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GROMOV'S CONVERGENCE THEOREM AND ITS APPLICATION

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One of the basic questions of Riemannian geometry is that "If two Riemannian manifolds are similar with respect to the Riemannian invariants, for example, the curvature, the volume, the first eigenvalue of the Laplacian, then are they topologically similar?". Initiated by H. Rauch, many works are developed to the above question. Recently M. Gromov showed a remarkable theorem ([7] 8.25, 8.28), which may be useful not only for the above question but also beyond the above. But it seems to the author that his proof is heuristic and it contains some gaps (for these, see § 1), so we give a detailed proof of 8.25 in [7]. This is the first purpose of this paper. Second purpose is to prove a differentiable sphere theorem for manifolds of positive Ricci curvature, using the above theorem as a main tool.

For a d-dimensional Riemannian manifold M, we denote by K_M the sectional curvature, by $\mathrm{vol}(M)$ the volume, by $\mathrm{diam}(M)$ the diameter, by $d_M(m, n)$ the distance between m and n induced from Riemannian metric g and by i_M the injectivity radius.

A subset B is called δ-dense when for any point $m \in M$, there exists a point $n \in B$ with $d_M(m, n) \leqq \delta$. A subset B is called δ-discrete if $n_1, n_2 \in B$ $(n_1 \neq n_2)$ implies $d_M(n_1, n_2) \geqq \delta$. Let $M(d, \Delta, i_0)$ (resp. $M(d, \Delta, \rho, v)$) be the category of all complete Riemannian manifolds M with dimension $= d$, $|K_M| \leqq \Delta$ and $i_M \geqq i_0$ (resp. dimension $= d$, $|K_M| \leqq \Delta$, $\mathrm{diam}(M) \leqq \rho$, $\mathrm{vol}(M) \geqq v$).

The following theorem is seemingly different from 8.25 in [7] but the inwardness is essentially same.

THEOREM 1 (Gromov's convergence theorem). *Given* $d, \Delta, i_0 > 0$, $0 < R < \min(1/2\sqrt{\Delta}, i_0/2)$, *for any* $\delta > 0$, *there exist* $a = a(d, \Delta, i_0, R; \delta) > 0$ *and*

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$\varepsilon=\varepsilon(d, \Delta, i_0, R; \delta)>0$ *such that if* M, $M' \in M(d, \Delta, i_0)$ *have an* ε*-dense,* $\varepsilon/10$*-discrete subset* $N[\varepsilon]=\{m_i\}_{i=1}^{N_\varepsilon} \subset M$ *and* $N'[\varepsilon]=\{m'_i\}_{i=1}^{N_\varepsilon} \subset M'$ *containing the same number of members with*

$$1-a \leqq \frac{d_M(m'_i, m'_j)}{d_M(m_i, m_j)} \leqq 1+a \qquad \text{for } 0<d_M(m_i, m_j) \leqq R\,,$$

then there exists a diffeomorphism $F: M \to M'$ *with* $||dF_m(\xi)|-1|<\delta$ *for* $\xi \in UM$, *where* UM *is the unit sphere bundle of* M.

We can estimate constants a, $\varepsilon>0$ explicitly, but we omit it to avoid non-essential complexity. Here we call it Gromov's convergence theorem because he proved a convergence theorem (8.18 in [7]) with respect to the Hausdorff distance using this theorem as a main tool.

An easy application of Theorem 1 and Dirichlet drawer principle is,

THEOREM 2 (Cheeger's finiteness theorem). *The number* N *of the diffeomorphism classes of the manifolds in* $M(d, \Delta, \rho, v)$ *is finite.*

This theorem was originally proved by J. Cheeger [2] except for $d=4$. After this, in Cheeger-Ebin's book [3], it was stated in the above form without proof. It was also given by M. Gromov [6]. S. Peters [12] gave another (simple) proof.

The following is the differentiable sphere theorem mentioned above. Let Ric_M be the Ricci curvature of M.

THEOREM 3. *Given* $d, \Delta>0$, *there exists* $\delta_0=\delta_0(d, \Delta)>0$ *such that if a compact d-dimensional Riemannian manifold* M *has the property that* $\mathrm{Ric}_M \geqq d-1$, $|K_M| \leqq \Delta$, $\mathrm{vol}\,(M) \geqq \omega_d-\delta_0$, *where* ω_d *is the volume of the d-dimensional unit sphere, then* M *is diffeomorphic to* S^d.

In [16], T. Yamaguchi obtained the same conclusion under a stronger assumption and in [9], Y. Itokawa showed that, under the essentially same assumption except for the estimate of the constant, M has the same homotopy type as S^d. (He only assumes the upper bound of K_M but under the condition of $\mathrm{Ric}_M \geqq d-1$, the lower bound of K_M is automatically derived.) But it should be remarked that in [15], K. Shiohama proved that M is homeomorphic to S^d under a weaker assumption than ours.

Finally we remark that for the diameter or the first eigenvalue of the Laplacian $\lambda_1(M)$, the following pinching theorem is obtained by using the above one and the results of C. B. Croke [5] and A. Kasue [10].

COROLLARY. *Given $d, \Delta, v > 0$ there exist $\delta_1 = \delta_1(d, \Delta, v) > 0$ and $\delta_2 = \delta_2(d, \Delta, v) > 0$ such that if a d-dimensional Riemannian manifold M with* $\mathrm{Ric}_M \geqq d-1$, $|K_M| \leqq \Delta$, $\mathrm{vol}(M) \geqq v$ *has the property that* $\mathrm{diam}(M) \geqq \pi - \delta_1$ *or* $\lambda_1(M) \leqq d + \delta_2$. *then M is diffeomorphic to* S^d.

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Remark. After the preparation of this paper the author learned that D. L. Brittain also got the same result as Corollary independently.

[Donald L. Brittain, A diameter pinching theorem for positive Ricci curvature. (preprint.)]

§1. Outline of the proof of Theorem 1

Firstly we observe the case when $M, M' \in M(d, \Delta, i_0)$ is compact. For an ε-dense, $\varepsilon/10$-discrete subset $N[\varepsilon] = \{m_i\}_{i=1}^{N_\varepsilon}$, we define a map $f: M \to \boldsymbol{R}^{N_\varepsilon}$ using the distance from m_i. If ε is sufficiently small, then f is an embedding (§ 2). We can estimate $\delta > 0$ such that the normal exponential map Exp is a diffeomorphism on the δ-tubular neighborhood of $f(M)$; $B_\delta(f(M))$ (§ 4). For $M' \in M(d, \Delta, i_0)$ and for $f': M' \to \boldsymbol{R}^{N_\varepsilon}$ which is defined similarly to f, we see that $f(M) \subset B_\delta(f'(M'))$ and $f'(M') \subset B_\delta(f(M))$. From this, the normal projection $P: f(M) \to f'(M')$ can be defined (§ 5). Nextly, we see that the tangent spaces $T_p f(M)$ and $T_{p'} f'(M')$ are almost parallel, where $p' = P(p)$ (§ 6). Using this, it can be shown that $P: f(M) \to f'(M')$ is a diffeomorphism (§ 7). For $F = f'^{-1} \circ P \circ f$, we estimate $dF(\xi)|$ (§ 8). In the case when M is non compact, the diffeomorphism is given by the approximation arguments (§ 9).

