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ON THE MODULAR VERSION OF ITO'S THEOREM ON CHARACTER DEGREES FOR GROUPS OF ODD ORDER*

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§ 1. Introduction

One of the most useful theorems in classical representation theory is a result due to N. Ito, which can be stated using the classification of the finite simple groups in the following way.

THEOREM (N. Ito, G. Michler). *Let* Irr (G) *be the set of all irreducible complex characters of the finite group G and q be a prime number. Then* $q \nmid \chi(1)$ *for* $\chi \in \mathrm{Irr}(G)$ *if and only if G has a normal, abelian Sylow-q-subgroup.*

Ito himself proved the "if-part" in [7] and the "only-if-part" for p-solvable groups in [6]. To prove the last one in general, it is sufficient to investigate simple groups G (cf. Issacs [5] 12.33). For those, G. Michler [8] was able to prove that for $q \neq 2$ they all have q-blocks of non-maximal defect, which implies the result. For $q = 2$ he could show that each non-cyclic simple group has at least one character the degree of which is even.

Now replace the field C of complex numbers by any algebraically closed field K of characteristic $p > 0$ and denote by IBr (G) the Brauer characters of G with respect to p. The question which arises is, whether there is an analogue to the theorem above for IBr (G) instead of Irr (G). But now there are two different cases to consider, namely $q = p$ and $q \neq p$. The answer to the first one is quite satisfactory.

THEOREM. *We have* $p \nmid \beta(1)$ *for all* $\beta \in \mathrm{IBr}(G)$ *if and only if G has a normal Sylow-p-subgroup.*

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We sketch the proof here. Clearly, the "if-part" is trivial, as the normal Sylow-p-subgroup is contained in the kernel of all irreducible KG-modules. Considering the "only-if-part", we first mention that by Michler [8] for p odd and Okuyama [9] for $p = 2$ one can assume that G is p-solvable. In this case, however, the proof is rather easy, namely take a minimal normal subgroup V of G. By induction, G/V has a normal Sylow-p-subgroup N/V. If V is a p-group, we are done. Therefore V is a p'-group. But now $P \in \mathrm{Syl}_p(N)$ has to centralize V, because otherwise N would have an irreducible Brauer character the degree of which would be divisible by p.

Unfortunately, a similar result does definitely not hold for $q \neq p$. To see this consider the permutation group S_4 on 4 letters, take $p = 3$ and $q = 2$. It's easy to see that the degrees of the Brauer characters of S_4 are 1 and 3, but S_4 has no normal Sylow-2-subgroup. Hence the best one could expect to prove is that G has q-length $l_q(G)$ at most 2, provided that $q \nmid \beta(1)$ for all $\beta \in \mathrm{IBr}(G)$. The purpose of this paper is to establish this assertion in the odd order case.

THEOREM. *Let $|G|$ be odd and q a prime number different from p such that $q \nmid \beta(1)$ for all $\beta \in \mathrm{IBr}(G)$. Then the q-length $l_q(G)$ is at most 2. Furthermore, the q-factors in the ascending and descending q-series of G are abelian.*

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§ 2. The main results

We'll prove a somewhat stronger result than stated in the introduction. For this purpose, we fix a set of primes π such that $p \notin \pi$. Furthermore we remark that the assertion "$\gamma(1) \in \pi'$ for all $\gamma \in \mathrm{IBr}(G)$" is inherited by factor groups and normal subgroups of G. We start with a certainly well-known lemma.

