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GENERALIZED RADON TRANSFORM AND LÉVY'S BROWNIAN MOTION, I*)

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§1. Introduction

In connection with a Gaussian system $X = \{X(x);\ x \in M\}$ called Lévy's Brownian motion (Definition 1), we shall introduce two integral transformations of special type—one is a generalized Radon transform R on a measure space (M, m), and the other is a dual Radon transform R^* on another measure space (H, ν) such that $H \subset 2^M$, the set of all subsets of M (Definition 2). To each Lévy's Brownian motion X, there is attached a distance $d(x, y) := E[(X(x) - X(y))^2]$ on M having a notable property named L^1-embeddability ([3]). The above measure ν on H is then chosen to satisfy

$$d(x, y) = \nu(B_x \triangle B_y) \qquad \text{with } B_x := \{h \in H;\ x \in h\},$$

where $\triangle$ stands for the symmetric difference.

It turns out that these transforms constitute a factorization of the covariance operator of X (Theorem 3); a more explicit link between X and R^* can be noticed in the somewhat informal expression

$$X(x) = (R^* W)(x),$$

where $W = \{W(dh);\ h \in H\}$ is a Gaussian random measure with mean 0 and variance $\nu(dh)$. In view of the quite simple probabilistic structure of W, an idea comes to mind: The deep study of R and R^* will yield fruitful results on X. Thus, we shall investigate the transforms R and R^* as well as the Lévy's Brownian motion X in the present and subsequent papers.

The main purpose of the present paper (I) is to obtain the singular value decomposition of R^* (Theorem 5), which gives us the Karhunen-

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Loève expansion of X (Theorem 6). The second paper (II) will concentrate on the investigation of the null spaces of R^*:

$$N_1(A) := \{g \in L^2(H, \nu);\ (R^*g)(x) \equiv 0,\ x \in A\}, \qquad A \subset M.$$

The structure of the closed linear span $[X(x); x \in A]$ in $L^2(\Omega, P)$ will be described in terms of $N_1(A)$ and W.

In order to give some interpretation to the representation of Chentsov type which is useful for our study, we begin with a familiar Brownian motion $X = \{X(x);\ x \in R^n\}$ with n-dimensional parameter. The variance of the increment $X(x) - X(y)$ is, by definition, equal to the Euclidean distance $|x - y|$ between x and y. The idea of Chentsov [6] (cf. [24] and [26]) now leads us to take the following measure space (H, ν): H is the set of all half-spaces $h_{t,\omega} := \{x \in R^n;\ (x, \omega) > t\}$ not containing the origin O; an element $h_{t,\omega} \in H$ is parametrized by the distance $t > 0$ and the direction $\omega \in S^{n-1} := \{\omega \in R^n;\ |\omega| = 1\}$. The measure ν is an invariant measure on H, explicitly given by

$$\nu(dh_{t,\omega}) = \frac{n-1}{|S^{n-2}|}\, dt\, d\omega\,.$$

Then it is easy to verify that $\nu(B_x \triangle B_y) = |x - y|$. We thus get at the conclusion that X is expressed in the form

$$X(x) = \int_{B_x} W(dh) = W(B_x)\,. \tag{1}$$

A general framework behind the representation (1) of Chentsov type consists of the following:

(i) A centered Gaussian system $X = \{X(x);\ x \in M\}$ with parameter space M; the variance of the increment is denoted by $d(x, y) := E[(X(x) - X(y))^2]$.

(ii) A Gaussian random measure $W = \{W(dh);\ h \in H\}$ based on a measure space (H, ν) such that $H \subset 2^M$ and $\nu(B_x) < \infty$ for all $x \in M$.

It follows from (1) that

$$d(x, y) = \int_H |\chi_{B_x}(h) - \chi_{B_y}(h)|\, \nu(dh) = \nu(B_x \triangle B_y)\,, \tag{2}$$

where χ_B denotes the indicator function of a subset $B \subset H$. Conversely, this equation (2) guarantees the existence of such a representation (1). The variance of X admitting a representation (1) of Chentsov type is therefore a (semi-)metric on M of the form $\|\chi_{B_x}-\chi_{B_y}\|_{L^1(H,\nu)}$; such a metric is said to be *L^1-embeddable* ([3]).

We are now in a position to introduce the following

DEFINITION 1. Let (M, d) be an L^1-embeddable metric space. Then a centered Gaussian system $X=\{X(x);\ x\in M\}$ with the variance $d(x,y)$ of the increment $X(x)-X(y)$ is called *Lévy's Brownian motion with parameter space (M, d)*.

With this terminology, our first conclusion (Theorem 1) is that every Lévy's Brownian motion admits of a representation of the form (1).

Another ingredient in our study is a pair of integral transformations associated with the expression (1).

