TENSOR PRODUCTS AND LOCALIZATIONS OF ALGEBRAS

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In a recent paper [5], it was shown that the tensor product of a finite number of fields over a common subfield satisfies the property that each localization at a prime ideal is a primary ring (in the sense that a zero-divisor is in fact a nilpotent element).

In the first section of this paper, we exploit the properties of associated primes and of flat extensions so as to generalize the above result to zero-dimensional algebras; a simple example shows that this is the best one can hope for. The converse situation is also investigated.

In the second section, a result of Grothendieck [1] is used to answer an open question in [2], so completing the programme of generalizing the work of Trung [7] and Nagata [4] on a conjecture of O'Carroll and Qureshi [6].

§ 1. Locally primary rings

In this paper, all rings are commutative with identity element. In fact only algebras over a field, k say, are considered, and all tensor products are taken over k. For the properties of flat extensions and associated primes we refer to [3, Chapters 2 and 3]. The basic theory of the tensor product of algebras is given in [9, Chapter III]; a brief discussion of the interaction with the theory of flat extensions is given in [5].

A *primary ring* is one in which each zero-divisor is a nilpotent element. A ring R is called *locally primary* if, for each prime ideal P in R, R_P is a primary ring. In [5, Theorem 3] it is shown that the tensor product of a finite number of fields over a common subfield is a locally primary ring. Our first result examines the converse situation.

PROPOSITION 1. *Let $R_1, \cdots, R_n$ be algebras over a field k, and suppose*

that $R_1 \otimes \cdots \otimes R_n$ is a locally primary ring. Then each R_i is also a locally primary ring.

Proof. We may restrict our attention to R_1, which we refer to as R; let $S = R_1 \otimes \cdots \otimes R_n$. Consider a prime ideal P in R. Since, by base change, S is a faithfully flat extension of R, there exists a prime ideal Q in S which contracts to P in R. Regarding S as an R-module we have $R_P \subseteq S_P$, and we also have the natural map $S_P \to S_Q$. Let $r \in R$. Since S is a flat extension of R, $0:_S r = (0:_R r)S$. Hence if $0:_S r \not\subseteq Q$, then $0:_R r \not\subseteq P$. It follows that R_P is isomorphically embedded in S_Q by the natural maps given above. By hypothesis, S_Q is a primary ring. It follows that R_P is also a primary ring.

However, the following example shows that the converse of Proposition 1 does not hold in general.

EXAMPLE. Let $R_1 = Q[X, Y]/(X^2 - 2Y^2)$, and let $R_2 = Q(\sqrt{2})$. Since R_1 and R_2 are domains, they are primary, and so locally primary, rings in a trivial way. However the ring

$$S := R_1 \otimes_Q R_2 = Q(\sqrt{2})[X, Y]/(X^2 - 2Y^2)$$

is not locally primary, since the maximal ideal $(X, Y)S$ contains two distinct minimal prime ideals $(X - \sqrt{2}\, Y)S$ and $(X + \sqrt{2}\, Y)S$. (Note that a primary ring contains a unique minimal prime ideal, consisting of all the zero-divisors.)

Note also that $\dim R_2 = 0$, but that $\dim R_1 = 1$.

We now wish to generalize [5, Theorem 3] in such a way as to provide a partial converse to Proposition 1; a zero-dimensional ring is of course locally primary. First we have two simple preliminary lemmas.

LEMMA 2. *Let T be a zero-dimensional algebra, over a field k, which is also a local ring, and let F be a finite subset of T. Then F is contained in a zero-dimensional subalgebra S of T, where S consists of a localization of an affine subalgebra of T at a prime ideal.*

Proof. Let $A = k[F]$ and let $S = A_P$, where P is the restriction to A of the maximal ideal of T. Since P consists of nilpotent elements, it is the unique minimal prime ideal of A. Moreover, $A \backslash P$ lies in the group of units of T, and in particular $A \backslash P$ consists of non-zero-divisors in A. Hence $F \subseteq A \subseteq S \subseteq T$, and the result follows.

