REPRESENTATION OF EUCLIDEAN RANDOM FIELD

SHIGEO TAKENAKA

P. Lévy introduced a notion of Brownian motion $\mathscr{X} \equiv \{X(p); p \in M\}$ with parameter in a metric space (M, d), which is a centered Gaussian system satisfying

$$E|X(p) - X(q)|^2 = d(p, q) \quad \text{and} \quad X(O) = 0, \ O \text{ being the origin} .$$

In the case of $M = \boldsymbol{R}^n$, S^n or the hyperbolic space H^n with usual geodesic metric, the Brownian motion above has the following representation

(1) $X(p) = Y(S_p)$, where $S_p = \{\text{hyperplanes intersect } \overline{Op}\}$ and $\mathscr{Y} = \{Y(\cdot)\}$ is the Gaussian random measure associated with a certain measure μ on the set of all hyperplanes.

In this paper we shall discuss Brownian motion that corresponds to a general metric. When the metric d on $\boldsymbol{R}^n$ is expressible as $d(p, q) = r(|p - q|)$ where r is a positive increasing continuous function, the Brownian motion is called a Euclidean random field (ERF). The main purpose of this paper is to establish the representations of the form (1) for some important classes of ERFs.

In Section 1 we will consider a simple and basic class of ERFs, denote it by $\{\mathscr{U}^\rho\}$, and their representations. The covariance function of the field $\mathscr{U}^\rho$ is of finite range and rotationally invariant. The form (1) for the ordinary Brownian motion with parameter $\boldsymbol{R}^n$ is obtained as the limit $\rho \to \infty$ of these fields $\mathscr{U}^\rho$.

In Section 2 the representation of type (1) will be considered for general ERF related to the $\{\mathscr{U}^\rho\}$. We will start with a special class of ERFs. If $r(t) = t^\alpha$ the random field is called the self-similar Euclidean random field (SERF) of index α. The representation of SERF of index $0 < \alpha < 1$ will be written as a superposition of the fields $\mathscr{U}^\rho$. For general ERF, two sufficient conditions for the existence of the representation of form (1) will be given as conditions on the function $r(\cdot)$.

Received June 19, 1985.

In Section 3, invariance properties of SERFs will be considered in connection with our representations.

§0. Introduction

0-1. P. Lévy defined a notion of Brownian motion with parameter space M equipped with a metric d as below:

DEFINITION 1. A Gaussian system $\mathscr{X} = \{X(p); p \in M\}$ is called a *Brownian motion* with parameter space (M, d) if it satisfies

$\mathscr{G}0)$ $X(O) = 0$ for a fixed origin $O \in M$,

$\mathscr{G}1)$ $X(p) - X(q) \overset{\mathscr{L}}{=} N(0, d(p, q))$,

that is the left-hand-side is subject to the Gaussian law with mean 0 and variance $d(p, q)$.

It is well known that the Brownian motion exists if and only if the covariance function $\sigma(p, q) = (1/2)(d(p, O) + d(q, O) - d(p, q))$ is positive definite. In this line, several investigations have been done on the positive definiteness of function σ ([3], [6], [7], [9], [10]). In the case of $M = \boldsymbol{R}^n$ with usual metric, N. N. Chentsov took more direct manner to obtain Brownian motion in [1] (P. Lévy also treated the case $M = S^n$, by the same manner in [5]). To describe Chentsov's method, we need the notion of Gaussian random measure attached to measure space $(E, \mathscr{B}, \mu)$.

DEFINITION 2. A Gaussian system $\mathscr{Y} = \{Y(B); B \in \mathscr{B}, \mu(B) < \infty\}$ is called a *Gaussian random measure* associated with $(E, \mathscr{B}, \mu)$ if it satisfies

$\mathscr{R}1)$ $Y(B) \overset{\mathscr{L}}{=} N(0, \mu(B))$

$\mathscr{R}2)$ $Y(B_1 \cup B_2) = Y(B_1) + Y(B_2)$,   a.e., if $B_1 \cap B_2 = \phi$.

Let $\mathscr{H}$ be the set of all hyperplanes of codimension 1 in $\boldsymbol{R}^n$. $\mathscr{H}$ is parametrized by $(q, r) \in S^{n-1} \times \boldsymbol{R}_+$ as $\mathscr{H} \ni H = \{y; \langle y, q \rangle = r\}$, and define a measure μ on $\mathscr{H}$ as $d\mu = dq \cdot dr$, where dq is the normalized uniform measure on S^{n-1}. Set

(12) $S_p = \{H \in \mathscr{H}; H \text{ intersects the line segment } \overline{Op}\}$,

then Chentsov's representation of the Brownian motion is

(1) $X(p) = Y(S_p)$, where $\mathscr{Y}$ in the definition 2 is the random measure associated with $(\mathscr{H}, \mu)$.

