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# AN EXTENSION OF ITO'S DIFFERENTIATION FORMULA

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INTRODUCTION 1. If $L_t^a$ denotes the local time of a continuous semimartingale $X$ at $a$ Bouleau and Yor [1] have obtained a form of Ito's differentiation formula which states that for absolutely continuous functions $F(x)$

$$F(X_t) = F(X_0) + \int_0^t \frac{\partial F}{\partial x}(X_s)dX_s - \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial F}{\partial x}(a)d_a L_t^a . \tag{1}$$

In [5] Yor uses this expression to discuss the approximations obtained by Yamada [4] to 'zero energy' processes. This article extends these ideas to suitable functions of the form $F(t, x)$. In fact, for a continuous semimartingale $X_t$, $t \geq 0$, with local time $L_t^a$ at $a$, (which may be taken to be jointly right continuous in $a$ and $t$, left limited in $a$ and continuous in $t$), and a function $F$ which is $C^1$ in $t$, and for which $F(t, x)$ and $(\partial F/\partial t)(t, x)$ are absolutely continuous in $x$, with bounded derivatives, the following differentiation formula holds:

$$\begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t \frac{\partial F}{\partial t}(s, X_s)ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s)dX_s \\ &\quad - \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial F}{\partial x}(t, a)d_a L_t^a + \frac{1}{2}\int_0^t \int_{-\infty}^{\infty} \frac{\partial^2 F}{\partial t \partial x}(s, a)d_a L_s^a \, ds . \end{aligned} \tag{2}$$

An advantage of this expression is that only differentiability to the first order in $x$ is required.

ASSUMPTIONS 2. In the sequel $X$ will denote a real, continuous semimartingale $\{X_t, t \geq 0\}$ defined on a filtered probability space $(\Omega, \underline{F}, \underline{\underline{F}}, P)$ which satisfies the usual conditions. Write $T_n = \inf(t: |X_t| \geq n)$. By localizing, that is by considering $X^{T_n}$, we can suppose that $X$ is bounded. We shall take the version of the local time $L_t^a$ with the above continuity properties in $a$ and $t$.

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*Remarks* 3. A key step in formulae (1) and (2) is the definition of the integrals with respect to $d_a L_t^a$ for fixed $t \geq 0$. Recall Tanaka's formula for the local time at $a$:

$$(X_t - a)^- = (X_0 - a)^- - \int_0^t I_{X_s \leq a} dX_s + \frac{1}{2} L_t^a . \tag{3}$$

By initially considering step functions of the form

$$f(u) = \sum_{i=1}^n f_i I_{]a_i, a_{i+1}]}(u) ,$$

and linear combinations of expression (3), Bouleau and Yor [1] show that if $F(x) = \int_0^x f(u) du$ then

$$F(X_t) = F(X_0) + \int_0^t f(X_s) dX_s - \frac{1}{2} \int_{-\infty}^{\infty} f(a) d_a L_t^a , \tag{4}$$

where the last integral is the sum

$$\sum_{i=1}^n f_i (L_t^{a_{i+1}} - L_t^{a_i}) .$$

It is shown this map can be extended to a vector measure on the Borel field of $R$ with values in $L^2(\underline{F}, P)$, so that if $f: R \rightarrow R$ is a locally bounded Borel measurable function and $F(x) = \int_0^x f(u) du$ then $F(X_t)$ is given by (4). Indeed, if $F(x)$ is any absolutely continuous function with a locally bounded derivative then $F(X_t)$ is given by (4), because, writing $G(x) = F(x) - F(0) = \int_0^t (\partial F / \partial x)(u) du$, the result is valid for $G(X_t)$.

