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ON SOME DIMENSION FORMULA FOR AUTOMORPHIC FORMS OF WEIGHT ONE, II

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§0. Introduction

Let Γ be a fuchsian group of the first kind and assume that Γ contains the element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $(= -I)$, and let χ be a unitary representation of Γ of degree 1 such that $\chi(-I) = -1$. Let $S_1(\Gamma, \chi)$ be the linear space of cusp forms of weight one on the group Γ with character χ. We shall denote by d_1 the dimension of the linear space $S_1(\Gamma, \chi)$. It is not effective to compute the number d_1 by means of the Riemann-Roch theorem. Because of this reason, it is an interesting problem in its own right to determine the number d_1 by some other method (for example, [5]).

When the group Γ has a compact fundamental domain in the upper half plane S[1], we have obtained the following dimension formula which is a slightly modified form of the previous result ([1]):

$$(1) \qquad d_1 = \frac{1}{2} \sum_{\{M\}} \frac{\chi(M)}{[\Gamma(M): \pm I]} \frac{\bar{\zeta}}{1 - \bar{\zeta}^2} + \frac{1}{2} \operatorname*{Res}_{s=0} \zeta^*(s)\,,$$[2]

where the sum over $\{M\}$ is taken over the distinct elliptic conjugacy classes of $\Gamma/\{\pm I\}$, $\Gamma(M)$ denotes the centralizer of M in Γ, $\bar{\zeta}$ is one of the eigenvalues of M, and $\zeta^*(s)$ denotes *the Selberg type zeta-function* defined by

$$\zeta^*(s) = \sum_{\alpha=1}^{\infty} \sum_{k=1}^{\infty} \frac{\chi(P_\alpha)^k \log \lambda_{0,\alpha}}{\lambda_{0,\alpha}^k - \lambda_{0,\alpha}^{-k}} |\lambda_{0,\alpha}^k + \lambda_{0,\alpha}^{-k}|^{-s}\,.$$

Here, $\lambda_{0,\alpha}$ denotes the eigenvalue ($\lambda_{0,\alpha} > 1$) of a representative P_α of the primitive hyperbolic conjugacy classes $\{P_\alpha\}$ in $\Gamma/\{\pm I\}$.

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1) In this case, $S_1(\Gamma, \chi)$ denotes simply the space of all holomorphic automorphic forms of weight one with χ.

2) For this modified formula of d_1, refer to Hiramatsu ([6], Remark 1 in §2).

The purpose of this paper is to give a similar formula of the number d_1 when the group Γ is of finite type *reduced at infinity and* $\chi^2 \neq 1$, by using the method of Selberg ([3], [4]). In this case, the operator ω_δ in [1] is not generally completely continuous on the space $L^2(\Gamma\backslash\tilde{S}, \chi)$ and a new term from cusp ought to have added to the formula (1). The notation used here will generally be those of [1].

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§ 1. The Selberg eigenspace $\mathfrak{M}(1, -3/2, \chi)$ for a group Γ of finite type

Let Γ be a fuchsian group of the first kind containing the element $-I$, and suppose that Γ has a non-compact fundamental domain in S. Let T be the real torus and put $\tilde{S} = S \times T$. Denote by $L^2(\Gamma\backslash\tilde{S}, \chi)$ the following set

$$\left\{f \in L^2(\Gamma\backslash\tilde{S})\colon f(g(z, \phi)) = \chi(g)f(z, \phi) \text{ for all } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\right\},$$

where

$$f(g(z, \phi)) = f\left(\frac{az+b}{cz+d}, \phi + \arg(cz+d)\right).$$

Moreover we denote by $\mathfrak{M}_\Gamma(k, \lambda, \chi) = \mathfrak{M}(k, \lambda, \chi)$ the set of all functions $f(z, \phi)$ satisfying the following conditions:

(i) $f(z, \phi) \in L^2(\Gamma\backslash\tilde{S}, \chi)$,

(ii) $\tilde{\Delta}f(z, \phi) = \lambda f(z, \phi), \dfrac{\partial}{\partial\phi}f(z, \phi) = -ikf(z, \phi)$.

Then we have the following

LEMMA. *To each function $f(z, \phi) \in \mathfrak{M}(1, \lambda, \chi)$, we associate a function on S by letting*

$$F(z) = e^{i\phi}y^{-1/2}f(z, \phi).$$

Then the function $F(z)$ belongs to $S_1(\Gamma, \chi)$ if and only if

$$f(z, \phi) \in \mathfrak{M}(1, -3/2, \chi).$$

Proof. For each $F(z) \in S_1(\Gamma, \chi)$ we define $f(z, \phi)$ on $\tilde{S}$ by

$$f(z, \phi) = e^{-i\phi}y^{1/2}F(z). \tag{1.1}$$

Then the function $f(z, \phi)$ satisfies the conditions:

(1.2) $f(g(z, \phi)) = \chi(g)f(z, \phi)$ for all $g \in \Gamma$;

(1.3) $(\partial/\partial\phi)f(z, \phi) = -if(z, \phi)$;

(1.4) $\tilde{\Delta}f(z, \phi) = -(3/2)f(z, \phi)$ by regularity of $F(z)$ on S;

(1.5) Since $y^{1/2}|F(z)|$ is bounded on S,

$$\|f\| = \frac{1}{\pi}\int_{\Gamma\backslash\tilde{S}} |e^{-i\phi}y^{1/2}F(z)|^2 \frac{dxdyd\phi}{y^2}$$
$$= \int_{\Gamma\backslash S} |y^{1/2}F(z)|^2 \frac{dxdy}{y^2} < \infty .$$

Therefore, by (1.2)$\sim$(1.5), the function $f(z, \phi)$ belongs to $\mathfrak{M}(1, -3/2, \chi)$. We now prove conversely that any function in $\mathfrak{M}(1, -3/2, \chi)$ must be of the form (1.1) with $F(z) \in S_1(\Gamma, \chi)$. Let $f(z, \phi)$ be a function in $\mathfrak{M}(1, -3/2, \chi)$. Put

$$F(z) = e^{i\phi}y^{-1/2}f(z, \phi) .$$

Then the Γ-invariance of $f(z, \phi)$ is equivalent to a transformation law for $F(z)$:

$$F(g(z)) = \chi(g)(cz + d)F(z)$$

for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Therefore, it is sufficient for the proof of the latter half of our lemma to show that $F(z)$ is holomorphic with respect to the complex variable z on S, and vanishes at every cusp of Γ.