0-2. In the case of $M = S^n$ and H^n — the n-dimensional hyperbolic space — the similar results were obtained ([10]):

Let $\mathscr{H}$ be the set of all hyperplanes of codimension 1 in M. There exists a canonical measure μ on $\mathscr{H}$ in the sense of integral geometry. Define a map S from M into the Borel sets of $\mathscr{H}$ as

(12) $$S_p = \{\text{hyperplanes intersect } \overline{Op}\} .$$

Then,

(1) $$X(p) = Y(S_p)$$

is the Brownian motion with parameter (M, d), where $\mathscr{Y} = \{Y(\cdot)\}$ is the Gaussian random measure associated with $(\mathscr{H}, \mu)$.

The essential parts of the representations above are

i) the pair $\{(E, \mathscr{B}, \mu), S\}$ of measure space and the map S from the metric space M to $\mathscr{B}$.

ii) the relation $d(p, q)$ $(= E|X(p) - X(q)|^2) = \mu(S_p \triangle S_q)$ between the metric and the measure, where $\triangle$ means the symmetric difference.

In this paper we will discuss the case of $M = \mathbf{R}^n$ with rotational invariant d and will establish the pairs $\{(E, \mathscr{B}, \mu), S\}$ for some important classes of random fields.

§1. Random field $\mathscr{U}^\rho$ and the Chentsov representation

We start with a simple class $\{\mathscr{U}^\rho\}$ of Brownian motions and their representations. The original Chentsov's representation will be obtained as the limit $\rho \to \infty$ of $\mathscr{U}^\rho$. Moreover, in the next section the field $\mathscr{U}^\rho$ will be used as the basic element to construct representations of certain classes of Brownian motions. In this sense $\{\mathscr{U}^\rho\}$ is an important and fundamental class of Brownian motions.

1-1. Let M be a set and $\mathscr{Y}$ be the Gaussian random measure associated with a measure space $(E, \mathscr{B}, \mu)$. Assume that there is a map

$$S: M \ni p \longrightarrow S_p \in \mathscr{B} .$$

Define a Gaussian system

(1) $$\mathscr{X} = \{X(p) \equiv Y(S_p); \ p \in M\} .$$

Then, since $(A \triangle B) \cup (B \triangle C) \supset (A \triangle C)$, the function $d(p, q) \equiv E|X(p) - X(q)|^2 = \mu(S_p \triangle S_q)$ is a pseudo-metric on M.

We call this representation a Chentsov type representation of the random field $\mathscr{X}$ with parameter (M, d).

1-2. The class $\{\mathscr{U}^\rho\}$. Let $\mathscr{Y} = \{Y(B); \, B \in \mathscr{B}\}$ be the Gaussian random measure with respect to $(R^n, \mathscr{B}, dx)$. Set,

$$(2.\rho) \qquad\qquad S_p^\rho \equiv \{x \in R^n; \, |x - p| \leqq \rho\} \,,$$

for a positive constant ρ, and define a Gaussian system

$$(1.\rho) \qquad\qquad \tilde{\mathscr{U}}^\rho \equiv \{\tilde{U}^\rho(p) \equiv Y(S_p^\rho)\} \,.$$

Then,

PROPOSITION 1. $\tilde{\mathscr{U}}^\rho$ is a Gaussian random field stationary under the action of the group $M(n)$ of n-dimensional Euclidean solid motions.

Let us calculate the metric function d^ρ related to $\tilde{\mathscr{U}}^\rho$. Set

$$r_\rho(|p - q|) \equiv d^\rho(p, q) \equiv E[\tilde{U}^\rho(p) - \tilde{U}^\rho(q)]^2 = |S_p^\rho \triangle S_q^\rho| \,.$$

Then,

$$(3) \qquad\qquad r_\rho(t) \equiv 4\sigma_{n-1} \int_0^{\min(t/2, \, \rho)} (\rho^2 - x^2)^{(n-1)/2} \, dx \,,$$

where σ_{n-1} is the volume of the $(n-1)$-dimensional unit ball.

PROPOSITION 2.

