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(H, C)-GROUPS WITH POSITIVE LINE BUNDLES

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§ 0. Introduction

Let G be a connected complex Lie group. Then there exists the smallest closed complex subgroup G^0 of G such that G/G^0 is a Stein group (Morimoto [8]). Moreover G^0 is a connected abelian Lie group and every holomorphic function on G^0 is a constant. G^0 is called an (H, C)-group or a toroidal group. Every connected complex abelian Lie group is isomorphic to the direct product $G^0 \times C^m \times C^{*n}$, where G^0 is an (H, C)-group ([7], [9]).

Recently, several interesting results with respect to (H, C)-groups have been obtained (Kazama [5], Kazama and Umeno [6], Vogt [13], [14] and [15]). The set of (H, C)-groups includes the set of complex tori. A complex torus is called an abelian variety if it satisfies Riemann condition. The definition of quasi-abelian variety for (H, C)-groups was given in [2]. In this paper we shall show that the concept of quasi-abelian variety is a natural generalization of abelian variety. Throughout this paper, we assume that $\dim H^1(X, \mathcal{O}) < \infty$ for (H, C)-groups X. Our main result is the following.

Let $X = C^n/\Gamma$ be an (H, C)-group. The following statements are equivalent:

(1) X has a positive line bundle;

(2) X is a quasi-abelian variety;

(3) X is a covering space on an abelian variety;

(4) X is embedded in a complex projective space as a locally closed submanifold.

The above result is well-known for complex tori. For the proof we use the theory of weakly 1-complete manifolds and results of Vogt. We note that implications (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) were obtained by Gherardelli

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and Andreotti [2]. Combining with a result of Gherardelli and Andreotti [2], we get the affirmative answer to a problem of the structure of weakly 1-complete manifolds in the case of (H, C)-groups (see § 6).

The author was inspired from dissertations of Pothering [12] and Vogt [13]. He is very grateful to Professor H. Kazama who told him the existence of their dissertations.

§ 1. Preliminaries

Let G be an n-dimensional connected complex Lie group without nonconstant holomorphic functions. Such a Lie group G is said to be an (H, C)-group or a toroidal group ([7], [8]). We recall that G is abelian and then G is isomorphic onto $\boldsymbol{C}^n/\Gamma$ for some discrete subgroup Γ of $\boldsymbol{C}^n$ as a Lie group ([8]). If $\boldsymbol{C}^n/\Gamma$ is an (H, C)-group, then the generators of Γ contains n vectors linearly independent over $\boldsymbol{C}$ and rank $\Gamma = n + q$ $(1 \leqq q \leqq n)$. When Γ is generated by $p_1, \cdots, p_{n+q} \in \boldsymbol{C}^n$, we write

$$P = (p_1, \cdots, p_{n+q}),$$

and we call it a period basis of Γ, or also of $X = \boldsymbol{C}^n/\Gamma$. Two period bases P and P' are equivalent if and only if there exist a non-singular matrix S and a unimodular matrix M such that

$$P' = SPM.$$

A period basis P is always equivalent to the following standard form

$$(I_n\ V),$$

where $V = (v_1, \cdots, v_q)$, $v_j = {}^t(v_{1j}, \cdots, v_{nj})$, $\det(\operatorname{Im} v_{ij}; 1 \leqq i, j \leqq q) \neq 0$ and I_n is the (n, n) unit matrix.

It is well-known that $\boldsymbol{C}^n/\Gamma$ is an (H, C)-group if and only if

$$\max\left\{\left|\sum_{k=1}^{n} v_{kj} m_k - m_{n+j}\right|; 1 \leqq j \leqq q\right\} > 0$$

for all $m = (m_1, \cdots, m_n, m_{n+1}, \cdots, m_{n+q}) \in \boldsymbol{Z}^{n+q} \backslash \{0\}$ (Kopfermann [7] and Morimoto [8]).