Let $\mathfrak{g}$ be the Lie algebra of $SL_2(\boldsymbol{R})$ $(=G)$. Then we can take the basis $\mathfrak{a}$ of $\mathfrak{g}$ such that the Lie derivatives associated with the elements of $\mathfrak{a}$ are given by the following invariant differential operators:

$$X = y\cos 2\phi\frac{\partial}{\partial x} - y\sin 2\phi\frac{\partial}{\partial y} + \frac{1}{2}(\cos 2\phi - 1)\frac{\partial}{\partial\phi} ,$$
$$Y = y\sin 2\phi\frac{\partial}{\partial x} + y\cos 2\phi\frac{\partial}{\partial y} + \frac{1}{2}\sin 2\phi\frac{\partial}{\partial\phi} ,$$
$$\Phi = \frac{\partial}{\partial\phi} .$$

It is easy to see that

$$\tilde{\Delta} = \left(X + \frac{1}{2}\Phi\right)^2 + Y^2 + \Phi^2 .$$

Now we put

$$A^- = 2\left(X + \frac{1}{2}\Phi\right) + 2iY.$$

Then, the function $F(z)$ is holomorphic on S if and only if

$$A^- f(z, \phi) = 0. \tag{1.6}$$

To prove (1.6), first note that the operation of A^- depends only on the representations of the Lie algebra $\mathfrak{g}$. Let $L^2_d(\Gamma\backslash G, \chi)$ be the discrete part of the space $L^2(\Gamma\backslash G, \chi)$. Then $f \in L^2_d(\Gamma\backslash G, \chi)$. Let

$$L^2_d(\Gamma\backslash G, \chi) = \sum_i V_i$$

be the irreducible splitting of the space $L^2_d(\Gamma\backslash G, \chi)$ and put

$$f = \sum_i f_i \quad (f_i \in V_i).$$

Then, if $f_i \neq 0$, we have

$$\tilde{\Delta} f_i = -\frac{3}{2} f_i, \frac{\partial}{\partial\phi} f_i = -\sqrt{-1} f_i.$$

Therefore, each subspace V_i such that $f_i \neq 0$ is isomorphic to the space H_1 of the irreducible representation of the limit of discrete series. Hence it is sufficient for the proof of (1.6), to show that for any highest weight vector φ in H_1,

$$A^- \varphi = 0. \tag{1.7}$$

For example, by Lemma 5.6 in [2], the formula (1.7) is well known.

Next we shall see the condition for $F(z)$ at every cusp of Γ. Let s be a cusp of $\Gamma \cap \operatorname{Ker} \chi$. We may assume that $s = \infty$ and the intersection of a fundamental domain for Γ and a neighborhood of ∞ is the following type

$$\{z = x + iy \colon 0 \leqq x \leqq 1, y \geqq M\},$$

where M denotes a positive constant. Then, by the given condition $f(z, \phi) \in L^2(\Gamma\backslash \tilde{S}, \chi)$,

$$\int_M^\infty \left\{\int_0^1 y |F(z)|^2 dx\right\} \frac{dy}{y^2} < \infty.$$

Let

$$F(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z}$$

be the Fourier expansion of F at ∞. Then, we have

$$\int_0^1 |F(z)|^2 dx = \int_0^1 (\sum_n a_n e^{2\pi i n z})(\sum_m \bar{a}_m e^{-2\pi i m \bar{z}}) dx$$
$$= \sum_{n,m} a_n \bar{a}_m \int_0^1 e^{2\pi i (n-m)x - 2\pi(n+m)y} dx$$
$$= \sum_n |a_n|^2 e^{-4\pi n y} .$$

Therefore,

$$\int_M^\infty y(\sum_n |a_n|^2 e^{-4\pi n y}) \frac{dy}{y^2} = \sum_n |a_n|^2 \int_M^\infty y^{-1} e^{-4\pi n y} dy .$$

If $n \leqq 0$, then

$$\int_M^\infty y^{-1} e^{-4\pi n y} dy = \infty ,$$

so that $a_n = 0$ for all $n \leqq 0$. Q.E.D.

§ 2. Eisenstein series and continuous spectrum

2.1. This section is essentially based upon the work of Selberg ([3], [4]). We shall review the definition and elementary properties of Eisenstein series for the cusp ∞, and the spectral decomposition of $L^2(\Gamma \backslash \tilde{S}, \chi)$ (abbreviated hereafter as $L^2(\Gamma \backslash \tilde{S})$). Let Γ be of finite type reduced at ∞, namely, ∞ is a cusp of Γ and the stabilizer Γ_∞ of ∞ in Γ is equal to $\pm \Gamma_0$ with $\Gamma_0 = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \boldsymbol{Z} \right\}$. The Eisenstein series $E_\chi(z, \phi; s)$ attached to the cusp ∞ and χ is then defined by

$$E_\chi(z, \phi; s) = \sum_{\substack{M \in \Gamma_\infty \backslash \Gamma \\ M = \binom{* \, *}{c \, d}}} \frac{\bar{\chi}(M) y^s}{|cz + d|^{2s}} e^{-i(\phi + \arg(cz+d))} , \tag{2.1}$$

where $s = \sigma + ir$ with $\sigma > 1$. It is easy to check that

(i) $E_\chi(M(z, \phi); s) = \chi(M) E_\chi(z, \phi; s)$ for $M \in \Gamma$;

(ii) $\tilde{\Delta} E_\chi(z, \phi; s) = \left\{ s(s-1) - \dfrac{5}{4} \right\} E_\chi(z, \phi; s)$;

(iii) $\dfrac{\partial}{\partial \phi} E_\chi(z, \phi; s) = -i E_\chi(z, \phi; s)$.

