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A REMARK ON HOMOGENEOUS CONVEX DOMAINS

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§0. Introduction

In this note, by a homogeneous convex domain in $\boldsymbol{R}^n$ we mean a convex domain Ω in $\boldsymbol{R}^n$ containing no complete straight lines on which the group $G(\Omega)$ of all affine transformations of $\boldsymbol{R}^n$ leaving Ω invariant acts transitively. Let Ω be a homogeneous convex domain. Then Ω admits a $G(\Omega)$-invariant Riemannian metric which is called the canonical metric (see [11]). The domain Ω endowed with the canonical metric is a homogeneous Riemannian manifold and we denote by $I(\Omega)$ the group of all isometries of it. A homogeneous convex domain Ω is called reducible if there is a direct sum decomposition of the ambient space $\boldsymbol{R}^n = \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$, $n_i > 0$, such that $\Omega = \Omega_1 \times \Omega_2$ with Ω_i a homogeneous convex domain in $\boldsymbol{R}^{n_i}$; and if there is no such decomposition, then Ω is called irreducible.

The purpose of this note is to prove the following:

THEOREM. *Let M be a homogeneous Riemannian manifold whose universal covering is isometric to a homogeneous convex domain Ω in $\boldsymbol{R}^n$ endowed with the canonical metric. If Ω is irreducible and not affinely equivalent to a convex cone, then M is simply connected, that is, M itself is isometric to Ω.*

It is already known in [2] that an analogous fact holds for a homogeneous bounded domain in $\boldsymbol{C}^n$.

We prove the above theorem along the same line as in [2] by using results of Tsuji [9], [10].

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§1. The center of a group of affine automorphisms of Ω

First we discuss the connection between the irreducibilities of a homogeneous convex domain and the cone fitted onto it. For the purpose we

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need the notion of T-algebras. The details for it can be found in [11].

Let Ω be a homogeneous convex domain in $\boldsymbol{R}^n$ and V the cone fitted onto it, that is,

$$V = \{(\lambda x, \lambda) \in \boldsymbol{R}^n \times \boldsymbol{R} \mid x \in \Omega,\ \lambda > 0\}\ .$$

Note that V is a homogeneous convex cone in $\boldsymbol{R}^{n+1}$ (cf. the proof of Proposition 2 in this section). By a theorem of Vinberg [11], we may assume that $\Omega = \Omega(\mathfrak{A})$ and $V = V(\mathfrak{A})$, where $\mathfrak{A} = \sum_{1 \leqq i,j \leqq r} \mathfrak{A}_{ij}$ is a T-algebra of rank r ($r \geqq 2$) and the notations $V(\mathfrak{A})$ and $\Omega(\mathfrak{A})$ bear the same meanings as in [9], [10]. We put $\dim \mathfrak{A}_{ij} = n_{ij}$. A criterion for Ω and V to be irreducible can be given in terms of the T-algebra $\mathfrak{A}$ as follows:

(i) (Tsuji [10]) $\Omega = \Omega(\mathfrak{A})$ is irreducible if and only if, for every pair (i, j) of indices with $1 \leqq i \leqq j \leqq r-1$, there exists a series $i_0, i_1, \cdots, i_p$ of indices such that $1 \leqq i_\alpha \leqq r-1$ $(0 \leqq \alpha \leqq p)$, $i_0 = i$, $i_p = j$ and $n_{i_{\alpha-1} i_\alpha} \neq 0$ $(1 \leqq \alpha \leqq p)$.

(ii) (Asano [1]) $V = V(\mathfrak{A})$ is irreducible if and only if, for every pair (i, j) of indices with $1 \leqq i \leqq j \leqq r$, there exists a series $i_0, i_1, \cdots, i_p$ of indices such that $1 \leqq i_\alpha \leqq r$ $(0 \leqq \alpha \leqq p)$, $i_0 = i$, $i_p = j$ and $n_{i_{\alpha-1} i_\alpha} \neq 0$ $(1 \leqq \alpha \leqq p)$.