LEMMA 4. *Suppose $f: R \rightarrow R$ is $C^1$. Then for any $t$:*

$$\int_{-\infty}^{\infty} f(a) d_a L_t^a = - \int_0^t \frac{\partial f}{\partial x}(X_s) d\langle X, X \rangle_s$$
$$= - \int_{-\infty}^{\infty} \frac{\partial f}{\partial x}(a) L_t^a da .$$

*Proof.* Write $F(x) = \int_0^x f(u) du$. Then applying the Ito differentiation formula to $F(X_t)$:

$$F(X_t) = F(X_0) + \int_0^t f(X_s) dX_s + \frac{1}{2} \int_0^t \frac{\partial f}{\partial x}(X_s) d\langle X, X \rangle_s . \tag{5}$$

Equating the final terms of (4) and (5) the result follows. However, we also have from [2], p. 368, that

$$\int_0^t \frac{\partial f}{\partial x}(X_s)d\langle X, X\rangle_s = \int_{-\infty}^{\infty} \frac{\partial f}{\partial x}(a)L_t^a da .$$

*Remark* 5. For absolutely continuous $f$

$$\int_{-\infty}^{\infty} f(a)d_a L_t^a = -\int_{-\infty}^{\infty} \frac{\partial f}{\partial x}(a)L_t^a da ,$$

and treating the $t$ in the function as a constant, we also have for functions $f(t, x)$ which are absolutely continuous in $x$,

$$\int_{-\infty}^{\infty} f(t, a)d_a L_t^a = -\int_{-\infty}^{\infty} \frac{\partial f}{\partial x}(t, a)L_t^a da .$$

The generalized differentiation formula is first established for a suitably smooth function $f(t, x)$.

THEOREM 6. *Suppose, for $(t, x) \in [0, \infty)\times R$, $F(t, x) \in R$ is continuously differentiable in $t$ and twice continuously differentiable in $x$. Then*

$$(6)\qquad \begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t \frac{\partial F}{\partial t}(s, X_s)ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s)dX_s \\ &\quad - \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial F}{\partial x}(t, a)d_a L_t^a + \frac{1}{2}\int_0^t\int_{-\infty}^{\infty} \frac{\partial^2 F}{\partial t \partial x}(s, a)d_a L_s^a\, ds . \end{aligned}$$

*Proof.* By Ito's differentiation formula:

$$(7)\qquad \begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t \frac{\partial F}{\partial t}(s, X_s)ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s)dX_s \\ &\quad + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s)d\langle X, X\rangle_s . \end{aligned}$$

Recall we are taking $X = X^{T_n}$ so $(\partial^2 F/\partial x^2)(s, X_s)$ is continuous and bounded for $s \leq t$. Again from [2], p. 368,

$$\int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s)\, d\langle X, X\rangle_s = \int_{-\infty}^{\infty}\int_0^t \frac{\partial^2 F}{\partial x^2}(s, a)d_s L_s^a\, da .$$

Integrating the inner integral by parts in $s$ this is

$$= \int_{-\infty}^{\infty}\left(L_t^a \frac{\partial^2 F}{\partial x^2}(t, a) - \int_0^t L_s^a \frac{\partial^3 F}{\partial t \partial x^2}(s, a)ds\right) da .$$

Using Fubini's Theorem to interchange the order of integration, ($L^a$ has compact support), and then integrating by parts in $a$ this equals:

$$-\int_{-\infty}^{\infty}\frac{\partial F}{\partial x}(t, a)d_aL_t^a + \int_0^t\int_{-\infty}^{\infty}\frac{\partial^2 F}{\partial t\partial x}(s, a)d_aL_s^a\,ds\,.$$

Substituting in (7) the result follows.

*Remarks* 7. When $X$ is Brownian motion Perkins, [3], has shown that $L_t^a$ is a semimartingale in $a$ for each $t \in [0, \infty)$. Yor, [5], has pointed out, using the monotone class theorem, that the integral with respect to $d_aL_t^a$ then equals the stochastic integral in $a$. The advantage of the differentiation formula in the form given by Theorem 6 is that, as stated, it requires only differentiability of order one in $x$. Following the usual mollifier techniques we show that the result holds under a weaker differentiability hypothesis.

COROLLARY 8. *Suppose that $F(t, x)$ is continuously differentiable in $t$ and absolutely continuous in $x$ with a locally bounded derivative $\partial F/\partial x$. Furthermore, suppose that $F(t, 0) = 0$ so that for all $t \geq 0$*

$$F(t, x) = \int_0^x \frac{\partial F}{\partial x}(t, y)dy\,.$$

*Similarly, suppose that for all $t \geq 0$*

$$\frac{\partial F}{\partial t}(t, x) = \int_0^x \frac{\partial^2 F}{\partial t\partial x}(t, y)dy$$

*where $\partial^2 F/\partial t\partial x$ is locally bounded. Then $F(t, x_t)$ is given by the differentiation formula* (6) *of Theorem* 6.