1) $r_\rho(t) = \rho^n r_1\left(\dfrac{t}{\rho}\right),$

2) $r_1(t) = O(t)$ when $t \to 0,$

3) $r_1(t) = 2\sigma_n,$ if $t \geqq 1,$

4) $r_\rho(t) = O(\rho^{n-1})$ when $\rho \to \infty.$

It is easy to see that the centered random field

$$\mathscr{U}^\rho \equiv \{U^\rho(p) \equiv \tilde{U}^\rho(p) - \tilde{U}^\rho(O)\}$$

is the Brownian motion corresponding to the metric $d^\rho(p, q) \equiv r_\rho(|p - q|)$.

1-3. Chentsov representation as the limit of $\mathscr{U}^\rho$. We note that

$$(4) \qquad\qquad r_\rho(t)/\rho^{n-1} = 4\rho\sigma_{n-1} \int_0^{\min(t/2\rho, \, 1)} (1 - x^2)^{(n-1)/2} dx$$

$$= 2\sigma_{n-1} \cdot t + o(1) \,, \quad \text{when } \rho \longrightarrow \infty \,.$$

It is natural to expect that the original Chentsov's representation of usual Brownian motion would be obtained as a limit of representations of $\mathscr{U}^\rho$. In fact, this is the case as is shown below. Take a normalization of the measure in 1-2 as to be

$$(5) \qquad d\tilde{\mu}_\rho \equiv \frac{dx}{2\sigma_{n-1}\rho^{n-1}} \quad \text{or} \quad = \frac{r^{n-1}}{2\cdot\rho^{n-1}} dr \cdot dq$$

in the polar coordinate $x = r \cdot q$, $r \in R_+$ and $q \in S^{n-1}$. Related to this normalization we have a series of Gaussian system

$$(6) \qquad \bar{\mathscr{U}}^\rho = \{U^\rho(p)/\sqrt{2\sigma_{n-1}\rho^{n-1}}\}, \qquad \rho > 0.$$

Set,

$$(7) \qquad \bar{\mathscr{U}}_R^\rho = \{\bar{U}^\rho(p); |p| \leq R\} \qquad \text{for fixed } R > 0.$$

For $\rho > R$, express the set $S_p^\rho \triangle S_0^\rho$ in the polar coordinate;

$$(8) \quad S_p^\rho \triangle S_0^\rho = \left\{(q, r) \in S^{n-1} \times R_+; \begin{array}{l} \rho \leq r \leq \rho + \langle p, q\rangle \\ + (\sqrt{\langle q, p\rangle^2 + \rho^2 - t^2} - \rho), \ \rho\langle p, q\rangle \geq t^2/2 \\ \rho + \langle q, p\rangle + (\sqrt{\langle q, p\rangle^2 + \rho^2 - t^2} - \rho) \\ \qquad \leq r \leq \rho, \qquad\qquad \rho\langle p, q\rangle < t^2/2 \end{array} \right\},$$

where $t = |p|$.

The term $(\sqrt{\langle q, p\rangle^2 + \rho^2 - t^2} - \rho)$ tends to zero when $\rho \to \infty$. So, let us substract the offset ρ from the radial part r and set $\tilde{r} \equiv r - \rho$. Then,

$$(8') \quad S_p^\rho \triangle S_0^\rho = \left\{(q, \tilde{r}) \in S^{n-1} \times [-R, R]; \begin{array}{l} 0 \leq \tilde{r} \leq \langle q, p\rangle + o(1), \ \rho\langle p, q\rangle \geq t^2/2 \\ o(1) + \langle p, q\rangle \leq \tilde{r} \leq 0, \ \rho\langle p, q\rangle < t^2/2 \end{array} \right\}.$$

On the other hand, for a Borel subset A of $S^{n-1} \times [-R, R]$,

$$(9) \qquad \tilde{\mu}_\rho(A) = \mu_\infty(A) + o(1), \qquad \text{where } d\mu_\infty = \tfrac{1}{2}dqd\tilde{r}.$$

Finally, set

$$(10) \qquad \chi_{S_p}(q, \tilde{r}) \equiv \lim_{\rho\to\infty} |\chi_{S_p^\rho}(q, \tilde{r}) - \chi_{S_0^\rho}(q, \tilde{r})|,$$

where χ_A means the indicator function of the set A. Then,

$$(11) \qquad S_p = \{(q, \tilde{r}); \min(0, \langle q, p\rangle) \leq \tilde{r} \leq (\max(0, \langle q, p\rangle)\}.$$