DEFINITION 2. Let $m(dx)$ be a reference measure on M. The integral transform

$$(3)\qquad (Rf)(h):=\int_h f(x)m(dx)\,,\qquad f\in L^1(M,m)\,,$$

(resp.

$$(4)\qquad (R^*g)(x):=\int_{B_x} g(h)\nu(dh)\,,\qquad g\in L^2(H,\nu)\,,)$$

is called a *generalized* (resp. *dual*) *Radon transform.*

The reason for using the symbol R^* lies in the obvious relation of duality:

$$(Rf, g)_{L^2(H,\nu)}=(f, R^*g)_{L^2(M,m)}\,.$$

In case X is a Brownian motion with n-dimensional parameter, the value $(Rf)(h_{t,\omega})$ is nothing but the integral of f over the half-space $h_{t,\omega}$ and hence the classical Radon transform, the integral over the hyperplane $\delta h_{t,\omega}$ (Radon's celebrated paper [31]; see also [8], [15] and [23]) can be derived from the first variation of R (cf. [19], p. 47). On the other hand, the dual Radon transform R^* is closely related to the one studied by Cormack and Quinto [7], because the set B_x is changed into the open ball $\tilde{B}_x$ with diameter $\overline{Ox}$ by means of the mapping

$h_{t,\omega}\in H\longmapsto y=t\omega\in R^n\backslash\{O\}$, the foot of the perpendicular from O to the hyperplane $\delta h_{t,\omega}$.

Another important example should be mentioned here; it is a Lévy's Brownian motion with parameter space (S^n, d_G), d_G being the geodesic distance on S^n. Due to Lévy [21] (cf. also [18]), the corresponding measure space (H, ν) is chosen to be the set of all hemispheres endowed with an invariant measure ν. In this case, the transforms R and R^* take the same form—the integral over a hemisphere. Since the integral over a great circle can also be derived from the first variation of R, the study of R and R^* has another origin in Funk [11] and [12].

In Section 2 we shall establish the representation (1) of Chentsov type, and give several examples of (M, d) and (H, ν), except the case of $M = R^n$, the usual parameter space of random fields. A variety of L^1-embeddable metrics d on R^n will be described in the second paper (II).

Section 3 is devoted to the study of fundamental properties of R and R^*. In particular, we shall obtain their singular value decompositions, which will be applied to show that X admits of the Karhunen-Loève expansion in terms of an i.i.d. sequence of standard Gaussian random variables.

Section 4 will concern the n-sphere $M = S^n$ equipped with the uniform probability measure σ. The Karhunen-Loéve expansion will be explicitly calculated for a certain class of Lévy's Brownian motions $X = \{X(x);\ x \in S^n\}$ including the one due to Lévy [21] mentioned above; all of them have probability laws invariant under every rotation on S^n.

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§ 2. Representations of Chentsov type

The purpose of this section is two-fold: to prove the representation (1) of Chentsov type for each Lévy's Borwnian motion X, and to give several examples of (H, ν) combined with (M, d) via the equality (2). Particular attention will be paid to the case of $M = S^n$.

Suppose that (M, d) is an L^1-embeddable metric space; by definition, there exist a measure space (T, μ) and a mapping $x \in M \mapsto f_x(t) \in L^1(T, \mu)$ such that $d(x, y) = \|f_x(t) - f_y(t)\|_{L^1(T,\mu)}$. Then, as was shown by Assouad and Deza [3], we can find another measure space (H, ν) satisfying $H \subset 2^M$ and

$$d(x, y) = \nu(B_x \triangle B_y) = \int_H \pi_h(x, y)\nu(dh)\,, \tag{2}$$

where we have used the notation

$$\pi_h(x, y) := |\chi_h(x) - \chi_h(y)| = |\chi_{B_x}(h) - \chi_{B_y}(h)|\,.$$

Among various kinds of possible realizations of the distance d, the above one in terms of the indicator function $\chi_{B_x}(h)$ in $L^1(H, \nu)$ is most convenient for us to associate the transforms R and R^* with the expression (1). It was called a *multiplicity realization* in [3]. The correspondence $(M, d) \mapsto (H, \nu)$ has a tiny fault, however; it is not one to one (see Example 1b below).

Having found a multiplicity realization $\chi_{B_x}(h)$ in $L^1(H, \nu)$ of a given L^1-embeddable metric d on M, our first conclusion follows immediately:

THEOREM 1. *A Lévy's Brownian motion X with parameter space (M, d) admits of the representation*

$$X(x) = \int_{B_x} W(dh) \tag{1}$$

in terms of a Gaussian random measure W based on the measure space (H, ν).

Now choose and fix a point $O \in M$ as the origin. In view of a simple fact that $\pi_{h^c} \equiv \pi_h$, we may change an element $h \in H$ with its complement h^c if $O \in h$, so that $H \subset (2^M)_O := \{h \subset M;\ O \notin h\}$. This choice of H implies that $B_O = \phi$, which leads to the assumption $X(O) = 0$ often added in the definition of Lévy's Brownian motion.

EXAMPLE 1. Let us mention a couple of examples in which (M, d) is induced by a graph G ([14]), i.e., M is the set of all vertexes and $d(x, y)$ is the number of edges in a shortest path between x and y.

