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Nagoya Math. J.
Vol. 100 (1985), 135-143

UMBILICAL POINTS ON SURFACES IN R^N

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Let $\varphi: M \rightarrow R^N$ be an isometric imbedding of a compact, connected surface M into a Euclidean space R^N. ψ is said to be umbilical at a point p of M if all principal curvatures are equal for any normal direction. It is known that if the Euler characteristic of M is not zero and $N = 3$, then ψ is umbilical at some point on M. In this paper we study umbilical points of surfaces of higher codimension. In Theorem 1, we show that if M is homeomorphic to either a 2-sphere or a 2-dimensional projective space and if the normal connection of ψ is flat, then ψ is umbilical at some point on M. In Section 2, we consider a surface M whose Gaussian curvature is positive constant. If the surface is compact and $N = 3$, Liebmann's theorem says that it must be a round sphere. However, if $N \geq 4$, the surface is not rigid: For any isometric imbedding Φ of R^3 into R^4 $\Phi(S^2(r))$ is a compact surface of constant positive Gaussian curvature $1/r^2$. We use Theorem 1 to show that if the normal connection of ψ is flat and the length of the mean curvature vector of ψ is constant, then $\psi(M)$ is a round sphere in some $R^3 \subset R^N$. When $N = 4$, our conditions on ψ is satisfied if the mean curvature vector is parallel with respect to the normal connection. Our theorem fails if the surface is not compact, while the corresponding theorem holds locally for a surface with parallel mean curvature vector (See Remark (i) in Section 3).

The author wishes to thank Professor Hung-Hsi Wu for his constant encouragement and valuable suggestions.

§ 1. Preliminaries

Let M be a connected n-dimensional C^∞ Riemannian manifold and let $\psi: M \rightarrow R^N$ be an isometric immersion of M into an N-dimensional Euclidean space R^N. Let D and $\bar{D}$ denote the covariant differentiations of M and R^N respectively. Let X, Y be tangent vector fields on M. Then

Received August 16, 1984.

(1.1) $$\bar{D}_X Y = D_X Y + B(X, Y)$$

where $B(X, Y)$ is the normal component of $\bar{D}_X Y$.

Let ξ be a normal vector field on M. We write

(1.2) $$\bar{D}_X \xi = -A_\xi X + D_X^\perp \xi$$

where $A_\xi X$ and $D_X^\perp \xi$ are the tangential and normal components of $\bar{D}_X \xi$. Then we have

(1.3) $$\langle A_\xi X, Y \rangle = \langle B(X, Y), \xi \rangle$$

where $\langle\ ,\ \rangle$ denotes the inner product of R^N. The linear transformation A_ξ on the tangent bundle TM is called the *shape operator* of M with respect to ξ. Since A_ξ is symmetric, i.e.

(1.4) $$\langle A_\xi X, Y \rangle = \langle X, A_\xi Y \rangle ,$$

all eigenvalues of A_ξ are real. An eigenvalue of A_ξ is called a *principal curvature* with respect to ξ. An eigenvector of A_ξ is called a *principal vector* with respect to ξ. The *mean curvature vector* H is defined by

(1.5) $$H = \frac{1}{n} \operatorname{trace}(B)$$

The length of H is called the *mean curvature.*

Let R and $R^\perp$ be the curvature tensors associated with D and $D^\perp$ respectively, i.e.

(1.6) $$R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z$$

(1.7) $$R^\perp(X, Y)\xi = D_X^\perp D_Y^\perp \xi - D_Y^\perp D_X^\perp \xi - D_{[X,Y]}^\perp \xi$$

where X, Y, Z are tangent to M and ξ is normal to M.