Take $E_R = (S^{n-1} \times [-R, R], d\mu_\infty)$ and let $\mathscr{Y}$ be the Gaussian random measure associated with E_R. Set $\mathscr{X}_R$ be the limit $\rho \to \infty$ of $\bar{\mathscr{U}}_R^\rho$;

$$\mathscr{X}_R \equiv \{X(p) \equiv Y(S_p); \ p \in S^{n-1} \times [-R, R]\}.$$

Then we obtain

THEOREM 3. *The projective limit $\mathscr{X} = \mathrm{proj}\lim_{R\to\infty}\mathscr{X}_R$ is the Brownian motion parametrized in n-dimensional Euclidean space that corresponds to*

the usual metric. Moreover, the representation $X(p) = Y(S_p)$ is essentially equal to the one of N. N. Chentsov in 0–2.

Proof. Let us identify a hyperplane $H(\tilde{r}, q) = \{x; \langle x, q \rangle = \tilde{r}\}$ with a point $(q, \tilde{r}) \in S^{n-1} \times \mathbf{R}_+$. Then the set

(12) $\{H \in \mathscr{H}; H$ intersects the line segment $\overline{Op}\}$

is identified with

(11′) $\{(q, \tilde{r}) \in S^{n-1} \times \mathbf{R}_+; 0 \leqq \tilde{r} \leqq \langle q, p \rangle\}$.

Compare (11′) with (11) then we see that our new representation is just the Chentsov's representation of multiplicity 2.

1-4. **Another examples of Brownian motions.** The rest of this paper is devoted to the case of rotationally invariant metrics. But our formulation works for some of general metrics without rotationally invariance property.

EXAMPLE A. Let $\mathscr{H}_i = \{h; $ hyperplanes in $\mathbf{R}^n, h \perp \overline{Oe_i}\}$, where e_i is the unit vector parallel to the i-th coordinate, and let $\tilde{\mu}_i$ be the normalized uniform measure on $\mathscr{H}_i$. Consider the measure space $(\mathscr{H}, \tilde{\mu})$, where $\mathscr{H} = \mathscr{H}_1 \cup \mathscr{H}_2 \cup \cdots \cup \mathscr{H}_n$ and $\tilde{\mu} = \tilde{\mu}_1 + \tilde{\mu}_2 + \cdots + \tilde{\mu}_n$. Define $S_p = \{h \in \mathscr{H}: h$ intersects $\overline{Op}\}$ and let $\mathscr{Y}$ be the random measure associated with $(\mathscr{H}, \tilde{\mu})$. Then $\{X(p) = Y(S_p)\}$ is the Brownian motion that corresponds to the metric $d(x, y) = \sum |x_i - y_i|$.

EXAMPLE B. In the case of $\mathbf{R}^2$, set $\mathscr{H}_+ = \{h:$ parallel with $\overline{O(e_1 + e_2)}\}$ and set $\mathscr{H}_- = \{h;$ parallel with $\overline{O(e_1 - e_2)}\}$. The measure $\mu = (1/\sqrt{2})(\mu_+ + \mu_-)$, the space $\mathscr{H} = \mathscr{H}_+ \cup \mathscr{H}_-$, the map S_p and the random measure $\mathscr{Y}$ are all defined similarly to those in example A. Then the field $\{X(p) = Y(S_p)\}$ is the Brownian motion that corresponds to the metric

$$d(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|).$$

The author have been informed by private communications with professor A. Noda that for general metrics—for example ℓ^p metric on $\mathbf{R}^n$ and its α-th power, $0 < \alpha \leq 1$—the Chentsov type representation can be obtained.

§ 2. Representation of Euclidean random field

In this section we treat the Chentsov type representations of an important class of random fields—self similar Gaussian random fields—and

a general class of Gaussian random fields with Euclidean invariance property.

2-1. Self-similar Euclidean random field.

DEFINITION 3. A Gaussian system $\mathscr{V}^\alpha = \{V^\alpha(p); \ p \in R^n\}$ is called a *self-similar Euclidean random field (SERF) of index* α if it satisfies

$\mathscr{S}0)$ $$V^\alpha(O) = 0 \, ,$$

$\mathscr{S}1)$ $$V^\alpha(p) - V^\alpha(q) \overset{\mathscr{L}}{=} N(0, |p - q|^\alpha) \, .$$

The SERF exists if and only if $0 < \alpha \leqq 2$ ([4]). It is easy to see that $|\cdot|^\alpha$ is a metric if and only if $0 < \alpha \leq 1$. Therefore, the Chentsov type representation of SERF may be obtained in the case. In fact, we do form the representation in the case $0 < \alpha < 1$. (excluding the usual Brownian motion, $\alpha = 1$).