(a) $G = T$, a tree. At each edge e of T, M is separated into the two complementary subsets h_e and h_e^c; the root O of T always belongs to h_e^c. Define

$$H = \{h_e \text{ for all edges } e\} \subset (2^M)_O \quad \text{with weight } \nu(h_e) \equiv 1\,,$$

to get the desired distance d on M. With this choice of (H, ν), the representation (1) of Chentsov type can be regarded as a simple extension of partial sums of a sequence of i.i.d. Gaussian random variables.

(b) $G = K_m$, the complete graph of m vertexes. It is possible to find several different kinds of (H, ν). Indeed, for each k, $1 \le k \le [m/2]$, take

$$H_k = \{\text{all subsets } h \text{ of } k \text{ vertexes}\} \quad \text{with weight } \nu_k(h) \equiv \left\{2\binom{m-2}{k-1}\right\}^{-1}.$$

Then it is easy to show that

$$d(x, y) := \sum_{h \in H_k} \pi_h(x, y) \nu_k(h) \equiv 1 \qquad \text{for any } x,\ y \in M.$$

We note that (M, d) induced by a cyclic graph C_m is a discrete analogue of (S^1, d_G) and hence the corresponding measure space (H, ν) can be constructed after the manner of the one described in Section 1.

EXAMPLE 2. In case M is the set of all natural numbers, an easy way to get an L^1-embeddable metric d on M is as follows: Take

$$H := \{h_m;\ m \geq 2\} \quad \text{with weight } \nu(h_m) \geq 0\,,$$

and define

$$d(x, y) := \sum_{m=2}^{\infty} \pi_{h_m}(x, y) \nu(h_m)\,,$$

where $h_m := \{mk;\ k = 1, 2, \cdots\}$ is the set of all multiples of m. The special choice of weight

$$\nu(h_m) = \log p \text{ if } m \text{ has only one prime factor } p, \ = 0 \text{ otherwise},$$

gives us the interesting distance $d(x, y) = \log(x \cup y / x \cap y)$ mentioned in [1] and [2], where $x \cup y$ (resp. $x \cap y$) denotes the L. C. M. (resp. G. C. M.) of x and y.

A generalized Radon transform of the form

$$(Rf)(h_m) := \sum_{k=1}^{\infty} f(mk)$$

was considered by Strichartz [34], who gave the inversion formula

$$f(x) = \sum_{k=1}^{\infty} \mu(k)(Rf)(h_{xk})\,, \tag{5}$$

where $\mu(k)$ is the Möbius function defined by

$$\mu(k) := \begin{cases} (-1)^l\,, & \text{if } k \text{ has } l \text{ distinct prime factors}\,, \\ 0\,, & \text{if } k \text{ is divisible by the square of a prime.} \end{cases}$$

The representation (1) for a Lévy's Brownian motion X with parameter space (M, d) now takes the form

$$X(x) = \sum_{m \mid x} W(h_m)\,, \quad x \geq 2, \quad \text{and} \quad X(1) = 0\,,$$

which is canonical ([16]) in the sense that

$$[X(2), \cdots, X(m)] = [W(h_2), \cdots, W(h_m)] \qquad \text{for every } m \geq 2\,.$$

To be more precise, we obtain the exact expression of W in terms of X:

$$W(h_m) = \sum_{x|m} \mu(m/x) X(x) . \tag{6}$$

The proof of (5) consists of an application of the inversion formula (5) to a general relation

$$\sum_{x=2}^{\infty} f(x)X(x) = \sum_{m=2}^{\infty} (Rf)(h_m) W(h_m) .$$

The rest of this section concentrates of the case of the n-sphere $(M, m) = (S^n, \sigma)$. For each $\rho \in (0, 2\pi)$, set

$$C_\rho(p) := \{x \in S^n ;\ (x, p) > \cos(\rho/2)\} .$$

This is an open cap with north pole $p \in S^n$ and in particular $C_\pi(p)$ is the hemisphere. Take $H_\rho := \{C_\rho(p);\ p \in S^n\}$ with an invariant measure

$$d\nu(C_\rho(p)) = c d\sigma(p) , \qquad c = \nu(H_\rho) > 0 .$$

Then the corresponding distance becomes

$$d_\rho(x, y) := c \int_{S^n} \pi_{C_\rho(p)}(x, y) \sigma(dp) = c\sigma(C_\rho(x) \triangle C_\rho(y)) , \tag{7}$$

which is rotation-invariant and hence of the form $cr_\rho(d_G(x, y))$, where $d_G(x, y) := \arccos(x, y)$. Since, $\pi_{C_{2\pi-\rho}(p)} = \pi_{C_\rho(-p)}$, we have $r_{2\pi-\rho}(t) \equiv r_\rho(t)$. Furthermore, a straightforward computation $(\rho = \pi)$ yields the explicit form of r_π: $r_\pi(t) = t/\pi$ (cf. [18] and [21]).

