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PRECISE PROPAGATION OF SINGULARITIES FOR A HYPERBOLIC SYSTEM WITH CHARACTERISTICS OF VARIABLE MULTIPLICITY

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Introduction

In this paper we consider the Cauchy problem for a hyperbolic system with characteristics of variable multiplicity and construct a certain solution whose wave front set propagates precisely along the so-called "broken null bicharacteristic flow", in other words, along the admissible trajectory if we use the terminology of [6].

Let L be a hyperbolic system of the form

$$\begin{bmatrix} D_t + \lambda_1(t, x, D_x) & & 0 \\ & \ddots & \\ 0 & & D_t + \lambda_\ell(t, x, D_x) \end{bmatrix} + B(t, x) \qquad \text{on} \quad R_t^1 \times R_x^n, \tag{0.1}$$

where $\lambda_j(t, x, \xi)$ are linear in ξ, that is,

$$\lambda_j(t, x, \xi) = \sum_{k=1}^{n} a_{j,k}(t, x)\xi_k, \qquad j \in \{1, \cdots, \ell\}. \tag{0.2}$$

Here $a_{j,k}(t, x) \in C^\infty$ are real-valued and polynomials of first order with respect to x;

$$a_{j,k}(t, x) = \sum_{m=1}^{n} C_{j,k}^m(t)x_m + C_{j,k}^0(t). \tag{0.3}$$

We assume that the term of order zero $B(t, x) = (b_{j,k}(t, x))$ satisfies

$$b_{j,k} \neq 0 \qquad \text{if} \quad j \neq k. \tag{0.4}$$

We consider the Cauchy problem

$$LU = 0, \quad U(0, x) = G, \quad G \in \mathscr{E}' \tag{C.P}$$

and show the precise propagation of the wave front set of a solution along

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the admissible trajectory, by assuming some hypotheses (See Theorem 1 in § 1). Hypotheses of Theorem 1 are fairly complicated, but they are satisfied for some cases discussed in [6] and [7], for instance, we can apply Theorem 1 to the proof of Theorem 2 of [7] (see also Theorem 2 of [6] and its §7).

In the last section, as another application of Theorem 1 we give an example of hyperbolic system L on $R_t^1 \times R_x^1$, for which there exists an initial value G such that

$$\text{sing supp}\, G = \{0\}\,, \qquad \text{sing supp}\, U(T) = [-1, 1]\,, \tag{0.5}$$

where U is the solution of (C.P) and T is a fixed positive. For a hyperbolic equation of second order, Ichinose-Kumano-go [5] gave a result similar to (0.5) by using the work of Taniguchi-Tozaki [12], though we do not know whether the Cauchy problem for their equation is well-posed until $t = T$ (see Theorem 3.5 of [5]).

In the next paper [11], the proof of Theorem 1 will be applied to show the precise propagation of wave front sets in Gevrey classes along "generalized null bicharacteristic flow" defined by Kumano-go-Taniguchi [9] and Wakabayashi [13] [14].

§ 1. Notations and main result

We say that a curve $\{t, x(t), \xi(t)\} \subset R_t^1 \times T^*R_x^n$ is the bicharacteristic curve with respect to λ_j through (s, y, η) if it satisfies equations

$$\begin{cases} dx/dt = d_\xi \lambda_j(t, x, \xi)\,, & d\xi/dt = -\, d_x \lambda_j(t, x, \xi) \\ (x, \xi)|_{t=s} = (y, \eta). \end{cases} \tag{1.1}$$

For the abbreviation we write bic. curve w.r.t. λ_j in what follows. We denote by $\chi_j(t, s)$ the transformation from the cotangent space $T^*R_x^n$ to itself such that $(t, \chi_j(t, s)(y, \eta))$ is bic. curve w.r.t. λ_j through (s, y, η). Since λ_j are linear in ξ and $a_{j,k}$ polynomials of the first order with respect to x, there exist a matrix $M_j(t, s)$ and a vector $d_j(t, s)$ such that

$$\chi_j(t, s)(y, \eta) = (M_j(t, s)y + d_j(t, s),\ {}^tM_j(t, s)^{-1}\eta)\,, \tag{1.2}$$

where ${}^tM^{-1}$ denotes the inverse of the transposed matrix of M.

For an integer $\nu \geqq 0$ let $J_\nu = (j_1, j_2, \cdots, j_q, \cdots, j_{\nu+1})$ be a $(\nu+1)$-repeated permutation of elements $\{1, 2, \cdots, \ell\}$, that is, $j_q \in \{1, \cdots, \ell\}$ for $1 \leqq q \leqq \nu + 1$. Let Π_ν be a set of all J_ν and let Π_ν^0 be a set of all J_ν satisfying $j_q \neq j_{q+1}$ for $1 \leqq q \leqq \nu$.

For a fixed $t_0>0$ let $\varDelta_\nu^0$ denote an open set of $R^\nu\ (\nu\geqq 1)$ as follows; $\varDelta_\nu^0=\{\tilde{t}_\nu=(t_1,\ \cdots,\ t_\nu);\ t_0>t_1>t_2>\cdots>t_\nu>0\}$. A continuous curve $\{(t, x(t), \xi(t));\ t\in[0, t_0]\}$ is called a trajectory of step ν, issuing from $\rho\in T^*R^n\backslash 0$ if for some $J_\nu\in\Pi_\nu^0$ and some $\tilde{t}_\nu\in\varDelta_\nu^0$ it is bic. curve w.r.t. λ_{j_q} when $t\in[t_q, t_{q-1}]$ $(q=1,\ \cdots,\ \nu+1,\ t_{\nu+1}=0)$ and $(x(0), \xi(0))=\rho$. We denote this trajectory by $C(J_\nu, \tilde{t}_\nu, \rho)$. A point

$$\chi_{j_1}(t_0, t_1)\chi_{j_2}(t_1, t_2)\cdots\chi_{j_{\nu+1}}(t_\nu, 0)\rho \tag{1.3}$$

is called the end point (at $t=t_0$) of the trajectory. It follows from (1.2) that the end point is equal to

$$(M_{J_\nu}(\tilde{t}_\nu)y+d_{J_\nu}(\tilde{t}_\nu)\,,\quad {}^tM_{J_\nu}(\tilde{t}_\nu)^{-1}\eta)\,,\quad \rho=(y, \eta)\,, \tag{1.4}$$

where a matrix M_{J_ν} and a vector d_{J_ν} are defined by, respectively,

$$\begin{aligned}M_{J_\nu}(\tilde{t}_\nu)&=M_{j_1}(t_0, t_1)M_{j_2}(t_1, t_2)\cdots M_{j_{\nu+1}}(t_\nu, 0)\\ d_{J_\nu}(\tilde{t}_\nu)&=d_{j_1}(t_0, t_1)+M_{j_1}(t_0, t_1)d_{j_2}(t_1, t_2)+\cdots\\ &\quad +M_{j_1}(t_0, t_1)\cdots M_{j_\nu}(t_{\nu-1}, t_\nu)d_{j_{\nu+1}}(t_\nu, 0)\,.\end{aligned}$$

DEFINITION 1.1. We say that a trajectory $C(J_\nu, \tilde{t}_\nu, \rho)$ is ε-admissible for a $\varepsilon\geqq 0$ if we have

$$|(\lambda_{j_q}-\lambda_{j_{q+1}})(t_q, x^q, \xi^q)|\leqq\varepsilon|\xi^q|\qquad\text{for any } q\in\{1,\ \cdots,\ \nu\}\,, \tag{1.5}$$

where $(x^q, \xi^q)=\chi_{j_{q+1}}(t_q, t_{q+1})\cdots\chi_{j_{\nu+1}}(t_\nu, 0)\rho$. We say only admissible in place of 0-admissible.

We remark that a bic. curve is always an adm. traj. of step 0. For the brevity we often write adm. traj. instead of admissible trajectory.