Set $E = R_+ \times R^n$ and introduce a coordinate in E as

(13) $$x = (x_0, \underline{x}) = (x_0, x_1, x_2, \cdots, x_n) \, , \qquad x_0 > 0 \, .$$

Consider a measure

(14) $$d\mu_\alpha(x) = M(\alpha) \cdot x_0^{\alpha - n - 1} dx_0 d\underline{x} \quad \text{on } E,$$

where $M(\alpha)$ is the normalizing constant which will be defined in (23). Let $\mathscr{Y}$ be the Gaussian random measure associated with (E, μ_α). Define the following maps;

(15) $$\tilde{S} \colon R^n \longrightarrow \mathscr{B}(E); \ p \longrightarrow \tilde{S}_p = \{x; |\underline{x} - p| \leqq x_0\}$$

and

(16) $$S_p = S_p \triangle \tilde{S}_O \, .$$

Then,

THEOREM 4. *For* $0 < \alpha < 1$,
1) $\mu_\alpha(S_p) < \infty$,
2) $V^\alpha(p) \equiv Y(S_p)$ *is the SERF of index* α.

Proof. 1)

$$\mu_\alpha(S_p) = M(\alpha) \int_0^\infty r_{x_0}(t) x_0^{\alpha - n - 1} dx_0 = M(\alpha) \int_0^\infty x_0^{\alpha - 1} r_1\left(\frac{t}{x_0}\right) dx_0 \, ,$$

where $t = |p|$. Because $r_1(t) = 2\sigma_n$ if $t \geqq 1$,

$$\left| \int_0^1 x_0^{\alpha-1} r_1\left(\frac{t}{x_0}\right) dx_0 \right| = \int_1^\infty y^{-\alpha-1} r_1(ty) dy \leqq 2\sigma_n \int_1^\infty y^{-\alpha-1} dy < \infty, \quad \text{if } \alpha > 0 .$$

And because $r_\rho(t) = O(\rho^{n-1})$, $\rho \to \infty$, we get

$$\left| \int_1^\infty r_{x_0}(t) x_0^{\alpha-n-1} dx_0 \right| \leqq K \int_1^\infty x_0^{n-1} x_0^{\alpha-n-1} dx_0 < \infty , \quad \text{if } \alpha < 1 .$$

Combining the two inequalities above, we get $\mu_\alpha(S_p) < \infty$ for $0 < \alpha < 1$.

2) $E \, | \, V^\alpha(p) - V^\alpha(q)|^2 = E \, | \, Y(S_p \triangle S_o) - Y(S_q \triangle S_o)|^2 = \mu_\alpha(S_p \triangle S_q)$. Note that the measure μ_α is invariant under the action

$$(17) \qquad\qquad\qquad gx \equiv (x_0, g\underline{x})$$

of the group of solid motions $M(n)$. Then there is a positive function s such that

$$(18) \qquad\qquad\qquad E \, | \, V^\alpha(p) - V^\alpha(q)|^2 = s(p - q|) .$$

For any $c > 0$, we have

$$(19) \qquad E \, | \, V^\alpha(cp)|^2 = M(\alpha) \int_0^\infty r_{x_0}(ct) x_0^{\alpha-n-1} dx_0 = M(\alpha) \int_0^\infty x_0^{\alpha-1} r_1\left(\frac{ct}{x_0}\right) dx_0$$

$$= c^\alpha M(\alpha) \int_0^\infty r_1\left(\frac{t}{x_0}\right) x_0^{\alpha-1} dx_0 = c^\alpha E[V^\alpha(p)]^2.$$

That is $s(ct) = c^\alpha s(t) = (ct)^\alpha s(1)$, and the fact $s(1) = 1$ is derived from (23) and (24).

The SERF of index 1 is nothing but the ordinary Brownian motion and we have already obtained the representation of this type in 1–2. Consider the section $\widetilde{S}_p \cap \{x_0 = \rho\}$ of the set in (16). This is the n-ball of radius ρ. Thus we can see that the field $\mathscr{V}^\alpha$ of index $0 < \alpha < 1$ is represented by a superposition of the fields $\{\mathscr{U}^\rho\}$, that is, in an intuitive manner

$$V^\alpha(p) = \int_0^\infty U^\rho(p) \rho^{(\alpha-n-1)/2} d\rho .$$

2-2. Euclidean random fields. Let us proceed to a more general class of random fields which can be obtained by superpositions.