Since $E_\chi(z+1, \phi; s) = E_\chi(z, \phi; s)$, we have a Fourier-Bessel expansion of the form

$$E_\chi(z, \phi; s) = \sum_{m=-\infty}^{\infty} a_{m,\chi}(y, \phi; s) e^{2\pi i m x} ,$$

where

$$a_{m,\chi}(y, \phi; s) = \int_0^1 E_\chi(z, \phi; s) e^{-2\pi i m x} dx .$$

Let us now try to find the constant term $a_{0,\chi}(y, \phi; s)$ explicitly. Put

$$a_{0,\chi}(y, \phi; s) = e^{-i\phi} a_{0,\chi}(y; s) ,$$

and

$$E_\chi(z; s) = e^{i\phi} E_\chi(z, \phi; s) ;$$

then

$$\begin{aligned}
a_{0,\chi}(y; s) &= \int_0^1 E_\chi(z; s) dx \\
&= \int_0^1 \sum_{\substack{M \in \Gamma_\infty \backslash \Gamma \\ M = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)}} \frac{\bar{\chi}(M) y^s}{|cz + d|^{2s}} \lambda(cz + d)^{-1} dx \quad \left(\lambda(z) = \frac{z}{|z|}, z \neq 0\right) \\
&= y^s + \int_0^1 \sum_{\substack{M \in \Gamma_\infty \backslash \Gamma \\ M = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \\ c > 0}} \frac{\bar{\chi}(M) y^s}{|cz + d|^{2s}} \lambda(cz + d)^{-1} dx \\
&= y^s + \int_{-\infty}^{\infty} \sum_{\substack{M \in \Gamma_\infty \backslash \Gamma / \Gamma_\infty \\ M = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \\ c > 0}} \frac{\bar{\chi}(M) y^s}{|cz + d|^{2s}} \lambda(cz + d)^{-1} dx \\
&= y^s + \sum_{\substack{c > 0 \\ d \bmod c \\ \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)}} \frac{\bar{\chi}(c, d)}{|c|^{2s}} \int_{-\infty}^{\infty} \frac{y^s}{\left|z + \dfrac{d}{c}\right|^{2s}} \lambda\left(z + \frac{d}{c}\right)^{-1} dx \\
&= y^s + y^s \varphi_{0,\chi}(s) \int_{-\infty}^{\infty} \frac{\lambda(z)^{-1}}{|z|^{2s}} dx \\
&= y^s + y^{1-s} \varphi_{0,\chi}(s) \int_{-\infty}^{\infty} \frac{\lambda(i + t)^{-1}}{(1 + t^2)^s} dt ,
\end{aligned}$$

where

$$\varphi_{0,\chi}(s) = \sum_{\substack{c > 0 \\ d \bmod c \\ \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma}} \frac{\bar{\chi}(c, d)}{|c|^{2s}}$$

Furthermore,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\lambda(i + t)^{-1}}{(1 + t^2)^s} dt &= \int_0^\pi \frac{(i + \cot\theta)^{-1}}{|\operatorname{cosec}\theta|^{2s-1}} \operatorname{cosec}^2\theta \, d\theta \\
&= \int_0^\pi \sin^{2s-2}\theta (\cos\theta - i \sin\theta) \, d\theta
\end{aligned}$$

$$= -i\int_0^\pi \sin^{2s-1}\theta\, d\theta$$
$$= -i\sqrt{\pi}\,\frac{\Gamma(s)}{\Gamma\left(s+\frac{1}{2}\right)}.$$

Therefore we have

$$a_{0,\chi}(y;s) = y^s - i\sqrt{\pi}\,\frac{\Gamma(s)}{\Gamma\left(s+\frac{1}{2}\right)}\varphi_{0,\chi}(s)y^{1-s}.$$

2.2. Since the group Γ is reduced at ∞, the integral operator ω_δ in [1] is not generally completely continuous on $L^2(\Gamma\backslash\tilde{S})$ and beside the discrete spectrum in $L^2(\Gamma\backslash\tilde{S})$, the operator ω_δ has one or more continuous spectra in $L^2(\Gamma\backslash\tilde{S})$. The space $L^2(\Gamma\backslash\tilde{S})$ has the following spectral decomposition

$$L^2(\Gamma\backslash\tilde{S}) = L_0^2(\Gamma\backslash\tilde{S})\oplus L_{sp}^2(\Gamma\backslash\tilde{S})\oplus L_{cont}^2(\Gamma\backslash\tilde{S}),$$

where L_0^2 is the space of cusp forms and is discrete, L_{sp}^2 is the discrete part of the orthogonal complement of L_0^2, and L_{cont}^2 is continuous part of the spectrum, In the following we shall only consider the case

$$\chi\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right) = 1.$$

By using the analytic continuation of the Eisenstein series (2.1) as a function of s for $s = 1/2 + ir$, we put

$$\tilde{H}_\delta(z,\phi;z',\phi') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} h(r)E_\chi\left(z,\phi;\frac{1}{2}+ir\right)\overline{E_\chi\left(z',\phi';\frac{1}{2}+ir\right)}dr,$$

where $h(r)$ denotes the eigenvalue of ω_δ in $\mathfrak{M}(1,\lambda,\chi)$ ([1], p. 217):

$$h(r) = 2^{2+\delta}\pi\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+\delta}{2}\right)}{\Gamma(\delta)\,\Gamma\left(1+\frac{\delta}{2}\right)}\Gamma\left(\frac{\delta}{2}+ir\right)\Gamma\left(\frac{\delta}{2}-ir\right) \tag{2.2}$$

with $\lambda = s(s-1) - 5/4$ and $s = 1/2 + ir$. Then the integral operator $\tilde{K}_\delta = K_\delta - \tilde{H}_\delta$ is now completely continuous on $L^2(\Gamma\backslash\tilde{S})$ and has all discrete spectra of K_δ, where

$$K_\delta(z,\phi;z',\phi') = 2\sum_{M\in\Gamma/\{\pm I\}}\chi(M)\omega_\delta(z,\phi;M(z',\phi')).$$

Furthermore, an eigenvalue of $f(z, \phi)$ in $L_0^2(\Gamma\backslash\tilde{S}) \oplus L_{\mathrm{sp}}^2(\Gamma\backslash\tilde{S})$ for $\tilde{K}_\delta$ is equal to that for K_δ and the image of $\tilde{K}_\delta$ on it is contained in $L_0^2(\Gamma\backslash\tilde{S})$. Considering the trace of $\tilde{K}_\delta$ on $L_0^2(\Gamma\backslash\tilde{S})$, we now obtain the following modified trace formula

$$\begin{aligned}\sum_{n=1}^{\infty} h(\lambda^{(n)}) &= \int_{\tilde{D}} \tilde{K}_\delta(z, \phi; z, \phi)\, d(z, \phi) \\ &= \int_{\tilde{D}} \{ 2\sum_{M \in \Gamma/\{\pm I\}} \chi(M)\omega_\delta(z, \phi; M(z, \phi)) - \tilde{H}_\delta(z, \phi; z, \phi)\}\, d(z, \phi)\,,\end{aligned} \tag{2.3}$$

where $\tilde{D}$ denotes a fundamental domain of Γ in $\tilde{S}$, and each of $\lambda^{(n)}$ denotes an eigenvalue corresponding to an orthogonal basis $\{f^{(n)}\}$ for $L_0^2(\Gamma\backslash\tilde{S})$.

