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EQUIVALENT CONDITIONS FOR THE TIGHTNESS OF A SEQUENCE OF CONTINUOUS HILBERT VALUED MARTINGALES

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1. Introduction

In [1] D. Aldous gave a sufficient condition for the tightness of a sequence $(X^n)_{n\geq 0}$ of right continuous (with left limits) processes taking their values in a separable complete metric space S. As already noted by Aldous this condition is far from being necessary when the *processes* X^n *are not* continuous. More precisely the Aldous-condition implies the left-quasi-continuity of all the weak limits of the sequence $(X^n)_{n\geq 0}$. (see [1] or [4]).

When the X^n's are real square integrable martingales (or more generally locally square integrable martingales), it has been shown by R. Rebolledo ([9, see also an exposition in [4]) that the Aldous-condition for the positive increasing Meyer-processes $(\langle X^n\rangle)$ implies the Aldous-condition for $(X^n)_{n\geq 0}$.

In the case of Hilbert valued martingales it has been shown in [6] that the Aldous-condition on $(\langle X^n\rangle)$ plus a tightness condition on the sequence $(\langle\!\langle X^n\rangle\!\rangle_T)_{n\geq 0}$ of operator valued random variables, $\langle\!\langle X^n\rangle\!\rangle$ being the "tensor-Meyer-process" of X^n (see [7]), is also sufficient for the tightness of $(X^n)_{n\geq 0}$.

But in general neither the Aldous-condition on $(\langle\!\langle X^n\rangle\!\rangle)_{n\geq 0}$ is necessary for the tightness of $(X^n)_{n\geq 0}$, nor the tightness of $(\langle\!\langle X^n\rangle\!\rangle)_{n\geq 0}$ alone implies the tightness of $(X^n)_{n\geq 0}$ (see J. Jacod, J. Mémin, M. Métivier [3]) *unless* some condition is assumed on the limits of the laws of the processes $\langle\!\langle X^n\rangle\!\rangle$. When the processes are real or finite dimensional, the fact that the limiting laws are carried by the subset of continuous paths in $D(\boldsymbol{R}_+, H)$ is sufficient. (see R. Rebolledo [9] and also [3] Theorem 1).

Considering only continuous processes, S. Nakao ([8]) recently proved

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an analogous result for Hilbert valued martingales. He showed that the tightness of $(X^n)_{n\geq 0}$ and the tightness of the operator-valued processes $(\langle\langle X^n\rangle\rangle)_{n\geq 0}$ are equivalent when the X^n are continuous Hilbert valued martingales. His proof is "direct", without reference to the Aldous-condition.

In this paper we prove that in the continuous case the tightness of $(X^n)_{n\geq 0}$ actually implies the Aldous-condition for $(\langle\langle X^n\rangle\rangle)_{n\geq 0}$ and the tightness of marginals of $(\langle\langle X^n\rangle\rangle_T)_{n\geq 0}$. As a consequence of a result in [6] we get a set of equivalent conditions for tightness containing in particular S. Nakao's result.

2. Definitions and statement of the theorem

Let $(X^n)_{n\geq 0}$ be a sequence of processes with values in a separable complete metric space S with distance d. We assume that each process X^n is defined on a probability space $(\Omega^n, (\mathscr{F}^n), P_n)$ with its own filtration $(\mathscr{F}^n_t)_{t\in[0,T]}$.

2.1. We say that the sequence $(X^n)_{n\geq 0}$ satisfies the Aldous condition, which, from now on, we designate by [A], if for any $\eta > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for every n and every $(\mathscr{F}^n_t)$-stopping time τ^n on Ω^n

$$\sup_{0\leq\theta\leq\delta} P_n\{d(X^n_{\tau^n+\theta}, X^n_{\tau^n}) > \eta\} \leq \varepsilon .$$

2.2. We say that for $t\in[0, T]$ the sequence satisfies the condition $[\mathrm{T}_t]$ if the sequence $(X^n_t)_{n\geq 0}$ of S-valued random variables is tight, i.e.: for every $\varepsilon>0$ there exists a compact K_ε in S such that:

$$P_n\{X^n_t \notin K_\varepsilon\} \leq \varepsilon .$$

Let us call $D(T, S)$ (resp. $C(T, S)$) the set of mappings from $[0, T]$ in S which are right continuous and have left limits in every $t\in[0, T]$ (resp. which are continuous), endowed with the Skorokhod topology (see Billingsley [2]) (resp. with the topology of uniform convergence). We call $\tilde{P}_n$ the law of $X^n : \tilde{P}_n$ is the image of P_n by the mapping $\omega \rightsquigarrow X^n(\omega, \cdot)$. If X^n is continuous, $\tilde{P}_n$ is carried by the closed subset $C(T, S)$ of $D(T, S)$.

