = \int_0^1 \frac{\partial a_0}{\partial z}(x, tz)z\, dt.$$

(ii) Without loss of the generality we may assume that $|\eta| \leqq |\eta'|$. According to simple calculations and (I)-(ii) in the assumption A, we see that

$$(1.5) \qquad \begin{aligned}&\sum_{j=1}^{n}(a_j(x,\eta)-a_j(x,\eta'))(\eta_j-\eta'_j)\\ &=\sum_{j=1}^{n}\int_0^1\frac{d}{dt}a_j(x,t\eta+(1-t)\eta')(\eta_j-\eta'_j)dt\\ &\geqq\int_0^{1/4}\sum_{i,j=1}^{n}\frac{\partial a_j}{\partial\eta_i}(x,t\eta+(1-t)\eta')(\eta_i-\eta'_i)(\eta_j-\eta'_j)dt\\ &\geqq\gamma\int_0^{1/4}(\kappa_0+|t\eta+(1-t)\eta'|^{p-2})|\eta-\eta'|^2dt.\end{aligned}$$

On the other hand it holds that

$$(1.6) \qquad \begin{aligned}|t\eta+(1-t)\eta'|&\geqq(1-t)|\eta'|-t|\eta|\\ &=(3/4-t)|\eta'|-(t+1/4)|\eta|+1/4(|\eta'|+|\eta|)\\ &\geqq(t+1/4)(|\eta'|-|\eta|)+1/4(|\eta'|+|\eta|)\\ &\geqq 1/4(|\eta'|+|\eta|),\end{aligned}$$

for all $t\in[0,1/4]$. The inequalities (1.5) and (1.6) imply (1.3). The estimate (1.4) can be proved more easily. Q.E.D.

Remark. In (1.6) we note that the estimate $|t\eta+(1-t)\eta'|\geqq(1/4)\cdot|\eta'-\eta|$ holds for all $t\in[0,1/4]$. Thus we have the following estimate in place of (1.3):

$$(1.7) \qquad \sum_{j=1}^{n}(a_j(x,\eta)-a_j(x,\eta'))(\eta_j-\eta'_j)\geqq\gamma_0(\kappa_0+|\eta-\eta'|^{p-2})|\eta-\eta'|^2.$$

This is due to K. L. Kuttler Jr. ([25]).

LEMMA 1.2. *For $u, v\in W_0^{1,p}(\Omega)(W_{(0)}^{1,q}(\Omega))$ we define the operator A in such a way that $(A(u),v)=\sum_{j=1}^{n}\langle a_j(x,\nabla u),\partial_{x_j}v\rangle+(a_0(x,u),v)$, then the operator A is pseudo-monotone and coercive from $W_0^{1,p}(\Omega)(W_{(0)}^{1,p}(\Omega))$ to its dual space.*

Proof. We write $W_0^{1,p}(\Omega)(W_{(0)}^{1,p}(\Omega))$ simply by V and its dual space by V'.

In order to prove the pseudo-monotonicity of the operator A, it is enough to show that the operator A is (i) bounded, (ii) hemi-continuous and (iii) monotone (J.L. Lions [29], p. 179).

(i) Boundedness.

According to the definition

$$(1.8) \qquad |(A(u),v)|\leqq\sum_{j=1}^{n}|\langle a_j(x,\nabla u),\partial_{x_j}v\rangle|+|(a_0(x,u),v)|.$$

We estimate each term on the right-hand side of (1.8).

At first we obtain

$$(1.9)\qquad |\langle a_j(x, \nabla u), \partial_{x_j} v\rangle| \leqq C\int_\Omega (\kappa_0 + |\nabla u|^{p-2})|\nabla u|\cdot|\nabla v|\, dx \leqq C(\|\nabla u\|_{p^*}\|\nabla v\|_p + \|\nabla u\|_p^{p-1}\|\nabla v\|_p),$$

by (1.1) and Hölder's inequality. Since $p \geqq 2$, p and p^* satisfy $1 < p^* \leqq 2 \leqq p$. Thus the inequality $\|\nabla u\|_{p^*} \leqq C\|\nabla u\|_p$ is verified. Therefore from (1.9) we deduce the estimate

$$(1.10)\qquad |\langle a_j(x, \nabla u), \partial_{x_j} v\rangle| \leqq C(\|u\|_V + \|u\|_V^{p-1})\|v\|_V .$$

On account of Lemma 1.1 and Hölder's inequality, we obtain similarly the inequality

$$(1.11)\qquad |(a_0(x, u), v)| \leqq \{C\|u\|_{p^*}\|v\|_p + (\|u\|_{(\alpha+1)p^*})^{\alpha+1}\|v\|_p\} .$$

By Poincaré's inequality, we get

$$(1.12)\qquad \|u\|_{p^*} \leqq C\|\nabla u\|_{p^*} \leqq C\|\nabla u\|_p \leqq C\|u\|_V ,$$

$$(1.13)\qquad \|v\|_p \leqq C\|\nabla v\|_p \leqq C\|v\|_V .$$

And Sobolev's imbedding theorem yields the inequality

$$(1.14)\qquad \|u\|_{(\alpha+1)p^*} \leqq C\|u\|_{1,p} ,$$

because $0 \leqq \alpha < n(p-1)/(n-p) - 1$ for $p < n$. From (1.11)–(1.14) it follows that

$$(1.15)\qquad |(a_0(x, u), v)| \leqq C(\|u\|_V + \|u\|_V^{\alpha+1})\|v\|_V .$$

Combining (1.8), (1.10) with (1.15), we derive the following estimate from the definition of the dual norm of $A(u)$ in V':

$$(1.16)\qquad \|A(u)\|_{V'} \leqq C(\|u\|_V^{p-1} + \|u\|_V^{\alpha+1} + \|u\|_V).$$

(ii) Hemi-continuity.

