

ON THE POWER SERIES REPRESENTATION OF SMOOTH CONFORMAL MARTINGALES

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We introduce here the notion of (stochastically) differentiable process with respect to a fixed conformal martingale and compute the remainder term of the Taylor expansion of the given process (Definition 1 and Proposition 3). An a-priori estimate in the L^2 -norm of the above mentioned remainder term is given and consequently a power series representation of smooth conformal martingales is obtained (Theorem 4).

The results in this note on the one hand extend those of our previous works on smooth martingales with respect to the stopped complex Brownian motion (see [3]) and on the other hand can be applied to approximate the solutions of complex Ito-diffusion equations (see Applications). Furthermore they give a positive answer to an open question raised by M. Yor (see [5]).

Power Series representation of smooth conformal martingales with respect to a fixed martingale

For the convenience of the reader let us recall briefly the notion of conformal martingales and their associated Ito formula. We refer to the original work of Gettoor and Sharpe for the complete literature of the theory of conformal martingales (see [2]). By probability system we always mean a collection $\Sigma = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ which consists of a probability space (Ω, \mathcal{F}, P) and an increasing family $(\mathcal{F}_t)_{t \geq 0}$ of sub σ -fields of \mathcal{F} with the following standard hypotheses:

- $(\mathcal{F}_t)_{t \geq 0}$ is right-continuous,
- \mathcal{F}_0 contains all null sets of \mathcal{F} .

We always denote by \mathcal{M}_c the class of continuous complex local martingales for Σ . Thus $W \in \mathcal{M}_c$ iff its real and imaginary parts are continuous

local martingales. In other words,

$$(1) \quad W = W_0 + U + iV \text{ where } W_0 \text{ is an } \mathcal{F}_0\text{-measurable complex random variable and where } U, V \text{ belong to the class } \mathcal{L}_c \text{ of continuous, real local martingales which are null at } t = 0.$$

Let \mathcal{A}_c be the class of continuous, real processes for Σ which are null at $t = 0$ and which are locally of integrable total variations.

For $W_1, W_2 \in \mathcal{M}_c$ the process $\langle W_1, W_2 \rangle$, defined by complex polarization, is the unique member of $\mathcal{A}_c + i\mathcal{A}_c$ such that

$$(2) \quad W_1 \cdot \overline{W_2} - \langle W_1, W_2 \rangle \text{ is an element of } \mathcal{M}_c.$$

One finds that $\langle W_1, W_2 \rangle = \langle W_1 - W_1(0), W_2 - W_2(0) \rangle$ and that $\langle W_1, W_2 \rangle$ is linear in W_1 , conjugate linear in W_2 . It follows at once from (2) that for an element $W = W_0 + U + iV \in \mathcal{M}_c$ we have:

$$(3) \quad \begin{cases} \langle W, W \rangle = \langle U, U \rangle + \langle V, V \rangle & \text{and,} \\ \langle W, \overline{W} \rangle = \langle U, U \rangle - \langle V, V \rangle + 2i\langle U, V \rangle \end{cases}$$

In the following we denote also by $\langle W \rangle$ (resp. $\langle U \rangle$) the process $\langle W, W \rangle$ (resp. $\langle U, U \rangle$).

DEFINITION A (see [2], (5.1)). An element $W = W_0 + U + iV$ of \mathcal{M}_c is called *conformal* provided that:

$$(4) \quad \langle U, U \rangle = \langle V, V \rangle \quad \text{and} \quad \langle U, V \rangle = 0.$$

The class of conformal martingales for the probability system Σ will be denoted by \mathcal{C} . Clearly $W \in \mathcal{C}$ iff $\overline{W} \in \mathcal{C}$ and the stopped process $W^T = (W_{T \wedge t})$ also belongs to \mathcal{C} for every (\mathcal{F}_t) -stopping time T .

Now fix an element $W = W_0 + U + iV \in \mathcal{M}_c$ and let $L_{\text{loc}}^2(W)$ be the class of previsible complex processes $C = (C_s)$ such that $\left(\int_0^t |C_s|^2 d\langle W \rangle_s \right)$ is locally integrable. If W is furthermore conformal then:

$$(5) \quad L_{\text{loc}}^2(W) = L_{\text{loc}}^2(U) + iL_{\text{loc}}^2(U)$$

(notice $L_{\text{loc}}^2(U) = L_{\text{loc}}^2(V)$).