For $J_\nu=(j_1,\ \cdots, j_{\nu+1})\in\Pi_\nu^0$ and $J'_\mu=(j'_1,\ \cdots,\ j'_{\mu+1})\in\Pi_\mu^0$ we write $J_\nu\supset J'_\mu$ if there exists a subset $\{q_1^0, q_2^0,\ \cdots,\ q_r^0,\ \cdots,\ q_{\mu+1}^0;\ q_r^0<q_{r+1}^0\}$ of $\{1,\ \cdots, \nu+1\}$ such that $j_{q_r^0}=j'_r$ for $r=1,\ \cdots,\ \mu+1$. Obviously, if $J_\nu\supset J'_\mu$ then $\nu\geqq\mu$.

DEFINITION 1.2. For a small $\varepsilon_0>0$ and a fixed trajectory $C_\mu=C(J'_\mu, \tilde{t}'_\mu, \rho)$ we say that a trajectory $C(J_\nu, \tilde{t}_\nu, \rho)$ is contained in ε_0-neighborhood of C_μ if and only if $J_\nu\supset J'_\mu$ and the following conditions are satisfied: Suppose that

$$J_\nu=(j_1,\ \cdots, j_q,\ \cdots, j_{\nu+1})\,,\qquad \tilde{t}_\nu=(t_1,\ \cdots,\ t_q,\ \cdots,\ t_\nu)\,.$$

When $\mu\geqq 1$, suppose that

$$J'_\mu=(j'_1,\ \cdots, j'_r,\ \cdots, j'_{\mu+1})\,,\qquad \tilde{t}'_\mu=(t'_1,\ \cdots,\ t'_r,\ \cdots,\ t'_\mu)\,.$$

Then there exists a subset $\{q_1, \cdots, q_\mu\}$ of $\{1, \cdots, \nu\}$ such that

(1.6) $$|t_{q_r} - t'_r| < \varepsilon_0, \qquad r = 1, \cdots, \mu.$$

And furthermore we have

(1.7) $$|t_q - t_{q-1}| < \varepsilon_0 \qquad \text{for any } q \in \{1, \cdots, \nu+1\} \text{ satisfying}$$

(1.8) $$q_r \geqq q > q-1 \geqq q_{r-1} \quad \text{and} \quad j_q \neq j'_r, \quad r = 1, \cdots, \mu+1,$$

where $q_0 = 0$, $q_{\mu+1} = \nu + 1$ and $t_{\nu+1} = 0$.

When $\mu = 0$, suppose that $J'_\mu = (j)$. Then $J_\nu = (j)$ if $\nu = 0$. If $\nu \geqq 1$ we have (1.7) for any $1 \leqq q \leqq \nu + 1$ satisfying $j_q \neq j$.

THEOREM 1. *We assume that*

(H-1) *There exists an admissible trajectory $C_\mu = C(J'_\mu, \tilde{t}'_\mu, \rho_0)$ of step μ issuing from $\rho_0 = (y^0, \eta^0) \in T^*R^n \backslash 0$ whose end point is $\delta_0 = (x^0, \xi^0) (= (M_{J'_\mu}(\tilde{t}'_\mu) y^0 + d_{J'_\mu}(\tilde{t}'_\mu), {}^t M_{J'_\mu}(\tilde{t}'_\mu)^{-1} \eta^0))$.*

(H-2) *If $\mu \geqq 1$ then it follows that*
$$\det(\partial_{t_p} \partial_{t_q} (\xi^0 \cdot (M_{J'_\mu}(\tilde{t}'_\mu) y^0 + d_{J'_\mu}(\tilde{t}'_\mu)))) \neq 0, \qquad (p, q = 1, \cdots, \mu).$$

(H-3) *Let V_ε be an open ε-conic neighborhood of ξ^0,*
$$V_\varepsilon = \{\xi \in R^n \backslash 0;\ |\xi/|\xi| - \xi^0/|\xi^0|| < \varepsilon\}.$$
There exist an open neighborhood V_0 of x^0 and a $0 < \sigma \leqq 1$ such that any ε-admissible trajectory issuing from ρ_0 whose end point belongs to $V_0 \times V_\varepsilon$ is contained in ε^σ-neighborhood of C_μ if $\varepsilon > 0$ is small enough.

Then, there exists an initial value G such that

(1.9) $$WF\, G = \{(y^0, c\eta^0);\ c > 0\}$$

and

(1.10) $$WF\, U(t_0) \ni \delta_0 = (x^0, \xi^0),$$

where U is the solution of (C.P).

§ 2. Proof of Theorem 1

Without loss of generality we may assume that the lower order term B of L satisfies

(0.4)′ $$b_{j,j} = 0, \quad b_{j,k} \neq 0 \quad \text{if} \quad j \neq k.$$

Indeed, let $\alpha_j(t, x)$ be solutions of equations

$$(D_t + \lambda_j(t, x, D_x))\alpha_j + b_{j,j}\alpha_j = 0\,, \qquad \alpha_j(0, x) = 1\,.$$

If we put a matrix

$$A(t, x) = \begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_\ell \end{bmatrix}$$

and put $\tilde{U}(t, x) = A^{-1}(t, x)U(t, x)$ for the solution U of (C.P), we get the Cauchy problem for $\tilde{U}(t, x)$

$$\text{(2.1)} \qquad \begin{cases} L_0\tilde{U} + A^{-1}(L_0 A + BA)\tilde{U} = 0 \\ \tilde{U}(0, x) = G, \quad G \in \mathscr{E}'\,, \end{cases}$$

where L_0 is the principal term of L. It is easy to see that $A^{-1}(L_0A + BA)$ satisfies the condition like (0.4)$'$. Since the multiplier A does not change the wave front set of U, it suffices to consider (2.1) instead of (C.P).

Let π be the natural projection from T^*R^n to R^n. If we put a transformation from R^n to itself

$$\text{(2.2)} \qquad \pi_j(s, t) = \pi\chi_j(s, t)\pi^{-1}$$

by means of (1.2) we get

$$\text{(2.3)} \qquad \pi_j(s, t)y = M_j(s, t)y + d_j(s, t) = M_j(t, s)^{-1}(y - d_j(t, s))\,.$$

We put $I_j(t, s)f(x) = f(\pi_j(s, t)x)$ for $f \in \mathscr{E}'$ and put

$$I(t, s) = \begin{bmatrix} I_1(t, s) & & 0 \\ & \ddots & \\ 0 & & I_\ell(t, s) \end{bmatrix}.$$

Then, by the method of iteration we get the fundamental solution $E(t, s)$ for L of the form

$$E(t, s) = I(t, s) + \int_s^t I(t, \theta)W(\theta, s)d\theta\,,$$

$$W(t, s) = \sum_{\nu=1}^{\infty} W_\nu(t, s)\,, \quad W_1(t, s) = B(t)I(t, s)\,,$$

$$W_{\nu+1}(t, s) = \int_s^t\int_s^{t_1}\cdots\int_s^{t_{\nu-1}} W_1(t, t_1)W_1(t_1, t_2)\cdots W_1(t_\nu, s)dt_\nu\cdots dt_2dt_1\,,$$

$$\nu \geqq 1\,.$$

Hence, if we denote by Δ_ν and $d\tilde{t}_\nu$ the closure of Δ_ν^0 and $dt_1 \cdots dt_\nu$, respectively, the solution $U(t_0, x) = {}^t(u_1(t_0, x), \cdots, u_\ell(t_0, x))$ of (C.P) for initial value $G = {}^t(g_1, \cdots, g_\ell)$ is given in the form

$$(2.4) \qquad u_j(t_0, x) = g_j(\pi_j(0, t_0)x) + \sum_{\nu=1}^{\infty} \sum_{\substack{J_\nu \in \Pi_\nu \\ j_1 = j}} \int_{\Delta_\nu} \beta^j_{J_\nu} g_{j_{\nu+1}}(\pi_{J_\nu}(\tilde{t}_\nu)x) d\tilde{t}_\nu$$
$$(j = 1, \cdots, \ell),$$

where

$$(2.5) \qquad \pi_{J_\nu}(\tilde{t}_\nu) = \pi_{j_{\nu+1}}(0, t_\nu)\pi_{j_\nu}(t_\nu, t_{\nu-1}) \cdots \pi_{j_1}(t_1, t_0),$$

$$(2.6) \qquad \begin{aligned} \beta_{J_\nu} &\equiv \beta_{J_\nu}(\tilde{t}_\nu, x) \\ &= \beta_{j_1}(t_0, x)\beta_{j_2}(t_0, t_1, x) \cdots \beta_{j_\nu}(t_0, \cdots, t_{\nu-1}, x) \end{aligned}$$

and

$$(2.7) \qquad \beta_{j_q}(t_0, \cdots, t_{q-1}, x) = -ib_{j_q, j_{q+1}}(t_q, \pi_{j_q}(t_q, t_{q-1}) \cdots \pi_{j_1}(t_1, t_0)x),$$
$$q = 1, \cdots, \nu.$$