Let us suppose that $|\lambda| \leqq \lambda_0$. For u, v and $w \in V$, it is valid that

$$(1.17)\qquad (A(u + \lambda v), w) = \int_\Omega \left\{\sum_{j=1}^n a_j(x, \nabla(u + \lambda v))\partial_{x_j} w + a_0(x, u + \lambda v)w\right\}dx ,$$

by the definition of the operator A. Applying the inequalities (1.1) and (1.2) to each term of the integrand in (1.17), we have

$$|a_j(x, \nabla(u+\lambda v))\partial_{x_j} w| \leqq C\{\kappa_0(|\nabla u| + \lambda_0|\nabla v|) + |\nabla u|^{p-1} + \lambda_0^{p-1}|\nabla v|^{p-1}\}|\nabla w|,$$
$$|a_0(x, u+\lambda v)w| \leqq C\{(|u| + \lambda_0|v|) + |u|^{\alpha+1} + \lambda_0^{\alpha+1}|v|^{\alpha+1}\}|w|.$$

By the same way as in the proof of (i), we can prove that each term on the right-hand sides is integrable. Hence $(A(u+\lambda v), w)$ is continuous with respect to λ by Lebesgue's convergence theorem.

(iii) Monotonicity.

From the remark after Lemma 1.1, it holds for any $u, v \in V$ that

$$(1.18) \qquad (A(u) - A(v), u - v) \geqq \gamma_0(\kappa_0\|\nabla(u-v)\|_2^2 + \|\nabla(u-v)\|_p^p) \geqq 0.$$

Consequently, we have shown the pseudo-monotonicity of the operator A.

Finally we prove that the operator A is coercive. Setting $v = 0$ in (1.18), we have

$$(A(u), u) \geqq \gamma_0(\kappa_0\|\nabla u\|_2^2 + \|\nabla u\|_p^p).$$

Hence

$$\frac{(A(u), u)}{\|u\|_V} \geqq C\gamma_0\|u\|_V^{p-1} \longrightarrow \infty \qquad \text{as } \|u\|_V \longrightarrow \infty.$$

Therefore the operator A is coercive. Q.E.D.

As we have mentioned at the beginning of this section, a solution of the variational inequality (3) exists for any $f \in V'$ (J. L. Lions [29, p. 247]). The uniqueness of solutions follows immediately from (1.18).

The estimate for the gradient of the solution u is carried out in the next lemma.

LEMMA 1.3. *Under the assumption* A *if* $f \in L^{p^*}(\Omega)$ (resp. $L^2(\Omega)$) *and* $\kappa_0 = 0$ (resp. $\kappa_0 \neq 0$), *the gradient of the solution u of the variational inequality* (3) *is estimated as follows*:

$$(1.19) \qquad \kappa_0\|\nabla u\|_2^2 + \|\nabla u\|_p^p \leqq C(1 + \|f\|_{p^*}^{p^*}) \qquad (\text{resp. } C(1 + \|f\|_2^2)).$$

Proof. We denote by u_0 the particular solution of (3) for $f = 0$. We easily see that

$$(1.20) \quad \sum_{j=1}^n \langle a_j(x, \nabla u) - a_j(x, \nabla u_0), \partial_{x_j}(u - u_0)\rangle + (a_0(x, u) - a_0(x, u_0), u - u_0) \leqq (f, u - u_0).$$

Applying the inequalities (1.4) and (1.7) to the left-hand side, we get

$$(1.21) \qquad \gamma_0(\kappa_0\|\nabla u - \nabla u_0\|_2^2 + \|\nabla u - \nabla u_0\|_p^p) \leqq |(f, u - u_0)| .$$

Using Hölder's inequality and Poincaré's inequality for the right-hand side of (1.21), we obtain the estimate (1.19). Q.E.D.

We give a sufficient condition to assure that a sequence of functions converges weakly in $L^q(\Omega)$.

LEMMA 1.4. *Let u be a distribution in Ω and let $\{u_\nu\}_{\nu=1}^\infty$ be a sequence in $L^q(\Omega)\,(1<q<\infty)$ such that the norms $\|u_\nu\|_q$ are uniformly bounded. If for any $\phi \in C_0^\infty(\Omega)$,*

$$(u_\nu, \phi) \longrightarrow (u, \phi) \qquad \text{as } \nu \longrightarrow \infty ,$$

then u belongs to $L^q(\Omega)$ and the sequence u_ν converges weakly to u in $L^q(\Omega)$.

Proof. From the assumption it holds for any $\phi \in C_0^\infty(\Omega)$,

$$(1.22) \qquad |(u, \phi)| \leqq (\varlimsup_{\nu\to\infty} \|u_\nu\|_q)\|\phi\|_{q^*} .^{4)}$$

For arbitrary $v \in L^{q^*}(\Omega)$ we take a sequence $\{\phi_k\}_{k=1}^\infty$ in $C_0^\infty(\Omega)$ such that $\|\phi_k - v\|_{q^*}\to 0$ as $k\to\infty$, then from (1.22) the sequence $\{(u, \phi_k)\}_{k=1}^\infty$ is a Cauchy sequence. Accordingly, $\lim_{k\to\infty}(u, \phi_k)$ exists and from (1.22) it is trivial that the limit depends only on v and does not depend on any choice of the sequence. Hence we can express $\lim_{k\to\infty}(u, \phi_k) = l_u(v)$. It is easy to verify that l_u is a linear functional on $L^{q^*}(\Omega)$, so there exists an element L_u in $L^q(\Omega)\,(=(L^{q^*}(\Omega))')$ such that $l_u(v) = (L_u, v)$ for all $v \in L^{q^*}(\Omega)$. The definition of $l_u(v)$ ensures the equality $(u, \phi) = (L_u, \phi)$ for any function ϕ in $C_0^\infty(\Omega)$, which is dense in $L^{q^*}(\Omega)$. Thus we can conclude that $u = L_u$ and therefore $u \in L^q(\Omega)$. The remainder of the proof is due to Theorem 3 in [44, p. 121]. Q.E.D.

2.

We introduce a coordinate transformation with the weight function $\Phi(x)$ and prepare some results with respect to it. We refer to some lemmas in [20].