§ 3. A formula for the dimension d_1

The purpose of this section is to obtain an explicit formula for the dimension d_1 by calculating the integrals in (2.3). We put

$$\begin{aligned}&\int_{\tilde{D}} \{ 2\sum_{M \in \Gamma/\{\pm I\}} \chi(M)\omega_\delta(z, \phi; M(z, \phi)) - \tilde{H}_\delta(z, \phi; z, \phi)\}\, d(z, \phi) \\ &\qquad = J(I) + J(P) + J(R) + J(\infty)\,,\end{aligned}$$

where $J(I)$, $J(P)$, $J(R)$, and $J(\infty)$ denote respectively the identity component, the hyperbolic component, the elliptic component, and the parabolic component of the traces. Then the components $J(I)$, $J(P)$, and $J(R)$ were obtained already in [1] and [6]. So in the following we shall calculate the component $J(\infty)$. Since Γ is reduced at ∞, the set $\Gamma/\{\pm I\} - \{I\}$ gives a complete system of representatives of the parabolic conjugacy classes in $\Gamma/\{\pm I\}$ and for each $\gamma \in \Gamma_\infty - \{\pm I\}$, the stabilizer of γ in Γ is always equal to Γ_∞. Put

$$P_\Gamma = \left\{\gamma^{-1}\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}\gamma :\ m \in Z,\, m \neq 0,\, \gamma \in \Gamma_\infty\backslash\Gamma\right\}.$$

Then P_Γ is the set consisting of all parabolic elements in $\Gamma/\{\pm I\}$. Hence

$$\begin{aligned}\int_{\tilde{D}} \sum_{\substack{M \in \Gamma/\{\pm I\}\\ M:\ \text{parabolic}}} \chi(M)\omega_\delta(z, \phi; M(z, \phi))\, d(z, \phi) &= \sum_{\substack{M \in \Gamma/\{\pm I\}\\ M:\ \text{parabolic}}} \int_{\tilde{D}} \chi(M)\omega_\delta(z, \phi; M(z, \phi))\, d(z, \phi) \\ &= \sum_{\substack{M \in \Gamma_0\\ M \neq I}} \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \int_{\tilde{D}} \omega_\delta(\gamma(z, \phi); M\gamma(z, \phi))\, d(z, \phi) \\ &= \sum_{\substack{M \in \Gamma_0\\ M \neq I}} \int_{\tilde{D}_{\Gamma_\infty}} \omega_\delta(z, \phi; M(z, \phi))\, d(z, \phi)\,,\end{aligned}$$

where $\tilde{D}_{\Gamma_\infty}$ denotes a fundamental domain of Γ_∞ in $\tilde{S}$. Therefore we have

$$J(\infty) = \lim_{Y\to\infty}\left\{\int_0^Y\int_0^1\int_0^\pi 2\sum_{\substack{M\in\Gamma_0\\ M\neq I}} \omega_\delta(z,\phi;M(z,\phi))\,d(z,\phi) - \int_{\tilde{D}_Y}\tilde{H}_\delta(z,\phi;z,\phi)\,d(z,\phi)\right\},$$

where $\tilde{D}_Y = \tilde{D} - \tilde{D}'_Y$ with the direct product $\tilde{D}'_Y$ of the real torus T and the subdomain of the strip determined by $\operatorname{Im} z > Y$ for a sufficiently large $Y > 0$. Furthermore

$$\text{(A)}\qquad \int_0^Y\int_0^1\int_0^\pi 2\sum_{\substack{M\in\Gamma_0\\ M\neq I}} \omega_\delta(z,\phi;M(z,\phi))\,d(z,\phi)$$

$$= 2\sum_{\substack{m\in\mathbf{Z}\\ m\neq 0}}\int_0^Y\int_0^1\int_0^\pi \omega_\delta(z,\phi;M_0^m(z,\phi))\,d(z,\phi) \qquad \left(M_0 = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)$$

$$= 2\pi\sum_{\substack{m\in\mathbf{Z}\\ m\neq 0}}\int_0^Y\left|\frac{y}{\dfrac{2yi-m}{2i}}\right|^\delta \frac{y}{\dfrac{2yi-m}{2i}}\frac{dy}{y^2}$$

$$= 2^{\delta+4}\pi\sum_{m=1}^\infty\int_0^Y \frac{1}{(4+(m/y)^2)^{\delta/2+1}}\frac{dy}{y^2}.$$

If we put

$$k(t) = \frac{1}{(4+t)^{\delta/2+1}} \quad\text{and}\quad f(x) = 2\int_0^Y k\left(\frac{x^2}{y^2}\right)\frac{dy}{y^2},$$

then

$$f(x) = \frac{1}{x}\int_{x^2/Y^2}^\infty \frac{k(t)}{\sqrt{t}}\,dt\,;$$

and hence

$$\int_0^Y\int_0^1\int_0^\pi 2\sum_{\substack{M\in\Gamma_0\\ M\neq I}} \omega_\delta(z,\phi;M(z,\phi))\,d(z,\phi) = 2^{\delta+3}\pi\sum_{m=1}^\infty f(m).$$