Then for any tangent vector fields X, Y, Z, W and normal vector fields ξ, η, we have the following equations:

(1.8) $$\langle R(X, Y)Z, W \rangle = -\langle B(X, Z), B(Y, W) \rangle + \langle B(Y, Z), B(X, W) \rangle$$
(Gauss equation)

(1.9) $$\langle R^\perp(X, Y)\xi, \eta \rangle = \langle (A_\xi A_\eta - A_\eta A_\xi)X, Y \rangle$$ (Ricci equation)

The normal connection $D^\perp$ is said to be *flat* if $R^\perp = 0$. (1.9) implies that $D^\perp$ is flat at $p \in M$ if and only if

(1.10) $$A_\xi A_\eta = A_\eta A_\xi$$

for any two normal vectors ξ and η at p. Thus if $D^{\perp}$ is flat at $p \in M$, there exists an orthonormal base $e_1, \cdots, e_n$ of T_pM such that each e_i ($i = 1, \cdots, n$) is a principal vector with respect to any normal vector at p.

A point p is said to be *umbilical with respect to* ξ if A_ξ is proportional to the identity transformation of T_pM. ψ is said to be *umbilical at* p if p is umbilical with respect to A_ξ for all normal vectors ξ at p. ψ is called *totally umbilical* if ψ is umbilical at every point of M. It is well known that if ψ is totally umbilical, then $\psi(M)$ is an open subset of either an n-dimensional affine subspace or an n-dimensional round sphere. (See, for instance, [3] for proof.)

§ 2. Umbilical points of surfaces in R^N

In this section we prove the following theorem.

THEOREM 1. *Let M be a compact surface which is homeomorphic to a 2-sphere or a 2-dimensional projective space and let $\psi: M \to R^N$ be an isometric imbedding. Suppose that the normal connection of ψ is flat. Then ψ is umbilical at some point $p_0 \in M$.*

Proof. Suppose that ψ does not have any umbilical point. Then at each point p of M there exists a neighborhood U_p of p and a normal vector field ξ on U_p such that A_ξ is not proportional to the identity transformation. We choose each U_p in such a way that U_p is simply connected and for any p and q $U_p \cap U_q$ is either empty or connected. Since M is compact, there exist a finite number of points $p_1, \cdots, p_k$ such that $M = U_{p_1} \cup \cdots \cup U_{p_k}$. We simply denote U_{p_i} by U_i. Let ξ_i be a normal vector field defined on U_i such that A_{ξ_i} is not proportional to the identity at each point of U_i. At each point of U_i, the eigenvectors of A_{ξ_i} form a pair of lines (i.e. 1-dimensional linear subspaces) in the tangent plane. Since U_i is simply connected, there exist continuous line fields L_1^i and L_2^i on U_i such that at each q in U_i $L_1^i(q)$ and $L_2^i(q)$ contain all eigenvectors of $A_{\xi_i(q)}$.

Suppose $U_i \cap U_j \neq \phi$. Let $q \in U_i \cap U_j$. Since $A_{\xi_i(q)}$ and $A_{\xi_j(q)}$ are not proportional to the identity transformation and the normal connection is flat, all eigenvectors of $A_{\xi_i(q)}$ and $A_{\xi_j(q)}$ coincide. This implies that either (i) $L_1^i(q) = L_1^j(q)$ and $L_2^i(q) = L_2^j(q)$ or (ii) $L_1^i(q) = L_2^j(q)$ and $L_2^i(q) = L_1^j(q)$. Since $U_i \cap U_j$ is simply connected, it follows from the continuity of the line fields that if (i) (or (ii)) occurs at one point of $U_i \cap U_j$, it must hold for all points of $U_i \cap U_j$. By renaming the line fields if necessary, we may assume that $L_1^i \equiv L_1^j$ and $L_2^i \equiv L_2^j$ on $U_i \cap U_j$. Let $\{U_{i_1}, \cdots, U_{i_s}\}$ be a chain of the elements of $\{U_i : i = 1, \cdots, k\}$, i.e. a subset of $\{U_i : i = 1, \cdots, k\}$ which satisfies $U_{i_t} \cap U_{i_{t+1}} \neq \phi$ for all $t = 1, \cdots, s-1$. Suppose that we obtain a line field $L_1^{i_s}$ on U_{i_s} by the continuation of $L_1^{i_1}$ along the chain. If $U_{i_s} \cap U_{i_1} \neq \phi$, it may well happen that $L_1^{i_s}$ coincides with $L_2^{i_1}$ rather than $L_1^{i_1}$ on $U_{i_s} \cap U_{i_1}$. But in the case when M is simply connected (i.e. homeomorphic to a 2-sphere), it follows from the standard monodromy argument that $L_1^{i_s}$ always coincides with $L_1^{i_1}$. This implies that a global continuous line field L_1 can be defined on M. This is a contradiction because there is no global continuous line field on a 2-sphere. Thus if M is homeomorphic to a 2-sphere, there exists at least one point on M where ψ is umbilical.