Let $\boldsymbol{h}$ be a non-zero vector in R^n with the length $h = |\boldsymbol{h}|$. Hereafter h is assumed to be sufficiently small. As mentioned in the introduction we put $\Phi(x) = \mathrm{dis}\,(x, \partial\Omega \cup E)$ (resp. $\mathrm{dis}\,(x, \partial\Omega)$) for $K = K_1$ (resp. $K = K_2$) and we consider the transformation of the coordinates:

4) q^* is the dual number of q, i.e., $q^*=q/(q-1)$.

$$\Phi_h\colon\ y = x + \Phi(x)\boldsymbol{h}\,. \tag{2.1}$$

We write $\boldsymbol{h} = (h_1, h_2, \cdots, h_n)$ and $\boldsymbol{J} = \partial(y_1, y_2, \cdots, y_n)/\partial(x_1, x_2, \cdots, x_n)$, then we have

$$\boldsymbol{J} = \begin{bmatrix} 1 + h_1\partial_{x_1}\Phi & h_2\partial_{x_1}\Phi & \cdots & h_n\partial_{x_1}\Phi \\ h_1\partial_{x_2}\Phi & 1 + h_2\partial_{x_2}\Phi & \cdots & h_n\partial_{x_2}\Phi \\ \vdots & \vdots & \ddots & \vdots \\ h_1\partial_{x_n}\Phi & h_2\partial_{x_n}\Phi & \cdots & 1 + h_n\partial_{x_n}\Phi \end{bmatrix}. \tag{2.2}$$

Let us put $\boldsymbol{e} = h^{-1}\boldsymbol{h}$ and let $\boldsymbol{e}$ be arbitrarily fixed. Noting that $|\partial_{x_j}\Phi| \leqq 1$ ([20, p. 57]), we see that the determinant J of $\boldsymbol{J}$ is not zero for sufficiently small h, therefore the mapping Φ_h and its inverse Φ_h^{-1} are both one-to-one from R^n onto itself. If we set $\Psi(y) = -\Phi(x)\,(= -\Phi(\Phi_h^{-1}(y)))$, it is written

$$\Phi_h^{-1}\colon\ x = y + \Psi(y)\boldsymbol{h}\,. \tag{2.3}$$

Here we remark that from (2.2) we can put $J = 1 + hJ_1$ and the determinant J^{-1} of the Jacobian $\boldsymbol{J}^{-1}$ connected with the inverse transformation Φ_h^{-1} can be described in the form $J^{-1} = 1 + hJ_2$, where J_1 and J_2 are uniformly bounded in $x \in \Omega$ and $\boldsymbol{h} \in R^n$. Furthermore the transformation Φ_h maps $x \in \Omega$ to $y \in \Omega$ and $x \in \Omega^c$ to $y \in \Omega^c$ respectively, so it is a one-to-one mapping from Ω onto itself.

Now we define

$$\begin{cases} (S_h u)(x) = u(x + \Phi(x)\boldsymbol{h}),\ (T_h u)(y) = u(y + \Psi(y)\boldsymbol{h})\,, \\ (P_h u)(x) = h^{-1}[(S_h u)(x) - u(x)]\,, \\ (Q_h u)(y) = h^{-1}[(T_h u)(y) - u(y)]\,. \end{cases} \tag{2.4}$$

Hereafter we write simply by $S_h u$, $T_h u$, $\cdots$ the functions $(S_h u)(x)$, $(T_h u)(y)$, $\cdots$, respectively.

LEMMA 2.1 ([20, p. 58, 59]).

(i) *We have*

$$\begin{cases} \nabla_x(S_h u) = S_h\nabla_x u + h(\boldsymbol{e}\cdot S_h\nabla_x u)\nabla_x\Phi\,, \\ \nabla_y(T_h u) = T_h\nabla_y u + h(\boldsymbol{e}\cdot T_h\nabla_y u)\nabla_y\Psi\,, \\ \nabla_x(P_h u) = P_h\nabla_x u + (\boldsymbol{e}\cdot S_h\nabla_x u)\nabla_x\Phi\,, \\ \nabla_y(Q_h u) = Q_h\nabla_y u + (\boldsymbol{e}\cdot T_h\nabla_y u)\nabla_y\Psi\,. \end{cases} \tag{2.5}$$

(ii) *If $u \in W^{1,q}(\Omega)\,(1 < q < \infty)$, there exists a constant C independent of $\boldsymbol{h}$ and u such that*

$$(2.6) \qquad \|P_h u\|_q, \qquad \|Q_h u\|_q \leqq C\|\nabla u\|_q .$$

We do not repeat the proof of the above lemma, since it is parallel to that of [20].

The following lemma is as important as Lemma 1.4 for the proof of our theorem.

LEMMA 2.2. *If* $u \in L^q(\Omega)\,(1 < q < \infty)$, *then for any function* ϕ *in* $C_0^\infty(\Omega)$ *it holds that*

$$(2.7) \qquad (P_h u - J_2 u, \phi) \longrightarrow ((e\cdot\nabla)(\Phi u), \phi) \qquad \text{as } h \longrightarrow 0 ,$$

where the derivative $(e\cdot\nabla)(\Phi u)$ *of* Φu *is in the sense of the distribution.*

Proof. For any $u \in L^q(\Omega)$ and $\phi \in L^{q^*}(\Omega)$ the formula

$$(2.8) \qquad (P_h u, \phi) - (J_2 u, \phi) = -(S_h u, P_h \phi)$$

holds. The function ϕ belongs to $C_0^\infty(\Omega)$ in this case, so we can prove the following convergence by the similar technique as in [20] (Lemma 5 in p. 59):

$$(2.9) \qquad P_h \phi \longrightarrow \Phi(e\cdot\nabla)\phi \quad \text{in } L^{q^*}(\Omega) \quad \text{as } h \longrightarrow 0 .$$

On the other hand it is an immediate consequence that

$$(2.10) \qquad S_h u \longrightarrow u \quad \text{in } L^q(\Omega) \quad \text{as } h \longrightarrow 0 ,$$

from the fact that $u \in L^q(\Omega)$. Therefore the right-hand side on (2.8) tends to $-(u, \Phi(e\cdot\nabla)\phi)$ by virtue of (2.9) and (2.10). In this way we arrive at the assertion of this lemma. Q.E.D.

3.

This section is devoted to the statement and the proof of the main proposition in this paper. It is very important for the proof of our theorem.