A discrete subgroup Γ of rank $n + q$ in $\boldsymbol{C}^n$ generates an $(n + q)$-dimensional real linear subspace $\boldsymbol{R}_\Gamma^{n+q}$ of $\boldsymbol{C}^n$. $\boldsymbol{R}_\Gamma^{n+q}$ contains the q-dimensional complex linear subspace $\boldsymbol{C}_\Gamma^q$ which is the maximal complex linear subspace contained in $\boldsymbol{R}_\Gamma^{n+q}$. If we take the standard form for the period basis of Γ, then $\operatorname{Im} v_1, \cdots, \operatorname{Im} v_q$ generate $\boldsymbol{C}_\Gamma^q$.

§ 2. Factors of automorphy

We introduce some results of Vogt ([13] and [14]). For the details, we refer the reader to [13].

Let Γ be a discrete subgroup of rank $n+q$ in $\boldsymbol{C}^n$, $X=\boldsymbol{C}^n/\Gamma$ and $\pi\colon \boldsymbol{C}^n \to X$ be the projection. If $p\colon L \to X$ is a holomorphic line bundle, then its pull-back π^*L is given by the following fibre product

$$\pi^*L = \boldsymbol{C}^n \times_X L = \{(z, v) \in \boldsymbol{C}^n \times L;\ \pi(z) = p(v)\}\,.$$

Since any holomorphic Line bundle over $\boldsymbol{C}^n$ is analytically trivial, we have a trivialization

$$\varphi\colon \pi^*L \longrightarrow \boldsymbol{C}^n \times \boldsymbol{C},\ (z, v) \longmapsto (z, \varphi_z(v))$$

of π^*L. We define

$$\alpha\colon \Gamma \times \boldsymbol{C}^n \longrightarrow \boldsymbol{C}^*,\ \alpha(\gamma, z) := \varphi_{z+\gamma}\varphi_z^{-1}\,.$$

Then α satisfies the following conditions:

(a) $\alpha_\gamma(z) := \alpha(\gamma, z)$ is holomorphic for all $\gamma \in \Gamma$;

(b) $\alpha(0, z) = 1$ for all $z \in \boldsymbol{C}^n$;

(c) $\alpha(\gamma+\gamma', z) = \alpha(\gamma, z+\gamma')\alpha(\gamma', z)$ for all $\gamma, \gamma' \in \Gamma$ and $z \in \boldsymbol{C}^n$.

DEFINITION. A map $\alpha\colon \Gamma \times \boldsymbol{C}^n \to \boldsymbol{C}^*$ is called a factor of automorphy for Γ on $\boldsymbol{C}^n$ if it satisfies the above conditions (a), (b) and (c).

Conversely, if $\alpha\colon \Gamma \times \boldsymbol{C}^n \to \boldsymbol{C}^*$ is a factor of automorphy, then we get a line bundle L over $\boldsymbol{C}^n/\Gamma$ defined as the quotient of $\boldsymbol{C}^n \times \boldsymbol{C}$ by the following action of Γ:

$$\gamma(z, v) := (z+\gamma, \alpha(\gamma, z)v) \ \text{for}\ \gamma \in \Gamma, z \in \boldsymbol{C}^n, v \in \boldsymbol{C}\,.$$

DEFINITION. Two factors of automorphy α, β are said to be equivalent if there exists a holomorphic function $h\colon \boldsymbol{C}^n \to \boldsymbol{C}^*$ such that

$$\beta(\gamma, z) = h(z+\gamma)\alpha(\gamma, z)h^{-1}(z)$$

for all $\gamma \in \Gamma$ and $z \in \boldsymbol{C}^n$.

PROPOSITION 1 (Vogt [13] and [14]). *The equivalent classes of factors of automorphy for Γ on $\boldsymbol{C}^n$ correspond one-to-one to the isomorphism classes of holomorphic line bundles over $\boldsymbol{C}^n/\Gamma$.*

PROPOSITION 2 (Vogt [13]). *Let L_1 and L_2 be holomorphic line bundles over $\boldsymbol{C}^n/\Gamma$. If factors of automorphy α_1 and α_2 give L_1 and L_2 respectively,* *then* $L_1 \otimes L_2$ *is given by the factor of automorphy*

$$\alpha_1\alpha_2 : \Gamma \times \boldsymbol{C}^n \longrightarrow \boldsymbol{C}^*, \ \alpha_1\alpha_2(\gamma, z) := \alpha_1(\gamma, z)\alpha_2(\gamma, z)$$

for all $\gamma \in \Gamma$ *and* $z \in \boldsymbol{C}^n$.