Here the author would like to comment on Gromov's proof in [7] 8.25. Firstly he says that it suffices to estimate $\delta > 0$ so that Exp is locally diffeomorphic but it really needs to estimate $\delta > 0$ so that it is globally diffeomorphic. (We add Lemma 4.3.) Secondly P may cut the two points of $f(M)$, for this possibility, he says "good" one can be chosen without detailed arguments. (We add Section 6.) Thirdly for the argument of the estimate of $|dF(\xi)|$, it needs more arguments than that given there.

Though almost all arguments owe to Gromov [7], we give a full proof for the sake of completeness. It should be noted that the author also referred to T. Sakai [13].

§ 2. Definition of the embedding $f: M \to R^{N_\varepsilon}$

We firstly prove the Theorem 1 in the case when M is compact.

Take constants $0 < r < R$ and $\kappa > 0$. Let $h: R \to [0, 1]$ be a C^∞ function such that

$$\begin{aligned} &h(t) = 1 \quad \text{if} \quad t \leqq 0,\ h(t) = 0 \quad \text{if} \quad t \geqq r \\ &\quad -\frac{4}{r} < h'(t) < -\frac{3}{r} \quad \text{if} \quad \frac{3r}{8} < t < \frac{5r}{8} \\ &\quad -\frac{4}{r} < h'(t) < 0 \quad \text{if} \quad \frac{2r}{8} < t \leqq \frac{3r}{8} \quad \text{or} \quad \frac{5r}{8} \leqq t < \frac{6r}{8} \\ &\quad -\kappa < h'(t) < 0 \quad \text{if} \quad 0 < t \leqq \frac{2r}{8} \quad \text{or} \quad \frac{6r}{8} \leqq t \leqq r\,. \end{aligned}$$

Note that we may take $\kappa > 0$ arbitrarily small, which is needed in Section 8.

Put

$$k = \max\left(\left|h'(t)\left(\frac{1}{t} + \frac{\Delta t}{2}\right)\right|,\ |h''(t)|\right) \quad \text{and} \quad A = \left(1 - \frac{1}{3d^2}\right)^{1/2}.$$

In the following, we remark that the constants $c_i > 0$, $\beta > 0, \cdots$ which appear in the proof, are depending only on $d, \Delta, i_0, r, \delta > 0$ and $h(t)$.

Put

$$\varepsilon_1 = \min\left(\frac{r}{16}, \frac{s_\Delta(r)}{2}(1 - A^2)^{1/2}, \left(\frac{1}{2} - \frac{r}{8s_\Delta(r/2)}\right)\left(r\Delta + \frac{16}{r}\right)^{-1}\right),$$

where $s_\tau(t)$ is the function

$$\begin{cases} \dfrac{1}{\tau^{1/2}} \sin(\tau^{1/2}t)\,, & \text{if } \tau > 0\,, \\ t\,, & \text{if } \tau = 0\,, \\ \dfrac{1}{(-\tau)^{1/2}} \sin h((-\tau)^{1/2}t)\,, & \text{if } \tau < 0\,. \end{cases}$$

Using this $h(t)$ and an ε-dense, $\varepsilon/10$-discrete subset $N[\varepsilon] = \{m_i\}_{i=1}^{N_\varepsilon}$ with $\varepsilon < \varepsilon_1$, we define a C^∞ map $f = f_\varepsilon$: $M \to R^{N_\varepsilon}$ by

$$f_\varepsilon(m) = (h(d_M(m_1, m)), \cdots, h(d_M(m_{N_\varepsilon}, m))) .$$

We show that f_ε is an embedding by the following two lemmas.

LEMMA 2.1. *f_ε has maximal rank at every point $m \in M$.*

Proof. Take an orthonormal basis $\{e_i\}_{i=1}^d$ of the tangent space T_mM to M at m and choose $\{m_{i_j}\}_{j=1}^d \subset N[\varepsilon]$ satisfying $d_M(\exp_m(r/2)e_j, m_{i_j}) < \varepsilon$. Put $t_j = |\exp_m^{-1} m_{i_j}|$ and $u_j = t_j^{-1} \exp_m^{-1} m_{i_j}$. Note that $3r/8 < d_M(m_{i_j}, m) < 5r/8$. Then, from the Rauch's comparison theorem (R. C. T.) (cf. [3] or [13] (1.2.20)), we see

$$\frac{s_\Delta(r)}{r} \cdot |(r/2)e_j - t_j u_j| \leqq d_M(m_{i_j}, \exp_m(r/2)e_j) < \varepsilon < \frac{s_\Delta(r)}{2}(1 - A^2)^{1/2}$$

and this implies $g(e_j, u_j) > A \geqq (1 - (1/3d^2))^{1/2}$. From this, we see $\{u_i\}_{i=1}^d$ are linearly independent. Since $\operatorname{grad} d_M|_{m_{i_j}} = u_j$, we can get the conclusion by

$$\begin{aligned} \text{the rank of } df \text{ at } m &= \operatorname{rank} df|_m \\ &= \operatorname{rank}(d\cdot h(d_M(m_{i_1}, \cdot))|_m, \cdots, d\cdot h(d_M(m_{i_d}, \cdot))|_m) \\ &\geqq \operatorname{rank}(h'(d_M(m_{i_1}, m))u_1, \cdots, h'(d_M(m_{i_d}, m))u_d) \\ &= d . \end{aligned}$$

q.e.d.

LEMMA 2.2. *f_ε is an embedding.*

Proof. If not, then there exist $m, n \in M$ with $m \neq n$ such that $f(m) = f(n)$. Since $d_M(m_i, m) = d_M(m_i, n)$ for all $m_i \in N[\varepsilon] \cap B_r(m) = N[\varepsilon] \cap B_r(n)$, we see $d_M(m, n) := \tilde{d} < 2\varepsilon < r/8$. Let γ be the minimal geodesic from m to n and put $z = \gamma((r/2) + \tilde{d})$. Then $z \in \bar{B}_{r/2}(n) - \bar{B}_{r/2}(m)$ and $B_{2\varepsilon}(z) \subset B_r(n) - \bar{B}_{r/4}(m)$, where $B_r(m)$ is the set of the point p with $d_M(p, m) < r$ and $\bar{B}$ is the closure of B. Take a point $p \in N[\varepsilon] \cap B_{2\varepsilon}(z)$ with $d' := d_M(p, n) \geqq r/2 - 2\varepsilon$, $d' < r/2$ and the vector $u \in T_nM$ that is the unit initial vector of the minimal geodesic λ from n to p. Now we estimate $g(u, \dot{\gamma}(\tilde{d}))$. From R.C.T., we get

$$\begin{aligned} |(r/2)\dot{\gamma}(\tilde{d}) - d'u| &= |\exp_n^{-1} z - \exp_n^{-1} p| \\ &\leqq \frac{r/2}{s_\Delta(r/2)} \cdot d_M(p, z) < \frac{r\varepsilon}{s_\Delta(r/2)} < \frac{r^2}{16 s_\Delta(r/2)} , \end{aligned}$$

from which follows

$$\begin{aligned}\frac{r}{2}\cdot g(\dot{\gamma}(\tilde{d}), u) &= g((r/2)\dot{\gamma}(\tilde{d}) - d'u, u) + d' \\ &\geqq d' - |(r/2)\dot{\gamma}(\tilde{d}) - d'u| > \frac{r}{2} - 2\varepsilon - \frac{r^2}{16s_\Delta(r/2)} \\ &\geqq \frac{r}{4}\cdot\left(1 - \frac{r}{4s_\Delta(r/2)}\right),\end{aligned}$$

namely

$$g(\dot{\gamma}(\tilde{d}), u) > \frac{1}{2}\cdot\left(1 - \frac{r}{4s_\Delta(r/2)}\right).$$