PROPOSITION. *Let us assume the assumption* A. *Then the following estimates hold for the solution* u *of the variational inequality* (3):

(i) *If* $\kappa_0 = 0$ *and* $f \in W^{1,p^*}(\Omega)$,

$$\||S_h\nabla u|^{(p-2)/2} P_h \nabla u\|_2^2 + \||\nabla u|^{(p-2)/2} P_h \nabla u\|_2^2 \leqq C(1 + \|f\|_{1,p^*}^{p^*} + \|f\|_{p^*}^{(\alpha+2)/(p-1)}) .$$

(ii) *If* $\kappa_0 \neq 0$ *and* $f \in L^2(\Omega)$,

$$\|P_h \nabla u\|_2^2 + \||S_h\nabla u|^{(p-2)/2} P_h \nabla u\|_2^2 + \||\nabla u|^{(p-2)/2} P_h \nabla u\|_2^2 \leqq C(1 + \|f\|_2^{2\beta/p}) ,$$

where $\beta = \max(p, \alpha + 2)$.

Proof. From now on we write simply $\partial_{x_j} = \partial_j$ or $\partial_{y_j} = \partial_j$ and we abbreviate the notation of sums.

As $S_h u$ and $T_h u$ belong to K_i ($i = 1$ or 2) (see p. 60 in [20]), so we can put $v = S_h u$ and $T_h u$ in (3). Hence we obtain the inequalities

$$\langle a_j(x, \nabla u),\ \partial_j(u - S_h u)\rangle + (a_0(x, u), u - S_h u) \leqq (f, u - S_h u)\,,$$
$$\langle a_j(y, \nabla u),\ \partial_j(u - T_h u)\rangle + (a_0(y, u),\ u - T_h u) \leqq (f, u - T_h u)\,.$$

Adding these two inequalities, we have

$$\begin{aligned} &\langle a_j(x, \nabla u), \partial_j(u - S_h u)\rangle + \langle a_j(y, \nabla u),\ \partial_j(u - T_h u)\rangle \\ &\quad + (a_0(x, u),\ u - S_h u) + (a_0(y, u),\ u - T_h u) \\ &\quad \leqq (f, u - S_h u) + (f, u - T_h u)\,. \end{aligned} \tag{3.1}$$

Denoting each term on the left-hand side by I_j ($j = 1, 2, 3, 4$) in turn from the left, we write I_1 and I_2 as follows by Lemma 2.1:

$$I_1 = \langle a_j(x, \nabla u),\ \partial_j u - S_h \partial_j u\rangle - h\langle a_j(x, \nabla u),\ (\boldsymbol{e}\cdot S_h \nabla u)\partial_j \Phi\rangle\,,$$
$$I_2 = \langle (1 + hJ_1) S_h a_j(x, \nabla u),\ S_h \partial_j u - \partial_j u\rangle - h\langle a_j(y, \nabla u),\ (\boldsymbol{e}\cdot T_h \nabla u)\partial_j \Psi\rangle\,.$$

Consequently, we get

$$\begin{aligned} I_1 + I_2 = &\langle S_h a_j - a_j,\ S_h \partial_j u - \partial_j u\rangle + h\langle J_1 S_h a_j,\ S_h \partial_j u - \partial_j u\rangle \\ &- h\langle a_j,\ (\boldsymbol{e}\cdot T_h \nabla u)\partial_j \Psi\rangle - h\langle a_j,\ (\boldsymbol{e}\cdot S_h \nabla u)\partial_j \Phi\rangle\,. \end{aligned} \tag{3.2}$$

After putting the right-hand side of (3.2) by $\sum_{j=5}^{8} I_j$, we estimate each term I_j. We rewrite I_5 in the form

$$\begin{aligned} I_5 = &\langle a_j(S_h x, S_h \nabla u) - a_j(x, S_h \nabla u),\ S_h \partial_j u - \partial_j u\rangle \\ &+ \langle a_j(x, S_h \nabla u) - a_j(x, \nabla u),\ S_h \partial_j u - \partial_j u\rangle\,. \end{aligned} \tag{3.3}$$

For the first term in (3.3), the following estimate holds:

$$\begin{aligned} &|\langle a_j(S_h x,\ S_h \nabla u) - a_j(x,\ S_h \nabla u),\ S_h \partial_j u - \partial_j u\rangle| \\ &\quad \leqq Ch^2 \int_\Omega (\kappa_0 |S_h \nabla u|\cdot|P_h \nabla u| + |S_h \nabla u|^{p-1} |P_h \nabla u|) dx \\ &\quad \leqq Ch^2 (\kappa_0 \|\nabla u\|_2 \|P_h \nabla u\|_2 + \|\nabla u\|_p^{p/2} \|\, |S_h \nabla u|^{(p-2)/2} P_h \nabla u\|_2)\,. \end{aligned} \tag{3.4}$$

Here we have used the equality

$$a_j(S_h x,\ S_h \nabla u) - a_j(x,\ S_h \nabla u) = \int_0^1 \sum_{k=1}^n \frac{\partial a_j}{\partial x_k}(x + \theta\Phi(x)\boldsymbol{h},\ S_h \nabla u) h_k \Phi(x) d\theta$$

and the inequality in the assumption A.

Secondly the remainding term in (3.3) is estimated from below by (1.3) as follows:

$$(3.5)\quad \langle a_j(x, S_h\nabla u) - a_j(x, \nabla u), S_h\partial_j u - \partial_j u\rangle \geqq h^2\gamma_0\{\kappa_0\|P_h\nabla u\|_2^2 + \||S_h\nabla u|^{(p-2)/2}P_h\nabla u\|_2^2 + \||\nabla u|^{(p-2)/2}P_h\nabla u\|_2^2\}.$$

In this way (3.3), (3.4) and (3.5) yield that

$$(3.6)\quad I_5 \geqq \frac{\gamma_0 h^2}{2}\{\kappa_0\|P_h\nabla u\|_2^2 + \||S_h\nabla u|^{(p-2)/2}P_h\nabla u\|_2^2 + \||\nabla u|^{(p-2)/2}P_h\nabla u\|_2^2\} - Ch^2(\kappa_0\|\nabla u\|_2^2 + \|\nabla u\|_p^p).$$