PROPOSITION 1. *In the above notation, if Ω is irreducible and not affinely equivalent to a convex cone, then V is irreducible.*

Proof. Since $\Omega = \Omega(\mathfrak{A})$ is not affinely equivalent to a convex cone by assumption, it follows from the definition of $\Omega(\mathfrak{A})$ that there exists an index i such that $1 \leqq i \leqq r-1$ and $n_{ir} \neq 0$. By (i) and (ii), this implies that V is irreducible. q.e.d.

Remark. If Ω is a convex cone, then the cone V fitted onto Ω is reducible. In fact, one has $V = \Omega \times \boldsymbol{R}_+$, where $\boldsymbol{R}_+$ denotes the cone of positive real numbers.

We fix notations. Let G be a group. For a subset H of G, $C_G(H)$ denotes the centralizer of H in G, and the center of G is denoted simply by $C(G)$. When G is a topological group, the connected component of G containing the identity element is denoted by G°. The unit element of a group is denoted by e. The identity matrix of degree n is denoted by 1_n. $A(n, \boldsymbol{R})$ denotes the group of all affine transformations of $\boldsymbol{R}^n$.

The aim of this section is to prove the following:

PROPOSITION 2. *Let Ω be an irreducible homogeneous convex domain* *in $\boldsymbol{R}^n$ which is not affinely equivalent to a convex cone. If a subgroup G of $G(\Omega)$ acts transitively on Ω, then one has $C_{A(n,\boldsymbol{R})}(G) = \{e\}$ and hence $C(G) = \{e\}$. In particular, one has $C(G(\Omega)) = C(G(\Omega)^\circ) = \{e\}$.*

For the proof, we need the following result:

(iii) (Rothaus [7]). Let V be an irreducible homogeneous convex cone in $\boldsymbol{R}^n$. If a subgroup G of $G(V)$ acts transitively on V, then one has $C_{GL(n,\boldsymbol{R})}(G) = \{\lambda 1_n \mid \lambda \in \boldsymbol{R}\}$.

Proof of Proposition 2. Let V be the cone fitted onto Ω. Let ρ denote the group homomorphism

$$A(n, \boldsymbol{R}) \ni a \longmapsto \begin{pmatrix} f(a) & q(a) \\ 0 & 1 \end{pmatrix} \in GL(n+1, \boldsymbol{R}) ,$$

where $f(a)$ and $q(a)$ denote, respectively, the linear and the translation parts of $a \in A(n, \boldsymbol{R})$. Then one has $\rho(G(\Omega)) \subset G(V)$. The pair (ρ, ι) of the group homomorphism $\rho\colon G(\Omega) \to G(V)$ and the natural embedding $\iota\colon \Omega \to V$ given by $\iota(x) = (x, 1)$ is equivariant, that is, $\iota(ax) = \rho(a)\iota(x)$ for all $a \in G(\Omega)$, $x \in \Omega$. Since G acts transitively on Ω by assumption, this shows that the subgroup $G' = \rho(G)\cdot\{\lambda 1_{n+1} \mid \lambda > 0\}$ of $G(V)$ acts transitively on V. By Proposition 1, V is an irreducible homogeneous convex cone in $\boldsymbol{R}^{n+1}$. Therefore, using (iii), we see $C_{GL(n+1,\boldsymbol{R})}(G') = \{\lambda 1_{n+1} \mid \lambda \in \boldsymbol{R}\}$. Let $a \in C_{A(n,\boldsymbol{R})}(G)$. Then one has $\rho(a) \in C_{GL(n+1,\boldsymbol{R})}(G')$. Hence $\rho(a)$ is a scalar matrix and this implies $a = e$ by the definition of ρ. q.e.d.

A homogeneous convex domain $\Omega(n)$ in $\boldsymbol{R}^n$ $(n \geqq 2)$ defined by

$$\Omega(n) = \{(x^1, \cdots, x^n) \in \boldsymbol{R}^n \mid x^1 > (x^2)^2 + \cdots + (x^n)^2\}$$

is called the elementary domain. Every elementary domain is irreducible and not affinely equivalent to a convex cone. The following result is known:

(iv) (Tsuji [9]). Let Ω be an irreducible homogeneous convex domain which is not affinely equivalent to the elementary domain. Then one has $I(\Omega)^\circ = G(\Omega)^\circ$.