To save spaces let us denote in the rest of the paper by $C * W$ the stochastic integral of an element of $C = (C_s) \in L_{\text{loc}}^2(W)$ with respect to $(W - W(0))$ and write:

$$(6) \quad C * W(t) = \int_0^t C_s \circ dW_s$$

$$(\text{instead of } C \cdot (W - W(0))(t) = \int_0^t C_s \cdot d(W - W(0))_s).$$

The notation $C_*^K W$ ($K \in \mathbb{N}$) will be used to denote the iterated stochastic integral:

$$(6') \quad C_*^K W(t) = C * W * \dots * W(t) \text{ (the operation } * \text{ is repeated } K \text{ times)}.$$

Notice that if $C \in L_{\text{loc}}^2(W)$ then by the continuity of the paths $C_*^K W$ also belongs to $L_{\text{loc}}^2(W)$ and furthermore it is not difficult to see that

$$\int_0^t |C_*^K W_s|^2 d\langle W \rangle_s = \int_0^t \left(\int_0^s |C_*^{K-1} W_u|^2 d\langle W \rangle_u \right) \cdot d\langle W \rangle_s + M_t \cdot \langle W \rangle_t$$

where (M_t) is the local martingale given by

$$M_t \equiv |C_*^{K-1} W_t|^2 - \langle C_*^{K-1} W \rangle_t.$$

The following Ito-formula for the class \mathcal{C} of conformal martingales of Gettoor and Sharpe will be used frequently in the sequel.

THEOREM B (see [2], (5.4)). *Let D be a domain in \mathbb{C}^n and let $Z^1, \dots, Z^n \in \mathcal{C}$ such that*

$$P((Z^1, \dots, Z^n) \in D \text{ for any } t \geq 0) = 1.$$

If F is holomorphic in a neighbourhood of D , then

$$(7) \quad F(Z_t^1, \dots, Z_t^n) = F(Z_0^1, \dots, Z_0^n) + \sum_{j=1}^n \int_0^t \frac{\partial F}{\partial z_j}(Z_s^1, \dots, Z_s^n) \cdot dZ_s^j$$

$$+ \frac{1}{2} \sum_{j,k=1}^n \int_0^t \frac{\partial^2 F}{\partial z_j \partial z_k}(Z_s^1, \dots, Z_s^n) \cdot d\langle Z^j, \bar{Z}^k \rangle_s$$

(Ito formula).

Consequently if $Z^1, \dots, Z^n \in \mathcal{C}$ and $\langle Z^j, \bar{Z}^k \rangle = 0$ for all j, k (notice $\langle Z^j, \bar{Z}^j \rangle = 0$ automatically) then under the same hypothesis on F , one has

$$F(Z_t^1, \dots, Z_t^n) = F(Z_0^1, \dots, Z_0^n) + \sum_{j=1}^n \int_0^t \frac{\partial F}{\partial z_j}(Z_s^1, \dots, Z_s^n) \cdot dZ_s^j,$$

i.e., $F(Z_t^1, \dots, Z_t^n)$ belongs again to \mathcal{C} .

DEFINITION 1. a) Let W be a fixed element of \mathcal{C} . A conformal martingale $\phi = (\phi(s))$ of \mathcal{C} is said to be *n-times differentiable* with respect

to W (abbrev. n -times W -differentiable) provided that there are n elements $\phi^{(i)} \in L_{\text{loc}}^2(W)$ ($1 \leq i \leq n$) such that:

$$(8) \quad \begin{cases} \phi - \phi(0) = \phi^1 * W \\ \phi^{(i)} - \phi^{(i)}(0) = \phi^{(i+1)} * W \quad (1 \leq i \leq n-1). \end{cases}$$

b) A conformal martingale $\phi \in \mathcal{C}$ is said to be *smooth with respect to* W if it is W -differentiable infinitely many times.

Remarks. 1) It is implicitly stated in the formula (8) of the Definition 1 that $\phi^{(i)} \in \mathcal{C}$ for $i = 1, 2, \dots, n-1$.

2) The particular case, where $W = (B_{T \wedge t})$ is chosen to be the complex Brownian motion stopped at the stopping time T , was introduced in an earlier work [3] and a smooth martingale (with respect to $W = (B_{T \wedge t})$) is called in [3] a locally analytic processes.