LEMMA 1. *Let $N \trianglelefteq G$ and M an irreducible KN-module for an arbitrary field K. We put $T = \boldsymbol{T}(M)$ the inertia subgroup of M. If W is an irreducible KT-module lying over M, then W^G is an irreducible KG-module.*

Proof. Let $G = \bigcup_{i=1}^n Tg_iN = \bigcup_{i=1}^n Tg_i$ be the double-coset-decomposition of (T, N) in G (w.l.o.g. $g_1 = 1$). Hence, by Mackey's lemma (cf. Huppert [3] V, 16.9),

$$(W^G)_N \cong \bigoplus_{i=1}^{n} ((W \otimes g_i)_{T^{g_i} \cap N})^N \cong \bigoplus_{i=1}^{n} (W \otimes g_i)_N .$$

As $W_N = e \cdot M$ for some positive integer e, we conclude $\dim_K W^G = e \cdot n \cdot \dim_K M$. Now let X be an irreducible submodule of W^G. Then Nakayama-reciprocity (cf. Huppert-Blackburn [4] VII, 4.10) yields

$$0 \neq \mathrm{Hom}_{KG}(X, W^G) \cong \mathrm{Hom}_{KT}(X_T, W) ;$$

in particular, there exists an KN-epimorphism from X_N onto $W_N = e \cdot M$. By Clifford's theorem, we at once get $e \cdot (M \otimes g_1 \oplus \cdots \oplus M \otimes g_n) \leq X_N$ and $\dim_K X \geq e \cdot n \cdot \dim_K M$. This finally yields that $X = W^G$ is irreducible. $\square$

From Lemma 1 we get informations about the orbit sizes of the action of G/N on the characters of N.

LEMMA 2. *Let $N \trianglelefteq G$ and suppose that for all $\beta \in \mathrm{IBr}(G)$ the degrees $\beta(1)$ are π'-numbers. Then we have $|G: \boldsymbol{T}(\alpha)| \in \pi'$ for all $\alpha \in \mathrm{IBr}(N)$.*

Proof. We choose $\gamma \in \mathrm{IBr}(\boldsymbol{T}(\alpha))$ lying over α. By Lemma 1, $\gamma^G \in \mathrm{IBr}(G)$ and the hypothesis forces $\gamma^G(1) = \gamma(1)|G: \boldsymbol{T}(\alpha)|$ to be a π'-number. $\square$

We premise two further lemmas.

LEMMA 3. *Let V be an irreducible G-module over $GF(q)$ for $q \neq p$. If G acts primitively on $\mathrm{IBr}(V)$, then also on V.*

Proof. Suppose we have an imprimitivity decomposition $V = U_1 \oplus \cdots \oplus U_n$. With the notation $U_i' = \{\beta \in \mathrm{IBr}(V) | \beta = 1 \text{ on } \bigoplus_{j \neq i} U_j\}$ we certainly obtain $\mathrm{IBr}(V) = U_1' \oplus \cdots \oplus U_n'$. To get a contradiction we'll show that this is an imprimitivity decomposition of $\mathrm{IBr}(V)$ as a G-module. Now let $\alpha \in U_i'$ and $g \in G$. By our assumption, we have that $U_i^g = U_j$ for some j. Hence, if $1 \neq u_k \in U_k$ $(k \neq j)$, we have $u_k^{g^{-1}} \in U_l$ $(l \neq i)$ and $\alpha^g(u_k) = \alpha(u_k^{g^{-1}}) = 1$. This means $(U_i')^g = U_j'$. As V and also $\mathrm{IBr}(V)$ are irreducible G-modules, the action of G on the U_i' is transitive and we have got the desired assertion. $\square$

LEMMA 4. *Let V be an irreducible G-module. Suppose that each element of V is stabilized by a π-Hall-subgroup of G and that $G = \boldsymbol{O}^{\pi'}(G)$. Suppose furthermore that $|G|$ is odd and let $S \in \mathrm{Syl}_{\pi'}(G)$. Then*