Now we consider the case when M is homeomorphic to a 2-dimensional projective space. Suppose that ψ does not have any umbilical point. Then, as we see in the above argument, there exists an open covering $\{U_i : i = 1, \cdots, k\}$ of M and continuous line fields L_1^i and L_2^i defined on U_i such that if $U_i \cap U_j \neq \phi$, either $L_1^i \equiv L_2^j$ or $L_1^i \equiv L_2^j$ on $U_i \cap U_j$. Let $\tilde{M}$ be the standard double covering of M which is homeomorphic to a 2-sphere and let $\pi : \tilde{M} \to M$ be the projection. Let U_{i1} and U_{i2} be the connected components of $\pi^{-1}(U_i)$. Let $L_1^{i\lambda}$ ($i = 1, \cdots, k$, $\lambda = 1, 2$) be the unique line field on $U_{i\lambda}$ which satisfies $d\pi(L_1^{i\lambda}) = L_1^i$. In a similar way, a continuous line field $L_2^{i\lambda}$ is defined. Now we have an open covering of $\tilde{M}$, $\{U_{i\lambda}\}$, and continuous line fields $L_1^{i\lambda}$ and $L_2^{i\lambda}$ on $U_{i\lambda}$. Moreover, if $U_{i\lambda} \cap U_{j\mu} \neq \phi$, we have either $L_1^{i\lambda} \equiv L_1^{j\mu}$ or $L_1^{i\lambda} \equiv L_2^{j\mu}$ on $U_{i\lambda} \cap U_{j\mu}$. Thus, using the standard monodromy argument again, we obtain a global continuous line field on $\tilde{M}$, which is a contradiction. Therefore, if M is homeomorphic to a 2-dimensional projective space, there exists at least one point of M where ψ is umbilical. This completes the proof of Theorem 1.

§ 3. Surfaces in R^N with positive constant curvature and constant mean curvature

In this section we use Theorem 1 to prove the following theorem.

THEOREM 2. *Let M be a compact surface with constant Gaussian curvature $c^2 > 0$ and let $\psi : M \to R^N$ be an isometric imbedding. Suppose that the mean curvature of ψ is constant, i.e. $|H|$ is constant, and* the *normal* *connection is flat. Then $\psi(M)$ is a round 2-sphere in a 3-dimensional affine space $\boldsymbol{R}^3 \subset \boldsymbol{R}^N$.*

Proof. First we define a function on M by

$$F(p) = |H(p)|^2 - K(p) \quad (p \in M) \tag{3.1}$$

where $H(p)$ is the mean curvature vector at p and $K(p)$ is the Gaussian curvature of M at p. We prove the following lemma:

LEMMA 3.1. *$F(p) = 0$ if and only if ψ is umbilical at p.*

Proof. Let $(\xi_1, \xi_2, \cdots, \xi_{N-2})$ be an orthonormal frame of $T_p^\perp M$, the normal space of M at p. Using (1.8), we obtain

$$K(p) = \sum_{\alpha=1}^{N-2} \det A_{\xi_\alpha}\,. \tag{3.2}$$

From (1.5) we have

$$H(p) = \frac{1}{2} \sum_{\alpha=1}^{N-2} (\text{trace } A_{\xi_\alpha}) \xi_\alpha \tag{3.3}$$

so that

$$|H(p)|^2 = \frac{1}{4} \sum_{\alpha=1}^{N-2} (\text{trace } A_{\xi_\alpha})^2\,. \tag{3.4}$$