Next we see by Lemma 1.1 that

$$(3.7)\quad |I_6| \leqq Ch^2\int_\Omega(\kappa_0 + |S_h\nabla u|^{p-2})|S_h\nabla u|\cdot|P_h\nabla u|\,dx \leqq Ch^2\{\kappa_0\|\nabla u\|_2\|P_h\nabla u\|_2 + \|\nabla u\|_p^{p/2}\||S_h\nabla u|^{(p-2)/2}P_h\nabla u\|_2\}.$$

Before proceeding to successive terms I_7 and I_8, we note that we can write $\nabla_y\Psi = \nabla_x\Psi({}^tJ^{-1}) = -(\nabla_x\Phi)(I + hH)$, where I is the unit matrix and each component of the matrix H is essentially bounded (see p. 64 in [20]). Consequently, it is seen that

$$\begin{aligned}(3.8)\quad I_7 + I_8 &= -h\{\langle a_j, (\boldsymbol{e}\cdot T_h\nabla u)[(-\nabla_x\Phi(x))(I + hH)]_j\rangle + \langle a_j, (\boldsymbol{e}\cdot S_h\nabla u)\partial_j\Phi\rangle\}^{5)}\\ &= -h\{-\langle(1 + hJ_1)S_h a_j, (\boldsymbol{e}\cdot\nabla u)\partial_j\Phi\rangle + \langle a_j, (\boldsymbol{e}\cdot S_h\nabla u)\partial_j\Phi\rangle\}\\ &\quad + h^2\langle a_j, (\boldsymbol{e}\cdot T_h\nabla u)(\nabla_x\Phi H)_j\rangle\\ &= h\{\langle S_h a_j, (\boldsymbol{e}\cdot\nabla u)\partial_j\Phi\rangle - \langle a_j, (\boldsymbol{e}\cdot S_h\nabla u)\partial_j\Phi\rangle\}\\ &\quad + h^2\langle J_1 S_h a_j, (\boldsymbol{e}\cdot\nabla u)\partial_j\Phi\rangle + h^2\langle a_j, (\boldsymbol{e}\cdot T_h\nabla u)(\nabla_x\Phi H)_j\rangle.\end{aligned}$$

Again we write $I_7 + I_8 = \sum_{j=9}^{11} I_j$ in (3.8). We rewrite I_9 in the same way as in (3.3), i.e.,

$$\begin{aligned}(3.9)\quad I_9 &= h\langle a_j(S_h x, S_h\nabla u) - a_j(x, S_h\nabla u), (\boldsymbol{e}\cdot\nabla u)\partial_j\Phi\rangle\\ &\quad + h\langle a_j(x, S_h\nabla u) - a_j(x, \nabla u), (\boldsymbol{e}\cdot\nabla u)\partial_j\Phi\rangle\\ &\quad + h\langle a_j(x, \nabla u), \boldsymbol{e}\cdot(\nabla u - S_h\nabla u)\partial_j\Phi\rangle.\end{aligned}$$

By applying the similar technique as in (3.4) to the first term of (3.9), we get the inequality

$$(3.10)\quad h|\langle a_j(S_h x, S_h\nabla u) - a_j(x, S_h\nabla u), (\boldsymbol{e}\cdot\nabla u)\partial_j\Phi\rangle| \leqq Ch^2(\kappa_0\|\nabla u\|_2^2 + \|\nabla u\|_p^p).$$

5) We write the j-th component of a vector U by U_j.

The second term of (3.9) is estimated as follows:

$$\begin{aligned}(3.11)\qquad & h|\langle a_j(x, S_h\nabla u) - a_j(x, \nabla u), (e\cdot\nabla u)\partial_j\Phi\rangle| \\ & \leqq Ch^2\int_\Omega (\kappa_0 + |\nabla u|^{p-2} + |S_h\nabla u|^{p-2})|P_h\nabla u|\cdot|\nabla u|dx \\ & \leqq Ch^2\{\kappa_0\|\nabla u\|_2\|P_h\nabla u\|_2 + \|\nabla u\|_p^{p/2}(\||\nabla u|^{(p-2)/2}P_h\nabla u\|_2 \\ & + \||S_h\nabla u|^{(p-2)/2}P_h\nabla u\|_2)\}.\end{aligned}$$

In the above estmiates we have used the assumption A and the equality

$$a_j(x, S_h\nabla u) - a_j(x, \nabla u) = \int_0^1 \sum_{i=1}^n \frac{\partial a_j}{\partial \eta_i}(x, \theta S_h\nabla u + (1-\theta)\nabla u)(S_h\partial_i u - \partial_i u)d\theta.$$

According to Lemma 1.1 the last term of (3.9) is estimated as follows:

$$\begin{aligned}(3.12)\qquad & h|\langle a_j, e\cdot(\nabla u - S_h\nabla u)\partial_j\Phi\rangle| \leqq Ch^2\int_\Omega (\kappa_0 + |\nabla u|^{p-2})|\nabla u|\cdot|P_h\nabla u|dx \\ & \leqq Ch^2(\kappa_0\|\nabla u\|_2\|P_h\nabla u\|_2 + \|\nabla u\|_p^{p/2}\||\nabla u|^{(p-2)/2}P_h\nabla u\|_2).\end{aligned}$$

By virtue of (3.9)–(3.12) we get

$$\begin{aligned}(3.13)\qquad |I_9| \leqq\ & Ch^2\{\kappa_0\|\nabla u\|_2\|P_h\nabla u\|_2 + \kappa_0\|\nabla u\|_2^2 + \|\nabla u\|_p^p \\ & + \|\nabla u\|_p^{p/2}(\||\nabla u|^{(p-2)/2}P_h\nabla u\|_2 + \||S_h\nabla u|^{(p-2)/2}P_h\nabla u\|_2)\}.\end{aligned}$$

On the other hand by Lemma 1.1, I_{10} and I_{11} are immediately estimated, that is,

$$(3.14)\qquad |I_{10}|, |I_{11}| \leqq Ch^2(\kappa_0\|\nabla u\|_2^2 + \|\nabla u\|_p^p).$$

Consequently (3.8), (3.13) and (3.14) yield that

$$\begin{aligned}(3.15)\qquad |I_7 + I_8| \leqq\ & Ch^2\{\kappa_0(\|\nabla u\|_2\|P_h\nabla u\|_2 + \|\nabla u\|_2^2) + \|\nabla u\|_p^p \\ & + \|\nabla u\|_p^{p/2}(\||\nabla u|^{(p-2)/2}P_h\nabla u\|_2 + \||S_h\nabla u|^{(p-2)/2}P_h\nabla u\|_2)\}.\end{aligned}$$