*Proof.* Write $f(t, y) = (\partial F/\partial x)(t, y)$. Suppose $g \in C_0^\infty(R)$ is such that $\int g(x)dx = 1$, and for each integer $n > 0$ put

$$\begin{aligned} F_n(t, x) &= n\int F(t, x - y)g(ny)dy \\ &= n\int F(t, y)g(n(x - y))dy\,. \end{aligned}$$

Then

$$\frac{\partial F_n}{\partial x}(t, x) = n\int f(t, x - y)g(ny)dy\,,$$

and

$$\frac{\partial^2 F_n}{\partial t \partial x}(t, x) = n \int \frac{\partial f}{\partial t}(t, x-y) g(ny) dy .$$

As $n \to \infty$, $\lim F_n(t, x) = F(t, x)$,

$$\lim \frac{\partial F_n}{\partial t}(t, x) = \frac{\partial F}{\partial t}(t, x) ,$$

$$\lim \frac{\partial F_n}{\partial x}(t, x) = f(t, x) \qquad \text{a.e.} ,$$

and

$$\lim \frac{\partial^2 F_n}{\partial t \partial x}(t, x) = \frac{\partial f}{\partial t}(t, x) \qquad \text{a.e.} .$$

Applying Theorem 6 to $F_n(t, x)$

$$F_n(t, X_t) = F_n(0, X_0) + \int_0^t \frac{\partial F_n}{\partial t}(s, X_s) ds + \int_0^t \frac{\partial F_n}{\partial x}(s, X_s) dX_s$$
$$- \frac{1}{2} \int_{-\infty}^{\infty} \frac{\partial F_n}{\partial x}(t, a) d_a L_t^a + \frac{1}{2} \int_0^t \left( \int_{-\infty}^{\infty} \frac{\partial^2 F_n}{\partial t dx}(s, a) d_a L_s^a \right) ds .$$

Letting $n \to \infty$ we have

$$F(t, X_t) = F(0, X_0) + \int_0^t \frac{\partial F}{\partial t}(s, X_s) ds + \int_0^t f(s, X_s)_s dX_s$$
$$- \frac{1}{2} \int_{-\infty}^{\infty} f(t, a) d_a L_t^a + \frac{1}{2} \int_0^t \left( \int_{-\infty}^{\infty} \frac{\partial f}{\partial t}(s, a) d_a L_s^a \right) ds .$$

*Remarks* 9. This corollary holds without the hypothesis that $F(t, 0) = 0$; suppose $F(t, x)$ satisfies the hypotheses of the corollary except possibly the condition $F(t, 0) = 0$. Then $G(t, x) = F(t, x) - F(t, 0)$ satisfies all the hypotheses, and so the result holds for $G$. However,

$$\frac{\partial G}{\partial t}(t, x) = \frac{\partial F}{\partial t}(t, x) - \frac{\partial F}{\partial t}(t, 0) ,$$

and the integral in $s$ then contributes an additional quantity

$$\int_0^t - \frac{\partial F}{\partial t}(s, 0) ds = F(0, 0) - F(t, 0) ,$$

so cancelling the extra terms.

The next result extends some formulae of Yamada [4], and Proposition 3.1 of Yor [5]. First we give a definition.

Suppose $B_t$, $t \geq 0$ is a standard Brownian motion and $F(t, x)$ is such that it is $C^1$ in $t$ and $\partial F/\partial x$ exists and belongs to $L^2_{\text{loc}}([0, \infty) \times R)$. Then the second derivative $\partial^2 F/\partial x^2$ exists in the sense of distribution theory.