a) *V is a primitive G-module.*

b) *V also is an irreducible and primitive S-module.*

Proof. a) Suppose there exists an imprimitivity decomposition $V =$

$U_1 \oplus \cdots \oplus U_n$ $(n \geq 2)$. Let $Y \lneqq G$ be the kernel of the permutation representation of G on the indices $\{1, \cdots, n\}$. Let $\{1, \cdots, k\} \subset \{1, \cdots, n\}$ and $0 \neq u_i \in U_i$ $(i = 1, \cdots, k)$. By our hypothesis, there is $H \in \mathrm{Syl}_\pi(G)$ such that $H \leq C_G(u_1 + \cdots + u_k)$. As $G = O^{\pi'}(G)$, G/Y is not a π'-group. Hence $Y \lneqq HY$ and for $hy \in HY$ we have

$$u_1^{hy} + \cdots + u_k^{hy} = (u_1 + \cdots + u_k)^{hy} = u_1^y + \cdots + u_k^y \in U_1 \oplus \cdots \oplus U_k$$

and hy fixes the set $\{1, \cdots, k\}$. We have just proved that each subset of $\{1, \cdots, n\}$ has a nontrivial stabilizer in G/Y, which contradicts Gluck [1], Corollary 1, as $|G/Y|$ is odd.

b) To prove the irreducibility of V as an S-module, we choose an arbitrary $0 \neq v \in V$ and $H \in \mathrm{Syl}_\pi(G)$ such that $v^H = v$. Hence $v^G = v^{HS} = v^S$ and v generates V as an S-module. Secondly, we suppose that $V = W_1 \oplus \cdots \oplus W_m$ is an imprimitivity decomposition of V as an S-module; in particular, S acts transitively on the W_i. If $w_j \in W_j$ and $g \in G$, we choose $H \in \mathrm{Syl}_\pi(G)$ such that $H \leq C_G(w_j)$. Hence we can write $g = hs$ $(h \in H, s \in S)$ and $w_j^g = w_j^s \in W_k$ (for some k, depending only on j, namely consider the sum of two w_j). Consequently, $V = W_1 \oplus \cdots \oplus W_m$ is an imprimitivity decomposition of V as a G-module, contradicting a). $\square$

Now we turn towards the proof of our main theorem.

THEOREM 5. *Let $|G|$ be odd (hence G is solvable). Suppose that $\beta(1) \in \pi'$ for all $\beta \in \mathrm{IBr}(G)$. Then the π-length $l_\pi(G)$ is at most 2. Furthermore, the π-factors in the ascending and descending π-series of G are abelian.*

Proof. Let G be a minimal counterexample.

(1) G has the following descending π-series:

$$\begin{array}{ll|l}
 & G = O^{\pi'}(G) & \\
\pi & & \text{abelian} \\
 & L = O^{\pi}(G) & \\
\pi' & & \\
 & M = O^{\pi'}(L) & \\
\pi & & \text{abelian} \\
 & N = O^{\pi}(M) & \\
\pi' & & \\
 & V = O^{\pi'}(N) & \\
q \in \pi\colon (q, \cdots, q) & & \text{minimal normal subgroup} \\
 & 1 &
\end{array}$$

To see that G/L and M/N are abelian we remind the reader that $p \notin \pi$.

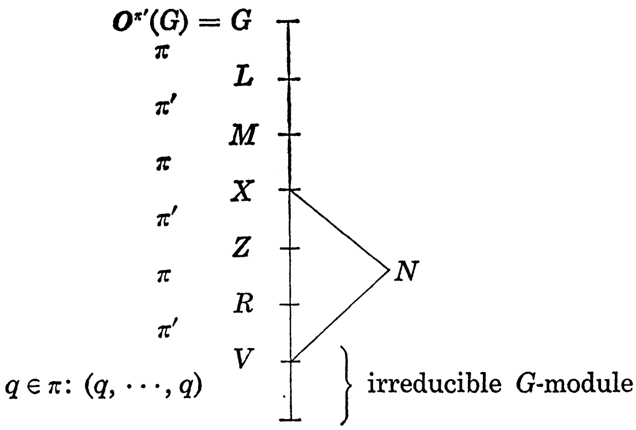
Hence all $\alpha \in \mathrm{Irr}(G/L)$ and $\alpha \in \mathrm{Irr}(M/N)$ have π'-degree and consequently are linear. This also proves the supplement in the assertion of the theorem.