On the other hand, note that $d_M(p, \gamma(t)) < r$ for $0 \leqq t \leqq \tilde{d}$ and $d_M(p, \gamma(0)) = d_M(p, \gamma(\tilde{d}))$, then from the Rolle's theorem, there exists a point $m_1 = \gamma(t_1)$ $(0 < t_1 < \tilde{d})$ with $g(\dot{\gamma}(t_1), u_{t_1}) = 0$, where u_t is the unit initial vector of the minimal geodesic from $\gamma(t)$ to p. Then we have

$$\begin{aligned}g(\dot{\gamma}(\tilde{d}), u) &= \int_{t_1}^{\tilde{d}} \frac{d}{dt} g(\dot{\gamma}(t), u_t)dt \\ &\underset{(*)}{=} \int_{t_1}^{\tilde{d}} \mathrm{Hess}\, d_{M,p}(\dot{\gamma}(t), \dot{\gamma}(t))dt \\ &\underset{(*)}{\leqq} \int_{t_1}^{\tilde{d}} \left(\frac{1}{d_M(p, \gamma(t))} + \frac{\Delta}{2} d_M(p, \gamma(t))\right)dt \\ &< 2\varepsilon\left(\frac{8}{r} + \frac{r\Delta}{2}\right).\end{aligned}$$

After all we get

$$\varepsilon\left(r\Delta + \frac{16}{r}\right) > \left(\frac{1}{2} - \frac{r}{8s_\Delta(r/2)}\right).$$

It contradicts the fact

$$\varepsilon \leqq \left(\frac{1}{2} - \frac{r}{8s_\Delta(r/2)}\right)\left(r\Delta + \frac{16}{r}\right)^{-1}.$$

Except for $(*)$ we get the conclusion.

To show the inequality $(*)$, we need following sublemma. Put $d_{M,p}(\cdot) = d_M(p, \cdot)$.

SUBLEMMA ([7] 8.23 or [13] (1.4.4), iii). *If $|K_M| \leqq \Delta$, then the hessian of $d_{M,p}$ at $x = \mathrm{Hess}\, d_{M,p}(x, x) \leqq |x|^2(1/d_M(p, m) + (\Delta/2)d_M(p, m))$ for $x \perp \mathrm{grad}\, d_{M,p}|_m$ and $d_M(p, m) < r$.* q.e.d.

§ 3. Estimate of *df*

The contents of this section are detailed arguments developed by Gromov's hints.

(i) Estimate of the number of the elements in $N[\varepsilon]$, which are nearly orthonormal.

Firstly, we take $c_1 > 0$ with

$$c_1 \leqq \inf_{0<\varepsilon<\varepsilon_1/10} \frac{b_\Delta(\varepsilon/20) b_\Delta(\varepsilon_1/4)}{b_{-\Delta}(4r)\cdot b_{-\Delta}(\varepsilon)},$$

where $b_\tau(t)$ is the volume of the ball with radius t in the space of the constant curvature τ. Note that c_1 can taken as positive because $\lim_{t\downarrow 0} b_\Delta(t/20)/b_{-\Delta}(t) = 20^{-d}$. Put $\tilde{N}_\varepsilon = \sup_m \sharp(B_{2r}(m) \cap N[\varepsilon])$, $\tilde{m}_i = \exp_m((r/2)e_i)$ and $D^i_m[\varepsilon] = B_{\varepsilon_1/2}(\tilde{m}_i) \cap N[\varepsilon]$.

LEMMA 3.1. *If* $\varepsilon \leqq \varepsilon_1/10$, *then* $c_1 \leqq \sharp(D^i_m[\varepsilon])/\tilde{N}_\varepsilon \leqq 1$.

Proof. From the fact

$$\bigcup_{q\in B_{\varepsilon_1/4}(\tilde{m}_i)\cap N[\varepsilon]} B_\varepsilon(q) \subset B_{\varepsilon_1/2}(\tilde{m}_i)$$
$$\bigcup_{q\in B_{2r}(m)\cap N[\varepsilon]} B_\varepsilon(q) \subset B_{4r}(m)$$

and the volume comparison theorem ([7] or [13]), we have

$$\sharp(D^i_m[\varepsilon]) \geqq \frac{b_\Delta(\varepsilon_1/4)}{b_{-\Delta}(\varepsilon)}$$
$$\tilde{N}_\varepsilon \leqq \frac{b_{-\Delta}(4r)}{b_\Delta(\varepsilon/20)}.$$

Combining these, we get the conclusion.

(ii) Estimate of *df*.

LEMMA 3.2. *For* $\varepsilon < \varepsilon_1$, *there exist* c_2, $c_3 > 0$ *such that*

$$c_2\tilde{N}_\varepsilon^{1/2} \leqq |df_\varepsilon(\xi)| \leqq c_3\tilde{N}_\varepsilon^{1/2} \qquad \text{for any } \xi\in UM.$$

Proof. From the definition of f_ε, we see

$$df_{\varepsilon,m}(\xi) = (a_1 g(u_1, \xi), \cdots, a_{N_\varepsilon} g(u_{N_\varepsilon}, \xi)),$$

where $a_i = h'(d_M(m, m_i))$. We may put $c_3 = \sup_{0\leqq t\leqq r} |h'(t)|$. For the existence of c_2, we take the representatives $m_{k_i} \in D^i_m[\varepsilon]$ and put $u_{k_i} = \exp_m^{-1} m_{k_i}/|\exp_m^{-1} m_{k_i}|$. Let $\ell = \ell_{(k_1,\dots,k_d)}: T_mM \to \boldsymbol{R}^d$ be a linear map defined by

$$\ell(\xi) = (a_{k_1} g(u_{k_1}, \xi), \cdots, a_{k_d} g(u_{k_d}, \xi)) .$$

Then we see that it satisfies the following estimate

$$\min_{|\xi|=1} |\ell(\xi)| \geqq \frac{3}{2r} > 0 .$$

In fact, if we put $\alpha_{ij} = g(u_{k_i}, e_j)$ and $\xi = \sum_j \xi_j e_j$, then from the proof of Lemma 2.1 $\alpha_{ii} \geqq A$, $|\alpha_{ij}| \leqq (1 - A^2)^{1/2} (i \neq j)$ and $4/r \geqq |a_{k_i}| \geqq 3/r$. Thus, we get

$$\begin{aligned} |\ell(\xi)|^2 &= \sum_{i,j,\ell} a_{k_i}^2 \xi_j \xi_\ell \alpha_{ij} \alpha_{i\ell} \\ &= \sum_i a_{k_i}^2 \xi_i^2 \alpha_{ii}^2 + (\text{the other terms}) \\ &\geqq \left(\frac{3}{r}\right)^2 A^2 - d^2 \left(\frac{4}{r}\right)^2 (1 - A^2) \geqq \left(\frac{3}{2r}\right)^2 > 0 . \end{aligned}$$