Combining (iv) with Proposition 2, we obtain

LEMMA. *Let Ω be an irreducible homogeneous convex domain which is affinely equivalent to neither a convex cone nor the elementary domain. If a connected Lie subgroup G of $I(\Omega)$ acts transitively on Ω, then one has $C(G) = \{e\}$.*

Remark. The above lemma remains valid for the elementary domain (cf. the proof of our theorem in the next section and Corollary 2 in Section 3).

§ 2. Proof of Theorem

First, suppose Ω is affinely equivalent to the elementary domain $\Omega(n)$. Then, since $\Omega(n)$ endowed with the canonical metric is of negative sectional curvature (see, e.g., [8]), M is a connected homogeneous Riemannian manifold of negative sectional curvature. Hence our assertion follows from [6, Theorem 8.3, p. 105].

Next, suppose Ω is not affinely equivalent to the elementary domain. We set $G = I(M)^{\circ}$. Then one has a natural identification $M = G/K$, where K is an isotropy subgroup of G at some point of M. Let $\tilde{G}'$ be the universal covering group of G and let π be the covering projection of $\tilde{G}'$ onto G. Then one has $M \simeq \tilde{G}'/\pi^{-1}(K)$ and $\Omega \simeq \tilde{G}'/\tilde{K}'$, where $\tilde{K}' = \pi^{-1}(K)^{\circ}$. We put

$$\begin{aligned} \varDelta_0 &= \{g \in \tilde{G}' \mid g \cdot y = y \text{ for all } y \in \Omega\}\,, \\ \varDelta &= \{g \in \tilde{G}' \mid g \cdot x = x \text{ for all } x \in M\}\,. \end{aligned}$$

It follows that $\varDelta_0 \subset \varDelta$. We note that, since G acts effectively on M, $\varDelta$ is a discrete subgroup of $\tilde{G}'$. Put $\tilde{G}'/\varDelta_0 = \tilde{G}$ and $\tilde{K}'/\varDelta_0 = \tilde{K}$. Then $\tilde{G}$ is a connected Lie subgroup of $I(\Omega)$, and one has $\Omega \simeq \tilde{G}/\tilde{K}$. Moreover, one has the following commutative diagram:

$$\begin{array}{ccc} & \tilde{G}' & \\ \swarrow & & \searrow^{\pi} \\ \tilde{G} = \tilde{G}'/\varDelta_0 & \xrightarrow[\pi']{} & \tilde{G}'/\varDelta \simeq G\,. \end{array}$$

Since $\ker \pi' \subset C(\tilde{G})$, we see by the lemma in the previous section that π' is an isomorphism of $\tilde{G}$ onto G. Therefore $\pi'^{-1}(K)$ is compact, because so is K. It is easy to see $\tilde{K} = \pi'^{-1}(K)^{\circ}$, and hence $\tilde{K}$ is compact. Since Ω is a cell (see [11]) and since $\Omega \simeq \tilde{G}/\tilde{K}$, $\tilde{K}$ is a maximal compact subgroup of $\tilde{G}$. Therefore one has $\tilde{K} = \pi'^{-1}(K)$, and this implies $\Omega \simeq \tilde{G}/\tilde{K} \simeq G/K = M$. q.e.d.

§ 3. Corollaries and Remarks

An affine manifold M of dimension n is a manifold which admits an atlas $\{(U_\alpha, \phi_\alpha)\}$ such that each coordinate change $\phi_\alpha \circ \phi_\beta^{-1}$ is an affine transformation of $\boldsymbol{R}^n$ (cf. [5]). A diffeomorphism f of M is called an affine transformation of M if it is affine with respect to the atlas $\{(U_\alpha, \phi_\alpha)\}$, that is, if each transformation $\phi_\alpha \circ f \circ \phi_\beta^{-1}$ is an affine transformation of $\boldsymbol{R}^n$, and M is called homogeneous if the group $G(M)$ of all affine transformations of M acts transitively on it. Note that a domain Ω in $\boldsymbol{R}^n$ is naturally an affine manifold and the group $G(\Omega)$ defined in the introduction coincides with the one defined above.