With the aid of (3.2), (3.6), (3.7) and (3.15) we obtain the inequality

$$\begin{aligned}(3.16)\qquad I_1 + I_2 \geqq\ & \frac{\gamma_0 h^2}{4}\{\kappa_0\|P_h\nabla u\|_2^2 + \||S_h\nabla u|^{(p-2)/2}P_h\nabla u\|_2^2 \\ & + \||\nabla u|^{(p-2)/2}P_h\nabla u\|_2^2\} - Ch^2(\kappa_0\|\nabla u\|_2^2 + \|\nabla u\|_p^p).\end{aligned}$$

Now we estimate the sum of I_3 and I_4. It is written in the form

$$\begin{aligned}(3.17)\qquad I_3 + I_4 &= (a_0, u - S_h u) + ((1 + hJ_1)S_h a_0, S_h u - u) \\ &= (a_0(S_h x, S_h u) - a_0(x, S_h u), S_h u - u) \\ &\quad + (a_0(x, S_h u) - a_0(x, u), S_h u - u) + h(J_1 S_h a_0, S_h u - u).\end{aligned}$$

Similarly as in (3.4) we can estimate the first term of (3.17) as follows:

$$
\begin{aligned}
(3.18)\qquad & |(a_0(S_h x, S_h u) - a_0(x, S_h u), S_h u - u)| \\
& \leqq Ch^2 \int_\Omega (1 + |S_h u|^{\alpha+1})|P_h u| dx \\
& \leqq Ch^2\{\|P_h u\|_p + \|P_h u\|_p(\|u\|_{(\alpha+1)p^*})^{\alpha+1}\} \\
& \leqq Ch^2(\|\nabla u\|_p + \|\nabla u\|_p^{\alpha+2}) .
\end{aligned}
$$

Here we have used Lemma 2.1 and (1.14).
Because of Lemma 1.1 the second term of (3.17) is non-negative, i.e.,

$$(3.19)\qquad (a_0(x, S_h u) - a_0(x, u), S_h u - u) \geqq 0 .$$

For the last term in (3.17) we attain the following estimate from Lemma 1.1 and Hölder's inequality:

$$
\begin{aligned}
(3.20)\qquad h|(J_1 S_h a_0, S_h u - u)| & \leqq Ch^2 \int_\Omega (1 + |S_h u|^\alpha)|S_h u| \cdot |P_h u| dx \\
& \leqq Ch^2\{\|P_h u\|_p \|S_h u\|_{p^*} + \|P_h u\|_p(\|S_h u\|_{(\alpha+1)p^*})^{\alpha+1}\} .
\end{aligned}
$$

Similarly as in (3.18) we obtain the following inequality from (3.20):

$$(3.21)\qquad h|(J_1 S_h a_0, S_h u - u)| \leqq Ch^2(\|\nabla u\|_p^2 + \|\nabla u\|_p^{\alpha+2}) .$$

In this way we arrive at the inequality

$$(3.22)\qquad I_3 + I_4 \geqq - Ch^2(\|\nabla u\|_p^{\alpha+2} + \|\nabla u\|_p + \|\nabla u\|_p^2)$$

from (3.17), (3.18), (3.19) and (3.21).

Finally we estimate the right-hand side of (3.1). By simple calculations it holds that

$$(3.23)\qquad (f, u - T_h u) + (f, u - S_h u) = h^2\{(P_h f, P_h u) + (J_1 S_h f, P_h u)\} .$$

Let us suppose that $\kappa_0 = 0$. From (3.23) and (ii) in Lemma 2.1 we have

$$(3.24)\qquad |\text{the right-hand side of } (3.1)| \leqq Ch^2 \|\nabla u\|_p \|f\|_{1,p^*} .$$

Combining (3.1), (3.16), (3.22) with (3.24), we deduce that if $\kappa_0 = 0$,

$$
\begin{aligned}
& \| |S_h \nabla u|^{(p-2)/2} P_h \nabla u\|_2^2 + \| |\nabla u|^{(p-2)/2} P_h \nabla u\|_2^2 \\
& \qquad \leqq C(\|\nabla u\|_p^2 + \|\nabla u\|_p^{\alpha+2} + \|\nabla u\|_p + \|\nabla u\|_p^p + \|f\|_{1,p^*}^{p^*}) .
\end{aligned}
$$

Therefore if $\kappa_0 = 0$, our proposition is correct in virtue of Lemma 1.3.

Secondly we suppose that $\kappa_0 \neq 0$. In this case we transform the first term in the brackets on the right-hand side of (3.23) into the following form by the expression (2.8) in the proof of Lemma 2.2:

$$(P_h f, P_h u) = -(S_h f, P_h P_h u) + (J_2 f, P_h u)\,.$$

Taking account of (3.23) and (ii) in Lemma 2.1, we obtain the inequality

$$(3.25)\qquad |\text{the right-hand side of (3.1)}| \leqq Ch^2\|f\|_2(\|P_h\nabla u\|_2 + \|\nabla u\|_2).$$

Hence from (3.1), (3.16), (3.22) and (3.25) it follows that

$$\begin{aligned}&\gamma_0\{\kappa_0\|P_h\nabla u\|_2^2 + \||S_h\nabla u|^{(p-2)/2}P_h\nabla u\|_2^2 + \||\nabla u|^{(p-2)/2}P_h\nabla u\|_2^2\}\\ &\qquad \leqq C(\kappa_0\|\nabla u\|_2^2 + \|\nabla u\|_p^p + \|\nabla u\|_p^{\alpha+2} + \|\nabla u\|_p + \|\nabla u\|_p^2 + \|f\|_2^2)\,.\end{aligned}$$

Thus the proof of our proposition is accomplished by the straight-forward application of Lemma 1.3. Q.E.D.

4.

In this section we prove our theorem.