LEMMA 2. a) *Let W be a fixed conformal martingale and let n be an integer then the process:*

$$(9) \quad W^n = (W(t) - W(0))^n$$

belongs to \mathcal{C} and furthermore, for every integer $1 \leq k \leq n$ we have

$$(10) \quad W^n = n(n-1) \cdots (n-k) (W^{(n-k)})^{\overset{k}{*}} W$$

(see (6) and (6') for the definition of $\overset{k}{} W$).*

b) *Let $\phi = (\phi(t))$ be a $(n+1)$ times W -differentiable martingale then ϕ can be written as follows:*

$$(11) \quad \phi(t) = \phi(0) + \sum_{i=1}^n \frac{\phi^{(i)}(0)}{i!} W^i(t) + (\phi^{(n+1)})^{\overset{(n+1)}{*}} W(t).$$

Proof. (a) Since $\langle W, \bar{W} \rangle = 0$, it follows from Ito-formula (7) applying to $f(z) = z^n$ that:

$$W^n(t) = nW^{n-1} * W(t).$$

From the above equality (10) can be obtained easily by iteration.

b) Suppose first that $n = 0$, then (11) says nothing else than the definition of W -differentiable martingale, i.e.,

$$\phi(t) = \phi(0) + \int_0^t \phi^{(1)}(s) \cdot dW_s.$$

Now suppose that (11) is true for $n = k$ and that ϕ is a $(k+1)$ -times W -

differentiable martingale then we have:

$$(12) \quad \phi(t) = \phi(0) + \sum_{i=1}^{k-1} \frac{\phi^{(i)}(0)}{i!} \cdot W^i(t) + (\phi^{(k)} * W)(t).$$

On the other hand since $\phi^{(k)}$ can be written as follows:

$$\phi^{(k)}(s) = \phi^{(k)}(0) + (\phi^{(k+1)} * W)(s),$$

thus the last term of the right hand side member of (12) turns out to be:

$$\begin{aligned} (\phi^{(k)} * W)(t) &= (\phi^{(k)}(0) + \phi^{(k+1)} * W) * W(t) \\ &= \phi^{(k)}(0) * W(t) + \phi^{(k+1)} * W(t) \\ &= \frac{\phi^{(k)}(0)}{k!} \cdot W^k(t) + \phi^{(k+1)} * W(t) \end{aligned}$$

by (10). Hence (11) is also true for $n = k + 1$.

q.e.d.

Remark. Notice that $\phi^{(k)}(0)$ ($0 \leq k \leq n$) is \mathcal{F}_0 -measurable hence they behave like a constant in the stochastic integration procedure and that if $\psi \in L^2_{\text{loc}}(W)$ then so is $\psi * W$.

Let $W \in \mathcal{C}$ be a fixed conformal martingale and let $\langle W \rangle = (\langle W \rangle_s)$ be the continuous increasing process associated with W . For $t \geq 0$ let us define the stopping-time $\tau(t)$ as follows:

$$\tau(t, \omega) = \begin{cases} \inf \{s > 0: \langle W \rangle_s(\omega) \geq t\} \\ +\infty \text{ if the set of } s \text{ such that } \langle W \rangle_s(\omega) \geq t \text{ is empty.} \end{cases}$$

PROPOSITION 3. a) *Let $\phi = (\phi(t, \cdot))$ be a $(n + 1)$ -times W -differentiable conformal martingale with derivatives $\phi^{(i)}$ ($1 \leq i \leq n + 1$). Suppose furthermore that $E(|\phi(\tau(t, \cdot))|^2) < +\infty$, for some fixed $t > 0$, then for every $t_0 < t$:*

$$(13) \quad \begin{aligned} E(|\phi(\tau(t, \cdot))|^2) &= E(|\phi(\tau(t_0, \cdot))|^2) + \int_{t_0}^t dt_n \cdots \int_{t_0}^{t_1} E(|\phi(\tau(s, \cdot))|^2) ds \\ &+ \sum_{i=1}^n E(|\phi^{(i)}(\tau(t, \cdot))|^2) \frac{(t - t_0)^i}{i!} \\ &- \sum_{i=1}^{n+1} E(|\phi^{(i)}(\infty, \cdot)|^2) \cdot \frac{(t - t_0 \vee \langle W \rangle_\infty)^i}{i!}; \langle W \rangle_\infty \leq t \end{aligned}$$