It follows from (0.4)′ that

$$(2.8) \qquad \begin{cases} \beta^j_{J_\nu} \neq 0 & \text{if } J_\nu \in \Pi^0_\nu \\ \beta^j_{J_\nu} = 0 & \text{otherwise}. \end{cases}$$

We take, as the initial value of (C.P),

$$(2.9) \qquad \begin{cases} G = {}^t(g_1, \cdots, g_\ell), \quad g_j = g \quad (j = 1, \cdots, \ell), \\ g(x) = \sum_{k=1}^{\infty} \tau_k^{-1}\phi(\tau_k^{1-\gamma}(x - y^0)) \exp i\tau_k(x - y^0)\cdot\eta^0, \end{cases}$$

where $\phi \in C_0^\infty$ such that $\hat{\phi}(0) = 1$ ($\hat{\phi}$ denotes the Fourier transform of ϕ). Here γ is a positive number such that

$$(2.10) \qquad 0 < \gamma < 1/2$$

and $\{\tau_k\}_{k=1}^\infty$ is a suitable increasing sequence of positive numbers such that τ_{k+1}/τ_k is sufficiently large in order to satisfy conditions demanded later on. The function g is a modification of that given by Hörmander (see Example 2.3 of [4]). Another modification was used in [10].

We have

$$(2.11) \qquad WF\, g = \{(y^0, c\eta^0);\ c > 0\}.$$

Indeed, since each term of g is in C^∞ and the supports shrink to y^0 it follows that

(2.12) $$\text{sing supp}\, g = \{y^0\}\,.$$

Now we have

(2.13) $$\hat{g}(\xi) = \sum_{k=1}^{\infty} \tau_k^{(\gamma-1)n-1} \hat{\phi}((\xi - \tau_k \eta^0)\tau_k^{\gamma-1}) \exp - (iy^0 \cdot \xi)\,.$$

We may assume that there exists a positive $\gamma_0 < \gamma < 1/2$ such that

(2.14) $$|\tau_k - \tau_{k'}|\tau_{k'}^{\gamma-1} > \tau_k^{\gamma_0} \qquad \text{if} \quad k \neq k'$$

because we have for any integer $m > 0$

$$\begin{aligned} |\tau - \tau'|\tau'^{\gamma-1} &= |\tau^{1/m} - \tau'^{1/m}| \sum_{r=0}^{m-1} \tau^{r/m} \tau'^{((m-1-r)/m)-(1-\gamma)} \\ &\geqq \tau^{[m-1-(1-\gamma)m]/m} + \sum_{r \neq [m-1-(1-\gamma)m]} \tau^{r/m} \tau'^{((m-1-r)/m)-(1-\gamma)} \end{aligned}$$

if $|\tau - \tau'|$ is large enough. Since we may assume $\sum \tau_k^{-1} < \infty$ and since for any $N > 0$ there exists a constant $C_N > 0$ such that

(2.15) $$|\hat{\phi}((\tau_k - \tau_{k'})\eta^0 \tau_{k'}^{\gamma-1})| \leqq C_N \tau_k^{-N} \qquad \text{for} \quad k \neq k'\,,$$

we have

(2.16) $$|\hat{g}(\tau_k \eta^0)| \geqq \tau_k^{(\gamma-1)n-1} \hat{\phi}(0)/2 \qquad \text{if} \quad \tau_k \text{ large enough.}$$

Let V be a conic neighborhood of η^0. If $\xi \notin V$, there exist two positive constants C_V and C'_V such that

(2.17) $$\begin{aligned} |\xi - \tau_k \eta^0| \tau_k^{\gamma-1} &\geqq C_V(|\xi| + |\tau_k \eta^0|)\tau_k^{\gamma-1} \\ &\geqq C'_V |\xi|^\gamma\,, \end{aligned}$$

where we used the Hölder inequality to obtain the second inequality. Then, for any $N > 0$ there exists a constant C'_N such that

(2.18) $$|\hat{g}(\xi)| \leqq C'_N |\xi|^{-N} \qquad \text{if} \quad \xi \notin V\,.$$

For the neighborhood of V_0 of x^0 given in the hypothesis (H-3) let $\psi(x)$ be $C_0^\infty(V_0)$ such that $\psi = 1$ near x^0. For the proof of Theorem 1 it suffices to show that for some $j \in \{1, \cdots, \ell\}$

(2.19) $$\begin{aligned} &|(\widehat{\psi u_j})(t_0, \tau_k \xi^0)| \geqq c_1 \tau_k^{(\gamma-1)n-1-\mu/2}\,, \\ &c_1 > 0, \qquad \text{if } \tau_k \text{ large enough.} \end{aligned}$$

If we calculate the Fourier transform of ψu_j by (2.4) in consideration of (2.8), we get

$$(2.20)\qquad (\widehat{\psi u_j})(t_0, \tau_k\xi^0) = \sum_{k'} \tau_{k'}^{(\gamma-1)n-1}\Big(w_{(j)}(\tau_k, \tau_{k'}) + \sum_{\nu=1}^{\infty}\sum_{\substack{J_\nu\in\Pi_\nu^0\\ j_1=j}}\int_{\Delta_\nu} w_{J_\nu}(\tau_k, \tau_{k'}; \tilde{t}_\nu)d\tilde{t}_\nu\Big),$$

where $w_{J_q}(\tau, \tau'; \tilde{t}_\nu) = G_{J_\nu}(\tau, \tau'; \tilde{t}_\nu)\exp i\tau F_{J_\nu}(\tilde{t}_\nu)$ $(\nu \geqq 0,\ J_0 = (j))$,

$$(2.21)\qquad F_{J_\nu}(\tilde{t}_\nu) = -\xi^0\cdot(M_{J_\nu}(\tilde{t}_\nu)y^0 + d_{J_\nu}(\tilde{t}_\nu)),$$

$$(2.22)\qquad G_{J_\nu}(\tau, \tau'; \tilde{t}_\nu) = \int \exp(iy\cdot(\tau'\eta^0 - \tau{}^tM_{J_\nu}(\tilde{t}_\nu)\xi^0)\tau'^{\gamma-1}) \times \psi(M_{J_\nu}(\tilde{t}_\nu)(\tau'^{\gamma-1}y + y^0) + d_{J_\nu}(\tilde{t}_\nu)) \times \phi(y)\beta'_{J_\nu}dy. \qquad (\beta'_{J_\nu} = \beta^j_{J_\nu}|\det M_{J_\nu}| \neq 0).$$

Here we used the change of variables x to

$$(2.23)\qquad \pi^{-1}_{J_\nu}(\tau_{k'}^{\gamma-1}y+y^0) = M_{J_\nu}(\tau_{k'}^{\gamma-1}y + y^0) + d_J.$$

We note that $M_{J_\nu}(\tilde{t}_\nu)$ is uniformly bounded with respect to ν and $\tilde{t}_\nu \in \Delta_\nu$ because $\pi_{J_\nu}(\tilde{t}_\nu)y$ is bounded for $|y| \leqq 1$ and so $d_{J_\nu}(\tilde{t}_\nu)$ is bounded. In the same way as in (2.14) we have for $0 < \gamma_0 < \gamma$

$$(2.24)\qquad |\tau'\eta^0 - \tau{}^tM_{J_\nu}(\tilde{t}_\nu)\xi^0|\tau'^{\gamma-1} \geqq \tau^{\gamma_0}$$

if $|\tau' - \tau|$ is large enough. The integration by parts with respect to y shows that for any $\nu \geqq 0$ and $N > 0$ there exists a constant C_N independent of ν such that

$$(2.25)\qquad |G_{J_\nu}(\tau, \tau'; \tilde{t}_\nu)| \leqq C_N^\nu\tau^{-N} \qquad \text{if } |\tau - \tau'| \text{ large enough.}$$

Since the volume of Δ_ν is equal to $t_0/\nu\,!$, for the proof of (2.19) it suffices to consider only the term with respect $k' = k$ in the sum of the right hand side of (2.20).

