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LIOUVILLIAN SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATION WITHOUT FUCHSIAN SINGULARITIES

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§ 0. Introduction

Consider a homogeneous linear differential equation of the second order whose coefficients are rational functions of the independent variable x over the field C of complex numbers. We assume that the coefficient of the first order derivative vanishes:

$$y'' + s(x)y = 0, \quad s \in C(x), \quad s' \neq 0.$$

THEOREM. *Suppose that none of the singular points is Fuchsian. Then it is reducible over $C(x)$ if it has a liouvillian solution.*

The condition that *none* of the singular points is Fuchsian can not be removed. The assumption that the coefficient of the first order derivative vanishes can not also be removed (cf. § 2).

Our proof is based on the discussions of Kaplansky's book [2] in Picard-Vessiot theory. The coefficient field C can be replaced by any algebraically closed field of constants of characteristic 0. The existence of Picard-Vessiot extension was proved by Kolchin [3].

If $s(x)$ is a polynomial the only singular point is the infinity and it is not Fuchsian. In this case our result was obtained by H. P. Rehm [6]. For liouvillian solutions of this equation we have several results due to R. R. Hailperin (formerly R. M. Roberts) [1] and H. P. Rehm himself.

§ 1. Proof of Theorem

The infinity is a singular point, since the coefficient of the first order derivative vanishes. It is not Fuchsian by our assumption. Hence, if the equation has an algebraic solution it is reducible over $C(x)$ because the conjugate elements over $C(x)$ are solutions of our equation. Therefore, we may assume that there is no algebraic solution. Suppose that our equation,

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having a liouvillian solution, is irreducible over $C(x)$. Then there is a rational function $a(x)$ over C which satisfies

$$(1) \qquad a'' = 3aa' + 2s' - 4as - a^3 .$$

This is due to Kaplansky [2, §25]. It takes the form:

$$a(x) = \Sigma \frac{e}{x - c} \neq 0 , \quad c, e \in C , \quad (e \neq 0)$$

due to Liouville [4] (cf. M. Matsuda [5, §13]). Let us express $s(x)$ as the sum of partial fractions over C:

$$(2) \qquad s(x) = P(x) + \Sigma \frac{e'}{x - c'} + \Sigma \frac{e''}{(x - c')^2} + \cdots , \qquad P \in C[x] .$$

Then either $P \neq 0$ or $P = 0$ and $\Sigma e' \neq 0$, since the infinity is not Fuchsian. First suppose that c is not a singular point, that is $c \neq c'$ for any c'. Comparing the coefficients of $(x - c)^{-3}$ in (1) we have

$$2e = - 3e^2 - e^3 ,$$

and e is either -1 or -2. Secondly suppose that c is a singular point, that is $c = c'$ for some c'. Since it is not Fuchsian, there is a term in (2):

$$\frac{g}{(x - c)^n} , \quad g = e^{(n)} \neq 0 , \quad n > 2$$

such that

$$e^{(j)} = 0 , \qquad j > n .$$

Comparing the coefficients of $(x - c)^{-n-1}$ in (1) we have

$$0 = g(- 2n - 4e) ,$$

and $e = - n/2$. If $P \neq 0$ we indicate $\deg P$ and its leading coefficient by m and b respectively. Let us multiply both sides of (1) by x^{1-m} and set $x = \infty$. Then we have

$$0 = b(2m - 4\Sigma e) ,$$

but it is impossible because each e is negative. If $P = 0$ we indicate $\Sigma e'$ by A, which does not vanish. Let us multiply both sides of (1) by x^2 and set $x = \infty$. Then we have

$$0 = A(- 2 - 4\Sigma e) ,$$

but it is impossible because each e is not greater than -1.

§ 2. Counter example

Consider a differential equation:

$$x^2y'' - \left[x^n + \frac{1}{4}\left(\frac{n^2}{4} - 1\right)\right]y = 0,$$

where n is an odd natural number. The origin is a Fuchsian singular point and the infinity is not Fuchsian. We have a liouvillian solution:

$$y = x^{(1-n/2)/2} \exp\left(\pm \frac{2}{n} x^{n/2}\right).$$

We shall show that the equation is irreducible over $C(x)$. To the contrary suppose that it is reducible. Then there is a rational function $v(x)$ over C which satisfies

$$v' + v^2 = -s = x^{-2}\left[x^n + \frac{1}{4}\left(\frac{n^2}{4} - 1\right)\right].$$

It takes the form:

$$v = R + \frac{Q'}{Q} + \frac{d}{x}, \quad Q, R \in C[x], \quad d \in C,$$

since at any pole c of $v(x)$ distinct from 0 and ∞ it is expressed as

$$v(x) = \frac{1}{x-c} + t(x)$$

such that c is not a pole of $t(x)$. We have

$$v' + v^2 = R^2 + R' + \frac{Q''}{Q} + \frac{2RQ'}{Q} + \frac{2d}{x}\left(R + \frac{Q'}{Q}\right) + \frac{d^2 - d}{x^2}.$$

The order of infinity is even and non-negative if $R \neq 0$, and less than -1 if $R = 0$. This contradicts our assumption that n is odd.

Consider a differential equation:

$$x^2z'' + z' - \left[x^n + \frac{1}{4}\left(\frac{n^2}{4} - 1\right) - \frac{1}{4}\frac{1}{x^2} + \frac{1}{x}\right]z = 0,$$

where n is an odd natural number. The origin and the infinity are the only singular points and they are not Fuchsian. If we change the dependent variable z into y by

$$z = y \exp \frac{1}{2x}, \tag{3}$$

the above equation is obtained for y. Hence, there is a liouvillian solution:

$$z = x^{(1-n/2)/2} \exp\left(\frac{1}{2x} \pm \frac{2}{n} x^{n/2}\right).$$

The equation for z is irreducible, since

$$\frac{z'}{z} = -\frac{1}{2} x^{-2} + \frac{y'}{y}$$

by (3) and the equation for y is irreducible.

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