b) *Consequently if ϕ is a W -smooth conformal martingale such that $E(|\phi(\tau(t, \cdot))|^2) < +\infty$, then the two following conditions are equivalent:*

$$(14) \quad \sum_{i=1}^{\infty} E(|\phi^{(i)}(\infty, \cdot)|^2) \cdot \frac{(t - \langle W \rangle_{\infty} \vee t_0)^i}{i!}; \langle W \rangle_{\infty} \leq t < +\infty,$$

$$(15) \quad \sum_{i=1}^{\infty} E(|\phi^{(i)}(\tau(t_0, \cdot))|^2) \cdot \frac{(t - t_0)^i}{i!} < +\infty.$$

Proof. a) Let $0 \leq t_0 \leq t$,

$$\begin{aligned} & E(|\phi(\tau(t, \cdot))|^2 - |\phi(\tau(t_0, \cdot))|^2 | \mathcal{F}_{\tau(t_0, \cdot)}) \\ &= E\left(\int_{\tau(t_0, \cdot)}^{\tau(t, \cdot)} |\phi^1(s, \cdot)|^2 d\langle W \rangle_s | \mathcal{F}_{\tau(t_0, \cdot)}\right) \\ &= E\left(\int_{t_0 \wedge \langle W \rangle_{\infty}}^{t \wedge \langle W \rangle_{\infty}} |\phi^1(\tau(s, \cdot))|^2 ds | \mathcal{F}_{\tau(t_0, \cdot)}\right). \end{aligned}$$

On the other hand since $\{\omega; \tau(t, \omega) = \infty\} = \{\omega; \langle W \rangle_{\infty} \leq t\}$ and since

$$\int_{t_0 \wedge \langle W \rangle_{\infty}}^{t \wedge \langle W \rangle_{\infty}} |\phi^1(\tau(s, \cdot))|^2 ds = \begin{cases} \int_{t_0}^t |\phi^1(\tau(s, \cdot))|^2 ds & \text{on } t < \langle W \rangle_{\infty} \\ \int_{t_0}^{\langle W \rangle_{\infty}} |\phi^1(\tau(s, \cdot))|^2 ds & \text{on } t_0 < \langle W \rangle_{\infty} \leq t, \\ 0 & \text{on } \langle W \rangle_{\infty} \leq t_0 \end{cases}$$

we have, by iterating on $E(|\phi^1(\tau(s, \cdot))|^2)$,

$$\begin{aligned} & E(|\phi(\tau(t, \cdot))|^2) - E(|\phi(\tau(t_0, \cdot))|^2) \\ &= E\left(\int_{t_0}^t |\phi^1(\tau(s, \cdot))|^2 ds; t < \langle W \rangle_{\infty}\right) \\ &\quad + E\left(\int_{t_0}^{\langle W \rangle_{\infty}} |\phi^1(\tau(s, \cdot))|^2 ds; t_0 < \langle W \rangle_{\infty} \leq t\right) \\ &= \int_{t_0}^t E(|\phi^1(\tau(s, \cdot))|^2) ds - E(|\phi^1(\infty, \cdot)|^2 (t - t_0 \vee \langle W \rangle_{\infty}); \langle W \rangle_{\infty} \leq t) \\ &= E(|\phi^1(\tau(t_0, \cdot))|^2) (t - t_0) + \int_{t_0}^t dt_{n-1} \int_{t_0}^{t_{n-1}} E(|\phi^2(t_n, \cdot)|^2) dt_n \\ &\quad - E(|\phi^1(\infty, \cdot)|^2 (t - t_0 \vee \langle W \rangle_{\infty}); \langle W \rangle_{\infty} \leq t) \\ &\quad - E\left(\int_{t_0}^t dt_n |\phi^2(\infty, \cdot)|^2 ds (t - t_0 \vee \langle W \rangle_{\infty}); \langle W \rangle_{\infty} \leq t_n\right). \end{aligned}$$

The last term of the right-hand side is:

$$\begin{aligned} & \int_{t_0}^t dt_n E(|\phi^2(\infty, \cdot)|^2 (t_n - t_0 \vee \langle W \rangle_{\infty}); \langle W \rangle_{\infty} \leq t_n) \\ &= E\left(|\phi^2(\infty, \cdot)|^2 \frac{(t - t_0 \vee \langle W \rangle_{\infty})^2}{2}; \langle W \rangle_{\infty} \leq t\right). \end{aligned}$$

Thus by iteration again on $E(|\phi^2(\tau(t_n, \cdot))|^2)$ and so on we get finally (13).

b) Suppose that (14) (resp. (15)) is true, by letting n to $+\infty$ in (13) we get (15) (resp. (14)). q.e.d.