For $J_\nu \in \Pi_\nu^0$ $(\nu \geqq 1)$ we put

$$(2.26)\qquad I_{J_\nu}(\tau) = \int_{\Delta_\nu} G_{J_\nu}(\tau; \tilde{t}_\nu)\exp i\tau F_{J_\nu}(\tilde{t})d\tilde{t}_\nu \qquad (\nu \geqq 1),$$

where

$$(2.22)'\qquad G_{J_\nu}(\tau; \tilde{t}_\nu) \equiv G_{J_\nu}(\tau, \tau; \tilde{t}_\nu) = \int \exp(i\tau^\gamma(\eta^0 - {}^tM_{J_\nu}(\tilde{t}_\nu)\xi^0)\cdot y) \times \psi(M_{J_\nu}(\tilde{t}_\nu)(\tau^{\gamma-1}y + y^0) + d_{J_\nu}(\tilde{t}_\nu))\phi(y)\beta'_{J_\nu}dy.$$

It is easy to see that for any multi-index α there exists a constant C_α independent of ν such that

$$|\partial_{\tilde{t}_\nu}^\alpha G_{J_\nu}(\tau;\tilde{t}_\nu)| \leqq C_\alpha^\nu \tau^{\gamma|\alpha|}, \qquad 0<\gamma<1/2\,. \tag{2.27}$$

When $\nu=0$ we also define $I_{J_\nu}(\tau)$ by the integrand of (2.26). The proof of Theorem 1 will be completed if we show the following:

LEMMA 2.1. *Suppose that $J_\nu \in \Pi_\nu^0$ and τ is large enough. If $J_\nu \supset J'_\mu$, for any $N>0$ there exists a constant C_1 independent of ν such that*

$$|I_{J_\nu}(\tau)| \leqq C_1^\nu \tau^{-\mu/2-\sigma\gamma(\nu-\mu)/4}/(\nu-\mu)\,!\,, \tag{2.28}$$

where σ is the positive in the hypothesis (H-3). *If $J_\nu \not\supset J'_\mu$, for any $N>0$ there exists a constant C_N independent of ν such that*

$$|I_{J_\nu}(\tau)| \leqq C_N^\nu \tau^{-N}/\nu\,! \tag{2.29}$$

Furtheromore, there exists a constant $c_0>0$ such that

$$|I_{J'_\mu}(\tau)| \geqq c_0 \tau^{-\mu/2}\,. \tag{2.30}$$

§ 3. Proof of Lemma 2.1

Suppose that τ is sufficiently large in what follows. In consideration of supports of $\phi(y)$ and $\psi(M_{J_\nu}(\tilde{t}_\nu)(\tau^{\gamma-1}y+y^0)+d_{J_\nu}(\tilde{t}_\nu))$ in the right hand side of (2.22)$'$ we see that for $\nu \geqq 1$

$$M_{J_\nu}(\tilde{t}_\nu)y^0 + d_{J_\nu}(\tilde{t}_\nu) \in V_0 \qquad \text{if} \quad \tilde{t}_\nu \in \operatorname{supp} G_{J_\nu} \subset \Delta_\nu\,, \tag{3.1}$$

and

$$M_j(t_0,0)y^0 + d_j(t_0,0) \in V_0 \qquad \text{if} \quad G_{(j)} \neq 0\,, \tag{3.1$'$}$$

where V_0 is the neighborhood in the hypothesis (H-3) of Theorem 1.

First we consider the case when $\mu \geqq 1$. Because the bicharacteristic curve is admissible and because we get $(j) \neq J'_\mu$, it follows from (3.1)$'$ and the hypothesis (H-3) that

$$\xi^0 \neq {}^tM_j(t_0,0)^{-1}\eta^0 \qquad \text{if} \quad G_{(j)} \neq 0. \tag{3.2}$$

Hence, by means of the integration by parts with respect to y we see that for any $N>0$ there exists a constant C_N such that

$$|G_{(j)}| \leqq C_N \tau^{-N}\,, \tag{3.3}$$

which gives the inequality (2.29) with $\nu=0$.

Let $\kappa > 0$ be a small parameter chosen later on. Let $h(s)$ be a C^∞ function such that $0 \leqq h \leqq 1$, $h = 1$ for $s \leqq 1$ and $h = 0$ for $s \geqq 2$. If we put

$$\psi_{J_\nu}(\tilde{t}_\nu) = h(\kappa^{-3\nu}\tau^{\gamma/2}|\eta^0 - {}^tM_{J_\nu}(\tilde{t}_\nu)\xi^0|)\,, \qquad \nu \geqq 1\,, \tag{3.4}$$

the integration by parts with respect to y shows that

$$|G_{J_\nu}(\tau; \tilde{t}_\nu)| \leqq C_N^\nu \tau^{-N} \qquad \text{if} \quad \tilde{t}_\nu \in \operatorname{supp}(1 - \psi_{J_\nu}) \tag{3.5}$$

because we have

$$|\eta^0 - {}^tM_{J_\nu}(\tilde{t}_\nu)\xi^0| \geqq \kappa^{3\nu}\tau^{-\gamma/2} \qquad \text{on} \quad \operatorname{supp}(1 - \psi_{J_\nu})\,.$$

Since the volume of Δ_ν is $t_0/\nu\,!$, in view of (3.5) it suffices to consider $G'_{J_\nu} = \psi_{J_\nu}G_{J_\nu}$ in place of G_{J_ν} in (2.26). For G'_{J_ν} we have the inequality of the same type as (2.27).

If $\tilde{t}_\nu \in \operatorname{supp} G'_{J_\nu} \cap \Delta_\nu$ then the end point of $C(J_\nu, \tilde{t}_\nu, \rho_0)$ belongs to $V_0 \times V_\varepsilon$ with $\varepsilon = \kappa^\nu\tau^{-\gamma/2}$. Indeed, it is the consequence of (3.1) and

$$|{}^tM_{J_\nu}^{-1}\eta^0 - \xi^0| \leqq \kappa^{2\nu}\tau^{-\gamma/2} \qquad \text{on} \quad \operatorname{supp}\psi_{J_\nu}\,, \tag{3.6}$$

where we used ${}^tM_{J_\nu}^{-1}\kappa^\nu \ll 1$ for κ small enough.