LEMMA 3. *Let A be a Noetherian ring. Then A is locally primary if and only if, for each prime ideal P in A,*

$$\{Q \in \operatorname{Spec} A \mid Q \subseteq P\} \cap \operatorname{Ass}_A(A)$$

is a singleton.

Proof. Let P be a prime ideal in A. Now A_P is a primary ring if and only if $\operatorname{Ass}_{A_P}(A_P)$ is a singleton. But by [3, Lemma, p. 50], $\operatorname{Ass}_{A_P}(A_P)$ contracts in A to $\{Q \in \operatorname{Spec} A \mid Q \subseteq P\} \cap \operatorname{Ass}_A(A)$, and the result follows.

We now come to the main result of this section.

THEOREM 4. *Let $R_1, \cdots, R_n$ be zero-dimensional algebras, over a field k. Then $R_1 \otimes \cdots \otimes R_n$ is a locally primary ring.*

Proof. Let $R = R_1 \otimes \cdots \otimes R_n$, and let P be a prime ideal in R. For $i = 1, \cdots, n$, let P_i denote the restriction of P to R_i, and let R_i' denote the localization of R_i at P_i. (Each R_i' is a zero-dimensional local ring.) Then R_P is a localization, at a prime ideal, of the ring $R_1' \otimes \cdots \otimes R_n'$. Hence, without loss of generality, we assume in addition that each R_i is a local ring.

To prove that R_P is a primary ring, it suffices to show that a zero-divisor of the form $x/1 \in R_P$ (where $x \in R$) is nilpotent. Thus, consider $x \in R$ such that $sxy = 0$ for some $y \in R$ and $s \in R \backslash P$, with $\operatorname{Ann}_R y \subseteq P$; we wish to show that $\operatorname{Ann}_R x^n \not\subseteq P$, for some positive integer n. Now each of s, x and y consists of a finite sum of tensors of the form $r_1 \otimes \cdots \otimes r_n$, where $r_i \in R_i$ $(i = 1, \cdots, n)$. For each such i, let F_i denote the finite set of the r_i which occur in this way, and let S_i be the corresponding sub-algebra of R_i given by Lemma 2 (with R_i playing the role of T).

Let $S = S_1 \otimes \cdots \otimes S_n$. Then S and each of the S_i are Noetherian (in fact, S is the localization of an affine algebra over k), and s, x and y lie in S; moreover, each S_i is a local, zero-dimensional, ring. Let $Q = P \cap S$. Then $\operatorname{Ann}_S y \subseteq Q$, and if $\operatorname{Ann}_S x^n \not\subseteq Q$ for some positive integer n, then $\operatorname{Ann}_R x^n \not\subseteq P$. Hence, without loss of generality, we assume also that R and each of the R_i are Noetherian. Recall that each R_i is also local and zero-dimensional, with maximal ideal P_i (using the same nota-tion as before).

For $i = 1, \cdots, n$, let F_i denote the field R_i/P_i, and let $D = F_1 \otimes \cdots \otimes F_n$. Now each P_i consists of nilpotent elements, and hence so does $(P_1, \cdots, P_n)R$, Thus $\operatorname{Spec} R \approx \operatorname{Spec} D$, under the map induced by the

natural projection $R \to D$.

Note that for $i = 1, \cdots, n$, $F_1 \otimes \cdots \otimes F_{i-1} \otimes R_i \otimes \cdots \otimes R_n$ is a flat R_i-algebra, by base change. By repeated use of [3, Corollary, p. 58] and [loc. cit., Proposition, p. 57], we deduce that $\mathrm{Ass}_R(R)$ corresponds to $\mathrm{Ass}_D(D)$ under the above isomorphism of spectra. However, by [5, Theorem 3], D is locally primary. Thus R is locally primary, by the criterion of Lemma 3.