D. Aldous proved that if $(X^n)_{n\geq 0}$ verifies the conditions [A] and $[\mathrm{T}_t]$ for a dense set of $t\in[0, T]$ then $(X^n)_{n\geq 0}$ is tight. The converse is not true (see [1]). However, when the processes X^n are continuous one has the following easy lemma:

2.3. LEMMA 1. *If the processes* $(X^n)_{n\geq 0}$ *are continuous and if their laws* $\tilde{P}_n$ *form a tight sequence in* $C(T, S)$, *then the conditions* [A] *and* $[\mathrm{T}_t]$ *for every* $t \in [0, T]$ *hold.*

This lemma is an easy consequence of the Ascoli theorem on the characterization of compact sets in $C(T, S)$.

2.4. DEFINITIONS. We recall the following definitions and notations. $\boldsymbol{H}$ being a real Hilbert space (the dual of which will be identified with $\boldsymbol{H}$ itself as long as there is no possible confusion), with scalar product $(\cdot, \cdot)$, we denote by $\mathscr{L}_\infty(\boldsymbol{H}, \boldsymbol{H})$ (resp. $\mathscr{L}_2(\boldsymbol{H}, \boldsymbol{H})$, resp. $\mathscr{L}_1(\boldsymbol{H}, \boldsymbol{H})$) the vector space of bounded linear operators in $\boldsymbol{H}$ with the operator norm (resp. the Hilbert space of Hilbert-Schmidt operators with the Hilbert-Schmidt norm $\|\ \|_2$, resp. the Banach space of nuclear operators with the nuclear norm $\|\ \|_1$).

Let M be an $\boldsymbol{H}$-valued right continuous square-integrable martingale. We denote by $\langle\langle M\rangle\rangle$ the unique (up to indistinguishability) predictable $\mathscr{L}_1(\boldsymbol{H}, \boldsymbol{H})$-valued process, with the following property: for every $f, g \in \boldsymbol{H}$ the process $Y^{f,g}$ defined by

$$Y_t^{f,g} := (M_t, f)(M_t, g) - (M_0, f)(M_0, g) - (f, \langle\langle M\rangle\rangle_t g)$$

is a martingale.

Actually $\langle\langle M\rangle\rangle$ takes its values in $\mathscr{L}_1^{+,s}(H, H)$, the cone of positive symmetric nuclear operators.

Now we write

$$\langle M\rangle := \text{trace of } \langle\langle M\rangle\rangle\,.$$

$\langle M\rangle$ is a predictable (continuous if M is continuous) positive increasing process with the property that $(\|M_t\|^2 - \|M_0\|^2 - \langle M\rangle_t)_{t\geq 0}$ is a martingale.

These definitions are easily extended to locally square integrable martingales.

The result of this paper is the following:

2.5. THEOREM. *Let* $(M^n)_{n\geq 0}$ *be a sequence of* $\boldsymbol{H}$*-valued continuous local martingales. Then the following properties are equivalent*:

a) *The laws* $(\tilde{P}_n)_{n\geq 0}$ *of the processes* M^n *form a tight sequence of probabilities on* $C(T, \boldsymbol{H})$.

b) *Conditions* [A] *and* $[\mathrm{T}_t]$, $t \in [0, T]$ *hold for the sequence* $(M^n)_{n\geq 0}$.

b′) J *being a dense subset of* $[0, T]$, *conditions* [A] *and* $\{[\mathrm{T}_t] : t \in J\}$ *hold for the sequence* $(M^n)_{n\geq 0}$.

c$_1$) *The laws* $(\tilde{Q}_n)_{n\geq 0}$ *of the processes* $(\langle\!\langle M^n\rangle\!\rangle^{1/2})_{n\geq 0}$ *form a tight sequence of probabilities on* $C(T, \mathscr{L}_2^+(H, H))$.