DEFINITION. A map $a: \Gamma \times \boldsymbol{C}^n \to \boldsymbol{C}$ is called a summand of automorphy, if

(a) $a_\gamma(z) := a(\gamma, z)$ is holomorphic for all $\gamma \in \Gamma$;

(b) $a(0, z) = 0$ for all $z \in \boldsymbol{C}^n$;

(c) $a(\gamma + \gamma', z) = a(\gamma, z + \gamma') + a(\gamma', z)$ for all $\gamma, \gamma' \in \Gamma$ and $z \in \boldsymbol{C}^n$.

Let a: $\Gamma \times \boldsymbol{C}^n \to \boldsymbol{C}$ be a summand of automorphy. We set $\alpha(\gamma, z) := \exp(a(\gamma, z))$. Then α is a factor of automorphy.

LEMMA 1 (Vogt [13]). *Let* Γ *be a discrete subgroup of rank* r *in* $\boldsymbol{C}^n$ *with a basis* $\{\gamma_1, \cdots, \gamma_r\}$. *If a map* b: $\Gamma \times \boldsymbol{C}^n \to \boldsymbol{C}$ *satisfies the following properties*:

(a) $b_j(z) := b(\gamma_j, z)$ *is holomorphic for all* $j = 1, \cdots, r$;

(b) $b(\gamma_i, z + \gamma_j) + b(\gamma_j, z) = b(\gamma_j, z + \gamma_i) + b(\gamma_i, z)$ *for all* $i, j = 1, \cdots, r$, *then there exists a summand of automorphy such that*

$$a(\gamma_i, z) = b(\gamma_i, z) \qquad \textit{for all } i = 1, \cdots, r\,.$$

DEFINITION. A factor of automorphy α: $\Gamma \times \boldsymbol{C}^n \to \boldsymbol{C}^*$ is called a theta factor for Γ on $\boldsymbol{C}^n$, if it is expressed as follows,

$$\alpha(\gamma, z) = \exp 2\pi\sqrt{-1}(\mathscr{L}_\gamma(z) + c(\gamma))\,,$$

where $\mathscr{L}_\gamma(z)$ is a linear polynomial and $c(\gamma)$ is a constant for all $\gamma \in \Gamma$.

The following proposition was mentioned in [15], and its proof is hidden in the proof of Theorem in [14].

PROPOSITION 3. *Let* $\boldsymbol{C}^n/\Gamma$ *be an* (H, C)*-group. For every line bundle* L *over* $\boldsymbol{C}^n/\Gamma$, *there exist a topologically trivial line bundle* L_0 *and a line bundle* L_1 *given by a theta factor such that* $L \cong L_0 \otimes L_1$.

Proof. The proof is along the argument of the implication 2) $\Rightarrow$ 1) of Theorem in [14]. Let α: $\Gamma \times \boldsymbol{C}^n \to \boldsymbol{C}^*$ be a factor of automorphy which gives L. There exists a map a: $\Gamma \times \boldsymbol{C}^n \to \boldsymbol{C}$ such that $\alpha(\gamma, z) = \exp a(\gamma, z)$. We have

$$a(\gamma, z + \gamma') + a(\gamma', z) = a(\gamma', z + \gamma) + a(\gamma, z) + 2\pi\sqrt{-1}n(\gamma, \gamma')\,, \quad n(\gamma, \gamma') \in \boldsymbol{Z}\,,$$

for all $\gamma, \gamma' \in \Gamma$ and $z \in \boldsymbol{C}^n$. We may assume that $P = (I_n V)$ is a period basis of Γ. Putting

$$\ell(e_j, z) := \pi\sqrt{-1} \sum_{k=1}^{n} n(e_j, e_k) z_k , \qquad j = 1, \cdots, n,$$