On the other hand, from Lemma 4.1, we see

$$\#\{(k_1, \cdots, k_d) \mid m_{k_i} \in D_m^i[\varepsilon]\} \geqq \inf \#(D_m^i[\varepsilon]) \geqq c_1 \tilde{N}_\varepsilon .$$

Combining these, we get

$$|df(\xi)|^2 \geqq \sum_{(k_1, \cdots, k_d)} |\ell_{(k_1, \cdots, k_d)}(\xi)|^2 \geqq c_1 \left(\frac{3}{2r}\right)^2 \tilde{N}_\varepsilon .$$

Therefore we may put

$$c_2 = c_1^{1/2} \left(\frac{3}{2r}\right) . \qquad \text{q.e.d.}$$

Remark. We discuss here the dependence of r on c_1, c_2, c_3 when r is sufficiently small, which is essential in Section 8. Since the function $f(t) = b_\Delta(t/20)/b_{-\Delta}(t)$ is decreasing and we may assume $\varepsilon_1 \geqq r/50d$, we can take

$$\begin{aligned} c_1 &= (10^5 d)^{-d} \leqq \left(\frac{1}{40} \cdot \frac{1}{1600d}\right)^d \leqq \frac{b_\Delta(\varepsilon_1/200) b_\Delta(\varepsilon_1/4)}{b_{-\Delta}(\varepsilon_1/10) b_{-\Delta}(4r)} \\ &\leqq \inf_{0<\varepsilon<\varepsilon_1/10} \frac{b_\Delta(\varepsilon/20) b_\Delta(\varepsilon_1)}{b_{-\Delta}(4r) b_{-\Delta}(\varepsilon)} \\ c_2 &= c_1^{1/2} \left(\frac{3}{2r}\right) = \frac{3}{2r} (10^5 d)^{-d/2} , \\ c_3 &= \frac{4}{r} . \end{aligned}$$

§4. The tubular neighborhood of $f(M)$ and the normal exponential mapping

Let $\mathrm{Exp}\colon Nf(M) \to \boldsymbol{R}^{N_\varepsilon}$ be the normal exponential map of the normal bundle $Nf(M)$. Put

$$B_\delta(Nf(M)) = \{(p, u) \in Nf(M) \mid |u| < \delta\} .$$

We estimate $\delta > 0$ such that $\mathrm{Exp}|_{B_\delta(Nf(M))}$ is a diffeomorphism.

(i) Local estimate.

The following Lemma 4.1 owe to [7] and [13].

LEMMA 4.1. *There exists $c_4 > 0$ such that if $\varepsilon \leqq \varepsilon_1$ and $\delta \leqq c_4 \tilde{N}_\varepsilon^{1/2}$, then $\mathrm{Exp}|_{B_\delta(Nf(M))}$ is an immersion.*

Proof. Suppose that $n \in \boldsymbol{R}^{N_\varepsilon}$ is a critical value of Exp. Namely there exists a curve $c(s) = f(m(s))$ in $f(M)$ and the normal vector field $n(s)$ along $c(s)$ such that $n = c(0) + n(0)$, $\dot{c}(0) + \dot{n}(0) = 0$. From $g(n(s), \dot{c}(s)) = 0$, we have

$$g(n(0), \ddot{c}(0)) = -g(\dot{n}(0), \dot{c}(0)) = |\dot{c}(0)|^2 .$$

Since $c(s)$ $(\cdots, h(d_M(m_i, m(s))), \cdots)$, we have

$$\ddot{c}(0) = \left(\cdots, h''(d_M(m_i, m(0)))\left(\frac{d}{ds}\Big|_{s=0} d_M(m_i, m(s))\right)^2 + h'(d_M(m_i, m(0)))\left(\frac{d^2}{ds^2}\Big|_{s=0} d_M(m_i, m(s))\right) \cdots\right) .$$

Recall that

$$\left|\frac{d}{ds}\Big|_{s=0} d_M(m_i, m(s))\right| = |g(\mathrm{grad}\, d_{M,m_i}, \dot{m}(0))| \leqq |\dot{m}(0)| ,$$

$$\left|\frac{d^2}{ds^2}\Big|_{s=0} d_M(m_i, m(s))\right| \leqq |\dot{m}(0)|^2\left(\frac{1}{d_M(m_i, m(0))} + \frac{\Delta}{2} d_M(m_i, m(0))\right) .$$

Note that $\max(|h'(t)(1/t + \Delta t/2)|, |h''(t)|) = k$. Then we see

$$|\dot{c}(0)|^2 \leqq |n(0)||\ddot{c}(0)| \leqq 2|n(0)||\dot{m}(0)|^2 k\tilde{N}_\varepsilon^{1/2} ,$$

and this implies,

$$\begin{aligned} d_M(n, f(M)) = |n(0)| &\geqq \frac{1}{2k\tilde{N}_\varepsilon^{1/2}} \cdot \frac{|\dot{c}(0)|^2}{|\dot{m}(0)|^2} \\ &\geqq \frac{1}{2k\tilde{N}_\varepsilon^{1/2}} |df_\varepsilon|^2 | \geqq \frac{1}{2k\tilde{N}_\varepsilon^{1/2}} c_2^2 \tilde{N}_\varepsilon = \frac{c_2^2}{2k} \cdot \tilde{N}_\varepsilon^{1/2} . \end{aligned}$$

Thus we get the conclusion by putting $c_4 = c_2^2/2k$.

Hereafter we denote by $\tilde{d}_M$, the distance on $f(M)$ defined by the induced Riemannian structure of $f(M)$ from $\boldsymbol{R}^{N_\varepsilon}$ and by d, the euclidean distance of $\boldsymbol{R}^{N_\varepsilon}$.

(ii) Relation between $\tilde{d}_M$ and d. (I)

LEMMA 4.2. *Fix* $\alpha > 0$. *If* $\varepsilon \leqq \min(\varepsilon_1/100, \alpha/100c_3)$, *then there exists* $\tilde{\alpha} > 0$ *such that if* $\tilde{d}_M(p, q) \geqq \alpha \cdot \tilde{N}_\varepsilon^{1/2}$, *then* $d(p, q) \geqq \tilde{\alpha} \cdot \tilde{N}_\varepsilon^{1/2}$. *For the case* $\alpha = c_4/10$, *we put* $\tilde{\alpha} = 3c_5$.

Proof. Since $\tilde{d}_M(p, q) \geqq \alpha \cdot \tilde{N}_\varepsilon^{1/2}$, we see $d_M(f^{-1}(p), f^{-1}(q)) \geqq \alpha/c_3$. Put $\varepsilon_2 = \min(r/10, \alpha/10c_3)$ and $\beta = |h(9\varepsilon_2) - h(\varepsilon_2)| > 0$.

Take the balls B_1, B_2 of radius ε_2 centered at $f^{-1}(p)$, $f^{-1}(q)$ respectively. By the method similar to Section 3-(i), we find that there exists $\tilde{\beta} > 0$ such that

$$\sharp(B_i \cap N[\varepsilon])/\tilde{N}_\varepsilon \geqq \tilde{\beta} \qquad (i = 1, 2)$$

Therefore we get

$$(d(p, q))^2 = \sum_{i=1}^{N_\varepsilon} \{h(d_M(f^{-1}(p), m_i)) - h(d_M(f^{-1}(q), m_i))\}^2 \geqq \beta^2 \tilde{\beta} \tilde{N}_\varepsilon .$$

We have done if we take $\tilde{\alpha} \leqq \tilde{\beta}^{1/2}\beta$.