DEFINITION 4. A Gaussian system $\mathscr{X}^r = \{X^r(p); \, p \in R^n\}$ is called a *Euclidean random field* (*ERF*) if it satisfies

$$\mathscr{E}0) \qquad\qquad\qquad X^r(O) = 0$$

and

$\mathscr{E}$1)
$$X^r(p) - X^r(q) \overset{g}{=} N(0, r(|p - q|)) ,$$

where r is a positive continuous function.

The SERF in 2-1 is a special class of ERFs with $r(t) = t^\alpha$. In this subsection we consider the conditions under which the ERF has a Chentsov type representation.

Suppose that $X^r(p)$ is decomposed into a superposition of the independent family of ERFs $\{\mathscr{U}^\rho\}$, as $X^r(p) = \int U^\rho(p)\nu(\rho)d\rho$ with a positive measure $\nu(\rho)d\rho$. Let $d\mu(x) = \nu^2(x_0)dx_0 d\underline{x}$ be a measure on $E = \mathbf{R}_+ \times \mathbf{R}^n$, S, be the same in (16) and $\mathscr{Y}$ be the random measure which corresponds to $d\mu$. Then the assumption above is equivalent to the existence of the representation

(20)
$$X^r(p) = Y(S_p) .$$

The relation between r and ν is given by

(21)
$$r(t) = E[X^r(p)]^2 = \int_0^\infty r_\rho(t)\nu^2(\rho)d\rho , \qquad \text{where } t = |p| .$$

Thus if the function r belongs to the image of the integral transform above, the ERF $\mathscr{X}^r$ has a Chentsov type representation.

The integral equation (21) can be solved as below;

THEOREM 5 (cf. [2]). *If* $\tilde{r}(t) \equiv (-d/dt)^{n+1}r'(2t) \geqq 0$ *and* $\int_1^\infty \tilde{r}(t) \cdot t^{n-1}dt < \infty$, *then the solution is obtained as*

(22)
$$\nu^2(\rho) = A \int_\rho^\infty \tilde{r}(t)\Big(\frac{t^2}{\rho^2} = 1\Big)^{(n-1)/2}dt ,$$

A being a positive constant.

We now consider another condition for the existence of the representation. We have already obtained a series of solutions of (21) in 2-1. Actually, set

(23)
$$M(\alpha) = \int_0^\infty \rho^{\alpha-1}r_1\Big(\frac{1}{\rho}\Big)d\rho ,$$

then

(24)
$$t^\alpha = \int_0^\infty \rho^n r_1\Big(\frac{t}{\rho}\Big)\frac{\rho^{\alpha-n-1}}{M(\alpha)}d\rho .$$

Thus the density $\nu^2(\rho)$ in question which corresponds to the metric $r(t) = t^\alpha$

is expressible as $\rho^{\alpha-n-1}/M(\alpha)$. Let us consider again a superposition of these special solutions by a positive density $m(\alpha)$, say $X(p) = \int_0^1 V^{\alpha}(p)m(\alpha)d\alpha$. That is, let us consider a measure μ on E as

$$d\mu(x) = \left(\int_0^1 \frac{x_0^{\alpha-n-1}}{M(\alpha)} m^2(\alpha)d\alpha \right) dx_0 d\underline{x} \quad \text{and} \quad \text{define}$$

(20′) $X(p) = Y(S_p),$

where $\mathscr{Y}$ is again the random measure associated with (E, μ). Then, the corresponding function r is

(21′) $r(t) = \int_0^1 t^{\alpha}m^2(\alpha)d\alpha .$

Set $R(x) = r(e^{-x})$. The Bernstein's theorem on Laplace transform tells us that the function $R(x)$ is completely monotone on $[0, \infty)$.

We then come to a class of functions r satisfying the conditions below

$\mathscr{A}0)$ $\lim_{t \to 0} r(t) = r(0) = 0,$

$\mathscr{A}1)$ $\tilde{d}(p, q) = r(|p - q|)$ defines a C^{∞}-metric on $\mathbf{R}^n$.

$\mathscr{A}2)$ $R(x) = r(e^{-x})$ is completely monotone.