A Lévy's Brownian motion X with parameter space (S^n, d_ρ) is then expressed in the form

$$X(x) = \sqrt{c} \int_{C_\rho(x)} W_0(dy) , \tag{1'}$$

where $W_0 = \{W_0(dy);\ y \in S^n\}$ is a Gaussian random measure based on the uniform probability space (S^n, σ). Instead of the pair of R and R^* associated with (1), it is more convenient to treat the following transform associated with (1′):

$$(R_\rho f)(x) := \int_{C_\rho(x)} f(y) \sigma(dy) , \tag{8}$$

which is a self-adjoint operator on $L^2(S^n, \sigma)$. The expression (1′) as well as the transform R_ρ will be further discussed in Section 4.

In the one-dimensional case $n = 1$, we can go further by making a superposition of $\{d_\rho:\ 0 < \rho \leq \pi\}$:

$$d(x, y) := \int_{(0,\pi]} d_\rho(x, y)\mu(d\rho)\,, \tag{9}$$

where μ is a probability measure on $(0, \pi]$. A measure space (H, ν) combined with this d is obviously taken as follows:

$$H := \{h_{\rho,p} := C_\rho(p);\ 0 < \rho \leq \pi,\ p \in S^1\} \quad \text{with} \quad \nu(dh_{\rho,p}) = c\mu(d\rho)\sigma(dp)\,.$$

Observe that the rotation-invariant distance d on S^1 takes the form $d(x, y) = r(d_G(x, y))$, where

$$r(t) = c\int_{(0,\pi]} r_\rho(t)\mu(d\rho) = 2c\int_{(0,\pi]} \min(t, \rho)\mu(d\rho)\,.$$

The right derivative $r'_+(t)$ is of the form $2c\mu((t, \pi])$ and therefore non-increasing in $0 \leq t < \pi$.

What we have just observed is summed up in the following

PROPOSITION 2. *Suppose that $r(t)$ is a continuous function on $[0, \pi]$, $r(0) = 0$ and has the right derivative $r'_+(t) \geq 0$, non-increasing on $[0, \pi)$. Then the distance $d(x, y) := r(d_G(x, y))$ on S^1 is L^1-embeddable.*

§3. Generalized Radon transform and its dual

This section is devoted to the study of basic properties of the generalized Radon transform R and the dual Radon transform R^*. The main fact we prove is the singular value decomposition of R^* regarded as a Hilbert-Schmidt operator from $L^2(H, \nu)$ to $L^2(M, \alpha(x)m(dx))$, where the density $\alpha(x)$ is chosen from among positive functions in $L^1(M, m)$ satisfying

$$\int_M \nu(B_x)\alpha(x)m(dx) := C < \infty\,.$$

The decomposition of R^* implies the Karhunen-Loève expansion of a Lévy's Brownian motion X with parameter space (M, d).

We shall begin by discussing the covariance operator of X. The representation (1) of X implies that the covariance function $\Gamma(x, y) := E[X(x)X(y)]$ is equal to $\nu(B_x \cap B_y)$. With a choice of α mentioned above, we consider the Hilbert space $L^2(M, \tilde{m})$, $\tilde{m}(dx) := \alpha(x)m(dx)$, instead of the usual $L^2(M, m)$. Then, the equation

$$(\Gamma f)(x) = \int_M \Gamma(x, y)f(y)\tilde{m}(dy)\,, \tag{10}$$

defines a positive, self-adjoint and trace class operator on $L^2(M, \tilde{m})$ (cf. [5], p. 294). The operator Γ is called the *covariance operator of X.*

We next consider the generalized Radon transform R. Observe that multiplication by α is a well-defined operator from $L^2(M, \tilde{m})$ to $L^1(M, m)$:

$$(T_\alpha f)(x) := \alpha(x) f(x)\,, \qquad f \in L^2(M, \tilde{m})\,.$$

So we can form the composition $R \circ T_\alpha$ to infer that it is a bounded operator from $L^2(M, \tilde{m})$ to $L^2(H, \nu)$. The proof of this assertion is an easy computation:

$$\begin{aligned}
&\|(R \circ T_\alpha f)(h)\|^2_{L^2(H,\nu)} \\
&\leq \int_H \nu(dh) \left\{ \iint_{M^2} \chi_h(x) \chi_h(y) |f(x)| |f(y)| \tilde{m}(dx) \tilde{m}(dy) \right\} \\
&= \iint_{M^2} \nu(B_x \cap B_y) |f(x)| |f(y)| \tilde{m}(dx) \tilde{m}(dy) \\
&\leq \left\{ \int_M \sqrt{\nu(B_x)}\, |f(x)| \,\tilde{m}(dx) \right\}^2 \leq C \|f\|^2_{L^2(M, \tilde{m})}\,.
\end{aligned}$$

A similar argument implies that the dual Radon transform R^* is bounded from $L^2(H, \nu)$ to $L^2(M, \tilde{m})$. We need one more step to get at the following

THEOREM 3. *We have a factorization of Γ:*

(11)
$$\Gamma = R^* \circ (R \circ T_\alpha)\,. \qquad \begin{tikzcd} L^2(M, \tilde{m}) \arrow[rr, mapsto, "\Gamma"] \arrow[dr, "R \circ T_\alpha"'] & & L^2(M, \tilde{m}) \\ & L^2(H, \nu) \arrow[ur, "R^*"'] & \end{tikzcd}$$