(iii) Global estimate.

LEMMA 4.3. *If* $\varepsilon < \min(\varepsilon_1/100, c_4/1000c_3)$ *and* $\delta < c_5\tilde{N}_\varepsilon^{1/2}$, *then* $\mathrm{Exp}|_{B_\delta(Nf(M))}$ *is a diffeomorphism.*

Proof. Suppose that there exist (p, u), $(q, v) \in B_\delta(Nf(M))$ with $(p, u) \neq (q, v)$ and $\mathrm{Exp}(p, u) = \mathrm{Exp}(q, v) := x$. Then from Lemma 4.2, we see $\tilde{d}_M(p, q) \leqq c_4/10 \cdot \tilde{N}_\varepsilon^{1/2}$ because

$$\begin{aligned} d(p, q) &\leqq d(\mathrm{Exp}(p, u), \mathrm{Exp}(q, v)) + d(\mathrm{Exp}(p, u), p) + d(\mathrm{Exp}(q, v), q) \\ &\leqq |u| + |v| \leqq 2c_5\tilde{N}_\varepsilon^{1/2} . \end{aligned}$$

Now we define a smooth map

$$F(s, t)\colon [0, 1] \times [0, 1] \longrightarrow \boldsymbol{R}^{N_\varepsilon}$$

by $F(s, t) = (1 - t)\gamma(s) + tx$, where $\gamma(s)$ is the minimal geodesic from p to q in $f(M)$.

Since

$$\begin{aligned} d(F(s, t), f(M)) &\leqq d(F(s, t), \gamma(s)) \leqq d(x, \gamma(s)) \\ &\leqq d(x, q) + d(q, \gamma(s)) \leqq d(x, q) + \tilde{d}_M(q, \gamma(s)) \end{aligned}$$

$$\leqq d(x, q) + \tilde{d}_M(p, q) \leqq c_5 \tilde{N}_\varepsilon^{1/2} + \frac{c_4}{10} \cdot \tilde{N}_\varepsilon^{1/2} \leqq \frac{c_4}{2} \cdot \tilde{N}_\varepsilon^{1/2} .$$

we observe

$$F(s, t) \subset B_{(c_4/2)\cdot \tilde{N}_\varepsilon^{1/2}}(f(M)) = \operatorname{Exp}(B_{(c_4/2)\cdot \tilde{N}_\varepsilon^{1/2}}(Nf(M)) .$$

The following sublemma is crucial in the proof. Put $B = B_{(c_4/2)\cdot \tilde{N}_\varepsilon^{1/2}}(Nf(M))$.

SUBLEMMA. *There exists a smooth map*

$$G(s, t)\colon [0, 1] \times [0, 1] \longrightarrow B$$

such that $\operatorname{Exp}(G(s, t)) = F(s, t)$.

Proof of the sublemma (cf. J. Schwartz [14] 1.23). Let I be the set of $t \in [0, 1]$ such that $G(s, t)$ can be defined for all $s \in [0, 1]$. Since $G(s, 0) = \gamma(s)$, $0 \in I \neq \phi$. It is sufficient to prove that I is open and closed.

We see that I is open by the following argument. Take $a \in I$. Since $\operatorname{Exp}|_B$ is an immersion and $\bigcup_s G(s, a)$ is compact, it can be covered by a family of finite open sets $\{U_i\}$, which are mapped by Exp diffeomorphically to open neighborhoods $\{V_i\}$ of $F(s, a_i)$ and $\bigcup_i V_i \supset \bigcup_s F(s, a)$. This implies $G(s, t)$ can be defined beyond a and I is open.

We show that I is closed. Since the closure of $B \subset B_{c_4 \tilde{N}_\varepsilon^{1/2}}(Nf(M))$ is compact, there exists $A > 0$ such that $|d \operatorname{Exp}| \geqq A$. Then for all $(s, t) \in [0, 1] \times I$,

$$|G_t(s, t)| = |d \operatorname{Exp}^{-1}||F_t(s, t)| \leqq A^{-1}|F_t(s, t)| = A_s < \infty$$

where G_t, F_t mean the derivative with respect to t.

Integrating this we get

$$|G(s, t_1) - G(s, t_0)| \leqq A_s |t_1 - t_0| .$$

It implies $\lim_{t \to \sup I} G(s, t)$ exists and $G(s, \sup I)$ can be defined. It means I is closed whence the conclusion.

From this sublemma, we see $\operatorname{Exp}(G(s, 1)) = x$. But this contradicts the fact that $\operatorname{Exp}|_B$ is an immersion. Therefore $\operatorname{Exp}|_{B_\delta(Nf(M))}$ is a diffeomorphism. q.e.d.

§ 5. Definition of the projection P

Take another $M' \in M(d, \Delta, i_0)$, which has an ε-dense $\varepsilon/10$-discrete subset

$N'[\varepsilon] = \{m'_i\} \subset M'$ such that

$$1 - a \leqq \frac{d_{M'}(m'_i, m'_j)}{d_M(m_i, m_j)} \leqq 1 + a \qquad \text{for } 0 < d_M(m_i, m_j) < R\,.$$

We define f' for M' in the same way as f for M. From the definition of f and f' we get

$$\begin{aligned} d(f(m_k), f'(m'_k)) &= \left(\sum_{i=1}^{N_\varepsilon} |h(d_M(m_i, m_k)) - h(d_{M'}(m'_i, m'_k))|^2\right)^{1/2} \\ &\leqq \left(\sum_{i=1}^{N_\varepsilon} (a \cdot \sup|h'(t)|)^2\right)^{1/2} \leqq \frac{4a}{r}\tilde{N}_\varepsilon^{1/2}\,. \end{aligned}$$

The last inequality follows from the fact $|h'(t)| = 0$ if $t \geqq r$. Therefore we see

$$\begin{aligned} d(f(m), f'(M')) &\leqq d(f(m), f(m_k)) + d(f(m_k), f'(m'_k)) \\ &\leqq \frac{4(a+\varepsilon)}{r}\tilde{N}_\varepsilon^{1/2}\,, \end{aligned}$$

where m_k is the point of $N[\varepsilon]$ with $d_M(m, m_k) \leq \varepsilon$. If a, $\varepsilon \leqq c_5 r/10$, then $f(M) \subset B_{c_5\tilde{N}_\varepsilon^{1/2}}(f'(M'))$ and similarly $f'(M') \subset B_{c_5\tilde{N}_\varepsilon^{1/2}}(f(M))$. From Lemma 4.3, the normal projection $P\colon B_{c_5\tilde{N}_\varepsilon^{1/2}}(f'(M')) \to f'(M')$ is well defined. In the later section, we show that for sufficiently small a, $\varepsilon > 0$ $P|_{f(M)}\colon f(M) \to f'(M')$ is a diffeomorphism.

§6. $T_p f(M)$ and $T_{p'} f'(M')$ are almost parallel

(i) Relation between $\tilde{d}_M$ and d. (II)

Firstly we investigate the relation between $\tilde{d}_M$ and d. We have already done in Lemma 4.2, but here, we need the estimate of $\tilde{d}_M/d$ in the case when $d_M(x, y)$ is small, which is different from previous one.