(2) We may assume that N/V acts faithfully on V: Define $C/V = C_{N/V}(V)$. Hence $C = V \times S$, where $S \in \mathrm{Syl}_{\pi'}(C)$. By factoring out S, assertion (2) follows.

(3) We may assume that $O_\pi(M/V) = 1$: Put $Q/V = O_\pi(M/V)$. As Q is a π-group and $p \notin \pi$, the argument used in (1) forces Q to be abelian. Hence we get the G-invariant decomposition $Q = C_Q(N/V) \times [Q, N/V]$, where $[Q, N/V] = V$. By factoring out $C_Q(N/V)$, the claim (3) holds.

(4) M/V acts faithfully on V: By (2), N/V acts faithfully on V. If $C_{M/V}(V) \neq 1$, we certainly had $O_\pi(M/V) \neq 1$, a contradiction to (3).

(5) In addition to (1), we fix some more notations: Let $X/N = \mathrm{Soc}(M/N)$ be the socle of M/N. Then X/N is the direct product of some elementary abelian groups for some primes in π. Furthermore, put $Z/V = O^{\pi'}(X/V)$ and $R/V = O_{\pi'}(Z/V)$. As X, Z, R are normal in G, we get the following normal series of G.



By construction, we have $O^{\pi'}(Z) = Z$, and by (4), Z/V acts faithfully on V.

(6) $V_Z = e \cdot W$ for some natural number e and some irreducible and faithful Z/V – module W: By Lemmas 2 and 4a), $\mathrm{IBr}(V)$ is a primitive G-module. By Lemma 3, also V is a primitive G-module, hence Clifford's theorem yields $V_Z = e \cdot W$, where W certainly is a faithful Z/V-module.

(7) W is a primitive Z-module and also irreducible and primitive as an R-module: The primitivity of W as a Z-module follows at once from Lemma 2 and Lemma 4a). The second assertion is a consequence of

Lemma 4b).

(8) R/V is not a nilpotent group: Suppose R/V is nilpotent. By (6) and (7), R/V acts faithfully, irreducibly and primitively on W. As $|R/V|$ is odd, a result of Roquette [10] forces R/V to be cyclic. Note that $R/V \neq 1$ (by (3)) and that the automorphism group of a cyclic group is abelian. Consequently $(G/V)/C_{G/V}(R/V)$ is abelian and a π-group (as $O^{\pi'}(G) = G$). This finally means $L/V = O^{\pi}(G/V) \leq C_{G/V}(R/V)$ and therefore $Z/V \leq C_{G/V}(R/V)$, contradicting $Z = O^{\pi'}(Z)$.

(9) Conclusion: Let in (9) $\bar{Z} = Z/V$ and choose $H \in \mathrm{Syl}_{\pi}(\bar{Z})$, hence $H = \langle x_1 \rangle \times \cdots \times \langle x_n \rangle$ for some groups $\langle x_i \rangle$ of prime order. Let $x = x_1$ and $o(x) = r \in \pi$. Then obviously $\bar{R} \cdot \langle x \rangle \trianglelefteq \bar{Z}$. If we furthermore put $\bar{Y} = O^{\pi'}(\bar{R} \cdot \langle x \rangle)$ and $\bar{S} = O^{\pi}(\bar{Y})$, we have $\bar{Y} \trianglelefteq \bar{Z}$ and $\bar{Y}/\bar{S} \cong \langle x \rangle$. By (7), $W_{\bar{Y}} = f \cdot U$ for some natural number f and some irreducible, faithful $\bar{Y}$-module U. Now, by Lemma 2, each element of $\mathrm{IBr}(U)$ is fixed by a Sylow-r-subgroup of $\bar{Y}$, and as $|\bar{Y}|$ is odd, the action of $\bar{Y}$ on $\mathrm{IBr}(U)$ satisfies the hypothesis of Wolf [11] Theorem 3.3, which forces $\bar{S}$ to be cyclic. Replacing x_1 by x_i, we see that the corresponding normal subgroups $\bar{S}_i \trianglelefteq \bar{Z}$ all are cyclic. By the construction of the $\bar{S}_i$, we have $[\bar{R}, x_i] \leq \bar{S}_i$ and consequently $[\bar{R}, H] \leq \bar{S}_1 \cdots \bar{S}_n$. As $O^{\pi'}(\bar{Z}) = \bar{Z}$, we can conclude that $\bar{R} = \bar{S}_1 \cdots \bar{S}_n$ is a product of cyclic normal subgroups, hence nilpotent.