THEOREM 4. *Let $\phi = (\phi(t, \cdot))$ be a W -smooth conformal martingale such that for every couple reals $t > t_0 \geq 0$:*

$$(16) \quad \begin{aligned} & E(|\phi(\tau(t, \cdot))|^2) < \infty \quad \text{and,} \\ & \sum_{i=1}^{\infty} E\left(|\phi^{(i)}(\infty, \cdot)|^2 \frac{(t - \langle W \rangle_{\infty} \vee t_0)^i}{i!}; \tau(t, \cdot) = +\infty\right) < \infty. \end{aligned}$$

Then ϕ (resp. $\phi^{(i)}$ ($i = 1, 2, \dots$)) admits a power series representation of the following form:

For every $t > 0$:

$$(17) \quad \phi(\tau(t, \cdot)) = \phi(0, \cdot) + \sum_{i=1}^{\infty} \frac{\phi^{(i)}(0, \cdot)}{i!} W_{\tau(t, \cdot)}^i$$

where the convergence of the right-hand side of (15) is to be in L^2 -norm

(resp. for $i = 1, 2, \dots$:

$$\phi^{(i)}(\tau(t, \cdot)) = \phi^{(i)}(0, \cdot) + \sum_{j=1}^{\infty} \frac{\phi^{(i+j)}(0, \cdot)}{j!} W_{\tau(t, \cdot)}^j.$$

b) *Let $\phi = (\phi(t, \cdot))$ be a W -smooth conformal martingale such that*

$$\sum_{i=1}^{\infty} E\left(|\phi^{(i)}(\infty, \cdot)|^2 \frac{(t - \langle W \rangle_{\infty} \vee t_0)^i}{i!}; \tau(t, \cdot) = +\infty\right) < \infty,$$

then for any $t \geq 0$:

$$(18) \quad \phi(\tau(t, \cdot)) = \phi(0, \cdot) + \sum_{i=1}^{\infty} \frac{\phi^{(i)}(0, \cdot)}{i!} W_{\tau(t, \cdot)}^i \quad \text{a.s. } P.$$

Proof. a) By Lemma 2, $\phi = (\phi(\tau(t, \cdot)))$ has the following Taylor expansion:

$$\phi(\tau(t, \cdot)) = \phi(0, \cdot) + \sum_{i=1}^n \frac{\phi^{(i)}(0, \cdot)}{i!} W_{\tau(t, \cdot)}^i + R_{n+1}(\tau(t, \cdot))$$

where

$$R_{n+1}(\tau(t, \cdot)) = \phi^{(n+1)} \binom{n+1}{*} W(\tau(t, \cdot)).$$

Let us estimate the L^2 -norm of the rest process $R_{n+1}(\tau(t, \cdot))$:

$$\begin{aligned}
(19) \quad E(|R_{n+1}(\tau(t, \cdot))|^2) &= E\left(\int_0^{\tau(t, \cdot)} |\phi^{n+1} * W|^2(s, \cdot) d\langle W \rangle_s\right) \\
&= E\left(\int_0^{\tau \wedge \langle W \rangle_\infty} |\phi^{n+1} * W|^2(\tau(s, \cdot)) ds\right)
\end{aligned}$$

By a computation similar to that used in the proof of Proposition 3, a), we get by iteration on the terms $E(|\phi^{n+1} * W|^2(\tau(t_i, \cdot)))$ ($0 \leq i \leq n$):