LEMMA 6.1. *There exists $c_6 > 0$ such that if $\varepsilon < \varepsilon_1/10$ and $d_M(m, n) < \varepsilon_1/10$, then*

$$1 \leqq \frac{\tilde{d}_M(f(m), f(n))}{d(f(m), f(n))} \leqq c_6\,.$$

Proof. Let γ be the minimal geodesic from m to n. Put $d_1 = d_M(m, n)$ and $z = \gamma((r/2) + d_1)$. For $p \in B_{\varepsilon_1}(z) \cap N[\varepsilon]$ with $d_M(n, p) < r/2 - (\varepsilon_1/10)$, if $p' \in B_{\varepsilon_1/10}(p) \cap N[\varepsilon]$, then $p' \in B_{\varepsilon_1}(z) \cap N[\varepsilon]$ and $d_M(n, p') < r/2$. Thus, by the argument of the proof of Lemma 2.2, we see

$$g(\dot{\gamma}(d_1), u') > \frac{1}{4}\left(1 - \frac{r}{4s_\Delta(r/2)}\right),$$
$$g(\dot{\gamma}(d_1), u') - g(\dot{\gamma}(t), u_t) = \int_t^{d_1} \frac{d}{dt} g(\dot{\gamma}(t), u_t)dt < \frac{\varepsilon_1}{10}\left(\frac{16}{r} + r\Delta\right),$$

where u', u_t are the unit initial vector of the minimal geodesic from n, $\gamma(t)$ to p' respectively. This implies

$$\inf_{0 \leqq t \leqq d_1} g(\dot{\gamma}(t), u_t) \geqq \frac{1}{4}\left(1 - \frac{r}{4s_\Delta(r/2)}\right) - \frac{\varepsilon_1}{10}\left(\frac{16}{r} + r\Delta\right) := \beta_1 > 0.$$

Since $|h'(t)| > 3/r$ for $t \in [3r/8, 5r/8]$, and $3r/8 \leqq d_M(p', \gamma(t)) \leqq 5r/8$,

$$|h(d_M(p', m)) - h(d_M(p', n))| = \left|\int_0^{d_1} h'(d_M(p, \gamma(t))g(\dot{\gamma}(t), u_t)dt\right|$$
$$\geqq \min\left(\frac{r}{10}, d_1\right) \cdot \beta_1 \cdot \frac{3}{r} \geqq \frac{3\beta_1 d_1}{10r}.$$

Combining this with the fact that there exists $c_7 > 0$ such that

$$\sharp(B_{\varepsilon_1/10}(p) \cap N[\varepsilon])/\tilde{N}_\varepsilon \geqq c_7,$$

which is obtained by the same method as Section 3-(i), we get, using the method similar to Section 4-(ii),

$$d(f(m), f(n)) \geqq c_7^{1/2} \cdot \frac{3\beta}{10r} \tilde{N}_\varepsilon^{1/2} d_1.$$

On the other hand, from Lemma 3.2, we get

$$\tilde{d}_M(f(m), f(n)) \leqq c_3 \tilde{N}_\varepsilon^{1/2} d_1.$$

These two estimates imply the conclusion.

For simplicity, we define some constants. For the later purpose, we introduce a new parameter $\sigma > 0$. For fixed $\sigma > 0$, we put

$$\mu = \max(8^{d-1}c_2^{-d}c_2^d c_6^d \sigma, 100\sigma(\Delta + 1)), \qquad \eta_1 = \frac{c_5}{100\mu}\tilde{N}_\varepsilon^{1/2}$$
$$\eta_2 = \frac{\eta_1}{1000\mu}, \qquad \eta_3 = \frac{\sigma c_6 \eta_3}{\mu c_2} \cdot \tilde{N}_\varepsilon^{-1/2},$$
$$\eta_4 = \frac{c_2\eta_3}{c_6} \cdot \tilde{N}_\varepsilon^{1/2} = \frac{\eta_1}{\mu}, \qquad \eta_5 = \frac{\eta_1}{c_3} \cdot \tilde{N}_\varepsilon^{-1/2}.$$

In the later parts, we denote by $B_\tau(p)$ the ball with radius τ and centered p in $\mathbf{R}^{N_\varepsilon}$ and $B_\tau^Q(p)$ is the τ-neighborhood of p in Q with respect to the induced metric of a subset Q in $\boldsymbol{R}^{N_\varepsilon}$. Let $\tilde{P}\colon \boldsymbol{R}^{N_\varepsilon} \to T_p f(M)$ be the normal projection.

(ii) The position of $f(M)$ and $T_p f(M)$.

For $p_0 \in f(M)$, put $\tilde{p}_0 = \tilde{P}(p_0)$.

LEMMA 6.2. *If* $d(p, p_0) \leqq \eta \leqq 2\eta_1$, *then* $d(p_0, \tilde{p}_0) \leqq \eta/1000$.

Proof. Let $B(t, n)$ be the $(d+1)$-dimensional ball centered at $\operatorname{Exp}(p, tn)$ with the radius t in the $(d+1)$-dimensional subspace of $\boldsymbol{R}^{N_\varepsilon}$ spanned by a unit vector n normal to $T_p f(M)$ and the vectors in $T_p f(M)$. Then $B(t, n)$ is tangent to $T_p f(M)$ at p. Put $\tilde{B}(t) = \bigcup_n B(t, n)$.

CLAIM: *If* $t \leqq c_5 \tilde{N}_\varepsilon^{1/2}$, *then* $\tilde{B}(t) \cap f(M) = \{p\}$.

Proof. Suppose that $\tilde{B}(t) \cap f(M)$ contains another point q. Let n be the unit vector normal to $T_p M$ such that $\partial B(t, n) \cap f(M) - \{p\} \neq \phi$. Put $x = \operatorname{Exp}(p, tn)$. Then there exists $q' \in f(M)$ such that $p \neq q'$, $d(x, q') = d(x, f(M)) := t' \leqq t$. Note that the vector $v = \overrightarrow{q'x}$ is perpendicular to $T_{q'} f(M)$. Since $\operatorname{Exp}(q', t'v/|v|) = x$, it contradicts that $\operatorname{Exp}|_{\tilde{B}(t)}$ is a diffeomorphism.

Then this lemma follows from the following elementary fact. In general, let B be the ball in euclidean space with the radius a, tangent to an affine subspace H at p. If we take a point $q \in H$ with $d(p, q) \leqq a/b$ ($b \geqq 1000$), then $d(q, q') \leqq a/b^2$, where q' is a point of ∂B which projects normally on q. q.e.d.

(iii) $\tilde{P}(B_{\eta_1}^{f(M)}(p))$ occupies a "large portion" in $B_{\eta_1}^{T_p f(M)}(p)$.

Let $\langle \cdot, \cdot \rangle$ be the standard inner product of $\boldsymbol{R}^{N_\varepsilon}$.