We extract two results from Theorem 5 corresponding to two different choices of the set π. The first one, namely $\pi = \{q\}$ where $q \neq p$, is the modular version of Ito's theorem, already stated in the introduction.

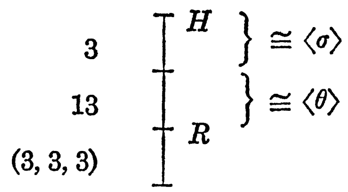
COROLLARY 6. *Let $|G|$ be odd and q be a prime number different from p such that $q \nmid \beta(1)$ for all $\beta \in \mathrm{IBr}(G)$. Then the q-length $l_q(G)$ is at most 2. Furthermore, the q-factors in the ascending and descending q-series of G are abelian.*

We continue looking in quite the other direction, namely where π equals the whole set of prime divisors of $|G|$ except of p.

COROLLARY 7. *Let $|G|$ be odd and suppose that all $\beta(1)$ are powers of p, where $\beta \in \mathrm{IBr}(G)$. Then the p-length $l_p(G)$ is at most* 3.

Now the question arises, whether the bounds given in the Corollaries above are best possible even in the odd order case. The following example shows that this is indeed true.

EXAMPLE 8. Let $R = GF(3^3)^+$ the additive group of the field with 27 elements. Let furthermore θ be an element of order 13 in $GF(3^3)^\times$ and $\langle\sigma\rangle = \mathrm{Gal}(GF(3^3)/GF(3))$. Hence σ has order 3 and $H = R\cdot\langle\theta\rangle\cdot\langle\sigma\rangle$ is a subgroup of the semilinear mappings $\Gamma(3^3)$ of order $3^4\cdot 13$.



Let $p = 13$, then it's not hard to check that all $\beta\in\mathrm{IBr}(H)$ have degree 1 or 13 (cf. Huppert [2] even for more examples of this type). Now take a faithful $GF(13)$ H-module V and consider the semidirect product V·H. As the normal 13-subgroup V is contained in the kernel of every 13-modular irreducible representation of V·H, certainly all $\beta\in\mathrm{IBr}(V\cdot H)$ also have degree 1 or 13. If finally Z denotes the cyclic group of order 13, the regular wreath product $G = (V\cdot H)\wr Z$ has only irreducible Brauer characters the degrees of which are powers of 13. On the other hand, $l_3(G) = 2$ and $l_{13}(G) = 3$.

Remark 9. Under the hypotheses of Corollary 6, a Sylow-q-subgroup Q of G clearly is metabelian. But it is not true in general that even the class of Q is at most 2. This can be seen considering the group H of example 8, where $R\cdot\langle\sigma\rangle\in\mathrm{Syl}_3(H)$ and $R\cdot\langle\sigma\rangle\cong Z_3\wr Z_3$ has class 3 (Z_3 denotes the cyclic group of order 3). If it's possible to construct groups of this type for arbitrarily large q (which finally reduces to a question about the existence of a special relation between the primes p and q; cf. Huppert [2]), the class of a Sylow-q-subgroup Q would be unbounded, because $Q\cong Z_q\wr Z_q$ and $\mathrm{cl}(Q) = q$ (cf. Huppert [3] III, 15.3).

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