DEFINITION 10. The process

$$A_t^F = \int_0^t \frac{\partial^2 F}{\partial x^2}(s, B_s)ds$$

is defined to be

$$2\left(F(t, B_t) - F(0, 0) - \int_0^t \frac{\partial F}{\partial x}(s, B_s)dB_s - \int_0^t \frac{\partial F}{\partial t}(s, B_s)ds\right).$$

THEOREM 11. *Suppose for* $(t, x) \in [0, \infty) \times R$ $F(t, x)$ *is continously differentiable in* $t$ *and twice continuously differentiable in* $x$ *outside the origin.*

*Write* $(\partial F/\partial x)(t, x) = f(t, x)$ *and, for some* $T > 0$, *suppose that*

$$f^*(x) = \sup_{t \leq T} |f(t, x)| \in L^2_{\text{loc}}(R)$$

*and*

$$\frac{\partial f^*}{\partial t}(x) = \sup_{t \leq T} \left|\frac{\partial f}{\partial t}(t, x)\right| \in L^1_{\text{loc}}(R).$$

*Then for all* $p \in [1, \infty)$

$$\lim_{\varepsilon \to 0} E\left[\sup_{t \leq T} \left|A_t^F - \left\{\int_0^t \frac{\partial^2 F}{\partial x^2}(s, B_s)I_{|B_s| \geq \varepsilon}ds + \int_0^t f(s, \varepsilon)d_s L_s^{\varepsilon} - \int_0^t f(s, -\varepsilon)d_s L_s^{-\varepsilon}\right\}\right|^p\right] = 0.$$

*Proof.* Without loss of generality suppose that $F(t, 0) = 0$ so

$$F(t, x) = \int_0^x f(t, y)dy$$

and

$$\frac{\partial F}{\partial t}(t, x) = \int_0^x \frac{\partial f}{\partial t}(t, y)dy.$$

Write $f_\varepsilon(t, y) = f(t, y)I_{|y| \geq \varepsilon}$ and

$$F_\varepsilon(t, x) = \int_0^x f_\varepsilon(t, y)dy.$$

Then

$$\frac{\partial F_\varepsilon}{\partial t}(t, x) = \int_0^x \frac{\partial f_\varepsilon}{\partial t}(t, y)dy$$

and applying Corollary 8 to $F_\varepsilon$ with $X$ a standard Brownian motion $B$

$$F_\varepsilon(t, B_t) = \int_0^t f_\varepsilon(s, B_s)dB_s + \int_0^t \frac{\partial F_\varepsilon}{\partial t}(s, B_s)ds$$
$$- \frac{1}{2}\int_{-\infty}^{\infty} f_\varepsilon(t, a)d_a L_t^a + \frac{1}{2}\int_0^t\int_{-\infty}^{\infty} \frac{\partial f_\varepsilon}{\partial t}(s, a)d_a L_s^a\, ds$$

Writing

$$A_t^{F_\varepsilon} = -\int_{-\infty}^{\infty} f_\varepsilon(t, a)d_a L_t^a + \int_0^t\int_{-\infty}^{\infty} \frac{\partial f_\varepsilon}{\partial s}(s, a)d_a L_s^a\, ds$$

we have

$$A_t^{F_\varepsilon} = -\left(\int_\varepsilon^\infty + \int_{-\infty}^{-\varepsilon} f(t, a)d_a L_t^a\right) + \int_0^t\left(\int_\varepsilon^\infty + \int_{-\infty}^{-\varepsilon} \frac{\partial f}{\partial t}(s, a)d_a L_s^a\right)ds\,,$$

and by parts (in $a$) this is

$$= L_t^\varepsilon f(t, \varepsilon) - L_t^{-\varepsilon} f(t, -\varepsilon) + \left(\int_\varepsilon^\infty + \int_{-\infty}^{-\varepsilon} \frac{\partial^2 F}{\partial x^2}(t, a)L_t^a da\right)$$
$$- \int_0^t\left(L_s^\varepsilon \frac{\partial f}{\partial t}(s, \varepsilon) - L_s^{-\varepsilon}\frac{\partial f}{\partial s}(s, -\varepsilon)\right)ds$$
$$- \int_0^t\left(\int_\varepsilon^\infty + \int_{-\infty}^{-\varepsilon} \frac{\partial^2 f}{\partial t\partial x}(s, a)L_s^a da\right)ds\,.$$