The proof of (11) is immediate:

$$\begin{aligned}
(R^* \circ R \circ T_\alpha f)(x) &= \int_M f(y) \tilde{m}(dy) \left\{ \int_H \chi_{B_x}(h) \chi_{B_y}(h) \nu(dh) \right\} \\
&= \int_M \Gamma(x, y) f(y) \tilde{m}(dy) = (\Gamma f)(x)\,, \qquad f \in L^2(M, \tilde{m})\,.
\end{aligned}$$

We are now going to give the singular value decompositions of the two factors, $R \circ T_\alpha$ and R^*, in Theorem 3. Positive eigenvalues λ_i^2 of the covariance operator Γ is enumerated by means of index $i \in I$, where I is a finite or countable infinite set and $\{\lambda_i\} \in l^2(I)$. Set

$$N_0 := \{f \in L^2(M, \tilde{m});\ (\Gamma f)(x) \equiv 0,\ x \in M\}\,, \qquad \text{the null space of } \Gamma\,.$$

Then we can select in $N_0^\perp$ a CONS $\{f_i(x);\ i \in I\}$ consisting of eigenfunctions of Γ:

$$(\Gamma f_i)(x) = \lambda_i^2 f_i(x)\,, \qquad i \in I\,.$$

Note that any non-negative function in $L^2(M, \tilde{m})$ cannot be in N_0 since $\Gamma(x, y) \geq 0$ for all $x, y \in M$.

Now, put

$$g_i(h) := (R \circ T_\alpha f_i)(h)/\lambda_i \in L^2(H, \nu),$$

and $N_1 := [g_i; i \in I]^\perp$, where $[g_i; i \in I]$ stands for the closed linear span of $\{g_i; i \in I\}$ in $L^2(H, \nu)$. The functions $g_i(h)$, $i \in I$, constitute a CONS in $N_1^\perp$; the proof of this assertion is carried out by using Theorem 3:

$$\begin{aligned}(g_i, g_j)_{L^2(H,\nu)} &= (\lambda_i\lambda_j)^{-1}(R^* \circ R \circ T_\alpha f_i, f_j)_{L^2(M,\tilde{m})} \\ &= (\lambda_i\lambda_j)^{-1}(\Gamma f_i, f_j)_{L^2(M,\tilde{m})} = \lambda_i\lambda_j^{-1}(f_i, f_j)_{L^2(M,\tilde{m})} = \delta_{i,j}.\end{aligned}$$

For our purpose we need the following

LEMMA 4. *We have an expansion*

$$\chi_h(x) = \chi_{B_x}(h) = \sum_{i \in I} \lambda_i f_i(x) g_i(h), \quad x \in M \quad \text{and} \quad h \in H. \tag{12}$$

Proof. We write the Fourier series of $\chi_{B_x}(h)$ as an element of $L^2(H, \nu)$:

$$\chi_{B_x}(h) = \sum_{i \in I} c_i(x) g_i(h) + g^0(h),$$

where $c_i(x) := (\chi_{B_x}(h), g_i(h))_{L^2(H,\nu)} = (R^* g_i)(x)$ and $g^0 \in N_1$. Since

$$(R^* g^0, f_i)_{L^2(M,\tilde{m})} = (g^0, R \circ T_\alpha f_i)_{L^2(H,\nu)} = \lambda_i (g^0, g_i)_{L^2(H,\nu)} = 0$$

for any $i \in I$, we have $R^* g^0 \in N_0$. Actually this function $(R^* g^0)(x)$ is constantly equal to 0, because it is a non-negative function in N_0:

$$(R^* g^0)(x) = (\chi_{B_x}(h), g^0(h))_{L^2(H,\nu)} = \|g^0\|^2_{L^2(H,\nu)} \geq 0.$$

We have thus proved that $g^0(h) \equiv 0$.

The next task is to calculate the Fourier coefficients $c_i(x) = (R^* g_i)(x)$, $i \in I$. Since

$$(R^* g_i, f^0)_{L^2(M,\tilde{m})} = (g_i, R \circ T_\alpha f^0)_{L^2(H,\nu)} = (f_i, \Gamma f^0)_{L^2(M,\tilde{m})}/\lambda_i = 0$$

for any $f^0 \in N_0$, we have $R^* g_i \in N_0^\perp$. Furthermore, the equality

$$(R^* g_i, f_j)_{L^2(M,\tilde{m})} = (g_i, R \circ T_\alpha f_j)_{L^2(H,\nu)} = \lambda_j \delta_{i,j}$$

shows that $c_i(x) = \lambda_i f_i(x)$, which completes the proof. We note that (12) is also the Fourier series of $\chi_h(x)$ as an element of $L^2(M, \tilde{m})$.

In view of the expressions

$$(R \circ T_\alpha f)(h) = (\chi_h(x), f(x))_{L^2(M,\tilde{m})} \quad \text{and} \quad (R^* g)(x) = (\chi_{B_x}(h), g(h))_{L^2(H,\nu)},$$

Lemma 4 immediately gives us their singular value decompositions having common positive singular values $\{\lambda_i;\ i\in I\}\in l^2(I)$.