By Bernstein's theorem there exists a positive measure $m^2(\alpha)d\alpha$ such that

(25) $R(x) = \int_{0+}^{\infty} e^{-x\alpha}m^2(\alpha)d\alpha \quad \text{or} \quad \text{equivalently}$

(26) $r(t) = \int_{0+}^{\infty} t^{\alpha}m^2(\alpha)d\alpha .$

Set $g(z) = \int_{0+}^{\infty} e^{i\alpha z}m^2(\alpha)d\alpha$ for $\mathrm{Im}(z) \geqq 0$. Then,

LEMMA 6.   *If $r(|p - q|)$ defines a metric on $\mathbf{R}^n$, then there are positive constants c_0 and c_1 such that $r(t) \leqq c_0 + c_1 t$, $t \in \mathbf{R}_+$.*

Using this lemma, we prove that $\int_0^{\infty} t^{\alpha}m^2(\alpha)d\alpha \leqq c_0 + c_1 t$ for any t, that is $\left| \int_0^{\infty} e^{\alpha z}m^2(\alpha)d\alpha \right| < \infty$ for any $z \in \mathbf{C}$. Thus the domain of definition of $g(z)$ can be extended to the whole complex plane.

THEOREM 7.   *The function $g(z)$ is of inferior exponential type of order 1, that is for any $\varepsilon > 0$ there is a constant C_{ε} such that*

$$|g(z)| \leqq C_{\varepsilon}e^{(1+\varepsilon)|\mathrm{Im}(z)|} , \qquad \text{for any } z \in \mathbf{C} .$$

Proof.

$$|g(z)| \leqq \int_{0+}^{\infty} e^{\alpha|\operatorname{Im}(z)|} m^2(\alpha) d\alpha = r(e^{|\operatorname{Im}(z)|}) \leqq c_0 + c_1 e^{|\operatorname{Im}(z)|} .$$

Hence, we can choose the desirable constant C_ε for any $\varepsilon > 0$.   Q.E.D.

By virtue of the Paley-Wiener theorem, the support of the measure $m^2(\alpha)d\alpha$ is included in $(0, 1]$. Finally we get;

THEOREM 8. *If r satisfies the conditions $\mathscr{A}0)$–$\mathscr{A}2)$ above, then the ERF $\mathscr{X}^r$ has Chentsov type representation,*

$$(1.r) \qquad\qquad X^r(p) = Y(S_p) ,$$

where the set S_p is the same as in the case of SERF, and the corresponding measure μ_r is

$$(27) \qquad d\mu_r(X) = \left(\int_{0+}^{1} \frac{x_0^{\alpha-n-1}}{M(\alpha)} m^2(\alpha) d\alpha \right) dx_0 d\underline{x} .$$

N. B.   The point mass at 1 corresponds to the original Chentsov representation.

2-3. *Remark.* In the remainder case of SERF of index α, $1 < \alpha < 2$, the covariance function σ is positive definite but $|p - q|^\alpha$ is not a metric function. It is impossible to get Chentsov type representation. However we can get another type of representation—in the form of the Wiener integral.

Let $\mathscr{Y}$ be the Gaussian random measure associated with (R^n, dx) and $I(\cdot)$ be the Wiener integral with respect to $\mathscr{Y}$.

THEOREM 9. *Set $f(x) \equiv |x|^{(\alpha-n)/2}$, for $\alpha \neq n$, and $\log|x|$ for $\alpha = n$, and set $F(t, x) \equiv f(x) - f(x - t)$, $x, t \in R^n$. Then for any α, $0 < \alpha < 2$,*
  1) *$F(t, x) \in L^2(R^n, dx)$, for any fixed t,*
  2) *$X^\alpha(t) \equiv I(F(t, \cdot))$ is proportional to the SERF of index α.*

The proofs are quite similar to that of Theorem 4.

§3. Invariance properties and representation

The SERF $\mathscr{V}^\alpha$ has the following invariance properties:

Set

$$(28) \qquad Q_g(p) \equiv V^\alpha(gp) - V^\alpha(gO) , \qquad g \in M(n) ,$$

and

(29) $Z^u(p) \equiv e^{-\alpha u/2} V^\alpha(e^u p) , \qquad u \in \mathbf{R}^1 ,$

then both Q_g and Z^u are again SERFs of the same index α. It is obvious that the actions of $M(n)$ and the multiplications related to Z^u form a group G.