c$_2$) *The laws* $(\tilde{Q}_n^1)_{n\geq 0}$ *of the processes* $(\langle\!\langle M^n\rangle\!\rangle)_{n\geq 0}$ *form a tight sequence of probabilities on* $C(T, \mathscr{L}_1^+(H, H))$.

d$_1$) *J being a dense subset of* $[0, T]$, *condition* [A] *holds for the sequence* $(\langle M^n\rangle)_{n\geq 0}$ *and* $\{[\mathrm{T}_t] : t\in J\}$ *holds for the sequence* $(\langle\!\langle M^n\rangle\!\rangle^{1/2})_{n\geq 0}$.

d$_2$) *J being a dense subset of* $[0, T]$, *condition* [A] *holds for the sequence* $(\langle M^n\rangle)_{n\geq 0}$ *and condition* $\{[\mathrm{T}_t] : t\in J\}$ *holds for the sequence* $(\langle\!\langle M^n\rangle\!\rangle)_{n\geq 0}$.

3. Proof of the Theorem

Lemma 1 gives a) $\Rightarrow$ b). Since, for $s\leq t$

$$\langle M^n\rangle_t - \langle M^n\rangle_s = \mathrm{trace}(\langle\!\langle M^n\rangle\!\rangle_t - \langle\!\langle M^n\rangle\!\rangle_s) = \|\langle\!\langle M^n\rangle\!\rangle_t - \langle\!\langle M^n\rangle\!\rangle_s\|_1$$

Lemma 1 also gives c$_2$) $\Rightarrow$ d$_2$).

The mapping $\Phi : u\rightarrow u\circ u$ from $\mathscr{L}_2^{+,s}(H, H)$ into $\mathscr{L}_1^{+,s}(H, H)$ being continuous, one to one and with continuous inverse (see appendix), the sequences $(\langle\!\langle M^n\rangle\!\rangle^{1/2})_{n\geq 0}$ and $(\langle\!\langle M^n\rangle\!\rangle)_{n\geq 0}$ are together tight or not. Therefore the following equivalences are trivial: d$_1$) $\Leftrightarrow$ d$_2$), c$_1$) $\Leftrightarrow$ c$_2$). Since b) $\Rightarrow$ b$'$) is also trivial that the implications b$'$) $\Rightarrow$ a) and d$_2$) $\Rightarrow$ c$_2$) are proved in [1] and the implication d$_1$) $\Rightarrow$ a) is proved in [6], we have only to show: a) $\Rightarrow$ d$_1$).

Let us set for any $\mathscr{F}_t^n$-stopping time τ_n:

$$Y_t^{n,\tau_n} := \sup_{\tau_n\leq s\leq \tau_n+t} \|M_s^n - M_{\tau_n}^n\|^2 .$$

For every stopping time σ

$$E(\langle M^n\rangle_{\tau_n+\sigma} - \langle M^n\rangle_{\tau_n}) = E(\|M_{\tau_n+\sigma}^n - M_{\tau_n}^n\|^2) . \tag{3.1}$$

We make use of the following particular case of a lemma due to Lenglart (see [5]).

LEMMA 2. *Let X be an adapted positive process on some stochastic basis* $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in R_+}, P)$ *and Y be a positive adapted increasing continuous process such that for every stopping time σ*

$$E(X_\sigma)\leq E(Y_\sigma) .$$

Then, for every stopping time σ, every $\eta>0$, $a>0$

$$P\{\sup_{s\leq\sigma} X_s>\eta\}\leq \frac{a}{\eta} + P\{Y_\sigma\geq a\} . \tag{3.2}$$

In view of (3.1) we apply this lemma with $X_t := \langle M^n\rangle_{\tau_n+t} - \langle M^n\rangle_{\tau_n}$ and $Y_t := Y_t^{n,\tau_n}$. We thus obtain for every a and η

$$P_n\{\langle M^n\rangle_{\tau_n+\delta} - \langle M^n\rangle_{\tau_n} > \eta\} \leq a/\eta + P_n\{Y_\delta^{n,\tau_n} \geq a\}. \tag{3.3}$$

η being fixed we choose a such that $a/\eta \leq \varepsilon/2$ and then, using the property [A] of $(M^n)_{n\geq 0}$ which holds as a consequence of Lemma 1, we can choose δ such that

$$P_n\{Y_\delta^{n,\tau_n} \geq a\} \leq \frac{\varepsilon}{2} \qquad \text{for all } n.$$