$$\ell(v, e_j) := \ell(e_j, v) + 2\pi\sqrt{-1} n(v, e_j) , \qquad v \in V,\ j = 1, \cdots, n,$$

we get a map $\ell \colon \Gamma \times \boldsymbol{C}^n \to \boldsymbol{C}$ such that $\ell(\gamma,\ ) \colon \boldsymbol{C}^n \to \boldsymbol{C}$ is $\boldsymbol{C}$-linear and

$$(*) \qquad \ell(\gamma, \gamma') = \ell(\gamma', \gamma) + 2\pi\sqrt{-1} n(\gamma, \gamma') \qquad \text{for all } \gamma, \gamma' \in P.$$

We define

$$b(\gamma, z) := a(\gamma, z) - \ell(\gamma, z) \qquad \text{for all } \gamma \in P.$$

We rewrite $P = (I_n V) = (\gamma_1, \cdots, \gamma_{n+q})$. By (*) we have

$$b(\gamma_i, z + \gamma_j) + b(\gamma_j, z) = b(\gamma_j, z + \gamma_i) + b(\gamma_i, z)$$

for all $i, j = 1, \cdots, n + q$. By Lemma 1 there exists a summand of automorphy $\tilde{b}$ such that

$$\tilde{b}(\gamma_i, z) = b(\gamma_i, z) , \qquad i = 1, \cdots, n + q, z \in \boldsymbol{C}^n.$$

Now we show the following (**).

(**) For any $\gamma \in \Gamma$ there exists a constant $c(\gamma) \in \boldsymbol{C}$ such that

$$\tilde{b}(\gamma, z) = a(\gamma, z) - \ell(\gamma, z) + c(\gamma) .$$

We use the proof of Lemma 1. Every $\gamma \in \Gamma$ can be expressed uniquely as

$$\gamma = \sum_{i=1}^{n+q} t_i \gamma_i , \qquad t_i \in \boldsymbol{Z} .$$

Let $|\gamma| = \sum_{i=1}^{n+q} |t_i|$. We show (**) by induction on $|\gamma|$. For $\gamma = 0$, we have

$$a(0, z) - \ell(0, z) = a(0, z) = 2\pi\sqrt{-1} n_0 , \qquad n_0 \in \boldsymbol{Z}.$$

Then we set $c(0) = -2\pi\sqrt{-1} n_0$. When $|\gamma| = 1$, $\gamma = \pm\gamma_i$. Since $\tilde{b}(\gamma_i, z) = b(\gamma_i, z)$, (**) holds for $\gamma = \gamma_i$ with $c(\gamma_i) = 0$. As in the proof of Lemma 1 ([13]), $\tilde{b}(-\gamma_i, z)$ is defined by

$$\tilde{b}(-\gamma_i, z) = -\tilde{b}(\gamma_i, z - \gamma_i) .$$

Hence it holds that

$$\tilde{b}(-\gamma_i, z) = -a(\gamma_i, z - \gamma_i) + \ell(\gamma_i, z - \gamma_i) .$$

And we have

$$-a(\gamma_i, z - \gamma_i) = a(-\gamma_i, z) + 2\pi\sqrt{-1}\{n(\gamma_i, -\gamma_i) - n_0\}.$$

Setting $c(-\gamma_i) = -\ell(\gamma_i, \gamma_i) + 2\pi\sqrt{-1}\{n(\gamma_i, -\gamma_i) - n_0\}$, we obtain (**). Assume that (**) holds for $|\gamma| \leqq N$. Let $|\gamma| = N + 1$. There exist $\gamma', \gamma'' \in \Gamma$ such that $\gamma = \gamma' \oplus \gamma''$, $|\gamma'|, |\gamma''| \leqq N$. By the definition of $\tilde{b}$ it holds that

$$\tilde{b}(\gamma, z) = \tilde{b}(\gamma', z + \gamma'') + \tilde{b}(\gamma'', z).$$