It follows from (3.2) and (3.4) that

$$F(p) = \frac{1}{4} \sum_{\alpha=1}^{N-2} \{(\text{trace } A_{\xi_\alpha})^2 - 4 \det A_{\xi_\alpha}\}\,. \tag{3.5}$$

Using elementary linear algebra, we can see that

$$(\text{trace } A_{\xi_\alpha})^2 - 4 \det A_{\xi_\alpha} \geq 0 \tag{3.6}$$

and the equality holds if and only if every A_{ξ_α} is proportional to the identity transformation. The lemma follows immediately.

Now we return to the proof of Theorem 2. Since M is compact, and the Gaussian curvature is positive, M is homeomorphic to either a 2-sphere or a 2-dimensional projective space. Hence, by Theorem 1, ψ is umbilical at some point p_0. By Lemma 3.1, $F(p_0) = 0$. On the other hand, since both $|H|$ and K are constant on M, F is a constant function on M. Thus $F = 0$ at every point of M. By Lemma 3.1 again, this implies that ψ is umbilical at every point of M. Since M is compact, $\psi(M)$ is a round sphere in some 3-dimensional affine space. This completes the proof of Theorem 2.

Remark. (i) If the mean curvature vector is parallel in the normal bundle, i.e. $D^{\perp}H=0$, then $|H|$ is constant and the normal connection $D^{\perp}$ is fiat unless M is either a minimal surface in $\boldsymbol{R}^4$ or a minimal surface in S^{N-1} ([2]). In [1], Chen and Ludden proved that if the Gaussian curvature of a surface in $\boldsymbol{R}^4$ is positive constant and the mean curvature vector is parallel in the normal bundle, it is an open piece of a round sphere. As we see in the following example, our theorem fails if M is not compact, while the Chen-Ludden theorem holds without global assumptions.

EXAMPLE 1. Let M be a surface of revolution in $\boldsymbol{R}^3$ which is obtained by rotating the curve

$$(x(s), z(s)) = \left(\alpha \cos s, \int_0^s [1-\alpha^2 \sin^2 t]^{1/2}\, dt\right) \tag{3.7}$$

around the z-axis where $s \in (-\varepsilon, \varepsilon)$ for some small $\varepsilon > 0$ and α is a positive number. Then M is a surface of constant Gaussian curvature 1 and if $\alpha \neq 1$, M is not totally umbilical. Let h be the mean curvature of M. h is a function of s only, which is given by

$$h = \frac{1+\alpha^2 \cos 2s}{2\alpha \cos s(1-\alpha^2 \sin s)^{1/2}}.$$

Now we define an isometric imbedding of $\boldsymbol{R}^3$ into $\boldsymbol{R}^4$. First we define a function $\kappa(s)$ by

$$\kappa(s) = \frac{2(\beta^2 - h^2)^{1/2}}{1-\alpha^2 \sin^2 s} \tag{3.8}$$

where β is any positive constant greater than

$$\sup h = \frac{1+\alpha^2 \cos 2\varepsilon}{2\alpha \cos \varepsilon(1-\alpha^2 \sin^2 \varepsilon)^{1/2}}.$$