$$\begin{aligned}
E(|R_{n+1}(\tau(t, \cdot))|^2) &= E\left(\int_0^t |\phi^{n+1} * W|^2(\tau(t_n, \cdot)) dt_n; \tau(t, \cdot) < +\infty\right) \\
&\quad + E\left(\int_0^{\langle W \rangle_\infty} |\phi^{n+1} * W|^2(\tau(t_n, \cdot)) dt_n; \tau(t_0, \cdot) < \tau(t, \cdot) = +\infty\right) \\
&= E\left(\int_0^t |\phi^{n+1} * W|^2(\tau(t_n, \cdot)) dt_n\right. \\
&\quad \left. - E\left(\int_{\langle W \rangle_\infty}^t |\phi^{n+1} * W|^2(\tau(t_n, \cdot)) dt_n; \tau(t_0, \cdot) < \infty, \tau(t, \cdot) = +\infty\right)\right) \\
&= \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_1} E(|\phi^{n+1}(\tau(s, \cdot))|^2) ds \\
&\quad - \sum_{i=0}^n \left(\int_0^{t_{n+1}} dt_n \cdots \int_0^{t_{i+2}} dt_{i+1}\right. \\
&\quad \left. \times E\left(\int_{\langle W \rangle_\infty}^{t_{i+1}} |\phi^{n+1} * W|^2(\tau(s, \cdot)) ds; t_0 < \langle W \rangle_\infty \leq t_{i+1}\right)\right).
\end{aligned}$$

Hence,

$$(20) \quad E(|R_{n+1}(\tau(t, \cdot))|^2) \leq E(|\phi^{n+1}(\tau(t, \cdot))|^2) \frac{t^{n+1}}{(n+1)!}.$$

Using the condition (16) we can apply Proposition 3 and we get the following by putting $t = 2t_0$ in (15):

$$\lim_{n \rightarrow \infty} E(|R_{n+1}(\tau(t, \cdot))|^2) \leq \lim_{n \rightarrow \infty} E(|\phi^{(n+1)}(\tau(t, \cdot))|^2) \frac{t^{n+1}}{(n+1)!} = 0.$$

b) By localization the local martingale $\phi = (\phi(t))$ with the sequence of $(\mathcal{F}_{\tau(t, \cdot)})$ -stopping time (T_n) , where

$$T_n = \inf\{t > 0: |\phi(\tau(t, \cdot))|^2 \geq n\}$$

we have by the part a):

$$\phi(\tau(t, \cdot) \wedge T_n, \cdot) = \lim_{m \rightarrow \infty} \sum_{i=0}^m \frac{\phi^{(i)}(0, \cdot)}{i!} \cdot W_{\tau(t, \cdot) \wedge T_n}^i \quad \text{in } L^2.$$

The desired formula (18) comes out by letting $n \rightarrow +\infty$.

q.e.d.

Remarks. 1) In the case where $\tau(t, \cdot) < +\infty$ a.s., i.e. $\langle W \rangle_\infty \geq t$, the condition (14) in Proposition 3 is automatically satisfied and hence (13) becomes:

$$\begin{aligned} E(|\phi(\tau(t, \cdot))|^2) &= E(|\phi(\tau(t_0, \cdot))|^2) + \sum_{i=1}^n E(|\phi^{(i)}(\tau(t_0, \cdot))|^2) \frac{(t - t_0)^i}{i!} \\ &\quad + \int_0^t dt_n \cdots \int_0^{t_1} E(|\phi(\tau(s, \cdot))|^2) ds. \end{aligned}$$

Thus we re-fined here known results of Caroli-Walsh (Lemma 9.13, [4]) and also that of M. Yor (Theorem 1.1, [5]). Furthermore our result answers also an open question raised by Yor in the above mentioned paper (see the remark following Theorem 1.1).

2) Instead of a computation on the expectation if we compute on paths of the increasing process $\langle R_{n+1} \rangle_{\tau(t, \cdot)}$, then we could obtain the following formula which gives rise to an estimation sharper than (20),

$$\begin{aligned} \langle R_{n+1} \rangle_{\tau(t, \cdot)} &= \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_1} \langle \phi^{n+1} \rangle_{\tau(s, \cdot)} ds \\ &\quad - \sum_{i=0}^n |\phi^{n+1}| * W|^2(\infty, \cdot) \cdot \frac{(t - \langle W \rangle_\infty \vee t_0)^i}{i!} \cdot \chi_{(\langle W \rangle_\infty \leq t)} + M_{\tau(t, \cdot)} \end{aligned}$$

where $M = (M(t, \cdot))$ is a process of the form $\sum M_i^i \cdot t^i / i!$ in which M^i are local martingales.