We have then proved the property [A] for the sequence $(\langle M^n\rangle)_{n\geq 0}$. Setting $\tau_n = 0$ in the formula (3.3) we get

$$P_n\{\langle M^n\rangle_t > \eta\} \leq a/\eta + P_n\{\sup_{0\leq s\leq T} \|M_s^n\|^2 \geq a\}. \tag{3.4}$$

In order to prove that condition [T_t] is valid for the sequence $(\langle\langle M^n\rangle\rangle_t^{1/2})_{n\geq 0}$, it is enough to prove (see [6] Proposition 1.3) that, for every $\varepsilon > 0$, $\eta > 0$ there exists a finite dimensional subspace $G_{\varepsilon,\eta}$ of $\mathscr{L}_2(\boldsymbol{H}, \boldsymbol{H})$ such that, for all n

$$P_n\{\| \langle\langle M^n\rangle\rangle_t^{1/2} - \prod_{G_{\varepsilon,\eta}} \langle\langle M^n\rangle\rangle_t^{1/2}\|_2 > \eta\} \leq \varepsilon \tag{3.5}$$

where $\prod_{G_{\varepsilon,\eta}}$ denotes the orthogonal projection on $G_{\varepsilon,\eta}$. But the tightness of $(M^n)_{n\geq 0}$ implies the existence of a finite dimensional subspace $\boldsymbol{H}_{\varepsilon,a}$ of $\boldsymbol{H}$ such that

$$P_n\{\sup_{s\leq T} \|M_s^n - \prod_{\boldsymbol{H}_{\varepsilon,a}} M_s^n\| \geq a\} \leq \varepsilon/2.$$

Observing that for every stopping time τ_n

$$E_n(\langle M^n - \prod_{\boldsymbol{H}_{\varepsilon,a}} M^n\rangle_{\tau_n}) \leq E_n(\sup_{s\leq \tau_n} \|M_s^n - \prod_{\boldsymbol{H}_{\varepsilon,a}} M_s^n\|^2)$$

and using again the Lenglart-inequality, we obtain

$$\begin{aligned} &P_n\{\langle M^n - \prod_{\boldsymbol{H}_{\varepsilon,a}} M^n\rangle_{\tau_n} > \eta\} \\ &\qquad \leq a/\eta + P_n\{\sup_{s\leq\tau_n} \|M_s^n - \prod_{\boldsymbol{H}_{\varepsilon,a}} M_s^n\|^2 \geq a\} \leq a/\eta + \varepsilon/2. \end{aligned}$$

The finite dimensional subset $\boldsymbol{H}_{\varepsilon,a}$ of $\boldsymbol{H}$ can therefore be chosen in such a way that for all $t \leq T$

$$P_n\{\langle M^n - \prod_{\boldsymbol{H}_{\varepsilon,a}} M^n\rangle_t > \eta\} \leq \varepsilon, \tag{3.6}$$

which can be read

$$P_n\{\langle \prod_{H_{\varepsilon,a}^{\perp}} M^n \rangle_t > \eta\} \leq \varepsilon . \tag{3.7}$$

Let us note that the orthogonal decomposition $H = H_{\varepsilon,a} + H_{\varepsilon,a}^{\perp}$ of H leads to an orthogonal decomposition of $\mathscr{L}_2(H, H)$ which we write $\mathscr{L}_2(H, H) = \sum_{i,j=1}^{2} H_i \hat{\otimes}_2 H_j$ with $H_1 := H_{\varepsilon,a}$ and $H_2 := H_{\varepsilon,a}^{\perp}$. Denoting by $\prod_{H_i \hat{\otimes}_2 H_j}$ (resp. $\prod_i$) the orthogonal projection on $H_i \hat{\otimes}_2 H_j$ in $\mathscr{L}_2(H, H)$ (resp. on H_i in H) one has the orthogonal decomposition in $\mathscr{L}_2(H, H)$:

$$\langle\!\langle M^n \rangle\!\rangle_t^{1/2} = \sum_{i,j=1}^{2} \prod_i \circ \langle\!\langle M^n \rangle\!\rangle_t^{1/2} \circ \prod_j . \tag{3.8}$$