Let $E(J_\nu)$ be a subset of Δ_ν defined by

$$E(J_\nu) = \{\tilde{t}_\nu \in \operatorname{supp} G'_{J_\nu} \cap \Delta_\nu;\ C(J_\nu, \tilde{t}_\nu, \rho_0) \text{ is } \kappa^\nu\tau^{-\gamma/2}\text{-admissible}\}\,.$$

It follows from (H-3) that if $J_\nu \not\supset J'_\mu$ then $E(J_\nu) = \emptyset$. In the case $J_\nu \supset J'_\mu$, if $\tilde{t}_\nu \in E(J_\nu)$ we have (1.6) and (1.7) with $\varepsilon_0 = (\kappa^\nu\tau^{-\gamma/2})^\sigma$. We note that the number of possibilities of subset $\{q_1, \cdots, q_\mu\}\,(\equiv Q_\mu)$ satisfying (1.6) is inferior to 2^ν. We denote by Q^j_μ $(j = 1, \cdots, n_\nu, n_\nu \leqq 2^\nu)$ all such subsets. For a fixed subset $Q_\mu = \{q_1, \cdots, q_\mu\}$ let q'_j $(j = 1, \cdots, n'_\nu)$ be all elements of $\{1, \cdots, \nu + 1\}$ satisfying (1.8). Clearly, the number n'_ν is superior to $[(\nu - \mu + 1)/2]$. We put

$$\phi_{J_\nu}(t_\nu) = \sum_{r=1}^{\mu} h(|t_{q_r} - t'_r|/\varepsilon_0) \sum_{j=1}^{n'_\nu} h(|t_{q'_j} - t_{q'_j-1}|/\varepsilon_0)\,, \tag{3.7}$$

where $\varepsilon_0 = (\kappa^\nu\tau^{-\gamma/2})^\sigma$. For each Q^j_μ if we define $\phi^j_{J_\nu}$ similarly, it follows from (H-3) that

$$E(J_\nu) \cap \left(\bigcap_{j=1}^{n_\nu} \operatorname{supp}(1 - \phi^j_{J_\nu})\right) = \emptyset\,. \tag{3.8}$$

For a moment we suppose $J_\nu \supset J'_\mu$. Put

$$(3.9)\quad \begin{cases} G^1_{J_\nu} = G'_{J_\nu}\phi^1_{J_\nu}, \quad G^j_{J_\nu} = G'_{J_\nu}\phi^j_{J_\nu}\prod_{m=1}^{j-1}(1-\phi^m_{J_\nu}), \quad 2 \leqq j \leqq n_\nu, \\ G''_{J_\nu} = G'_{J_\nu}\prod_{m=1}^{n_\nu}(1-\phi^m_{J_\nu}). \end{cases}$$

For G^j_J and G''_J we have the inequalities of the same type as (2.27). We have a decomposition

$$(3.10)\qquad I'_{J_\nu}(\tau) = \sum_{j=1}^{n_\nu} I^j_{J_\nu}(\tau) + I''_{J_\nu}(\tau),$$

where I'_{J_ν}, $I^j_{J_\nu}$, I''_{J_ν} are defined by (2.26) with G_{J_ν} replaced by G'_{J_ν}, $G^j_{J_\nu}$, G''_{J_ν}, respectively. It follows from (3.8) that if $\tilde{t}_\nu \in \operatorname{supp} G''_{J_\nu}$ then there exists a $q \in \{1, \cdots, \nu\}$ such that

$$(3.11)\qquad |\partial_{t_q} F_J(\tilde{t}_\nu)| \geqq \kappa^\nu \tau^{-\gamma/2}/2.$$

In order to show this inequality we prepare a proposition which is the immediate consequence of Theorem 2.3 of Kumano-go-Taniguchi-Tozaki [8].

PROPOSITION 3.1. *Let $\phi_j(t, s) = \phi_j(t, s; x, \xi)$ be solutions of eiconal equations*

$$(3.12)\qquad \partial_t\phi_j + \lambda_j(t, x, d_x\phi_j) = 0, \quad \phi_j(t, s)\big|_{t=s} = x\cdot\xi \qquad j = 1, \cdots, \ell.$$

For a sequence of phase-functions

$$\phi_{j_1}(t_0, t_1), \quad \phi_{j_2}(t_1, t_2), \cdots, \phi_{j_{\nu+1}}(t_\nu, 0)$$

($J_\nu = (j_1, \cdots, j_{\nu+1})$, $\tilde{t}_\nu \in \Delta_\nu$) if we denote by $\Phi_{J_\nu}(t_0, \tilde{t}_\nu, 0; x, \xi)$ the multi-product

$$(\phi_{j_1}(t_0, t_1)\#\phi_{j_2}(t_1, t_2)\#\cdots\#\phi_{j_{\nu+1}}(t_\nu, 0))(x, \xi),$$

then we get

$$(3.13)\quad \partial_{t_q}\Phi_{J_\nu}(t_0, \tilde{t}_\nu, 0; x, \xi)|_{\substack{x=z^\nu\\ \xi=\eta}} = (\lambda_{j_q} - \lambda_{j_{q+1}})(t_q, x^q, \xi^q), \qquad q = 1, \cdots, \nu,$$

where

$$(x^q, \xi^q) = \chi_{j_{q+1}}(t_q, t_{q+1}) \cdots \chi_{j_{\nu+1}}(t_\nu, 0)(y, \eta)$$

and

$$z^\nu = \pi\chi_{j_1}(t_0, t_1) \cdots \chi_{j_{\nu+1}}(t_\nu, 0)(y, \eta).$$

Because $\phi_j(t, s; x, \xi) = (\pi_j(s, t)x)\cdot\xi = M_j^{-1}(t, s)(x - d_j(t, s))\cdot\xi$ we have

$$\begin{aligned}\Phi_{J_\nu}(t_0, \tilde{t}_\nu, 0; x, \xi) &= (\pi_{J_\nu}(\tilde{t}_\nu)x)\cdot\xi \\ &= M_{J_\nu}(t_\nu)^{-1}(x - d_{J_\nu}(\tilde{t}_\nu))\cdot\xi.\end{aligned}$$

Then we get

$$\partial_{t_q}\Phi_{J_\nu}|_{\substack{x=M_{J_\nu}y^0+d_{J_\nu}\\ \xi=\eta^0}} = (\xi^0 - {}^tM_{J_\nu}^{-1}\eta^0)\cdot\partial_{t_q}(d_{J_\nu} + M_{J_\nu}y^0) + \partial_{t_q}F_{J_\nu}\,. \tag{3.14}$$

In view of (3.13) and (3.6) we get (3.11) because the first term of the right hand side of (3.14) is smaller than $\kappa^\nu\tau^{-\gamma/2}/2$ if we choose κ sufficiently small.

We shall show that for any $N>0$ there exists a constant C_N such that

$$|I''_{J_\nu}(\tau)| \leqq C_N^\nu \tau^{-N}/\nu\,!\,. \tag{3.15}$$

For a fixed τ, let $\sum_j \psi^j_{J_\nu}(\tilde{t}_\nu) \equiv 1$ be a partition of unity on $\operatorname{supp} G''_{J_\nu} \cap \Delta_\nu$ such that the number of $\psi^j_{J_\nu}$ whose supports superpose each other is inferior to 2^ν and we have

$$|\tilde{t}_\nu - \tilde{t}'_\nu| \leqq \kappa^{2\nu}\tau^{-\gamma/2} \qquad \text{if} \quad \tilde{t}_\nu, \tilde{t}'_\nu \in \operatorname{supp}\psi^j_{J_\nu}\,, \tag{3.16}$$

$$|\partial^\alpha_{\tilde{t}_\nu}\psi^j_{J_\nu}| \leqq C_\alpha^\nu \tau^{\gamma|\alpha|/2} \tag{3.17}$$

for a constant C_α independent of τ and ν. In fact, we can make such a partition of unity by putting

$$\psi^j_J = h_j/\sum_j h_j, \qquad h_j(\tilde{t}_\nu) = h(|\tilde{t}_\nu - \tilde{t}^j_\nu|\kappa^{-2\nu}\tau^{\gamma/2})$$

for suitable points $\tilde{t}^j_\nu \in \operatorname{supp} G''_J \cap \Delta_\nu$. By (3.11) and (3.16) we see that

$$|\partial_{t_q}F_{J_\nu}| \geqq \kappa^\nu\tau^{-\gamma/2}/3 \qquad \text{on} \quad \operatorname{supp}\psi^j_{J_\nu} \tag{3.18}$$

because $\kappa^{2\nu} \ll \kappa^\nu$. Note that

$$\int_{\Delta_\nu} dt_\nu = \int_{\Delta_{\nu-1}} dt_1 \cdots dt_{q-1}dt_{q+1}\cdots dt_\nu \int_{t_{q+1}}^{t_{q-1}} dt_q\,. \tag{3.19}$$