Applying Fubini's Theorem to the final term and integrating by parts in $s$

$$\left(\int_\varepsilon^\infty + \int_{-\infty}^{-\varepsilon}\right)\left(\int_0^t \frac{\partial^2 f}{\partial t dx}(s, a)L_s^a ds\right)da$$
$$= \left(\int_\varepsilon^\infty + \int_{-\infty}^{-\varepsilon}\right)\left(L_t^a \frac{\partial^2 F}{\partial x^2}(t, a) - \int_0^t \frac{\partial^2 F}{\partial x^2}(s, a)d_s L_s^a\right)da\,.$$

Therefore,

$$A_t^{F_\varepsilon} = -\int_{-\infty}^{\infty} f_\varepsilon(t, a)d_a L_t^a + \int_0^t\left(\int_{-\infty}^{\infty} \frac{\partial f_\varepsilon}{\partial t}(s, a)d_a L_s^a\right)ds$$
$$= L_t^\varepsilon f(t, \varepsilon) - L_t^{-\varepsilon} f(t, -\varepsilon) - \int_0^t L_s^\varepsilon \frac{\partial f}{\partial t}(s, \varepsilon)ds$$
$$+ \int_0^t L_s^{-\varepsilon}\frac{\partial f}{\partial t}(s, -\varepsilon)ds + \int_0^t \frac{\partial^2 F}{\partial x^2}(s, B_s)I_{|B_s|\geq\varepsilon}ds$$
$$= \int_0^t f(s, \varepsilon)d_s L_s^\varepsilon - \int_0^t f(s, -\varepsilon)d_s L_s^{-\varepsilon} + \int_0^t \frac{\partial^2 F}{\partial x^2}(s, B_s)I_{|B_s|\geq\varepsilon}ds\,.$$

For the function $F(t, x)$ the process $A_t^F$ is defined by

$$A_t^F = 2\Big(F(t, B_t) - \int_0^t f(s, B_s)dB_s - \int_0^t \frac{\partial F}{\partial t}(s, B_s)ds\Big).$$

Therefore,

$$A_t^F - A_t^{F_\varepsilon} = 2\Big(\int_0^{B_t} f(t, y)I_{|y|\le\varepsilon}dy - \int_0^t f(s, B_s)I_{|B_s|\le\varepsilon}dB_s - \int_0^t\int_0^{B_s} \frac{\partial f}{\partial t}(s, y)I_{|y|\le\varepsilon}dyds\Big),$$

and for $p \in [1, \infty)$, $T > 0$,

$$E[\sup_{t\le T}|A_t^F - A_t^{F_\varepsilon}|^p] \le \text{Const}\, E\Big[\sup_{t\le T}\Big|\int_0^{B_t} f(t, y)I_{|y|\le\varepsilon}dy\Big|^p + \sup_{t\le T}\Big|\int_0^t f(s, B_s)I_{|B_s|\le\varepsilon}dB_s\Big|^p + \sup_{t\le T}\Big|\int_0^t\Big(\int_0^{B_s}\frac{\partial f}{\partial t}(s, y)I_{|y|\le\varepsilon}dy\Big)ds\Big|^p\Big].$$

Denote the three terms in the expectation by $I^{(1)}$, $I^{(2)}$ and $I^{(3)}$, respectively.

Then

$$E[I^{(1)}] \le E\Big[\sup_{t\le T}\Big(\int_{-\varepsilon}^{\varepsilon}|f(t, y)|dy\Big)^p\Big] \le \Big(\int_{-\varepsilon}^{\varepsilon} f^*(y)dy\Big)^p,$$

and this converges to 0 as $\varepsilon \to 0$.

$$\begin{aligned} E[I^{(2)}] &\le C_p E\Big[\Big|\int_0^T f(s, B_s)I_{|B_s|\le\varepsilon}dB_s\Big|^p\Big] \le \text{Const}\, E\Big(\int_0^T f^2(s, B_s)I_{|B_s|\le\varepsilon}ds\Big)^{p/2} \\ &= \text{Const}\, E\Big(\int_{-\infty}^{\infty}\Big(\int_0^T f^2(s, a)I_{|a|\le\varepsilon}d_sL_s^a\Big)da\Big)^{p/2} \\ &\le \text{Const}\, E\Big(\int_{-\varepsilon}^{\varepsilon} f^*(a)^2L_T^a da\Big)^{p/2} \le \text{Const}\Big(E(L_T^*)^{p/2}\Big)\Big(\int_{-\varepsilon}^{\varepsilon} f^*(a)^2da\Big)^{p/2}, \end{aligned}$$

which again converges to 0 as $\varepsilon \to 0$.