Remarks. Similar ideas can be used to derive an alternative proof of [5, Theorem 3] from an extension of Vámos' result (cf. [8, Corollary 4]), viz. that in a finite tensor product of fields any prime ideal contains a unique minimal prime ideal. (This extension is a corollary of [5, Theorem 3], of course.) As above, one reduces to the case where the field extensions involved are finitely generated. It follows from [9, Corollary 1, p. 191] that the zero ideal in the tensor product of fields, which is now a Noetherian ring, is unmixed. The extension of the Vámos result, together with Lemma 3, now yield [5, Theorem 3].

Similarly, in any example like that given after Proposition 1 involving a Noetherian tensor product of two algebras which are domains, we know from [9, Corollary 1, p. 191] that the zero ideal in the tensor product will be unmixed. Hence (cf. Lemma 3), in constructing the example, the property to aim for is that some prime (and hence, some maximal) ideal should contain more than one minimal prime.

§ 2. Localizations yielding equidimensional Hilbert rings

Trung [7], and independently Nagata [4], confirmed a conjecture by O'Carroll and Qureshi [6] (which had been verified in special cases), that the tensor product of a finite number of fields over a common subfield, where at most one of the field extensions has infinite transcendence degree, is an equidimensional Hilbert ring. Subsequently this work was generalized by Howie and O'Carroll [2]. The result of this section answers a question left open in [2]; this completes the programme of generalizing the result of Trung and Nagata, without having to resort to the awkward device of shifting to a companion result in order to cover the case where the base field is finite. (See [2, § 3, Remarks 3, 4 and 5].) We refer to [2] for a discussion of the connections between the various results mentioned above, and also Theorem 5 below.

(As a point of notation, if D is a domain and B is a subring of D,

tr.d. $(D \mid B)$ denotes the transcendence degree of the quotient field of D over that of B. Moreover, an *equidimensional ring* is one in which all maximal ideals have the same rank.)

THEOREM 5. *Let D be a domain and let $B_1, \cdots, B_n$ be subrings of D, where $n \geq 1$. Suppose that B_1 is chosen so as to satisfy*

$$\text{tr.d. } (D \mid B_1) = \min_i \{\text{tr.d. } (D \mid B_i)\} \,.$$

Suppose further that
(i) *D is a finitely generated B_i-algebra, $1 \leq i \leq n$; and*
(ii) *$B_2, \cdots, B_n$ are integrally closed.*

Let S be the multiplicatively closed set generated by $\bigcup_i \{B_i \backslash 0\}$.

Then $S^{-1}D$ is a Noetherian equidimensional Hilbert ring of dimension

$$d = \text{tr.d. } (D \mid B_1) \,.$$

Proof. Let $B = \bigcap_i B_i$. In [2] (cf. Theorem 2 there) this result has been established in the case where B is infinite. Thus we now suppose that B is finite, and reduce to the former case. We may assume that $d \geq 1$.

Hence suppose that B is a finite field, which has algebraic closure K, say. Consider the effect of applying $K \otimes -$, where the tensor product is taken over B.

Let $\mathfrak{q}$ be a minimal prime in $K \otimes D$. Now the extension $D \subseteq K \otimes D$ is (faithfully) flat, by base change, and D is a domain. Thus $\mathfrak{q} \cap D = 0$, by "Going Down".

Fix i such that $1 \leq i \leq n$. Consider the extension $K \otimes B_i \subseteq K \otimes D$. Let

$$\mathfrak{q}_i = \mathfrak{q} \cap (K \otimes B_i) \,,$$

and let

$$\overline{D} = (K \otimes D)/\mathfrak{q} \quad \text{and} \quad \overline{B}_i = (K \otimes B_i)/\mathfrak{q}_i \,,$$

where we take $\overline{B}_i \subseteq \overline{D}$, $D \subseteq \overline{D}$ and $B_i \subseteq \overline{B}_i$ (since $\mathfrak{q}_i \cap B_i = 0$).