By means of (3.18) it follows from the integration by parts that

$$\begin{aligned}\int_{t_{q+1}}^{t_{q-1}} e^{i\tau F_\nu}\psi^j_\nu G''_\nu dt_q &= [-i\tau^{-1}e^{i\tau F_\nu}\psi^j_\nu G''_\nu(\partial_{t_q}F_\nu)^{-1}]^{t_q=t_{q-1}}_{t_q=t_{q+1}} \\ &\quad + i\tau^{-1}\int_{t_{q+1}}^{t_{q-1}} e^{i\tau F_\nu}\partial_{t_q}(\psi^j_\nu G''_\nu(\partial_{t_q}F_\nu)^{-1})dt_q\,,\end{aligned} \tag{3.20}$$

where for the brevity we wrote ν instead of J_ν. Noting that

$$F_{J_\nu}(\tilde{t}_\nu)|_{t_q=t_{q+1}} = F_{J^q_{\nu-1}}(t_1, \cdots, t_{q-1}, t_{q+1}, \cdots, t_\nu) \tag{3.21}$$

for $J^q_{\nu-1} = (j_1, \cdots, j_{q-1}, j_{q+1}, \cdots, j_{\nu+1})$, in view of (3.17)–(3.20) we have

$$I''_{J_\nu}(\tau) = \tau^{\gamma/2-1}\big(\sum_q \tilde{I}_{J^q_{\nu-1}}(\tau) + \tau^\gamma \tilde{I}_{J_\nu}(\tau)\big)\,,$$

where the first term of the right hand side is the sum with respect to all q satisfying (3.18) for each $\psi^j_{J_\nu}$. Here $\tilde{I}_{J^q_{\nu-1}}(\tau)$, $I_{J_\nu}(\tau)$ are defined, as in (2.26), with phases $F_{J^q_{\nu-1}}$, F_{J_ν} and amplitudes $G_{J^q_{\nu-1}}$, $\tilde{G}_{J_\nu}$ satisfying inequalities of the same type as (2.27), respectively, because the number of the superposition of $\psi^j_{J_\nu}$ is inferior to 2^ν and because we have (3.17) and (3.18). If $j_{q-1} \neq j_{q+1}$ in $J^q_{\nu-1}$, we can use the above procedure for $\tilde{I}_{J^q_{\nu-1}}(\tau)$ instead of $I''_{J_\nu}(\tau)$ because we have the inequality similar to (3.11) if $\tilde{t}_{\nu-1} \in \operatorname{supp} \tilde{G}_{J^q_{\nu-1}} \cap \Delta_{\nu-1}$. Note that for $J_\nu \in \Pi_\nu$ we have

$$F_{J_\nu}(\tilde{t}_\nu) = F_{J'_{\nu-1}}(\tilde{t}_{\nu-1}) \qquad \text{if } j_q = j_{q+1}\,, \tag{3.22}$$

where $J'_{\nu-1}$ and $\tilde{t}_{\nu-1}$ are defined by removing j_q and t_q in J_ν and $\tilde{t}_\nu$, respectively. If we define (2.26) also for $J_\nu \in \Pi_\nu \backslash \Pi^0_\nu$ then we get

$$I_{J_\nu}(\tau) = I_{J'_{\nu-1}}(\tau) \qquad \text{if } j_q = j_{q+1}\,, \tag{3.23}$$

where the amplitude of $I_{J'_{\nu-1}}(\tau)$ is given in the form

$$\int_{t_{q+1}}^{t_{q-1}} G_{J_\nu}(\tau, \tilde{t}_\nu) dt_q\,.$$

It follows from the formula (3.23) with $J_\nu = J^q_{\nu-1}$ that we can use the above procedure also for $I_{J^q_{\nu-1}}(\tau)$ when $j_{q-1} = j_{q+1}$ in $J^q_{\nu-1}$. Repeating the integration by parts with respect to t_q, in view of $3\gamma/2 - 1 < 0$ we get (3.15) because (2.29) with $\nu = 0$ is verified.

When $J_\nu \not\supset J'_\mu$, the method which gives (3.15) is applicable to show (2.29) for $I_{J_\nu}(\tau)$ because $E(J_\nu) = \emptyset$ and so we have (3.11) if $\tilde{t}_\nu \in \operatorname{supp} G'_{J_\nu}$.

To get (2.28) it suffices to show

$$|I^j_J(\tau)| \leqq C^\nu \tau^{-\mu/2-\sigma\gamma(\nu-\mu)/4}/\nu\,! \tag{3.24}$$

for a constant C independent of τ and ν, because $n_\nu \leqq 2^\nu$. We consider a $I^j_J(\tau)$. For the simplicity we write $j = 0$. Noting the hypothesis (H-3) and support of ϕ_{J_ν} defined by (3.7), we can see that there exist two subsets $\{p_1, \cdots, p_\mu\}$ and $\{p'_1, \cdots, p'_\mu\}$ of $\{1, \cdots, \nu\}$ such that

$$\begin{cases} p_j \leqq q_j \leqq p'_j & (j = 1, \cdots, \mu) \\ p'_j < p_{j+1} & (j = 1, \cdots, \mu - 1) \end{cases} \tag{3.25}$$

and with $\varepsilon_0 = (\kappa^\nu \tau^{-\gamma/2})^\sigma$ we have

$$|t_{p_r} - t_{p'_r}| \leqq \nu\varepsilon_0, \qquad r = 1, \cdots, \mu \tag{3.26}$$

$$\begin{cases} |t_{p_r} - t_{p_r - 1}| \geqq \varepsilon_0 \\ |t_{p'_r} - t_{p'_r + 1}| \geqq \varepsilon_0 \quad r = 1, \cdots, \mu, \quad t_{\nu+1} = 0 \end{cases} \tag{2.27}$$

$$\text{if} \quad \tilde{t}_\nu \in \operatorname{supp} G^0_{J_\nu} \cap \Delta_\nu .$$

Note that the number of possibilities of subsets $\{p_1, \cdots, p_\mu\} (\equiv P_\mu)$ and $\{p'_1, \cdots, p'_\mu\} (\equiv P'_\mu)$ is inferior to 4^ν. We denote by (P^j_μ, P'^j_μ) $(j = 1, \cdots, n''_\nu$, $n''_\nu \leqq 4^\nu)$ all couples of two subsets.

For a (P_μ, P'_μ) we put

$$f_{J_\nu}(\tilde{t}_\nu) = \prod_{r=1}^{\mu} (1 - h)(|t_{p_r} - t_{p_r - 1}|/\varepsilon_0)(1 - h)(|t_{p'_r} - t_{p'_r + 1}|/\varepsilon_0) . \tag{3.28}$$

If we define $f^j_{J_\nu}$ for each (P^j_μ, P'^j_μ) similarly as f_{J_ν}, then we get by means of (H-3)

$$\operatorname{supp} G^0_{J_\nu} \cap \Delta_\nu \cap \left(\bigcap_{j=1}^{n''_\nu} \operatorname{supp}(1 - f^j_{J_\nu}) \right) = \emptyset .$$

Then we have a decomposition

$$I^0_{J_\nu}(\tau) = \sum_{j=1}^{n''_\nu} I^{0,j}_{J_\nu}(\tau) , \tag{3.29}$$

where $I^{0,j}_{J_\nu}$ are defined by (2.26) with G_{J_ν} replaced by

$$G^{0,1}_{J_\nu} = G^0_{J_\nu} f^1_{J_\nu}, \quad G^{0,j}_{J_\nu} = G^0_{J_\nu} f^j_{J_\nu} \sum_{m=1}^{j-1} (1 - f^m_{J_\nu}), \quad j \geqq 2 ,$$

for which we have inequalities of the same type as (2.26).