Now we make use of a summation formula due to Euler-MacLaurin:

$$\text{(3.1)}\qquad \sum_{m=1}^\infty f(m) = \frac{1}{2}f(1) + \int_1^\infty f'(x)\{x\}\,dx + \int_1^\infty f(x)\,dx,$$

where $[x]$ denotes the greatest integer in x and $\{x\} = x - [x] - 1/2$. Then we have

$$\frac{1}{2}f(1) \longrightarrow 2^{-(\delta+2)}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\delta+1}{2}\right)}{\Gamma\left(1+\frac{\delta}{2}\right)},$$

and

$$\int_1^\infty f'(x)\{x\}\,dx = 2^{-(\delta+1)}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\delta+1}{2}\right)}{\Gamma\left(1+\frac{\delta}{2}\right)}\left(C-\frac{1}{2}\right)+o(1)\,,$$

as $Y\to\infty$, where C is Euler's constant. As for the third integral of (3.1), we have first the following by [3]:

$$\begin{aligned}&\int_1^\infty f(x)\,dx\\ &\quad = 2^{-(\delta+1)}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\delta+1}{2}\right)}{\Gamma\left(1+\frac{\delta}{2}\right)}\log Y+\frac{1}{2}\int_0^\infty\frac{\log t\cdot k(t)}{\sqrt{t}}\,dt+o(1)\,,\end{aligned}$$

as $Y\to\infty$. Furthermore,

$$\begin{aligned}\frac{1}{2}\int_0^\infty\frac{\log t\cdot k(t)}{\sqrt{t}}\,dt &= \frac{1}{2}\int_0^\infty\frac{\log t}{\sqrt{t}\,(4+t)^{\delta/2+1}}\,dt\\ &= 2^{-(\delta+1)}\log 2\int_0^\infty\frac{1}{\sqrt{t}\,(1+t)^{\delta/2+1}}\,dt+2^{-(\delta+2)}\int_0^\infty\frac{\log t}{\sqrt{t}\,(1+t)^{\delta/2+1}}\,dt\\ &= 2^{-(\delta+1)}\log 2\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\delta+1}{2}\right)}{\Gamma\left(1+\frac{\delta}{2}\right)}\\ &\quad + 2^{-(\delta+2)}\left\{-\sum_{k=0}^\infty\binom{-\delta/2-1}{k}\frac{1}{\left(\frac{1}{2}+k\right)^2}+\sum_{k=0}^\infty\binom{-\delta/2-1}{k}\frac{1}{\left(\frac{\delta+9}{2}+k\right)^2}\right\},\end{aligned}$$

where $-2<\delta<4$. Summing up the above results, we obtain for the first half of $J(\infty)$,

$$\int_0^Y\int_0^1\int_0^\pi 2\sum_{\substack{M\in\Gamma_0\\ M\neq I}}\omega_\delta(z,\phi;M(z,\phi))\,d(z,\phi)$$

$$= 2^2\pi \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\delta+1}{2}\right)}{\Gamma\left(1+\frac{\delta}{2}\right)}(\log Y + C + \log 2)$$

$$+ 2\pi\left\{-\sum_{k=0}^{\infty}\binom{-\delta/2-1}{k}\frac{1}{\left(\frac{1}{2}+k\right)^2}\right.$$

$$\left.+\sum_{k=0}^{\infty}\binom{-\delta/2-1}{k}\frac{1}{\left(\frac{\delta+9}{2}+k\right)^2}\right\} + o(1)\,,$$

as $Y \to \infty$.

(B) We define the following the compact part of $E_\chi(z, \phi; s)$:

$$\tilde{E}_\chi(z, \phi; s) = \begin{cases} E_\chi(z, \phi; s) & \text{for } y \leqq Y\,, \\ E_\chi(z, \phi; s) - e^{-i\phi}(y^s + \varphi_\chi(s)y^{1-s}) & \text{for } y > Y\,, \end{cases}$$

with $\varphi_\chi(s) = -i\sqrt{\pi}\dfrac{\Gamma(s)}{\Gamma\left(s+\frac{1}{2}\right)}\varphi_{0,\chi}(s)$. Then,

$$\int_{\tilde{D}_Y} \tilde{H}_\delta(z, \phi; z, \phi)\, d(z, \phi)$$

$$= \frac{1}{4\pi^2}\int_{\tilde{D}_Y}\int_{-\infty}^{\infty} h(r)E_\chi\left(z, \phi; \frac{1}{2}+ir\right)\overline{E_\chi\left(z, \phi; \frac{1}{2}-ir\right)}\,dr\,d(z, \phi)$$

$$= \frac{1}{4\pi^2}\lim_{\sigma\to\frac{1}{2}}\int_{\tilde{D}}\int_{-\infty}^{\infty} h(r)|\tilde{E}(z, \phi; \sigma+ir)|^2 dr\,d(z, \phi) + o(1)\,,$$

as $Y \to \infty$. Moreover,

$$\int_{\tilde{D}}\int_{-\infty}^{\infty} h(r)|\tilde{E}_\chi(z, \phi; s)|^2 dr\,d(z, \phi) = \int_{\tilde{D}_Y}\int_{-\infty}^{\infty} h(r)\,E_\chi(z, \phi; s)\overline{E_\chi(z, \phi; s)}\,dr\,d(z, \phi)$$

$$+ \int_{\tilde{D}_Y'}\int_{-\infty}^{\infty} h(r)\{E_\chi(z, \phi; s) - e^{-i\phi}(y^s + \varphi_\chi(s)y^{1-s})\}$$

$$\times \{\overline{E_\chi(z, \phi; s)} - e^{i\phi}(y^{\bar{s}} + \overline{\varphi_\chi(s)}y^{1-\bar{s}})\}dr\,d(z, \phi)\;;$$

$$\int_{\tilde{D}_Y}\int_{-\infty}^{\infty} h(r)\,E_\chi(z, \phi; s)\overline{E_\chi(z, \phi; s)}\,dr\,d(z, \phi)$$

$$= \int_{\cup M\tilde{D}_Y}\int_{-\infty}^{\infty} h(r)E_\chi(z, \phi; s)y^{\bar{s}}e^{i\phi}dr\,d(z, \phi)$$

$$= \int_0^Y\int_0^1\int_0^\pi\int_{-\infty}^{\infty} h(r)E_\chi(z, \phi; s)y^{\bar{s}-2}e^{i\phi}dr\,d\phi\,dx\,dy$$