But

$$\| \prod_i \circ \langle\!\langle M^n \rangle\!\rangle_t^{1/2} \circ \prod_j \|_2^2 \leq \| \prod_i \circ \langle\!\langle M^n \rangle\!\rangle_t^{1/2} \|_2^2 = \text{trace} \prod_i \circ \langle\!\langle M^n \rangle\!\rangle_t \circ \prod_i = \langle \prod_i M \rangle_t .$$

The inequality (3.7) then leads to

$$P_n\{\| \prod_i \circ \langle\!\langle M^n \rangle\!\rangle_t^{1/2} \circ \prod_{H_{\varepsilon,a}^{\perp}} \|_2^2 > \eta\} \leq \varepsilon \quad i = 1, 2$$
$$P_n\{\| \prod_{H_{\varepsilon,a}^{\perp}} \circ \langle\!\langle M^n \rangle\!\rangle_t^{1/2} \circ \prod_i \|_2^2 > \eta\} \leq \varepsilon \quad i = 1, 2$$

and according to the orthogonal decomposition (3.8) this gives

$$P_n\{\| \langle\!\langle M^n \rangle\!\rangle_t^{1/2} - \prod_{H_{\varepsilon,a} \hat{\otimes}_2 H_{\varepsilon,a}} \langle\!\langle M^n \rangle\!\rangle_t^{1/2} \|_2^2 > \eta\} \leq 3\varepsilon. \tag{3.9}$$

This proves (3.5) with $G_{\varepsilon,\eta} = H_{\varepsilon,a} \hat{\otimes}_2 H_{\varepsilon,a}$ and therefore the theorem.

Appendix

For the convenience of the reader we give here a proof of the continuity of the mapping $v \rightsquigarrow v^{1/2}$ from $\mathscr{L}_1^{+,s}(H, H)$, the set of positive symmetric nuclear operators on H (with the nuclear norm) into $\mathscr{L}_2^{+,s}(H, H)$, the set of symmetric positive Hilbert-Schmidt operators with the Hilbert-Schmidt norm. To this effect we consider a sequence u_n in $\mathscr{L}_2^{+,s}(H, H)$ such that $\lim_{n\to\infty} \| u_n \circ u_n - u \circ u \|_1 = 0$. Since

$$\| u_n \|_2^2 = \| u_n \circ u_n \|_1 \quad \text{and} \quad \| u \|_2^2 = \| u \circ u \|_1$$

the following holds:

$$\lim_{n\to\infty} \| u_n \|_2^2 = \| u \|_2^2 .$$

Therefore we have only to prove that u_n converges weakly to u in the Hilbert space $\mathscr{L}_2^{+,s}(H, H)$. But, since $\sup_n \| u_n \|_2 < \infty$, the sequence (u_n) is weakly compact and has weak limits. We have only to show that if u' is any limit then $u' = u$.

By definition, for every $\varphi \in \mathscr{L}_2(\boldsymbol{H}, \boldsymbol{H})$

$$\lim_{n\to\infty} \operatorname{trace}((u' - u_n)\circ\varphi) = 0 .$$

Therefore

$$\begin{aligned}
&\limsup_{n\to\infty} |\operatorname{trace}(u_n \circ u_n - u' \circ u') \circ \varphi| \\
&\quad \leq \limsup_{n\to\infty} [|\operatorname{trace} u_n \circ (u_n - u') \circ \varphi| + |\operatorname{trace}(u_n - u') \circ u' \circ \varphi|] \\
&\quad \leq \limsup_{n\to\infty} [\sup_n \| u_n \|_2 |\operatorname{trace}(u_n - u') \circ \varphi| + |\operatorname{trace}(u_n - u') \circ u' \circ \varphi|] \\
&\quad = 0 .
\end{aligned}$$

This shows that $(u_n \circ u_n)_{n\geq 0}$ converges to $u' \circ u'$ weakly in $\mathscr{L}_2(\boldsymbol{H}, \boldsymbol{H})$. But, since $(u_n \circ u_n)_{n\geq 0}$ converges to $u \circ u$ in $\mathscr{L}_1(\boldsymbol{H}, \boldsymbol{H})$ and therefore in $\mathscr{L}_2(\boldsymbol{H}, \boldsymbol{H})$, one has $u' \circ u' = u \circ u$. The u_n's being symmetric positive the same is true for u'. Then $u = u'$. This finishes the proof of the convergence of the sequence $(\mathrm{u}_n)_{n\geq 0}$ to u in $\mathscr{L}_2(\boldsymbol{H}, \boldsymbol{H})$.

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