We consider a $I^{0,j}_{J_\nu}(\tau)$. For the brevity we write $j = 0$. We take the change of variables $\tilde{t}_\nu = (t_1, \cdots, t_\nu)$ to $(\tilde{s}_\mu, \tilde{w}_{\nu_1}, \tilde{v}_{\nu_2})$ as follows: We put

$$\tilde{s}_\mu = (s_1, \cdots, s_\mu)$$
$$s_r = (t_{p_r} + t_{p_r + 1} + \cdots + t_{p'_r})/(p'_r - p_r + 1) .$$

For any pair $(q, q-1)(q = 1, \cdots, \nu)$ except $(p_r, p_r - 1)$ $(r = 1, \cdots, \mu)$ we put

$$\begin{cases} w_q = t_{q-1} - t_q & \text{if} \quad |t_{q-1} - t_q| \leqq \nu\varepsilon_0 \quad \text{on} \quad \operatorname{supp} G^{0,0}_{J_\nu} , \\ v_q = t_{q-1} - t_q & \text{otherwise} , \end{cases}$$

and put

$$\tilde{w}_{\nu_1} = (w_1, \cdots, w_{\nu_1}) , \qquad \tilde{v}_{\nu_2} = (v_1, \cdots, v_{\nu_2}) .$$

It follows from (3.7) that

$$\nu_1 \geqq n'_\nu \geqq [(\nu-\mu+1)/2]\,. \tag{3.30}$$

By the above change of variables we get

$$I^{0,0}_{J_\nu}(\tau)=\int_{D_1} d\tilde{w}_{\nu_1}\int_{D_2} d\tilde{v}_{\nu_2}\int \tilde{G}^{0,0}_{J_\nu}\exp i\tau\tilde{F}_{J_\nu}d\tilde{s}_\mu\,, \tag{3.31}$$

where $\tilde{G}^{0,0}_{J_\nu}(\tau;\,\tilde{s}_\mu,\,\tilde{w}_{\nu_1},\,\tilde{v}_{\nu_2})=G^{0,0}_{J_\nu}(\tau;\,\tilde{t}_\nu)$ and $\tilde{F}_{J_\nu}(\tilde{s}_\mu,\,\tilde{w}_{\nu_1},\,\tilde{v}_{\nu_2})=F_{J_\nu}(\tilde{t}_\nu)$. Here

$$D_1=\{w_j;\;0\leqq w_j\leqq \nu(\kappa^\nu\tau^{-\gamma/2})^\sigma\}\,,$$
$$D_2=\Big\{v_j\geqq 0;\;\sum_{j=1}^{\nu_2}v_j\leqq t_0\Big\}\,.$$

In view of (3.21) and (3.22) it follows from (H-3) that

$$\tilde{F}_{J_\nu}(\tilde{s}_\mu,\,0,\,\tilde{v}_{\nu_2})=F_{J'_\mu}(\tilde{s}_\mu)\,. \tag{3.32}$$

Noting (3.13) and (3.14), we can see by (H-2) and (H-3) that

$$\tilde{s}_\mu=\tilde{t}'_\mu \Longleftrightarrow \partial_{s_q}F_{J'_\mu}(\tilde{s}_\mu)=0 \qquad \text{for}\quad q=1,\,\cdots,\,\mu\,, \tag{3.33}$$

if $\tilde{s}_\mu\in\pi^*\operatorname{supp}\tilde{G}^{0,0}_{J_\nu}$, where π^* is the natural projection from $R_{\tilde{s}}\times R_{\tilde{w}}\times R_{\tilde{v}}$ to $R_{\tilde{s}}$. By (H-2) we get

$$\det(\partial_{s_p}\partial_{s_q}\tilde{F}_{J_\nu}(\tilde{s}_\mu,\,\tilde{w}_{\nu_1},\,\tilde{v}_{\nu_2}))\neq 0 \qquad \text{on}\quad \operatorname{supp}\tilde{G}^{0,0}_{J_\nu}\cap R_{\tilde{s}}\times D_1\times D_2 \tag{3.34}$$

if we choose κ small enough. On account of (3.33) and (3.34), the implicit function theorem shows the existence of the unique solution $\tilde{s}_\mu(\tilde{w}_{\nu_1},\,\tilde{v}_{\nu_2})$ of the equation

$$\partial_{s_q}\tilde{F}_{J_\nu}(\cdot\,,\,\tilde{w}_{\nu_1},\,\tilde{v}_{\nu_2})=0\,,\qquad q=1,\,\cdots,\,\mu$$

if $(\tilde{w}_{\nu_1},\,\tilde{v}_{\nu_2})$ varies on $D_1\times D_2$. Regarding $(\tilde{w}_{\nu_1},\,\tilde{v}_{\nu_2})$ as parameters we see by means of the stationary phase method (see § 1–2 of [2]) that

$$\begin{aligned}I_{J_\nu}(\tau;\,\tilde{w},\,\tilde{v})=\int \tilde{\tilde{G}}^{0,0}_{J_\nu}\exp i\tau\tilde{F}_{J_\nu}d\tilde{s}_\mu &\sim (2\pi/\tau)^{\mu/2}|\det Q|^{-1/2}e^{(\pi i/4)\operatorname{sgn}Q}\\ &\times\sum_{j=0}^{\infty}(R^j\tilde{\tilde{G}}^{0,0}_{J_\nu})\tau^{-j}/j\,!,\qquad \tau\to\infty\,,\end{aligned} \tag{3.35}$$

where

$$\begin{aligned}Q&\equiv Q(\tilde{w},\,\tilde{v})\\ &=\Big(\partial_{s_p}\partial_{s_q}\tilde{F}_{J_\nu}(\tilde{s}_\mu(\tilde{w},\,\tilde{v}),\,\tilde{w},\,\tilde{v});\;\begin{matrix}p=1,\,\cdots,\,\mu\rightarrow\\ q=1,\,\cdots,\,\mu\downarrow\end{matrix}\Big)\,,\end{aligned}$$

$R = i < Q^{-1}\partial_{\tilde{s}'_\mu}, \partial_{\tilde{s}'_\mu} > /2$ and $\tilde{\tilde{G}}^{0,0}_{J_\nu}(\tilde{s}'_\mu, \tilde{w}, \tilde{v}) = \tilde{G}^{0,0}_{J_\nu}(\tilde{s}_\mu, \tilde{w}, \tilde{v})$. Here $(\tilde{s}_\mu, \tilde{w}, \tilde{v}) \to \tilde{s}'_\mu$ is a C^∞ mapping such that

$$\tilde{s}'_\mu(\tilde{s}_\mu, \tilde{w}, \tilde{v}) = \tilde{s}_\mu - \tilde{s}_\mu(\tilde{w}, \tilde{v}) + 0(|\tilde{s}_\mu - \tilde{s}_\mu(\tilde{w}, \tilde{v})|^2)\,,$$
$$\tilde{F}_{J_\nu}(\tilde{s}_\mu, \tilde{w}, \tilde{v}) = \tilde{F}_{J_\nu}(\tilde{s}_\mu(\tilde{w}, \tilde{v}), \tilde{w}, \tilde{v}) + < Q\tilde{s}'_\mu, \tilde{s}'_\mu > /2\,.$$

Since volumes of D_1 and D_2 are estimated by $\tau^{-\sigma\gamma\nu_1/2}/\nu_1!$ and $t_0^{\nu_2}/\nu_2!$ respectively, in view of (3.35) and (3.30) it is easy to see that $I^{0,0}_{J_\nu}(\tau)$ is inferior to the right hand side of (3.24). It follows from (3.29) that we get (3.24), which shows (2.28).

When $J_\nu = J'_\mu$ we note that n_ν of (3.10) and n''_ν of (3.29) are 1 and that the change of variables $\tilde{t}_\nu \to (\tilde{s}_\mu, \tilde{w}_{\nu_1}, \tilde{v}_{\nu_2})$ is not necessary, that is, $\tilde{s}_\mu = \tilde{t}_\mu$. Hence the formula (3.35) gives (2.30) if we show that there exists a constant $c_1 > 0$ such that

$$(3.36) \qquad |G_{J'_\mu}(\tau; \tilde{t}'_\mu)| \geqq c_1 > 0\,.$$

Since $\xi^0 = {}^tM_{J'_\mu}(\tilde{t}'_\mu)\eta^0$ it follows from (2.22)′ that

$$(3.37) \qquad G_{J'_\mu}(\tau; \tilde{t}'_\mu) = \int \psi(M_{J'_\mu}(\tilde{t}'_\mu)(\tau^{\gamma-1}y + y^0) + d_{J'_\mu}(\tilde{t}'_\mu))\phi(y)\beta'_{J_\nu}dy\,.$$

Noting (2.8) and the change of variables (2.23), we see that $\beta^j_{J_\nu}$ is almost equal to a non-zero constant on supp ϕ when τ is sufficiently large (note the argument of ψ in (3.37)). Then we get (3.36) if τ is large enough.