$$- \int_{\substack{\cup M\tilde{D}_Y \\ M\in\Gamma_\infty\backslash\Gamma, M\neq I}} \int_{-\infty}^{\infty} h(r)E_\chi(z, \phi; s)y^{\bar{s}}e^{i\phi}dr\,d(z, \phi)$$

$$= \pi\int_0^Y\int_{-\infty}^{\infty} h(r)(y^s + \varphi(s)y^{1-s})y^{\bar{s}-2}drdy$$

$$- \int_{\tilde{D}_{Y'}}\int_{-\infty}^{\infty} h(r)E_\chi(z, \phi; s)\{\overline{E_\chi(z, \phi; s)} - e^{i\phi}y^s\}dr\,d(z, \phi)$$

$$= \pi\int_0^Y\int_{-\infty}^{\infty} h(r)y^{2\sigma-2}drdy + \pi\int_0^Y\int_{-\infty}^{\infty} h(r)\varphi_\chi(s)y^{-1-2ir}drdy$$

$$- \int_{\tilde{D}_{Y'}}\int_{-\infty}^{\infty} h(r)\{E_\chi(z, \phi; s) - e^{-i\phi}(y^s + \varphi_\chi(s)y^{1-s})\}$$

$$\times \{\overline{E_\chi(z, \phi; s)} - e^{i\phi}(y^{\bar{s}} + \overline{\varphi_\chi(s)}y^{1-\bar{s}})\}dr\,d(z, \phi)$$

$$- \pi\int_Y^\infty\int_{-\infty}^{\infty} h(r)\overline{\varphi_\chi(s)}y^{-1+2ir}dr\,dy - \pi\int_Y^\infty\int_{-\infty}^{\infty} h(r)|\varphi_\chi(s)|^2drdy\,.$$

Therefore we have

$$\int_{\tilde{D}_Y} \tilde{H}_\delta(z, \phi; z, \phi)d(z, \phi) = \frac{1}{4\pi}\lim_{\sigma\to\frac{1}{2}}\Big\{\int_0^Y\int_{-\infty}^{\infty} h(r)y^{2\sigma-2}drdy$$

$$- \int_Y^\infty\int_{-\infty}^{\infty} h(r)|\varphi_\chi(s)|^2y^{-2\sigma}drdy + \int_0^Y\int_{-\infty}^{\infty} h(r)\varphi_\chi(s)y^{-1-2ir}drdy$$

$$- \int_Y^\infty\int_{-\infty}^{\infty} h(r)\overline{\varphi_\chi(s)}y^{-1+2ir}drdy\Big\} + o(1)\,,$$

as $Y\to\infty$. Here we calculate the integrals appearing in the above expression.

(i) $$\int_0^Y\int_{-\infty}^{\infty} h(r)y^{2\sigma-2}drdy = \frac{Y^{2\sigma-1}}{2\sigma-1}\int_{-\infty}^{\infty} h(r)dr$$

$$= \frac{Y^{2\sigma-1} - Y^{1-2\sigma}}{2\sigma-1}\int_{-\infty}^{\infty} h(r)dr + \frac{Y^{1-2\sigma}}{2\sigma-1}\int_{-\infty}^{\infty} h(r)dr\,;$$

$$\int_{-\infty}^{\infty} h(r)\,dr = 2^{2+\delta}\pi\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+\delta}{2}\right)}{\Gamma(\delta)\Gamma\left(1+\frac{\delta}{2}\right)}\int_{-\infty}^{\infty}\left|\Gamma\left(\frac{\delta}{2}+ir\right)\right|^2dr$$

$$= 2^{2+\delta}\pi\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+\delta}{2}\right)}{\Gamma(\delta)\Gamma\left(1+\frac{\delta}{2}\right)}\cdot\sqrt{\pi}\,\Gamma\left(\frac{\delta}{2}\right)\Gamma\left(\frac{\delta+1}{2}\right)$$

$$= 2^3\pi^{5/2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+\delta}{2}\right)}{\Gamma\left(1+\frac{\delta}{2}\right)}\,;$$

$$\lim_{\sigma\to\frac{1}{2}} \frac{Y^{2\sigma-1} - Y^{1-2\sigma}}{2\sigma - 1} = 2\log Y.$$

(ii) $$\int_0^Y\int_{-\infty}^{\infty} h(r)\varphi_\chi(s)y^{-1-2ir}drdy = \int_0^Y\int_{-\infty}^{\infty} h(r)\varphi_\chi(s)e^{-2\pi i\cdot\log y}dr\,d(\log y)$$
$$= \int_{-\infty}^{\log Y}\int_{-\infty}^{\infty} h(r)\varphi_\chi(s)e^{-ir(2y')}dr\,dy'.$$
$$= \pi h(0)\varphi_\chi\Big(\frac{1}{2}\Big)$$

as $Y\to\infty$. By a similar calculation as in the above,

$$\int_Y^\infty\int_{-\infty}^{\infty} h(r)\overline{\varphi_\chi(s)}y^{-1+2ir}dr\,dy = \int_{\log Y}^\infty\int_{-\infty}^{\infty} h(r)\overline{\varphi_\chi(s)}e^{ir(2y')}drdy'$$

as $Y\to\infty$. Hence we have

$$\lim_{\sigma\to\frac{1}{2}}\left\{\int_0^Y\int_{-\infty}^{\infty} h(r)\varphi_\chi(s)y^{-1-2ir}drdy - \int_Y^\infty\int_{-\infty}^{\infty} h(r)\overline{\varphi_\chi(s)}y^{-1+2ir}dr\,dy\right\}$$
$$= \pi h(0)\varphi_\chi\Big(\frac{1}{2}\Big)$$

as $Y\to\infty$. We note that $\varphi_\chi\Big(\dfrac{1}{2}\Big) = \pm 1$.