COROLLARY 1. *Let M be a homogeneous affine manifold whose universal covering is affinely equivalent to a homogeneous convex domain Ω in $\boldsymbol{R}^n$. If Ω is irreducible and not affinely equivalent to a convex cone, then M is simply connected, that is, M itself is affinely equivalent to Ω.*

Proof. Let Γ be the covering transformation group of the covering $\Omega \to M$. By assumption, Γ is a subgroup of $G(\Omega)$, and hence the canonical metric of Ω is Γ-invariant. With respect to the induced Riemannian metric, M is a homogeneous Riemannian manifold. Indeed, since every element of $G(M)$ lifts to an element of $G(\Omega) \subset I(\Omega)$, $G(M)$ acts as an isometry group, and its action on M is transitive by assumption. Thus the theorem shows that M is simply connected. q.e.d.

COROLLARY 2. *Let Ω be an irreducible homogeneous convex domain which is not affinely equivalent to a convex cone. If a Lie subgroup G of $I(\Omega)$ acts transitively on Ω, then one has $C(G) = \{e\}$. In particular, one has $C(I(\Omega)) = \{e\}$.*

Proof. If Ω is affinely equivalent to the elementary domain, then this is a direct consequence of [6, Theorem 8.4, p. 107] (cf. Proof of Theorem). Otherwise, the proof goes as follows: Since $C(G) \subset C(\overline{G})$, where $\overline{G}$ is the closure of G in $I(\Omega)$, we may assume that G is a closed subgroup of $I(\Omega)$. The subgroup $C(G)$ of $I(\Omega)$ is discrete. Indeed, using the lemma in Section 1, we see $C(G)^\circ \subset C(G^\circ) = \{e\}$. The same reasoning as in the proof of [6, Theorem 8.4] yields that $C(G)$ acts properly discontinuously and freely on Ω and the quotient space $C(G)\backslash\Omega$ is a homogeneous Riemannian manifold with respect to the induced Riemannian metric. By the theorem, $C(G)\backslash\Omega$ is simply connected. Hence we conclude that $C(G) = \{e\}$. q.e.d.

Remark 1. In our theorem and Corollary 1, the assumption that Ω is not affinely equivalent to a convex cone can not be removed. Indeed, let Ω be a homogeneous convex cone in $\boldsymbol{R}^n$ and put $M = \Gamma\backslash\Omega$, where $\Gamma =$

$\{2^k 1_n \mid k \in Z\} \subset G(\Omega)$. Since $\Gamma \subset C(G(\Omega))$, the transitive action of $G(\Omega)$ on Ω induces a transitive action of $G(\Omega)$ on $M = \Gamma \backslash \Omega$ as an affine transformation group. This implies that M is a homogeneous affine manifold whose universal covering is affinely equivalent to Ω. Therefore M is also a homogeneous Riemannian manifold whose universal covering is isometric to Ω endowed with the canonical metric. However M is clearly not simply connected.

Remark 2. Consider the following problem:

Let M be an n-dimensional homogeneous affine manifold which is projectively hyperbolic in the sense of Kobayashi [4]. Then, is M a homogeneous convex domain in $\boldsymbol{R}^n$?

This is an affine analogue of Kobayashi's problem concerning homogeneous hyperbolic (complex) manifolds (cf. [3, Problem 12, p. 133]).

Since the intrinsic distance of M is complete, the universal covering of M is affinely equivalent to a convex domain in $\boldsymbol{R}^n$ containing no complete straight lines (see [5]). Therefore Corollary 1 shows that the answer to the above problem is affirmative when the universal covering Ω of M is irreducible (note that Ω is necessarily homogeneous) and not affinely equivalent to a convex cone.

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