Since $z(s) = z(s')$ if and only if $s = s'$, $\kappa(s)$ can be regarded as a function of z. $\kappa(z)$ is defined on $(z(-\varepsilon), z(\varepsilon))$ and we may assume, by taking ε small enough, that

$$\int_{z(-\varepsilon)}^{z(\varepsilon)} \kappa(z)dz < \frac{\pi}{2}.$$

We extend $\kappa(z)$ to a non-negative function which is defined on $(-\infty, \infty)$ and satisfies

$$\int_{-\infty}^{\infty} \kappa(z)dz < \pi . \tag{3.9}$$

Then there exists an isometric immersion of $\boldsymbol{R}$ into $\boldsymbol{R}^2$, φ: $z \mapsto (\varphi_1(z), \varphi_2(z))$, whose curvature is equal to $\kappa(z)$ at each z. φ does not have any self-intersection (i.e. is an imbedding) due to (3.9). Using φ, we define a map Φ: $\boldsymbol{R}^3 \to \boldsymbol{R}^4$ by $\Phi(x, y, z) = (x, y, \varphi_1(z), \varphi_2(z))$. Then Φ is an isometric imbedding of $\boldsymbol{R}^3$ into $\boldsymbol{R}^4$. We will show that $\Phi(M)$ is a surface in $\boldsymbol{R}^4$ with constant mean curvature and flat normal connection.

Let ξ be a unit normal vector to M in $\boldsymbol{R}^3$ and ξ' be a unit normal vector to $\Phi(\boldsymbol{R}^3)$ in $\boldsymbol{R}^4$. Let X_1 be a unit tangent vector to the generating curve $(x(s), z(s))$ and X_2 be a unit tangent vector to the circle $z = \text{const.}$ Then X_1 and X_2 are principal vectors of M and hence $d\Phi(X_1)$ and $d\Phi(X_2)$ are principal vectors of $\Phi(M)$ with respect to $d\Phi(\xi)$. $d\Phi(X_1)$ and $d\Phi(X_2)$ are also principal vectors of $\Phi(M)$ with respect to ξ'. Since each normal space of $\Phi(M)$ is spanned by $d\Phi(\xi)$ and ξ', $d\Phi(X_1)$ and $d\Phi(X_2)$ are principal vectors for all normal vectors to $\Phi(M)$ in $\boldsymbol{R}^4$. This implies that the normal connection of $\Phi(M)$ is flat. Let H be the mean curvature vector of $\Phi(M)$. Then

$$H = h\, d\Phi(\xi) + \frac{1}{2}\kappa\left(\frac{dz}{ds}\right)^2 \xi'$$

and from (3.7) and (3.8), we have

$$|H|^2 = h^2 + \frac{1}{4}\kappa^2\left(\frac{dz}{ds}\right)^4 = \beta^2 .$$

(ii) If a compact surface in $\boldsymbol{R}^4$ with positive (not necessarily constant) Gaussisn curvature has parallel mean curvature vector, the surface must be a round sphere ([6]). However, as we see in the following example, there exists a compact surface in $\boldsymbol{R}^4$ with positive Gaussian curvature which has constant mean curvature, but is not a round sphere. This contradicts Theorem 5 on p. 361 of [8]. (A possible source of error in the calculations in [8] might be the formula (4.6) on p. 354 of [8] which is used to give the formula (6.2) in the proof of Theorem 5. The formula (4.6) holds for $\alpha_j = 4$ only when either M is minimal or M has a parallel mean curvature vector). The method of construction of this example is similar to the one in Remark (i).

EXAMPLE 2. Let M be a surface of revolution defined by

$$(s, \theta) \longmapsto (x(s)\cos\theta,\ x(s)\sin\theta,\ z(s)) \quad (0 \leq \theta \leq 2\pi)$$

where, for technical reasons to be explained below, $x(s)$ and $z(s)$ are required to satisfy the following conditions:

(a) $(x(s), z(s))$ is defined on $\left[-\frac{7}{12}\pi, \frac{7}{12}\pi\right]$

(b) $x\left(\frac{7}{12}\pi\right) = x\left(-\frac{7}{12}\pi\right) = 0,\ z\left(-\frac{7}{12}\pi\right) = -z\left(\frac{7}{12}\pi\right)$