At first we show that $\Phi(x)a_j(x, \nabla u)$ belongs to $W^{1,p^*}(\Omega)$ and we estimate its $W^{1,p^*}(\Omega)$ norm. Considering Lemma 1.4 and Lemma 2.2, we first give the uniform $L^{p^*}(\Omega)$ estimate for the sequence $\{P_h(a_j(x, \nabla u)) - J_2 a_j(x, \nabla u)\}_{h>0}$, where h is sufficiently small. We write $P_h(a_j(x, \nabla u))$ in the form

$$\begin{aligned}(4.1)\qquad P_h(a_j(x, \nabla u)) &= [a_j(S_h x, S_h\nabla u) - a_j(x, S_h\nabla u)]h^{-1}\\ &\quad + [a_j(x, S_h\nabla u) - a_j(x, \nabla u)]h^{-1}\,.\end{aligned}$$

Similarly as in (3.4) we see that

$$(4.2)\qquad \|[a_j(S_h x, S_h\nabla u) - a_j(x, S_h\nabla u)]h^{-1}\|_{p^*}^{p^*} \leqq C(\kappa_0^{p^*}\|\nabla u\|_{p^*}^{p^*} + \|\nabla u\|_p^p)\,.$$

For the second term of (4.1), we get the inequality

$$\begin{aligned}(4.3)\qquad &\|[a_j(x, S_h\nabla u) - a_j(x, \nabla u)]h^{-1}\|_{p^*}^{p^*}\\ &\leqq C\left\{\kappa_0^{p^*}\|P_h\nabla u\|_{p^*}^{p^*} + \int_\Omega (|P_h\nabla u|^{p^*}|S_h\nabla u|^{p^*(p-2)} + |P_h\nabla u|^{p^*}|\nabla u|^{p^*(p-2)})dx\right\}\end{aligned}$$

by the same way as in (3.11).

Because of $1 < p^* \leqq 2 \leqq p$, we have $\|P_h\nabla u\|_{p^*} \leqq C\|P_h\nabla u\|_2$. And the integral terms are estimated in the following way:

$$\begin{aligned}&\int_\Omega |P_h\nabla u|^{p^*}|S_h\nabla u|^{p^*(p-2)}dx\\ &\leqq \left[\int_\Omega\left\{|P_h\nabla u|^{p^*}|S_h\nabla u|^{p^*(p-2)/2}\right\}^{2/p^*}dx\right]^{p^*/2}\left[\int_\Omega |S_h\nabla u|^{(p^*(p-2)/2)(1-p^*/2)^{-1}}dx\right]^{(1-p^*/2)}\end{aligned}$$

$$= \left(\int_\Omega |P_h \nabla u|^2 |S_h \nabla u|^{p-2} dx\right)^{p^*/2} \left(\int_\Omega |S_h \nabla u|^p dx\right)^{(1-p^*/2)\,[6]}$$
$$\leqq C\{\| |S_h \nabla u|^{(p-2)/2} P_h \nabla u\|_2^2 + \|\nabla u\|_p^p\},$$

where we use Hölder's inequality and Minkowski's inequality. Another integral term is similarly estimated.

Hence we conclude from (4.3) that

$$\text{(4.4)} \quad \|[a_j(x, S_h \nabla u) - a_j(x, \nabla u)]h^{-1}\|_{p^*}^{p^*}$$
$$\leqq C\{\kappa_0^{p^*}\|P_h \nabla u\|_2^{p^*} + \|\nabla u\|_p^p + (\| |S_h \nabla u|^{(p-2)/2} P_h \nabla u\|_2^2 + \| |\nabla u|^{(p-2)/2} P_h \nabla u\|_2^2)\}.$$

Because of (4.1), (4.2) and (4.4) it holds that

$$\text{(4.5)} \quad \|P_h(a_j(x, \nabla u))\|_{p^*}^{p^*} \leqq C\{\kappa_0^{p^*}(\|\nabla u\|_{p^*}^{p^*} + \|P_h \nabla u\|_2^{p^*}) + \|\nabla u\|_p^p$$
$$+ (\| |S_h \nabla u|^{(p-2)/2} P_h \nabla u\|_2^2 + \| |\nabla u|^{(p-2)/2} P_h \nabla u\|_2^2)\}.$$

Therefore if $\kappa_0 = 0$,

$$\|P_h(a_j(x, \nabla u))\|_{p^*}^{p^*} \leqq C(1 + \|f\|_{1,p^*}^{p^*} + \|f\|_{p^*}^{(\alpha+2)/(p-1)}),$$

from (4.5), Proposition and Lemma 1.3. We write the right-hand side by $C\gamma_f^0$. Moreover if $\kappa_0 \neq 0$, we have similarly

$$\|P_h(a_j(x, \nabla u))\|_{p^*}^{p^*} \leqq C(1 + \|f\|_2^{2\beta/p}),$$

where $\beta = \max(p, \alpha + 2)$. The right-hand side is written by $C\gamma_f$.

On the other hand the functions J_2 are uniformly bounded in x and h. Thus by Lemma 1.1

$$\text{(4.6)} \quad \|J_2 a_j(x, \nabla u)\|_{p^*}^{p^*} \leqq C(\kappa_0^{p^*}\|\nabla u\|_{p^*}^{p^*} + \|\nabla u\|_p^p).$$

Applying the estimate for ∇u in Lemma 1.3 to each term on the right-hand side of (4.6), the $L^{p^*}(\Omega)$ norms of the functions $J_2 a_j(x, \nabla u)$ are estimated as follows: if $\kappa_0 = 0$ (resp. $\kappa_0 \neq 0$),

$$\|J_2 a_j(x, \nabla u)\|_{p^*}^{p^*} \leqq C\gamma_f^0 \qquad (\text{resp. } C\gamma_f).$$

From the above the $L^{p^*}(\Omega)$ norms of the sequence $\{P_h(a_j(x, \nabla u)) - J_2 a_j(x, \nabla u)\}_{h>0}$ are estimated. That is, if $\kappa_0 = 0$ (resp. $\kappa_0 \neq 0$),

$$\text{(4.7)} \quad \|P_h(a_j(x, \nabla u)) - J_2 a_j(x, \nabla u)\|_{p^*}^{p^*} \leqq C\gamma_f^0 \qquad (\text{resp. } C\gamma_f).$$