Applications

1) *Approximation of the solution of a complex Ito-diffusion equation.*

Let $B = (B_s)$ be the complex Brownian motion, i.e. B is a special conformal martingale of the form $B_s = U_s + iV_s$ where $U = (U_s)$ and $V = (V_s)$ are two independent 1-dimensional Wiener processes for the given probability system $\Sigma = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$. Consider the complex Ito-diffusion equation:

$$(21) \quad \phi(t) = \phi(0) + \int_0^t a^1(\cdot, \phi(s)) \cdot dB_s$$

where the process $a^1 = (a^1(\cdot, \phi(s)))$ is supposed to belong to $L_{loc}^2(B)$.

If the equation (21) has a unique solution $\phi = (\phi(s))$ then we can inspired from our Definition 1 to say that the diffusion ϕ admits a^1 as its 1st derivative.

Suppose now that the process $a^1 = (a^1(\cdot, \phi(s)))$ is itself the unique solution of a complex Ito-diffusion equation of the type:

$$a^1(\cdot, \phi(t)) = a^1(\cdot, \phi(0)) + \int_0^t a^2(\cdot, a^1(\cdot, \phi(s)))dB_s$$

then the process $a^2 = (a^2(\cdot, a^1(\cdot, \phi(s))))$ is called *the 2^{nd} derivative of ϕ* .

By iteration we say that the solution $\phi = (\phi(s))$ of the equation (21) is *smooth* if it has derivative (a^i) of arbitrary order which can be defined recursively as the unique solutions of the following sequence of diffusion equations:

$$(22) \quad \begin{aligned} \phi(t) &= \phi(0) + \int_0^t a^1(\cdot, \phi(s)) \cdot dB_s \\ a^1(\cdot, \phi(t)) &= a^1(\cdot, \phi(0)) + \int_0^t a^2(\cdot, a^1(\cdot, \phi(s))) \cdot dB_s \\ &\vdots \\ a^i(\cdot, \phi(t)) &= a^i(\cdot, \phi(0)) + \int_0^t a^{i+1}(\cdot, a^i(\cdot, s)) \cdot dB_s. \end{aligned}$$

It turns out that if the solution $\phi = (\phi(s))$ is smooth then it is also B -smooth in the sense of Definition 1 and furthermore $\phi^{(i)} = a^i$ ($i = 1, 2, \dots$) and $\langle B \rangle_t = 2t$. Hence by Theorem 4, ϕ admits the following power series representation:

$$(23) \quad \phi(t) = \phi(0) + \sum_{i=0}^{\infty} \frac{a^{(i)}(0)}{i!} \cdot B_t^i \quad \text{a.s. } P.,$$

where $(a^{(i)}(0))$ is a sequence of \mathcal{F}_0 -measurable functions such that $a^{(1)}(0) = a^1(0, \phi(0))$ and $a^{(i)}(0) = a^i(0, a^{i-1}(0))$ ($i = 2, 3, \dots$).

On the one hand we can approximate in the L^2 -norm the solution $\phi = (\phi(t))$ of (21) by its Taylor expansion $\sum_{i=0}^n (a^{(i)}(0)/i!) \cdot B_t^i$ and on the other hand the rate of convergence for the approximation procedure could be controlled by the following inequality:

$$\lim_{n \rightarrow \infty} E(R_{n+1}(t)) \leq \lim_{n \rightarrow \infty} E \left(\frac{|a^{n+1}(t)|^2}{(n+1)!} \right) \cdot (2t)^{n+1},$$

in which the limit in the right hand-side should be zero by (15).

2) Power series expansion of finely holomorphic functions.

Let $U \subset \mathbf{C}$ be a finely open and finely connected subset of the complex plane \mathbf{C} , i.e., U is a fine domain. A complex function $f: U \rightarrow \mathbf{C}$ is said

to be finely holomorphic (see [1]) if for any $x \in U$ there is a compact fine neighbourhood $K_x \subset U$ of x such that $(f|_{K_x}) \in R(K_x)$. Recall that $R(K_x)$ is the algebra of functions whose elements can be uniformly approximated by a sequence of functions holomorphic in the neighbourhoods of K_x . Following [3] we will denote by $\mathcal{H}_f(U)$ the algebra of finely holomorphic functions in U . In our previous works (see [3]) we showed that there is a fine complex differential $\partial_U: \mathcal{H}_f(U) \rightarrow \mathcal{H}_f(U)$, such that for any $f \in \mathcal{H}_f(U)$ and for any $x \in U$ there is a compact fine neighbourhood K_x of x :

$$(24) \quad f(B_{T \wedge t}) = f(B_0) + \int_0^t \partial_U f(B_{T \wedge s}) \cdot dB_{T \wedge s} \quad \text{a.s. } P^x (x \in K^f),$$

where K^f is the fine interior of K_x and where T is the exit-time of K^f .