LEMMA 6.3. *For any* $x \in U_p f(M)$, *there exists* $p_0 \in B_{\eta_1}^{f(M)}(p)$ *such that*

$$\langle \tilde{p}_0, x \rangle \geqq \eta_4 .$$

Proof. Put $A_{\eta_4}^x = \{v = tx + y \mid v \in B_{\eta_1}^{T_p f(M)}(p),\ |t| \leqq \eta_4,\ \langle x, y \rangle = 0\}$. It suffices to prove that $\tilde{P}(B_{\eta_1}^{f(M)}(p))$ is not contained in $A_{\eta_4}^x$. From Lemma 3.2, we see $B_{\eta_1}^{f(M)}(p) \supset f(B_{\eta_5}^M(f^{-1}(p)))$, where $B_\eta^M(\cdot)$ is the ball with radius η in M. Take a maximal η_3-discrete subset $\{n_i\}$ in $B_{\eta_5}^M(f(p))$. From the volume comparison theorem, we have

$$\#\{n_i\} \geqq \frac{b_\Delta(\eta_5)}{b_{-\Delta}(\eta_3/2)} \geqq \left(\frac{\eta_5}{\eta_3}\right)^d ,$$

because

$$s_\Delta(\eta_5) > \frac{\eta_5}{2\Delta^{1/2}} \quad \text{and} \quad s_{-\Delta} < \frac{\eta_3}{(2\Delta)^{1/2}} .$$

From Lemma 3.2, we observe that $\{f(n_i)\}$ is a $c_2\tilde{N}_\varepsilon^{1/2}\eta_3$-discrete subset with respect to $\tilde{d}_M$ in $f(B^M_{\eta_5}(f^{-1}(p)))$. From Lemma 6.1, it is an η_4-discrete subset with respect to d in $B_{\eta_2}(B^{T_pf(M)}_{\eta_1}(p))$. On the other hand, we consider η_4-discrete set $\{n'_i\}$ in $B_{\eta_2}(A^x_{\eta_4})$. Since $\eta_2 \leqq \eta_4/1000$, we easily see that $\{\tilde{P}(n'_i)\}$ is $\eta_4/2$-discrete in $A^x_{\eta_4+\eta_4} \subset A^x_{2\eta_4}$. Then,

$$\sharp\{n'_i\} = \sharp\{\tilde{P}(n'_i)\} \leqq \frac{\mathrm{vol}\,(A^x_{2\eta_4})}{b_0(\eta_4/2)} \leqq \left(\frac{4\eta_1}{\eta_4}\right)^{d-1},$$

From

$$\left(\frac{4\eta_1}{\eta_4}\right)^{d-1} = (4\mu)^{d-1} < \left(\frac{\mu c_2}{c_6 c_3}\right)^d = \left(\frac{\eta_5}{\eta_3\sigma}\right)^d,$$

there exists $(n_i) \notin B_{\eta_2}(A^x_{\eta_4})$, whence the conclusion.

(iv) Estimate of the "angle" between $T_pf(M)$ and $T_{p'}f(M')$.

Put $p' = P(p)$ and take $a \leqq \varepsilon < c_8 := \eta_2 r/10 \cdot \tilde{N}_\varepsilon^{-1/2}$. Hereafter we assume this. Then, for $\eta(\varepsilon) := (10\varepsilon/r)\tilde{N}_\varepsilon^{1/2} < \eta_2$,

$$f(M) \subset B_{\eta(\varepsilon)}(f'(M')) \quad \text{and} \quad f'(M') \subset B_{\eta(\varepsilon)}(f(M)) .$$

For $v \in U_pf(M)$ and $v' \in U_{p'}f'(M')$, let $\measuredangle\,(v, v')$ be the angle between v and v', which is equal to $\cos^{-1}\langle v, v'\rangle$.

LEMMA 6.4. *For any $v \in U_pf(M)$, there exists $v' \in U_{p'}f'(M')$ such that*

$$\measuredangle\,(v, v') \leqq \sin^{-1}\left(\frac{1}{25\sigma}\right) := \mu_\sigma .$$

Proof. If not, then there exists $v_0 \in U_pf(M)$ such that

$$\inf_{v' \in U_{p'}f'(M')} \measuredangle\,(v_0, v') = \max_{v \in U_pf(M)} \Big(\inf_{v' \in U_{p'}f'(M')} \measuredangle\,(v, v') \Big) > \mu_\sigma .$$

Let $H_{p'}$ be the plane through p' parallel to $T_pf(M)$ and $H = H_{p'} \cap T_{p'}f'(M')$. Then v_0 is perpendicular to H. In fact, let P': $T_pf(M) \to T_{p'}f'(M')$ be the normal projection and decompose v_0 as $v_0 = \lambda_1 v_1 + \lambda_2 v_2$, where $\lambda_1^2 + \lambda_2^2 = 1$, $v_1 \perp H$ and $v_2 \in H$. Since $|\tilde{P}'(\lambda_1 v_1 + \lambda_2 v_2)| = |\tilde{P}'(\lambda_1 v_1) + \lambda_2 v_2| \geqq |\tilde{P}'(v_1)|$ and $|\tilde{P}'(v_0)|$ is minimal, we see $\lambda_2 = 0$ and therefore v_0 is perpendicular to H. For $x = v_0$, we take $p_0 \in B^{f(M)}_{\eta_1}(p)$ satisfying $\langle \tilde{p}_0, v_0\rangle \geqq \eta_4$, by Lemma 6.3. Translate $\tilde{p}_0$ to $p'_1 \in H_{p'}$ and decompose $p'_0 = p'_1 + p'_2 + p'_3$, where p'_1 is v_0-component, $p'_2 \in H$ and p'_3 belongs to the orthogonal complement. Put $\tilde{P}'(p'_i) = q_i$. Then,

$$d(p_0, T_{p'}f'(M')) > d(\tilde{p}_0, T_{p'}f'(M')) - d(\tilde{p}_0, p_0)$$
$$= |p_0' - q_0| - \eta_2 - \eta_2 \geqq |p_1' - q_1| - 2\eta_2$$
$$\geqq \eta_4 \sin(\mu_\sigma) - 2\eta_2 \geqq 5\eta_2 - 2\eta_2 = 3\eta_2 .$$

On the other hand, from $d(p, p_0) \leqq \tilde{d}_M(p, p_0) \leqq \eta_1$, we get

$$d(P(p_0), p') \leqq d(P(p_0), p_0) + d(p_0, p) + d(p, p') \leqq 2\eta_2 + \eta_1 \leqq 2\eta_1 .$$

Therefore, since Lemma 6.2 can be applied,

$$d(p_0, T_{p'}f'(M')) \leqq d(p_0, P(p_0)) + d(P(p_0), T_{p'}f'(M'))$$
$$\leqq \eta_2 + \eta_2 = 2\eta_2 .$$

It is a contradiction. q.e.d.

§7. The diffeomorphism from M to M'

(i) $P|_{f(M)}$ is an injection.

LEMMA 7.1. *$P|_{f(M)}$ is injective.*

Proof. Suppose $P(p) = P(q) = p'$ with $p \neq q$. Note that the vector $\overrightarrow{pq}$ is perpendicular to $T_{p'}f'(M')$. From Lemma 6.4, there exists a unit normal vector n, which is parallel to the orthogonal complement of $T_{p'}f'(M')$ of $\overrightarrow{pq}$, such that

$$\measuredangle (n, \overrightarrow{pq}) \leqq \mu_\sigma .$$

Now, put $x = \operatorname{Exp}(p, c_5\tilde{N}_\varepsilon^{1/2}n)$. Since $\operatorname{Exp}|_{Bc_5\tilde{N}_\varepsilon^{1/2}}(Nf(M))$ is diffeomorphic, we see $d(x, p) < d(x, q)$. Let r be the point of the through x and q and $\overrightarrow{pr} \perp \overrightarrow{qx}$. Note that $d(p, r) \leqq d(p, q)$ and $\mu := \measuredangle (n, \overrightarrow{pq})$. Therefore,

$$d(p, q) \geqq d(p, r) \geqq c_5\tilde{N}_\varepsilon^{1/2} \cos(\mu) > 3\eta_3 .$$

On the other hand, since $f(M) \subset B_{\eta_2}(f'(M'))$ and $P(p) = P(q) = p'$,

$$d(p, q) \leqq d(p, p') = d(p', q) \leqq 2\eta_2 .$$

This is a contradiction. q.e.d.