Then we obtain

$$\begin{aligned}\tilde{b}(\gamma, z) = a(\gamma' + \gamma'', z) - 2\pi\sqrt{-1}n(\gamma', \gamma'') - \ell(\gamma' + \gamma'', z) - \ell(\gamma', \gamma'') \\ + c(\gamma') + c(\gamma'').\end{aligned}$$

Thus (**) holds with

$$c(\gamma) = -2\pi\sqrt{-1}n(\gamma', \gamma'') - \ell(\gamma', \gamma'') + c(\gamma') + c(\gamma'').$$

We define the factor of automorphy by $\tilde{\beta} := \exp \tilde{b}$. Then the line bundle $L_{\tilde{\beta}}$ given by $\tilde{\beta}$ is topologically trivial (Vogt [13] and [14]). Put

$$\rho(\gamma, z) := \exp(\ell(\gamma, z) - c(\gamma)).$$

Then ρ is a theta factor and $\alpha(\gamma, z) = \tilde{\beta}(\gamma, z)\rho(\gamma, z)$. By Proposition 2 we obtain $L \cong L_{\tilde{\beta}} \otimes L_\rho$, where L_ρ is the line bundle given by ρ.

§ 3. Theta functions

DEFINITION. Let Γ be a discrete subgroup of rank $n + q$ in C^n. A holomorphic function $\theta(z)$ on C^n is called a theta function with theta factor $\rho(\gamma, z)$ if it satisfies

$$\theta(z + \gamma) = \rho(\gamma, z)\theta(z), \qquad \text{for all } \gamma \in \Gamma \text{ and } z \in C^n.$$

PROPOSITION 4 (Kopfermann [7]). *Let Γ be a discrete subgroup of rank $n + q$ in C^n and $\rho(\gamma, z)$ be a theta factor for Γ on C^n. Then there exist a hermitian form $\mathscr{H}: C^n \times C^n \to C$ with $\mathscr{A} := \operatorname{Im} \mathscr{H}$ Z-valued on $\Gamma \times \Gamma$, a C-bilinear symmetric form $\mathscr{Q}$, a C-linear form $\mathscr{L}$ and a semi-character ψ of Γ associated with $\mathscr{A}|_{\Gamma\times\Gamma}$ such that*

$$\rho(\gamma, z) = \psi(\gamma) \exp 2\pi\sqrt{-1}\left[\frac{1}{2\sqrt{-1}}(\mathscr{H} + \mathscr{Q})(\gamma, z) + \frac{1}{4\sqrt{-1}}(\mathscr{H} + \mathscr{Q})(\gamma, \gamma) + \mathscr{L}(\gamma)\right]$$

for all $\gamma \in \Gamma$ and $z \in C^n$. If rank $\Gamma = 2n$, then this expression is unique.

A theta factor ρ with the expression as in Proposition 4 is called a theta factor of type $(\mathscr{H}, \psi, \mathscr{Q}, \mathscr{L})$. A theta function with theta factor ρ of type $(\mathscr{H}, \psi, \mathscr{Q}, \mathscr{L})$ is called a theta function of type $(\mathscr{H}, \psi, \mathscr{Q}, \mathscr{L})$. A trivial theta function is a theta function of type $(0, 1, \mathscr{Q}, \mathscr{L})$. A theta function of type $(\mathscr{H}, \psi, 0, 0)$ is said to be reduced. Every theta function can be expressed as the product of a reduced theta function and a trivial theta function.

Let $\mathscr{H}$ be a hermitian form on $C^n \times C^n$. We set

$$\mathrm{Ker}(\mathscr{H}) := \{z \in C^q_\Gamma;\ \mathscr{H}(z', z) = 0 \quad \text{for all } z' \in C^q_\Gamma\}.$$

PROPOSITION 5 (Kopfermann [7]). *Let Γ be a discrete subgroup of rank $n+q$ in C^n. If $\theta(z)$ is a theta function for Γ of type $(\mathscr{H}, \psi, \mathscr{Q}, \mathscr{L})$ and $\theta(z) \not\equiv 0$, then*

(1) *$\mathscr{H}$ is positive semi-definite on $C^q_\Gamma \times C^q_\Gamma$,*

(2) *θ is constant on* $\mathrm{Ker}(\mathscr{H})$, *if θ is reduced.*

§ 4. Quasi-abelian varieties

Let Γ be a discrete subgroup of rank $n+q$ in C^n and C^n/Γ be an (H, C)-group. We consider the following condition.