Since $\partial_U^i f \equiv \partial_U(\partial_U^{i-1} f) \in \mathcal{H}_f(U)$ ($i = 1, 2, \dots$) it follows from (24) that the process $\phi = (f(B_{T \wedge t}))$ is smooth with respect to $W = (B_{T \wedge t})$ and its i^{th} derivative is equal to $(\partial_U^i f(B_{T \wedge t}))$. Since $P^x(T > 0) = 1$ for any $x \in K^f$ and since $\langle B_{T \wedge \cdot} \rangle_t = T \wedge t$, we have $\tau(t, \omega) = T(\omega) \wedge t$. By a known result on bounded derivation of the algebra $R(K_x)$ (see [6]) there is a constant c such that $|\partial_U^i f(y)| \leq c \cdot \|f\|_K$ for every $f \in \mathcal{H}_f(U)$ and $y \in K^f$ ($i = 1, 2, \dots$), consequently the condition (16) of the Theorem 4 is satisfied for $f(B_{T \wedge t})$. The processes $f(B_{T \wedge t})$ and $\partial_U^i f(B_{T \wedge t})$ ($i = 1, 2, \dots$) admit power series representations, thus we recover some results of [3].

3) Singularities of smooth conformal martingales.

Let $\phi = \{\phi(t); t > 0\}$ be a continuous complex process adapted to $(\mathcal{F}_t)_{t \geq 0}$ with parameter set $]0, +\infty[$. ϕ is said to be *smooth with respect to* $W \in \mathcal{C}$ *off the time* "t = 0" provided that for every real $t_0 > 0$ the shifted process $\phi(\theta_{t_0}) \equiv \{\phi(t + t_0)\}_{t \geq 0}$ is smooth with respect to the process $W(\theta_{t_0}) \equiv \{W(t + t_0)\}_{t \geq 0}$ which, by hypotheses, is conformal for the probability system $\Sigma_{t_0} \equiv (\Omega, \mathcal{F}, (\mathcal{F}_{t+t_0})_{t \geq 0}, P)$. A process $\phi \in \mathcal{M}_c$ is said to *have an isolated zero at time* "t = 0" provided that

$$P(\phi(0) = 0) = 1 \text{ and } P(\phi(t) = 0) = 0 \text{ for any } t > 0.$$

The proof of the following theorem is similar to that of Theorem 7 in [3].

THEOREM 5. *Let $W \in \mathcal{C}$ be a fixed conformal martingale with only isolated zero at "t = 0" and let $\phi = \{\phi(t)\}_{t > 0}$ be a smooth process off the time "t = 0" with respect to W . Then one of the following three cases must occur:*

a) The process ϕ can be extended to a smooth process with respect to W on the whole interval $[0, +\infty[$. In this case ϕ is said to have a W -removable singularity at the time " $t = 0$ ".

b) There are m complex numbers c_1, \dots, c_m where m is a positive integer and $c_m \neq 0$ such that the process,

$$\phi - \sum_{i=1}^m c_i W^{-i},$$

can be extended to be W -smooth on $[0, +\infty[$. In this case ϕ is said to have a W -pole of order m at time " $t = 0$ ".

c) for each $\delta > 0$ the set,

$$\{(\phi(\omega, t)); (\omega, t) \in]0, \delta[\}$$

is dense in the complex plane \mathbb{C} . In this case ϕ is said to have a W -essential singularity at time " $t = 0$ ".

Remarks. 1) Note that since we have only to study the process near the time " $t = 0$ " thus by localization we can suppose without loss of generalities that the process satisfies the condition (16) of Theorem 4.

2) Our result generalizes that of Walsh in [7] in the sense that if the conformal process is smooth of the time " $t = 0$ " then we can classify its pole at " $t = 0$ ".

ACKNOWLEDGEMENT. The author would like to thank the referee for pointing to him an error in the proof of Proposition 3 and for giving him many interesting comments.

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