Now the extension $B_i \subseteq K \otimes B_i$ is integral, with B_i a domain and $\mathfrak{q}_i \cap B_i = 0$; hence $\mathfrak{q}_i$ is a minimal prime in $K \otimes B_i$. Note also that $D \subseteq \overline{D}$ is an integral extension of domains, as is $B \subseteq \overline{B}_i$; thus tr.d. $(D \mid B_i) =$ tr.d. $(\overline{D} \mid \overline{B}_i)$. Moreover, $\overline{D}$ is a finitely generated $\overline{B}_i$-algebra.

We are given an element $a \in D \backslash 0$, and we wish to produce a non-zero prime ideal P' in D such that $a \notin P'$ and $P' \cap B_i = 0$, $1 \leq i \leq n$; as in [2], this will suffice to prove the result. We consider a as a non-zero element of $\bar{D}$.

The proof of [2, Theorem 2] shows (on replacing F, B_i and D there by K, $\bar{B}_i$ and $\bar{D}$ respectively) that

(i)   there exists an element $x \in \bar{D}$ which is transcendental over each $\bar{B}_i$;

(ii)   there exist an element $b \in K$ and a prime ideal P in $\bar{D}$ such that P is minimal over $(x - b)\bar{D}$;

(iii) $a \notin P$ and $P \cap \bar{B}_1[x] = (x - b)\bar{B}_1[x]$; and

(iv)   for each $i = 2, \cdots, n$, there exists an element c_i in a maximal extension $C^{(i)}$ of $\bar{B}_i[x]$ (in $\bar{D}$) by a finite number of algebraically independent elements, with $C_{c_i}^{(i)} \subseteq \bar{D}_{c_i}$ an integral extension and $c_i \notin P$.

Fix i such that $2 \leq i \leq n$, and let $P_i = P \cap \bar{B}_i$. Pull P_i back to a prime ideal $\mathfrak{p}_i$ of $K \otimes B_i$ which contains $\mathfrak{q}_i$. By [1, Corollary (6.14.2)] (cf. [loc. cit., (5.13.5)]), $(K \otimes B_i)_{\mathfrak{p}_i}$ is an integrally closed domain. In particular, $(\mathfrak{q}_i)_{\mathfrak{p}_i} = 0$, so $(K \otimes B_i)_{\mathfrak{p}_i} = (\bar{B}_i)_{P_i}$, and the latter is an integrally closed domain. Let $Q_i = P \cap C^{(i)}$. Then $(C^{(i)})_{Q_i}$ is an integrally closed domain, being a localization of a polynomial extension of $(\bar{B}_i)_{P_i}$. Since $c_i \notin Q_i$, the extension $(C^{(i)})_{Q_i} \subseteq \bar{D}_{Q_i}$ is integral (here we regard $\bar{D}$ as a $C^{(i)}$-module). By the "Cohen-Seidenberg" theorem, the extension $(C^{(i)})_{Q_i} \subseteq \bar{D}_{Q_i}$ satisfies "Going Down". Moreover the extension $\bar{B}_i[x] \subseteq (C^{(i)})_{Q_i}$ also satisfies "Going Down", since it is a flat extension, being a composite of flat extensions. Thus the composite extension $\bar{B}_i[x] \subseteq \bar{D}_{Q_i}$ satisfies "Going Down".

We are now in a position to apply the argument of [2, § 2]. Note that P survives in $\bar{D}_{Q_i}$. By the "Going Down" property, the prime $P \cap \bar{B}_i[x]$ is minimal over the ideal $(x - b)\bar{B}_i[x]$. But the latter is itself a prime ideal. We deduce that

$$P \cap \bar{B}_i[x] = (x - b)\bar{B}_i[x], \qquad 1 \leq i \leq n.$$

Hence, for $i = 1, \cdots, n$, $P_i = P \cap \bar{B}_i = 0$; moreover $a \notin P$.

Finally, set $P' = P \cap D$. Note that $a \notin P'$, and that $P' \neq 0$ by the "Incomparability" property in the integral extension $D \subseteq \bar{D}$ of domains. Moreover $P' \cap B_i = 0$, $1 \leq i \leq n$, and the result follows.

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