(R) There exists a hermitian form $\mathscr{H}$ on $C^n \times C^n$ such that

(1) $\mathrm{Im}\,\mathscr{H}$ is Z-valued on $\Gamma \times \Gamma$;

(2) $\mathscr{H}$ is positive definite on $C^q_\Gamma \times C^q_\Gamma$.

When rank $\Gamma = 2n$, it is well-known that C^n/Γ is an abelian variety if and only if the above condition (R) is satisfied. The following definition is due to Gherardelli and Andreotti [2].

DEFINITION. An (H, C)-group C^n/Γ is called a quasi-abelian variety if it satisfies the condition (R).

PROPOSITION 6 (Gherardelli and Andreotti [2]). *Let Γ be a discrete subgroup of rank $n+q$ in C^n. Suppose that $\mathscr{H}$ is a hermitian form on $C^n \times C^n$ satisfying the following properties:*

(a) $\mathrm{Im}\,\mathscr{H}$ *is Z-valued on $\Gamma \times \Gamma$,*

(b) *$\mathscr{H}$ is positive definition on $C^q_\Gamma \times C^q_\Gamma$.*

Then there exist $\gamma \in C^n$ and a hermitian form $\mathscr{Q}$ symmetric on $R^{n+q}_\Gamma \times R^{n+q}_\Gamma$ such that

(1) *$\Gamma_1 = \Gamma + Z\gamma$ is a discrete subgroup of rank $n+q+1$ in C^n,*

(2) $\mathrm{Im}(\mathscr{H} + \mathscr{Q})$ *is Z-valued on $\Gamma_1 \times \Gamma_1$,*

(3) *$\mathscr{H} + \mathscr{Q}$ is positive definite on $C^n \times C^n$.*

Let C^n/Γ be a quasi-abelian variety with a discrete subgroup Γ of rank $n+q$. Using Proposition 6 successively, we obtain a discrete subgroup $\tilde{\Gamma}$ of rank $2n$ such that $\tilde{\Gamma} \supset \Gamma$ and $\boldsymbol{C}^n/\tilde{\Gamma}$ is an abelian variety. Hence the following proposition holds.

PROPOSITION 7 (Gherardelli and Andreotti [2]). *If $\boldsymbol{C}^n/\Gamma$ is a quasi-abelian variety, then it is a covering space on an n-dimensional abelian variety.*

§ 5. (*H*, *C*)-groups with positive line bundles

Let X be a complex manifold. X is called a weakly 1-complete manifold if there exists a C^∞ plurisubharmonic exhaustion function on X (Nakano [11]). It is well-known that an (H, C)-group $\boldsymbol{C}^n/\Gamma$ is weakly 1-complete (cf. Kazama [4]).

Let X be an n-dimensional weakly 1-complete manifold and ψ be its C^∞ plurisubharmonic exhaustion function. We set $X_c = \{x \in X;\ \psi(x) < c\}$ for all $c \in \boldsymbol{R}$. Let $E \to X$ be a holomorphic vector bundle over X. We denote by $\Omega^p(E)$ the sheaf of germs of all E-valued holomorphic p-forms. We need the following theorems.

THEOREM A (Kazama [3]). *Let X be an n-dimensional weakly 1-complete manifold and $E \to X$ be a holomorphic vector bundle over X which is positive in the sense of Nakano [10]. Then for any $c \in \boldsymbol{R}$, the restriction map*

$$\rho\colon H^0(X, \Omega^n(E)) \to H^0(X_c, \Omega^n(E))$$

has a dense image with respect to the topology of uniform convergence on all compact sets in X_c.