We consider the case when $\mu = 0$. If $J'_\mu = (j)$, that is, the admissible trajectory C_μ is bic. curve w.r.t., λ_j then there exists a constant $c_2 > 0$ such that

$$(3.36)' \qquad |G_{(j)}(\tau)| \geqq c_2 > 0 \quad \text{if } \tau \text{ large enough,}$$

which shows (2.30). Inequality (2.29) is obtained in the same manner as in the case $\mu \geqq 1$. If we replace the definition of $\phi_{J_\nu}(\tilde{t}_\nu)$ by

$$(3.7)' \qquad \phi_{J_\nu}(\tilde{t}_\nu) = \prod_{j=1}^{n'_\nu} h(|t_{q'_j} - t_{q'_{j-1}}|/\varepsilon_0) \quad \text{with} \quad \varepsilon_0 = (\kappa^\nu\tau^{-\gamma/2})^\sigma\,,$$

the second term of the right hand side of (3.10) is estimated by the right hands side of (2.29), and the first term of the right hand side of (3.10) estimated by the right hand side of (2.28) because the volume of $\Delta_\nu \cap$ supp ϕ_{J_ν} is inferior to $t_0^\nu\tau^{-\sigma\gamma\nu}/\nu!$ Then we have (2.28) when $\mu = 0$.

Remark 1. It follows from Lemma 2.1 that if U is the solution of

(C.P) for the initial value G defined by (2.9) we have

$$c_3 \leqq |(\widehat{\psi U})(t_0;\ \tau_k\xi^0)|\tau^{(1-\gamma)n+1+\mu/2} \leqq c_3^{-1} \quad \text{if}\quad k \geqq k_0,\quad k_0 \text{ large enough,} \tag{3.38}$$

where the function $\psi \in C_0^\infty(V_0)$ was taken before (2.19) and c_3 is a positive constant independent of τ_k. By means of the formula (3.35) and the argument in getting (3.36) we see that

$$c_3 = c_4^\mu|\det(\partial_{t_p}\partial_{t_q}\xi^0\cdot(M_{J'_\mu}(\tilde{t}'_\mu)y^0 + d_{J'_\mu}(\tilde{t})))|^{-1/2}, \tag{3.39}$$

where c_4 is a positive constant independent of μ and the choice of $\psi \in C_0^\infty(V_0)$ such that $\psi = 1$ near x^0. We note that the number k_0 of (3.38) depends on the choice of ψ.

Remark 2. The proof of Theorem 1 is also valid when there exist plural admissible trajectories which link ρ_0 with $\delta_0 = (x^0, \xi^0)$ if numbers of step are different each other, the condition (H-2) is satisfied for each adm. traj. and the following condition is verified:

(H-3)′ Let V_ε be a ε-conic neighborhood of ξ^0. There exist an open neighborhood V_0 of x^0 and $0 < \sigma \leqq 1$ such that any ε-admissible trajectory issuing from ρ_0 whose end point belongs to $V_0 \times V_\varepsilon$ is contained in ε^σ-neighborhood of some admissible trajectory given above if $\varepsilon > 0$ small enough.

Indeed, it is clear if we note that the contribution with respect to the admissible trajectory of the smallest step is dominant.

§ 4. Application of Theorem 1

We consider an example of L with $l = 2$ on $R_t^1 \times R_x^1$ whose λ_1 and λ_2 are defined

$$\lambda_1 = \alpha(t)D_x, \qquad \lambda_2 = -\lambda_1, \tag{4.1}$$

where $\alpha(t)$ is a C^∞ function such that

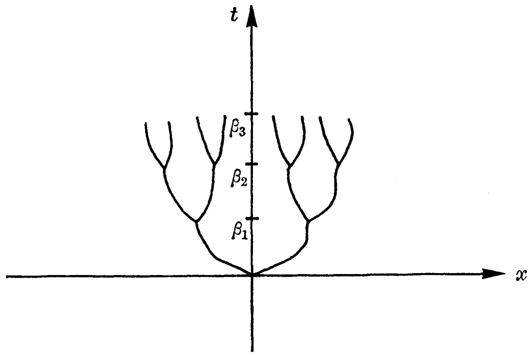
$$\begin{cases} \alpha(\beta_j) = 0, & (-1)^j\alpha(t) > 0 \quad \text{if}\quad t \in (\beta_j, \beta_{j+1}) \\ \alpha'(\beta_j) \neq 0, & \displaystyle\int_{\beta_j}^{\beta_{j+1}}|\alpha(t)|dt = 2^{-(j+1)} \end{cases} \tag{4.2}$$

for a sequence $\{\beta_j\}_{j=0}^\infty$ satisfying

$$0=\beta_0<\beta_1<\cdots<\beta_j<\cdots<T, \quad \beta_j\to T \quad (j\to\infty).$$

Here T is a fixed positive. We can show the existence of $\alpha(t)$ in the same way as in Proposition 1.3 of [5].

If we take β_j $(j=1,2,\cdots)$ as a fixed positive t_0 in the definition of the trajectory, we have 2^j admissible trajectories issuing from $(0,\eta^0)$, $\eta^0\neq 0$ whose end points are $(\pm(2k-1)2^{-j},\eta^0)$, $k=1,\cdots,2^{j-1}$ (see Fig.) Each



adm. traj. satisfies hypotheses of Theorem 1. In fact, (H-1) and (H-2) are verified because each adm. traj. is isolated in the sense of the Definition 1.2. We can take arbitrary positive σ smaller than 1 in (H-3) because $\alpha'(\beta_k)\neq 0$ $(k<j)$. It is easy to check the hypothesis (H-2) because M_j $(j=1,2)$ in (1.2) are identity and because

$$\begin{aligned}\partial_{t_q}d_{J'_\mu}(\tilde t_\mu)&=\partial_{t_q}d_{j_q}(t_{q-1},t_q)+\partial_{t_q}d_{j_{q+1}}(t_q,t_{q+1})\\&=\pm\alpha(t_q)-(\mp\alpha(t_q))=\pm 2\alpha(t_q).\end{aligned}$$

It follows from Theorem 1 that there exists an initial value G such that

$$WF\,G=\{(0,c\eta^0);\ c>0\}$$

$$WF\,U(\beta_j)=\{(\pm(2k-1)2^{-j},c\eta^0);\ k=1,\cdots,2^{j-1},\ c>0\}, \tag{4.3}$$

where U is the solution of (C.P) for the initial value G. Here we used Theorem 3.4 of [9] to show another direction of inclusion in (4.3). Because we take G uniformly with respect to β_j $(j=1,2,\cdots)$ and because $WF\,U\subset T^*(R^1_t\times R^1_x)$ is closed, we have

THEOREM 2. *Let L be the hyperbolic system of this section. Then there exists an initial value G such that*

$$WF\,G=\{(0,c\eta^0);\ c>0\} \qquad WF\,U(T)=[-1,1]\times\{c\eta^0;\ c>0\}, \tag{4.4}$$

where U is the solution of (C.P).

COROLLARY. *For G and U in Theorem 2 we have*

$$\begin{cases} \text{sing supp } G = \{0\} \\ \text{sing supp } U(T) = [-1, 1]. \end{cases} \tag{4.5}$$

As stated in Introduction, the result analogous to (4.4) had been given by [5] by using the work of [12], that is, the precise propagation of singularities for the equation

$$\partial_t^2 - t^{2s}\partial_x^2 - at^{s-1}\partial_x,$$

where a is a constant and $s = 1, 2, \cdots$, (cf. [1] [3]).

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