THEOREM 5. (i) *The operator* $R\circ T_\alpha$ *is a Hilbert-Schmidt operator from* $L^2(M,\tilde{m})$ *to* $L^2(H,\nu)$ *and has the singular value decomposition*

$$(R\circ T_\alpha f)(h)=\sum_{i\in I}\lambda_i(f,\ f_i)_{L^2(M,\tilde{m})}g_i(h). \tag{13}$$

The null space of $R\circ T_\alpha$ *is* $N_0=[f_i;\ i\in I]^\perp$.

(ii) *The dual Radon transform* R^* *is a Hilbert-Schmidt operator from* $L^2(H,\nu)$ *to* $L^2(M,\tilde{m})$ *and has the singular value decomposition*

$$(R^*g)(x)=\sum_{i\in I}\lambda_i(g,g_i)_{L^2(H,\nu)}f_i(x)\,. \tag{14}$$

The null space of R^* *is* $N_1=[g_i;\ i\in I]^\perp$.

An application of Theorem 5 (ii) to the representation (1) is now in order. Let us define

$$\xi_i:=\int_H g_i(h)W(dh)\,,$$

to get an i.i.d. sequence $\xi=\{\xi_i;\ i\in I\}$ of standard Gaussian random variables. Since (1) is rewritten as $X(x)=(R^*W)(x)$, the decomposition (14) yields

$$X(x)=\sum_{i\in I}\lambda_i\xi_i f_i(x)\,, \tag{15}$$

which is nothing but the Karhunen-Loéve expansion usually derived from Mercer's theorem (cf. [5] and [17]):

$$\Gamma(x,y)=\sum_{i\in I}\lambda_i^2 f(x)f_i(y)\,. \tag{16}$$

Moreover, orthonormality of the system $\{f_i;\ i\in I\}$ in $L^2(M,\tilde{m})$ implies the inverse expression of ξ in terms of X:

$$\xi_i=(X(x),f_i(x))_{L^2(M,\tilde{m})}/\lambda_i\,.$$

Summing up what we have just proved, we get

THEOREM 6. *Every Lévy's Brownian motion* X *with parameter space* (M,d) *admits of the Karhunen-Loève expansion* (15) *in terms of* ξ, *and moreover we have*

$$[X(x);\ x\in M]=[\xi_i;\ i\in I]=\left\{\int_H g(h)W(dh);\ g\in N_1^\perp\right\}. \tag{17}$$

As a direct consequence of (15), we obtain another useful expression of the L^1-embeddable metric $d(x, y) = \nu(B_x \triangle B_y)$ on M:

$$d(x, y) = \sum_{i \in I} \lambda_i^2 (f_i(x) - f_i(y))^2 , \tag{18}$$

which is equivalent to (16).

§4. Lévy's Brownian motion with parameter space (S^n, d_ρ)

The final section concerns the concrete examples on S^n discussed in Section 2. We shall calculate explicitly the eigenvalues and eigenfunctions of the self-adjoint operator R_ρ on $L^2(S^n, \sigma)$; then an application of the decomposition of R_ρ to the expression (1′) will yield a new representation for a Lévy's Brownian motion X with parameter space (S^n, d_ρ). In addition, we shall investigate the $M(t)$-process of X introduced by Lévy [20].

We recall some known facts about spherical harmonics (cf. [10] and [33]). Let SH_m denote the set of all spherical harmonics of degree m; then the dimension of SH_m is

$$h(m) := \frac{2m + n - 1}{m + n - 1} \binom{m + n - 1}{m} .$$

We get the direct sum decomposition $L^2(S^n, \sigma) = \sum_{m=0}^{\infty} \oplus SH_m$, as well as a CONS $\{S_{m,k}(x); (m, k) \in \Delta\}$ consisting of spherical harmonics, where $\Delta :=$ $\{(m, k);\ m \geq 0 \text{ and } 1 \leq k \leq h(m)\}$. In the sequel we shall make use of the addition formula

$$\frac{1}{h(m)} \sum_{k=1}^{h(m)} S_{m,k}(x) S_{m,k}(y) = C_m^\lambda((x, y)) / C_m^\lambda(1) ,$$

where $C_m^\lambda(u)$ is the Gegenbauer polynomial of degree m with $\lambda := (n-1)/2$.

Let us proceed to prove the explicit form of (12) in the present situation where $d = d_\rho$ and $(M, m) = (S^n, \sigma) \sim (H_\rho, \nu)$ by the mapping $x \in S^n \mapsto C_\rho(x) \in H_\rho$.