THEOREM B (Hironaka, cf. Fujiki [1]). *Let X be a weakly 1-complete manifold and $B \to X$ be a positive line bundle over X. Then, for any $c \in \boldsymbol{R}$ there exist natural numbers m, N and $\varphi^{(0)}, \cdots, \varphi^{(N)} \in H^0(X_d, \mathcal{O}(B^m))$ for $d > c$ such that $\Phi = (\varphi^{(0)} : \cdots : \varphi^{(N)})$ embeds X_c into $\boldsymbol{P}^N$ as a locally closed submanifold and $\Phi^*[e] = B^m$, where $\boldsymbol{P}^N$ is the N-dimensional complex projective space and $[e]$ is the hyperplane bundle of $\boldsymbol{P}^N$.*

PROPOSITION 8. *Let $X = \boldsymbol{C}^n/\Gamma$ be an (H, C)-group. Suppose that X has a positive theta bundle $L \to X$ with theta factor ρ. Then, for any $x, y \in \boldsymbol{C}^n$ with $x \not\equiv y \pmod{\Gamma}$ there exist a natural number m and theta functions, θ_1, θ_2 with theta factor ρ^m such that $f(x) \neq f(y)$, where $f = \theta_1/\theta_2$.*

Proof. Let $\pi\colon \boldsymbol{C}^n \to X$ be the projection. For any $x, y \in \boldsymbol{C}^n$ with $\pi(x)$

$\neq \pi(y)$, there exists a real number c such that $X_c \ni \pi(x), \pi(y)$. Take $c' > c$. By Theorem B there exist natural numbers m, N and an embedding map $\Phi: X_{c'} \to \boldsymbol{P}^N$, where $\Phi = (\varphi^{(0)}: \cdots : \varphi^{(N)})$ and $\varphi^{(j)} \in H^0(X_d, \mathcal{O}(L^m))$, $d > c'$. Since the canonical bundle K of X is analytically trivial, we have $\mathcal{O}(L^m) \cong \Omega^n(K^{-1} \otimes L^m)$ and $K^{-1} \otimes L^m$ is positive. Applying Theorem A to $\Omega^n(K^{-1} \otimes L^m)$, we can approximate any element in $H^0(X_d, \mathcal{O}(L^m))$ by elements in $H^0(X, \mathcal{O}(L^m))$ uniformly on $\bar{X}_{c'}$. Therefore there exist $\tilde{\varphi}^{(0)}, \cdots, \tilde{\varphi}^{(N)} \in H^0(X, \mathcal{O}(L^m))$ such that a holomorphic map $\tilde{\Phi} = (\tilde{\varphi}^{(0)}: \cdots : \tilde{\varphi}^{(N)}): X_c \to \boldsymbol{P}^N$ separates points $\pi(x)$ and $\pi(y)$. There exist hyperplanes H_1 and H_2 of $\boldsymbol{P}^N$ such that $H_1 \ni \tilde{\Phi}(\pi(x))$, $H_1 \not\ni \tilde{\Phi}(\pi(y))$ and $H_2 \not\ni \tilde{\Phi}(\pi(x)), \tilde{\Phi}(\pi(y))$. Let $\ell_j = 0$ be the homogeneous equation of H_j for $j = 1, 2$. We set

$$f := \frac{\ell_1(\tilde{\varphi}^{(0)}, \cdots, \tilde{\varphi}^{(N)})}{\ell_2(\tilde{\varphi}^{(0)}, \cdots, \tilde{\varphi}^{(N)})}.$$

Then f is a meromorphic function on X and $f(\pi(x)) \neq f(\pi(y))$. Each section $\ell_j(\tilde{\varphi}^{(0)}, \cdots, \tilde{\varphi}^{(N)})$ corresponds to a theta function θ_j with theta factor ρ^m for $j = 1, 2$. Then we have $f \circ \pi = \theta_1/\theta_2$.

PROPOSITION 9. *Let $X = \boldsymbol{C}^n/\Gamma$ be an (H, C)-group. If X has a positive line bundle, then it is a quasi-abelian variety.*

Proof. Let $L \to X$ be a positive line bundle over X. By Proposition 3, we have $L \cong L_0 \otimes L_1$, where L_0 is a topologically trivial line bundle and L_1 is a theta bundle. Under our assumption, every topologically trivial line bundle over X is given by a representation of Γ ([14]). Hence we may assume that L is a positive theta bundle. Suppose that L is given by a theta factor ρ of type $(\mathcal{H}, \psi, \mathcal{Q}, \mathcal{L})$.