(c) the curvature $\kappa(s)$ of $(x(s), z(s))$ satisfies the following conditions:

(c1) $\kappa(s) = \kappa(-s)$

(c2) $0 < \kappa(s) < 1$ if $|s| < \frac{\pi}{6}$

(c3) $\kappa(s) = 1$ if $\frac{\pi}{6} \leq |s| \leq \frac{7}{12}\pi$

(c4) $\int_0^{7\pi/12} \kappa(s)ds = \frac{\pi}{2}$

By (c1) and (c4), M becomes a compact surface in $\boldsymbol{R}^4$. By (c2) and (c3), M has a positive Gaussian curvature at every point. Let h be the mean curvature of M. Then h is a function of s only and we have $h(s) = 1$ if $\pi/6 \leq |s| \leq (7/12)\pi$ and $h(s) < 1$ if $|s| < \pi/6$. We define a function $\kappa(s)$ by

$$\kappa(s) = \frac{2(1 - h(s)^2)^{1/2}}{\left(\frac{dz}{ds}\right)^2}.$$

We regard κ as a function of z. Since $\kappa = 0$ if $z(\pi/6) \leq |z| \leq z((7/12)\pi)$, we can extend $\kappa(z)$ to a continuous function on $\boldsymbol{R}$ by setting $\kappa(z) = 0$ for all z such that $|z| > z((7/12)\pi)$. Then there exists an isometric imbedding of $\boldsymbol{R}$ into $\boldsymbol{R}^2$, $\varphi\colon z \mapsto (\varphi_1(z), \varphi_2(z))$ whose curvature is equal to $\kappa(z)$ at each z. We define a map $\Phi\colon \boldsymbol{R}^3 \to \boldsymbol{R}^4$ by $(x, y, z) = (x, y, \varphi_1(z), \varphi_2(z))$. By a similar argument to Remark (i), we can show that the mean curvature of $\Phi(M) \subset \boldsymbol{R}^4$ is constant and the normal connection is flat. Moreover, since we have

$$\int_{-\infty}^{\infty} \kappa(z)dz < \pi\,,$$

φ does not have any self-intersection and $\Phi|_M$ is an imbedding.

(iii) If $\dim M \geq 4$ and the codimension is two, then we have the following theorem which is the analogue of Theorem 1. (The case of $\dim M = 3$ is open.)

THEOREM 3. *Let M be a compact Riemanniam manifold of dimension $n \geq 4$ with positive constant sectional curvature $c^2 > 0$ and let $\psi: M \to \mathbf{R}^{n+2}$ be an isometric imbedding. Suppose that the mean curvature is constant. Then $\psi(M)$ is an n-dimensional round sphere in an $(n+1)$-dimensional affine space.*

Proof. Since the sectional curvature is positive constant and dim $M \geq 4$, there exists a global orthonormal frame field (ξ_1, ξ_2) of the normal bundle of M such that

$$A_{\xi_1} = -cI \quad \text{and} \quad \text{rank}\, A_{\xi_2} \leq 1 \tag{3.10}$$

where I is the identity transformation of TM. (This was found by Henke and Erbacher independently [4], [5].) Let $\lambda = \text{trace}\, A_{\xi_2}$. Then we have

$$H = c\xi_1 + \frac{\lambda}{n}\xi_2 \,. \tag{3.11}$$

Since $|H|^2 = c^2 + \lambda^2/n^2$ is constant, λ is constant.

On the other hand, due to a result obtained by O'Neill [7], there exists at least one point p_0 on M where ψ is umbilical. Since rank $A_{\xi_2} \leq 1$, $A_{\xi_2} = 0$ at p_0. Thus $\lambda = 0$ at p_0 and hence $\lambda \equiv 0$ on M. This implies that ψ is totally umbilical and since M is compact, M is an n-dimensional round sphere in some $\mathbf{R}^{n+1} \subset \mathbf{R}^{n+2}$.

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