(iii) $$-\int_Y^\infty\int_{-\infty}^{\infty} h(r)|\varphi_\chi(s)|^2y^{-2\sigma}drdy + \frac{Y^{1-2\sigma}}{2\sigma-1}\int_{-\infty}^{\infty} h(r)dr$$
$$= -\frac{Y^{1-2\sigma}}{2\sigma-1}\int_{-\infty}^{\infty} h(r)|\varphi_\chi(s)|^2dr + \frac{Y^{1-2\sigma}}{2\sigma-1}\int_{-\infty}^{\infty} h(r)dr$$
$$= -Y^{1-2\sigma}\int_{-\infty}^{\infty} h(r)\frac{|\varphi_\chi(s)|^2-1}{2\sigma-1}dr.$$

By the Maass-Selberg relation ([3], [4]), we know that

$$|\varphi_\chi(s)| \longrightarrow 1 \quad\text{as}\quad \sigma\longrightarrow\frac{1}{2};$$

and hence

$$\varphi_\chi\Big(\frac{1}{2}+ir\Big)\varphi_\chi\Big(\frac{1}{2}-ir\Big) = 1,$$

namely,

$$\lim_{\sigma\to\frac{1}{2}}\int_{-\infty}^{\infty} h(r)\frac{|\varphi_\chi(s)|^2-1}{2\sigma-1}dr = \int_{-\infty}^{\infty} h(r)\frac{\varphi'_\chi\Big(\frac{1}{2}+ir\Big)}{\varphi_\chi\Big(\frac{1}{2}+ir\Big)}dr.$$

By (i), (ii) and (iii), we obtain for the second half of the expression for $J(\infty)$,

$$\int_{\tilde{D}_Y} \tilde{H}_\delta(z, \phi; z, \phi)\, d(z, \phi)$$

$$= 2^2\pi \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\delta+1}{2}\right)}{\Gamma\left(1+\frac{\delta}{2}\right)} \log Y - \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r) \frac{\varphi_\chi'\left(\frac{1}{2}+ir\right)}{\varphi_\chi\left(\frac{1}{2}+ir\right)} dr$$

$$+ \frac{1}{4} h(0)\varphi_\chi\left(\frac{1}{2}\right) + o(1)\,,$$

as $Y \to \infty$. Summing up the above results (A) and (B), we obtain

$$J(\infty) = 2^2\pi \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\delta+1}{2}\right)}{\Gamma\left(1+\frac{\delta}{2}\right)} (C + \log 2) + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r) \frac{\varphi_\chi'\left(\frac{1}{2}+ir\right)}{\varphi_\chi\left(\frac{1}{2}+ir\right)} dr$$

$$- \frac{1}{4} h(0)\varphi_\chi\left(\frac{1}{2}\right)$$

$$+ 2\pi\left\{ -\sum_{k=0}^{\infty} \binom{-\delta/2-1}{k} \frac{1}{\left(\frac{1}{2}+k\right)^2} \right.$$

$$\left. + \sum_{k=0}^{\infty} \binom{-\delta/2-1}{k} \frac{1}{\left(\frac{\delta+9}{2}+k\right)^2} \right\}.$$

(C) We shall now calculate the limit $\lim_{\delta\to 0} \delta J(\infty)$.

(iv) we use the formula:

$$\binom{-\delta/2-1}{k} = (-1)^k \frac{\Gamma\left(k+\frac{\delta}{2}+1\right)}{k!\,\Gamma\left(\frac{\delta}{2}+1\right)}.$$

Then,

$$\lim_{\delta\to 0} \delta\left\{ -\sum_{k=0}^{\infty} \binom{-\delta/2-1}{k} \frac{1}{\left(\frac{1}{2}+k\right)^2} + \sum_{k=0}^{\infty} \binom{-\delta/2-1}{k} \frac{1}{\left(\frac{\delta+9}{2}+k\right)^2} \right\}$$

$$= -\lim_{\delta\to 0}\frac{\delta}{\Gamma\left(\frac{\delta}{2}+1\right)}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma\left(k+\frac{\delta}{2}+1\right)}{k!\left(\frac{1}{2}+k\right)^2}$$

$$+ \lim_{\delta\to 0}\frac{\delta}{\Gamma\left(\frac{\delta}{2}+1\right)}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma\left(k+\frac{\delta}{2}+1\right)}{k!\left(\frac{\delta+9}{2}+k\right)^2}$$

$$= -\lim_{\delta\to 0}\delta\sum_{k=0}^{\infty}\frac{(-1)^k}{\left(\frac{1}{2}+k\right)^2}+\lim_{\delta\to 0}\delta\sum_{k=0}^{\infty}\frac{(-1)^k}{\left(\frac{9}{2}+k\right)^2}=0\,.$$

(v) By the expression (2.2) of $h(r)$, we have

$$\lim_{\delta\to 0}\delta h(0)=16\pi^2\,.$$

Therefore

$$\lim_{\delta\to 0}\delta\left(-\frac{1}{4}h(0)\varphi_\chi\left(\frac{1}{2}\right)\right) = -4\pi^2\varphi_\chi\left(\frac{1}{2}\right).$$

By (2.2), we also have

$$h(r)\sim\frac{c(\delta)|r|^3}{|r|e^{\pi|r|}}$$

as $r\to\infty$, where $c(\delta)$ is independent of r and $\lim_{\delta\to 0}c(\delta)$ is finite. On the other hand, if we put

$$f(r)=\varphi_\chi'\left(\frac{1}{2}+ir\right)\Big/\varphi_\chi\left(\frac{1}{2}+ir\right),$$

then

$$\lim_{\delta\to +0}\delta\int_{-\infty}^{\infty}h(r)\frac{\varphi_\chi'}{\varphi_\chi}\left(\frac{1}{2}+ir\right)dr$$
$$=\lim_{\delta\to+0}\delta\left\{\int_{-\infty}^{-N}h(r)\frac{\varphi_\chi'}{\varphi_\chi}\left(\frac{1}{2}+ir\right)dr+\int_{-N}^{N}h(r)\frac{\varphi_\chi'}{\varphi_\chi}\left(\frac{1}{2}+ir\right)dr\right.$$
$$\left.+\int_{N}^{\infty}h(r)\frac{\varphi_\chi'}{\varphi_\chi}\left(\frac{1}{2}+ir\right)dr\right\}.$$