It suffices to show that $\mathrm{Ker}(\mathcal{H}) = \{0\}$. If $\mathrm{Ker}(\mathcal{H}) \neq \{0\}$, then there exist $x, y \in \mathrm{Ker}(\mathcal{H})$ such that $x \neq y \pmod{\Gamma}$. By Proposition 8 there exist a natural number m and theta functions θ_1, θ_2 with theta factor ρ^m such that $f(x) \neq f(y)$, where $f = \theta_1/\theta_2$. We may assume that θ_1 and θ_2 are reduced theta functions of type $(m\mathcal{H}, \psi^m, 0, 0)$. Since $\mathrm{Ker}(m\mathcal{H}) = \mathrm{Ker}(\mathcal{H})$, θ_1 and θ_2 must be constant on $\mathrm{Ker}(\mathcal{H})$ (Proposition 5). It is a contradiction.

From Propositions 7 and 9 the following theorem holds.

THEOREM 1. *Let $X = \boldsymbol{C}^n/\Gamma$ be an (H, C)-group. Then the following statements* (1), (2) *and* (3) *are equivalent.*

(1) *X has a positive line bundle.*

(2) *X is a quasi-abelian variety.*

(3) *X is a covering space on an n-dimensional abelian variety.*

§6. Some remarks

H. Kazama proposed the following problem of the structure of weakly 1-complete manifolds (in Sûgaku 32 (1980), Iwanami Shoten): Let X be a weakly 1-complete manifold. Is it possible to explain X by Stein manifolds and projective algebraic compact manifolds (for example, as a fibre space) when there exists a positive line bundle over X and $H^0(X, \mathcal{O}) = \boldsymbol{C}$?

In this section we shall give the affirmative answer to the above problem in the case of (H, C)-groups.

Let $X = \boldsymbol{C}^n/\Gamma$ be an (H, C)-group with rank $\Gamma = n + q$. If X is a quasi-abelian variety, there exists a hermitian form $\mathscr{H}$ on $\boldsymbol{C}^n \times \boldsymbol{C}^n$ such that

(a) Im $\mathscr{H}$ is $\boldsymbol{Z}$-valued on $\Gamma \times \Gamma$,

(b) $\mathscr{H}$ is positive definite on $\boldsymbol{C}^q_\Gamma \times \boldsymbol{C}^q_\Gamma$.

Let $\mathscr{A} = (\mathrm{Im}\, \mathscr{H})|_{\boldsymbol{R}^{n+q}_\Gamma \times \boldsymbol{R}^{n+q}_\Gamma}$. Since $\mathscr{A}$ is an alternating form, rank $\mathscr{A}$ is an even number. We set rank $\mathscr{A} = 2r$. Then $2q \leqq 2r \leqq n + q$. The following definition is due to Gherardelli and Andreotti [2].

DEFINITION. When rank $\mathscr{A} = 2(q + p)$, we say that a quasi-abelian variety $X = \boldsymbol{C}^n/\Gamma$ is of kind p.

Using the proof of Proposition 6 and a result of period basis of abelian variety, we obtain the following theorem.

THEOREM 2 (Gherardelli and Andreotti [2]). *Let $X = \boldsymbol{C}^n/\Gamma$ be a quasi-abelian variety of kind p with rank $\Gamma = n + q$. Then X is a fibre bundle over a $(q + p)$-dimensional abelian variety with fibres $\boldsymbol{C}^p \times (\boldsymbol{C}^*)^{n-q-2p}$.*

By Theorems 1 and 2, the problem given the beginning in this section is affirmative in the case of (H, C)-groups.

We do not know whether a weakly 1-complete manifold is globally embeddable in a complex projective space or not if it has a positive line bundle. But an (H, C)-group with positive line bundle is embedded in a complex projective space by Theorems 1 and 2.

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