LEMMA 7 (cf. [32]). *We have an expansion*

$$\begin{aligned} \chi_{C_\rho(x)}(y) &= \sum_{(m, k) \in \Delta} \lambda_m(\rho) S_{m,k}(x) S_{m,k}(y) \\ &= \sum_{m=0}^{\infty} \lambda_m(\rho) h(m) C_m^\lambda((x, y)) / C_m^\lambda(1) , \end{aligned} \tag{19}$$

where

$$\lambda_m(\rho) = \begin{cases} \dfrac{|S^{n-1}|}{|S^n|} \displaystyle\int_{\cos(\rho/2)}^1 (1-u^2)^{\lambda-1/2}\,du\,, & m=0\,, \\ \dfrac{|S^{n-1}|}{|S^n|\,n} \dfrac{C_{m+1}^{\lambda+1}(\cos(\rho/2))}{C_{m-1}^{\lambda+1}(1)} \sin^{2\lambda+1}(\rho/2)\,, & m \geq 1 \end{cases}$$

Proof. Appealing to the Funk-Hecke theorem ([10]), we have (19) with

$$\lambda_m(\rho) = \frac{|S^{n-1}|}{|S^n|} \int_{\cos(\rho/2)}^1 \frac{C_m^\lambda(u)}{C_m^\lambda(1)} (1-u^2)^{\lambda-1/2}du\,.$$

To compute the integral $\int_{\cos(\rho/2)}^1 C_m^\lambda(u)(1-u^2)^{\lambda-1/2}du$ for $m \geq 1$, we use the formula

$$C_m^\lambda(u) = b_m^\lambda(1-u^2)^{-\lambda+1/2}\frac{d^m}{du^m}(1-u^2)^{m+\lambda-1/2}\,,$$
$$b_m^\lambda = (-1)^m(2\lambda)_m/(2m)!!\,(\lambda+1/2)_m\,,$$

where $(a)_m := \prod_{j=0}^{m-1}(a+j)$. Since

$$\begin{aligned} \lambda_m(\rho) &= \frac{|S^{n-1}|}{|S^n|} \frac{b_m^\lambda}{C_m^\lambda(1)} \left[\frac{d^{m-1}}{du^{m-1}}(1-u^2)^{m+\lambda-1/2}\right]_{\cos(\rho/2)}^1 \\ &= \frac{|S^{n-1}|\,2\lambda}{|S^n|\,m(2\lambda+m)C_m^\lambda(1)} C_{m-1}^{\lambda+1}(\cos(\rho/2))\sin^{2\lambda+1}(\rho/2) \\ &= \frac{|S^{n-1}|}{|S^n|\,n} \frac{C_{m-1}^{\lambda+1}(\cos(\rho/2))}{C_{m-1}^{\lambda+1}(1)} \sin^{2\lambda+1}(\rho/2)\,, \end{aligned}$$

the proof is completed.

The generalized (or dual) Radon transform R_ρ associated with (1′) is a self-adjoint and Hilbert-Schmidt operator on $L^2(S^n, \sigma)$, and the factorization of the covariance operator Γ (Theorem 3) takes the simpler form

$$\Gamma = (\sqrt{c}\,R_\rho)^2\,. \qquad \begin{tikzcd} L^2(S^n,\sigma) \arrow[rr, mapsto, "\Gamma"] \arrow[dr, "\sqrt{c}R_\rho"'] & & L^2(S^n,\sigma) \\ & L^2(S^n,\sigma) \arrow[ur, "\sqrt{c}R_\rho"'] & \end{tikzcd}$$

In order to state the decomposition of R_ρ, we set

$$\varDelta_\rho := \{(m,k)\in\varDelta;\ \lambda_m(\rho)=0\} = \{(m,k)\in\varDelta;\ m\geq 2,\ C_{m-1}^{\lambda+1}(\cos(\rho/2))=0\}\,,$$

which corresponds to the null space N of R_ρ, and $I_\rho := \varDelta\backslash\varDelta_\rho$. Recalling that $(R_\rho f)(x) = (\chi_{C_\rho(x)}(y), f(y))_{L^2(S^n,\sigma)}$, Lemma 7 implies the following

THEOREM 8. *We have*

$$(R_\rho f)(x) = \sum_{(m,k)\in I_\rho} \lambda_m(\rho)(f, S_{m,k})_{L^2(S^n,\sigma)} S_{m,k}(x)\,, \tag{20}$$

and the null space $N = [S_{m,k}(x);\ (m, k) \in \varDelta_\rho]$.

By applying (20) to the expression $X(x) = \sqrt{c}\,(R_\rho W_0)(x)$, we obtain the Karhunen-Loève expansion of X in terms of the i.i.d. sequence

$$\xi = \left\{\xi_{m,k} := \int_{S^n} S_{m,k}(y) W_0(dy);\ (m, k) \in I_\rho\right\}$$

of standard Gaussian random variables.