Finally,

$$E[I^{(3)}] \le E\Big[\sup_{t\le T}\Big|\int_0^t\int_{-\varepsilon}^{\varepsilon}\Big|\frac{\partial f}{\partial t}(s, y)\Big|dyds\Big|^p\Big] \le T^p\Big(\int_{-\varepsilon}^{\varepsilon}\frac{\partial f^*}{\partial t}(y)dy\Big)^p,$$

which converges to 0 as $\varepsilon \to 0$, so the result is proved.

EXAMPLES 12. Suppose $B_t$, $t \ge 0$, is a standard Brownian motion.

1) Taking $F(t, B_t) = \exp(\lambda B_t - \lambda^2 t/2)$, for $\lambda \in R$, from the identity obtained in Theorem 6

$$\int_0^t \lambda \exp(\lambda B_s - \lambda^2 s/2) ds$$
$$= - e^{-\lambda^2 t/2} \int_{-\infty}^{\infty} e^{\lambda a} d_a L_t^a - \lambda^2/2 \int_0^t e^{-\lambda^2 s/2} \left( \int_{-\infty}^{\infty} e^{\lambda a} d_a L_s^a \, ds \right).$$

2) With $F(t, x) = \begin{cases} \phi(t)(x \log x - x) & \text{for } x > 0 \\ 0 & \text{for } x \leq 0, \end{cases}$

where $\phi$ is $C^1$ in $t$

$$\frac{\partial^2 F}{\partial^2 x}(t, x) = \phi(t)/x \qquad \text{for } x > 0$$

and Theorem 11 implies that in $L^p$, $p \in [1, \infty)$,

$$A_t^F = \text{Principal value of } \int_0^t \frac{\phi(s)}{(B_s)_+} ds$$
$$= \lim_{\varepsilon \to 0} \left\{ \int_0^t \frac{\phi(s)}{B_s} I_{B_s \geq \varepsilon} ds + \log \varepsilon \int_0^t \phi(s) d_s L_s^{\varepsilon} \right\}.$$

3) With $F(t, x) = \begin{cases} \phi(t)|x|^{\lambda+2}/(\lambda+1)(\lambda+2) & \text{for } x > 0 \\ 0 & \text{for } x \leq 0, \end{cases}$

where $-3/2 < \lambda < -1$ and $\phi$ is $C^1$ in $t$, we have from Theorem 11 that in $L^p$, $p \in [1, \infty)$,

$$A_t^F = \text{Finite part of } \int_0^t \phi(s)|B_s|^{\lambda} ds$$
$$= \lim_{\varepsilon \to 0} \left\{ \int_0^t \phi(s)|B_s|^{\lambda} I_{|B_s| \geq \varepsilon} ds + \frac{\varepsilon^{\lambda+1}}{(\lambda+1)} \int_0^t \phi(s) d_s L_s^{\varepsilon} - \frac{\varepsilon^{\lambda+1}}{(\lambda+1)} \int_0^t \phi(s) d_s L_s^{-\varepsilon} \right\}.$$

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## REFERENCES

[1] Bouleau, N. and Yor, M., Sur la variation quadratique des temps locaux de certaines semimartingales, C.R. Acad. Sci. Paris, **292** (1981), 491–494.

[2] Meyer, P. A., Un cours sur les integrales stochastiques, Sem de Probabilités X, Lec. Notes in Math., **511**, 245–400.

[3] Perkins, E., Local time is a semimartingale, Z. Wahrsch. Verw. Gebiete, **60** (1982), 79–117.

[ 4 ] Yamada, T., On some representations concerning stochastic integrals, to appear.
[ 5 ] Yor, M., Sur la transformation de Hilbert des temps locaux Browniens, et une extension de la Formule d'Ito, Sem de Probabilités XVI, Lec. Notes in Math., **920**, 238–247.

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