Since the function $f(r)$ is bounded on $[-N,N]$, we have

$$\int_{-N}^{N}h(r)f(r)dr=O\left(\int_{-\infty}^{\infty}h(r)dr\right),\ \text{i.e.,}$$

$$\lim_{\delta\to+0}\delta\int_{-N}^{N}h(r)f(r)dr=0\,.$$

Moreover, Since the operator $\tilde{K}_\delta$ is completely continuons on $L^2(\Gamma\backslash\tilde{S})$, there exists some constant δ_1 such that

$$\int_N^\infty |r|^{\delta_1} \frac{f(r)}{|r|e^{\pi|r|}} dr < +\infty .$$

Then, for any δ such that $0 \leqq \delta < \delta_1$, the function $|r|^\delta(f(r)/|r|e^{\pi|r|})$ is integrable on $[N, \infty)$ and its convergence is uniform for δ. Thus

$$\lim_{\delta\to+0} \delta \int_N^\infty |r|^\delta \frac{f(r)}{|r|e^{\pi|r|}} dr = 0 .$$

Therefore, we have

$$\lim_{\delta\to+0} \delta \int_{-\infty}^\infty h(r) \frac{\varphi_\chi'}{\varphi_\chi}\Big(\frac{1}{2} + ir\Big) dr = 0 .$$

Remark 1. The function $(\varphi_\chi'/\varphi_\chi)(1/2 + ir)$ satisfies

$$\left|\frac{\varphi_\chi'}{\varphi_\chi}\Big(\frac{1}{2} + ir\Big)\right| < c \log(2 + |r|)$$

for some constant c. By this estimation we have again

$$\lim_{\delta\to0} \delta \int_{-\infty}^\infty h(r) \frac{\varphi_\chi'}{\varphi_\chi}\Big(\frac{1}{2} + ir\Big) dr = 0 .$$

It is now clear that the above result, combined with the formula (1), proves the following.

THEOREM F. *Let Γ be a fuchsian group of the first kind containing the element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $(= -I)$ and suppose that Γ is reduced at infinity. Let χ be a one-dimensional unitary representation of Γ such that $\chi(-I) = -1$, $\chi\Big(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\Big) = 1$ and $\chi^2 \neq 1$, and denote by d_1 the dimension for the linear space consisting of all cusp forms of weight one with respect to Γ with χ. Then the dimension d_1 is given by*

$$(3.2) \qquad d_1 = \frac{1}{2} \sum_{\{M\}} \frac{\chi(M)}{[\Gamma(M)\colon \pm I]} \frac{\bar{\zeta}}{1 - \bar{\zeta}^2} + \frac{1}{2} \operatorname*{Res}_{s=0} \zeta^*(s) - \frac{1}{4} \varphi_\chi\Big(\frac{1}{2}\Big) ,$$

where the sum over $\{M\}$ is taken over the distinct elliptic conjugacy classes of $\Gamma/\{\pm I\}$, $\Gamma(M)$ denotes the centralizer of M in Γ, ζ is one of the eigenvalues of M, and $\zeta^*(s)$ denotes the Selberg type zeta-function defined in Section 0.

We may call the formula (3.2) a kind of Riemann-Roch type theorem for automorphic forms of weight one.

Example. Let p be a prime number such that $p \equiv 3 \bmod 4$, $(p \neq 3)$, and let $\Phi_0(p)$ be the group generated by the group $\Gamma_0(p)$ and the element $K = \begin{pmatrix} 0 & -\sqrt{p}^{-1} \\ \sqrt{p} & 0 \end{pmatrix}$. Let ε be the Legendre symbol on $\Gamma_0(p)$: $\varepsilon(L) = (d/p)$ for $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p)$. Since $\varepsilon(K^2) = \varepsilon(-I) = -1$, we can define the odd characters $\varepsilon^{\pm}$ on the Fricke group $\Phi_0(p)$ such that $\varepsilon^{\pm}(K) = \pm i$. Then we have

$$S_1(\Gamma_0(p), \varepsilon) = S_1(\Phi_0(p), \varepsilon^+) \oplus S_1(\Phi_0(p), \varepsilon^-)\,.$$

We put

$$\mu_1^{\pm} = \dim S_1(\Phi_0(p), \varepsilon^{\pm})\,.$$

Then

$$d_1 = \dim S_1(\Gamma_0(p), \varepsilon) = \mu_1^+ + \mu_1^-\,.$$

If $\sigma^*(p)$ is the parabolic class number of $\Phi_0(p)/\{\pm I\}$, then $\sigma^*(p) = 1$. As shown in [6], the contribution from elliptic classes to $\mu_1^{\pm}$ is given by

$$\frac{1}{2} \sum_{\{M\}} \frac{1}{[\Gamma(M) \colon \pm I]} \frac{\bar{\zeta}}{1-\bar{\zeta}^2} \varepsilon^{\pm}(M) = \mp \frac{1}{4} h\,.$$

We also have $\varphi_{\varepsilon^{\pm}}(1/2) = \mp 1$. Let P_α $(\alpha = 1, 2, 3, \cdots)$ be a complete system of representatives of the primitive hyperbolic conjugacy classes in $\Gamma_0(p)/\{\pm I\}$ and let $\lambda_{0,\alpha}$ be the eigenvalue $(\lambda_{0,\alpha} > 1)$ of a representative P_α. We put

$$Z^*(s) = \sum_{\alpha=1}^{\infty} \sum_{k=1}^{\infty} \frac{\varepsilon(P_\alpha)^k \log \lambda_{0,\alpha}}{\lambda_{0,\alpha}^k - \lambda_{0,\alpha}^{-k}} |\lambda_{0,\alpha}^k + \lambda_{0,\alpha}^{-k}|^{-s}\,.$$

Then, we have the following formula for d_1 which is our conclusion

$$d_1 = \mu_1^+ + \mu_1^- = \frac{1}{2} \operatorname*{Res}_{s=0} Z^*(s)\,.$$

Remark 2. For a general discontinuous group Γ of finite type containing the element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, we obtain, in the same way as in the case of a group reduced at ∞, the contribution from parabolic classes to d_1.

Addendum

Taking this opportunity, I would like to comment on our previous paper:

T. Hiramatsu and Y. Mimura,

The modular equation and modular forms of weight one,

Nagoya Math. J., **100** (1985), 145-162.

By an exchange of letters, it has been shown that the paper "Hohere Reziprozitatggesetze und Modulformen von Gewicht Eins, Jour. reine angew. Math., **361** (1985), 11-22", remarkably overlapping with our one, was written after its author had read a preprint of ours.

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