Because $a_j(x, \nabla u) \in L^{p^*}(\Omega)$, we conclude on account of Lemma 2.2 that for any $\phi \in C_0^\infty(\Omega)$

[6] Indeed, $(p^*(p-2)/2)(1-p^*/2)^{-1}=p$.

$$(P_h(a_j(x, \nabla u)) - J_2 a_j(x, \nabla u), \phi) \longrightarrow ((e\cdot\nabla)(\Phi a_j(x, \nabla u)), \phi)\,,$$
$$\text{as } h \longrightarrow 0\,.$$

Hence Lemma 1.4 yields that the distribution $(e\cdot\nabla)(\Phi a_j(x, \nabla u))$ belongs to $L^{p^*}(\Omega)$ and that

$$(4.8) \qquad P_h(a_j(x, \nabla u)) - J_2 a_j(x, \nabla u) \rightharpoonup (e\cdot\nabla)(\Phi a_j(x, \nabla u)) \quad \text{in } L^{p^*}(\Omega)\,,$$
$$\text{as } h \longrightarrow 0\,,$$

where "$\rightharpoonup$" means the weak convergence.

Therefore from (4.7) and (4.8) we derive that if $\kappa_0 = 0$ (resp. $\kappa_0 \neq 0$),

$$(4.9) \qquad \|(e\cdot\nabla)(\Phi a_j(x, \nabla u))\|_{p^*}^{p^*} \leqq C\gamma_f^0 \qquad (\text{resp. } C\gamma_f)\,.$$

The assertion with respect to $\Phi(x)|\nabla u|^{p/2}$ can be treated more easily. By simple calculations we deduce the inequality

$$|P_h(|\nabla u|^{p/2})| \leqq C|P_h\nabla u|(|S_h\nabla u|^{(p-2)/2} + |\nabla u|^{(p-2)/2})\,.$$

From this inequality and the estimate in Proposition we have for the case of $\kappa_0 = 0$ (resp. $\kappa_0 \neq 0$)

$$\|P_h(|\nabla u|^{p/2})\|_2^2 \leqq C\gamma_f^0 \qquad (\text{resp. } C\gamma_f)\,.$$

Since the functions J_2 are uniformly bounded, we get from Lemma 1.3

$$\|J_2|\nabla u|^{p/2}\|_2^2 \leqq C\gamma_f^0 \qquad (\text{resp. } C\gamma_f)\,,$$

if $\kappa_0 = 0$ (resp. $\kappa_0 \neq 0$).

Thus the $L^2(\Omega)$ norms of the sequence $\{P_h(|\nabla u|^{p/2}) - J_2|\nabla u|^{p/2}\}_{h>0}$ are estimated as follows: if $\kappa_0 = 0$ (resp. $\kappa_0 \neq 0$),

$$(4.10) \qquad \|P_h(|\nabla u|^{p/2}) - J_2|\nabla u|^{p/2}\|_2^2 \leqq C\gamma_f^0 \qquad (\text{resp. } C\gamma_f)\,.$$

With the aid of a priori estimate (4.10) we can select a sequence $\{h_\nu\}_{\nu=1}^\infty$ with $h_\nu \to 0$ ($\nu \to \infty$) and choose a function $v \in L^2(\Omega)$ such that

$$(4.11) \qquad P_{h_\nu}(|\nabla u|^{p/2}) - J_2|\nabla u|^{p/2} \rightharpoonup v \quad \text{in } L^2(\Omega) \quad \text{as } \nu \longrightarrow \infty\,,$$

and besides if $\kappa_0 = 0$ (resp. $\kappa_0 \neq 0$),

$$(4.12) \qquad \|v\|_2^2 \leqq C\gamma_f^0 \qquad (\text{resp. } C\gamma_f)\,.$$

By Lemma 2.2 we see that for any function ϕ in $C_0^\infty(\Omega)$

$$(4.13) \qquad (P_h(|\nabla u|^{p/2}) - J_2|\nabla u|^{p/2}, \phi) \longrightarrow ((e\cdot\nabla)(\Phi|\nabla u|^{p/2}), \phi)$$
$$\text{as } h \longrightarrow 0\,.$$

From (4.11), (4.12) and (4.13) it holds that $v=(\boldsymbol{e}\cdot\nabla)(\Phi|\nabla u|^{p/2})$ and if $\kappa_0=0$ (resp. $\kappa_0\neq 0$),

$$\|(\boldsymbol{e}\cdot\nabla)(\Phi|\nabla u|^{p/2})\|_2^2 \leqq C\gamma_f^0 \qquad (\text{resp. } C\gamma_f)\,. \tag{4.14}$$

By virtue of (4.9) and (4.14) we have proved the required for $\Phi(x)a_j(x,\nabla u)$ and $\Phi(x)|\nabla u|^{p/2}$. And the proof of the part for $\Phi(x)\nabla u$ is left. We briefly explain it. On account of the estimate (ii) in Proposition, $\|P_h u\|_{1,2}$ are uniformly bounded in h, more precisely,

$$\|P_h u\|_{1,2} \leqq C\gamma_f^{1/2}\,,$$

if $\kappa_0\neq 0$.

Accordingly there are a sequence $\{h_\mu\}_{\mu=1}^{\infty}$ with $h_\mu\to 0(\mu\to\infty)$ and an element w in $W^{1,2}(\Omega)$, to which the sequence $P_{h_\mu}u$ converges weakly in $W^{1,2}(\Omega)$ as $\mu\to\infty$. On the other hand the sequence $P_{h_\mu}u$ converges strongly to $\Phi(\boldsymbol{e}\cdot\nabla)u$ in $L^2(\Omega)$ as $\mu\to\infty$, which is shown by the same way as in Lemma 5 in [20]. Consequently we have that $w=\Phi(\boldsymbol{e}\cdot\nabla)u$ and

$$\|\Phi(\boldsymbol{e}\cdot\nabla)u\|_{1,2} \leqq C\gamma_f^{1/2}\,.$$

Thus the proof of our theorem is finished. Q.E.D.

Remark. The same conclusion is obtained if we replace $a_j(x,\nabla u)$ by $|\nabla u|^{p-2}\partial_{x_j}u$ or $|\partial_{x_j}u|^{p-2}\partial_{x_j}u$ in our theorem.

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