Let us consider the relation between this invariance properties under G and the representation of the Brownian motion. To simplify our discussion, we modify our representation of SERF as follows; set

(30) $\tilde{V}^\alpha(S_p) \equiv \lim_{R\to\infty} \{ Y(\tilde{S}_p \cap D_R) - Y(\tilde{S}_o \cap D_R) \}$, with $D_R = \{ x = (x_0, \underline{x}); x_0 \leqq R \}$,

and write the left hand side as

(31) $\tilde{Y}(F_p(x))$, where $F_p(x) = \chi_{\tilde{S}_p}(x) - \chi_{\tilde{S}_o}(x) .$

N.B. In this section we consider invariance properties. If parallel transforms are considered, it is natural to treat stationary fields. But such a field $Y(S_p)$ does not exist. So we consider $\tilde{Y}(F_p(x))$ as a substitute of $Y(S_p)$. Note that $F_{gp}(x) \neq F_p(g^{-1}x)$ for $g \in M(n)$, $g \neq$ id.

The fields $\tilde{\mathscr{V}}^\alpha$ and $\mathscr{V}^\alpha$ has the same covariance. However, from the view point of group action, $\tilde{\mathscr{V}}^\alpha$ behaves more naturally than $\mathscr{V}^\alpha$. In the rest of this paper, we use the modified field $\tilde{\mathscr{V}}^\alpha$ instead of $\mathscr{V}^\alpha$. So in the definition (28) and (29), V^α has to be changed to $\tilde{V}^\alpha$.

Let us fix α between 0 and 1. It is obvious that the group G is the subgroup of continuous point transforms of $\mathbf{R}_+ \times \mathbf{R}^n$ which act as automorphisms on the family of subsets $\{\tilde{S}_p; p \in \mathbf{R}^n\}$. Any element $\bar{g} \in G$ is written as $\bar{g} = (g, u)$, where g is the part of solid motion and u is the dilation part of $\bar{g}$. The action of $\bar{g}$ on the point $x \in \mathbf{R}_+ \times \mathbf{R}^n$ is defined as

$$\bar{g}x = (e^u x_0, e^u g\underline{x}) .$$

Then we have;

THEOREM 10.

$$Q_g(p) = \tilde{Y}(F_{gp}(x)) = \tilde{Y}(F_p(g^{-1}x)) - \tilde{Y}(F_{go}(x))$$

and

$$Z^u(p) = e^{-\alpha u/2} \tilde{Y}(F_p(e^{-u}x)) , \quad a.a. \ \omega .$$

The theorem above describes the invariance properties of the SERF from the viewpoint of our representation and tells us why we have introduced $\tilde{Y}$.

ADDEMDUM. SERF is called fractional Brownian motion and plays a central roll in the theory of fractals. The representation treated in 2-3 and its application were appeared in the paper of Mandelbrot [11].

REFERENCES

[1] Chentsov, N. N., Lévy's Brownian motion of several parameters and generalized white noise, Theory Probab. Appl., **2** (1957), 265–266.

[2] Deans, S. R., A unified Radon inversion formula, J. Math. Phys., **19** (1978), 2346–2349.

[3] Gangolli, R., Positive definite kernels on homogeneous spaces and certain stochastic processes related to Lévy's Brownian motion of several parameters, Ann. Inst. H. Poincaré Sect. B, **3** (1967), 121–226.

[4] Inoue, K. and Noda, A., Independence of the increments of Gaussian random fields, Nagoya Math. J., **85** (1982), 251–268.

[5] Lévy, P., Processus stochastiques et mouvement brownien, Gauthier-Villars (1965).

[6] Lifshits, M. A., On representation of Lévy's field by indicators, Theory Probab. Appl., **24** (1979), 629–633.

[7] Molcan, G. M., Markov property of Lévy's fields on spaces of constant curvature, Dokl. Akad. Nauk CCCP, tom. **221** (1975), 1276–1279.

[8] Noda, A., Integral transformation, associated with Lévy's Brownian motion, Tech. Rep. Univ. Erlangen (1984).

[9] Pflug, G., A statistically important Gaussian process, Stochastic Process. Appl., **13** (1982), 45–57.

[10] Takenaka, S., Kubo, I. and Urakawa, H., Brownian motion parametrized with metric space of constant curvature, Nagoya Math. J., **82** (1981), 131–140.

[11] Mandelbrot, B., Fonctions aléatoires plurei-temporelles; approximation poissonienne de cas brownien et généralisations, C.R. Acad. Sc. Paris, **280A** (1975), 1075–1078.

[12] Noda, A., Generalized Radon transform and Lévy's Brownian motion I, II, Nagoya Math. J., **105** (1987), 71–87, 89–107.

[13] Ossiander, M. and Pyke, R., Lévy's Brownian motion as a set-indexed process and a related central limit theorem, Stochastic Process. Appl., **21** (1985), 133–145.

Department of Mathematics
Nagoya University
Chikusa-ku, Nagoya 464
Japan