THEOREM 9. *A Lévy's Brownian motion X with parameter space (S^n, d_ρ) admits of a representation*

$$X(x) = \sqrt{c} \sum_{(m,k)\in I_\rho} \lambda_m(\rho)\xi_{m,k} S_{m,k}(x)\,. \tag{21}$$

Moreover, we have

$$[X(x);\ x \in S^n] = [\xi_{m,k};\ (m, k) \in I_\rho] = \left\{\int_{S_n} g(y) W_0(dy);\ g \in N^\perp\right\}, \tag{22}$$

and

$$d_\rho(x, y) = 2c \sum_{m=1}^{\infty} \lambda_m^2(\rho) h(m)\{1 - C_m^\lambda((x, y))/C_m^\lambda(1)\}\,. \tag{23}$$

We now focus our attention on the case $\rho = c = \pi$, i.e., X is a Lévy's Brownian motion with the geodesic distance d_G. In this case,

$$\varDelta_\pi = \{(2j, k);\ j = 1, 2, \cdots \text{ and } 1 \le k \le h(2j)\}$$

and

$$\lambda_m(\pi) = \begin{cases} 1/2\,, & m = 0\,, \\ \dfrac{\Gamma((n+1)/2)(n-2)!!}{\Gamma(n/2)\sqrt{\pi}} (-1)^j \dfrac{(2j-1)!!}{(2j+n)!!}\,, & m = 2j+1\,. \end{cases}$$

With the help of these values, one can compute the coefficients $2\pi\lambda_m^2(\pi)h(m)$ in (23), to find the formula of d_G due to Gangoli [13] and Molčan [25] (cf. also [27], p. 143) who proved it via an entirely different approach. In view of the special form of $\varDelta_\pi$, it is natural to assume that $X(x)$ is odd, i.e., $X(x) + X(-x) = 0$; the expression (21) then becomes

$$X(x) = \frac{\Gamma((n+1)/2)(n-2)!!}{\Gamma(n/2)} \sum_{j=0}^{\infty} (-1)^j \frac{(2j-1)!!}{(2j+n)!!} \times \sum_{k=1}^{h(2j+1)} \xi_{2j+1,k} S_{2j+1,k}(x)\,. \tag{21'}$$

In connection with the $M(t)$-process ([20]), we need another transform $(T_\rho f)(x)$ (cf. [15]) which is given by the mean value of f over the small (or great in case $\rho=\pi$) circle $\delta C_\rho(x)$, $0<\rho<2\pi$.

DEFINITION 3. For $f\in C(S^n)$, the set of all continuous functions on S^n, the integral transformation defined by

$$(24)\qquad (T_\rho f)(x):=\int_{\delta C_\rho(x)} f(y)s(dy)/s(\delta C_\rho(x))\,,$$

is called the *mean value operator over* $\delta C_\rho(x)$, where s denotes the $(n-1)$-dimensional surface measure on $\delta C_\rho(x)$. For each fixed $x_0\in S^n$, the Gaussian process

$$(25)\qquad M(t):=(T_{2t}X)(x_0)-X(x_0)\,,\qquad 0<t<\pi\,,$$

is called the *$M(t)$-process.*

By appealing, again, to the Funk-Hecke theorem, we get the decomposition of T_ρ.

PROPOSITION 10. *The mean value operator T_ρ on $C(S^n)$ is extended to be a self-adjoint, compact operator on $L^2(S^n,\sigma)$, and it has the decomposition*

$$(26)\qquad (T_\rho f)(x)=\sum_{(m,k)\in \tilde I_\rho}\frac{C_m^\lambda(\cos(\rho/2))}{C_m^\lambda(1)}(f,S_{m,k})_{L^2(S^n,\sigma)}S_{m,k}(x)\,,$$

where $\tilde I_\rho:=\Delta\backslash\tilde\Delta_\rho$ and $\tilde\Delta_\rho:=\{(m,k)\in\Delta;\ C_m^\lambda(\cos(\rho/2))=0\}$. Moreover, the null space of T_ρ is $[S_{m,k}(x);(m,k)\in\tilde\Delta_\rho]$.

By the combination of (21) and (26), we write

$$\begin{aligned} M(t)&=\sqrt{c}\sum_{(m,k)\in I_\rho}\left\{\frac{C_m^\lambda(\cos t)}{C_m^\lambda(1)}-1\right\}\lambda_m(\rho)\xi_{m,k}S_{m,k}(x_0)\\ &=\sqrt{c}\sum_{m\in J_\rho}\lambda_m(\rho)\sqrt{h(m)}\,\eta_m\{1-C_m^\lambda(\cos t)/C_m^\lambda(1)\}\,,\end{aligned}$$

where we have put

$$\eta_m:=\frac{-1}{\sqrt{h(m)}}\sum_{k=1}^{h(m)}\xi_{m,k}S_{m,k}(x_0)\qquad \text{for } m\in J_\rho:=\{m\geq 1;\ \lambda_m(\rho)\neq 0\}\,.$$

It is shown that the η_m form an i.i.d. sequence of standard Gaussian random variables.

PROPOSITION 11 (cf. [27] in case $\rho=\pi$). *The $M(t)$-process of a Lévy's Brownian motion X with parameter space (S^n,d_ρ) is expressed in the form*

$$M(t)=\sqrt{c}\sum_{m\in J_\rho}\lambda_m(\rho)\sqrt{h(m)}\,\eta_m\{1-C_m^\lambda(\cos t)/C_m^\lambda(1)\}\,. \tag{27}$$

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