(ii) $P|_{f(M)}$ is an immersion.

It sufficies to show the following.

LEMMA 7.2.

$$\frac{1 - \sin(\mu_\sigma)}{1 + \lambda} \leqq |dP(\xi)| \leqq \frac{1 + \sin(\mu_\sigma)}{1 - \lambda} \qquad \textit{for } \xi \in UM ,$$

where $\lambda = 2\eta(\varepsilon)r/c_2^2\tilde{N}_\varepsilon^{1/2}$.

Proof. Firstly, we estimate the principal curvature of $f(M)$. For $x \in U_p f(M)$, let $c(s) = f(m(s))$ be the curve with $\dot{c}(0) = x$, $m(0) = m$. From the definition, the second fundamental form $H(x, x)$ is the normal component of $d^2/ds^2|_{s=0}c(s)$. Let $v^\perp$ be the normal component of the vector v.

$$\begin{aligned} H(x, x) &= \left(\frac{d^2}{ds^2}\Big|_{s=0} c(s)\right)^\perp = \left(\frac{d^2}{ds^2}\Big|_{s=0} f(m(s))\right)^\perp \\ &= \left(\cdots, h'(d_M(m_i, m)) \operatorname{Hess} d_{M,m_i}\left(\frac{\dot{m}(0)}{|\dot{c}(o)|}, \frac{\dot{m}(0)}{|\dot{c}(0)|}\right)\right. \\ &\quad \left. + h''(d_M(m, m))\left(g\left(\operatorname{grad} d_{M,m_i}, \frac{\dot{m}(0)}{|\dot{c}(0)|}\right)\right)^2, \cdots\right)^\perp . \end{aligned}$$

By the argument similar to Lemma 4.1,

$$|H(x, x)| \leqq 2k\tilde{N}_\varepsilon^{1/2} \cdot \frac{|\dot{m}(0)|^2}{|\dot{c}(0)|^2} \leqq \frac{2k}{c_2^2}\tilde{N}_\varepsilon^{-1/2} .$$

Nextly, let $x(s)$ be the curve on $f(M)$ with $\dot{x}(0) = \xi$ and put $y(s) = P(x(s))$. Then it can be written as $x(s) - y(s) = \ell(s)n(s)$, where $n(s)$ is the unit normal vector field along $y(s)$. Since $\xi - dP(\xi) = \dot{x}(0) - \dot{y}(0) = \dot{\ell}(0)n(0) + \ell(0)\dot{n}(0)$, we get

$$\begin{aligned} \tilde{P}'(\xi) &= \tilde{P}'(dP(\xi) + \dot{\ell}(0)n(0) + \ell(0)\dot{n}(0)) \\ &= dP(\xi) + \ell(0)\tilde{P}'(\dot{n}(0)) , \end{aligned}$$

where $\tilde{P}'$ is the normal projection to $T_{p'}f'(M')$.

Note that $\tilde{P}'(\dot{n}(0))$ is the tangential component of $\dot{n}(0)$. The above estimate implies,

$$\begin{aligned} |\tilde{P}'(\xi) - dP(\xi)| &= |\ell(0)\tilde{P}'(\dot{n}(0))| \leqq \frac{2k}{c_2^2}\eta(\varepsilon)\tilde{N}_\varepsilon^{-1/2}|dP(\xi)| \\ &= \lambda|dP(\xi)| . \end{aligned}$$

On the other hand, from Lemma 6.4, if we denote by $\tilde{\xi}$ the parallel translation from p to p' of ξ, then

$$|\tilde{\xi} - P'(\xi)| \leqq \sin(\mu_\sigma) .$$

Therefore

$$\begin{aligned} |dP(\xi) - \tilde{\xi}| &\leqq |dP(\xi) - \tilde{P}'(\xi)| + |\tilde{P}'(\xi) - \tilde{\xi}| \\ &\leqq \sin(\mu_\sigma) + \lambda|dP(\xi)| . \end{aligned}$$

From this, we get a conclusion.

Finally, we get the diffeomorphism $F\colon M \to M'$ by $F = f'^{-1} \circ P \circ f$.

§ 8. Estimate of *dF*

We show that $|dF|$ is close to 1, if we take sufficiently small $r > 0$, a, $\varepsilon > 0$.

(i) Triangle comparison theorem.

Following lemma is an easy consequence of triangle comparison theorem in [3] Chap. 2.

Let $\Delta(a, b, c) \subset M$ be the geodesic triangle whose segments are a, b, c and $\ell(a)$ be the length of a and $\sphericalangle(a, b)$ is the angle between a and b.

LEMMA 8.1. *For any $\delta' > 0$, there exist c_9, $c_{10} > 0$ such that if $\Delta(a, b, c) \subset M$ and $\Delta(a', b', c') \subset M'$ satisfy the following,*

i) $c_9 \geqq \ell(a),\ \ell(b),\ \ell(a'),\ \ell(b') \geqq c_9/10$,

ii) $|\ell(a) - \ell(a')|,\ |\ell(b) - \ell(b')|,\ |\ell(c) - \ell(c')| \leqq c_{10}$,

then $|\sphericalangle(a, b) - \sphericalangle(a', b')| \leqq \delta'$.

(ii) Estimate of $|d_M(m_i, m) - d_{M'}(m'_i, F(m))|$.

LEMMA 8.2. *There exist c_{11}, $c_{12} > 0$ such that if $a \leqq \varepsilon < c_{12}$, then*

$$|d_M(m_i, m) - d_{M'}(m'_i, F(m))| \leqq c_{11}\varepsilon\ .$$

Proof. Take $m_j \in N[\varepsilon]$ and $m'_k \in N'[\varepsilon]$ satisfying

$$d_M(m, m_j) \leqq \varepsilon \quad \text{and} \quad d_{M'}(F(m), m'_k) \leqq \varepsilon\ .$$

From this,

$$\begin{aligned} d(f'(m'_j), f'(m'_k)) &\leqq d(f'(m'_j), f(m_j)) + d(f(m_j), f(m)) \\ &\qquad + d(f(m), P\circ f(m)) + d(P\circ f(m), f'(m'_k)) \\ &\leqq \frac{4a}{r}\tilde{N}_\varepsilon^{1/2} + \varepsilon c_3 \tilde{N}_\varepsilon^{1/2} + \eta(\varepsilon) + \varepsilon c_3 \tilde{N}_\varepsilon^{1/2} \\ &\leqq \left(\frac{4}{r} + c_3 + \frac{10}{r} + c_3\right)\tilde{N}_\varepsilon^{1/2}\varepsilon := c_{13}\tilde{N}_\varepsilon^{1/2}\ . \end{aligned}$$

We recall Lemma 4.2 and take $\alpha = c_2\varepsilon_1/10$. For sufficiently small $a, \varepsilon > 0$, we see $c_{13}\varepsilon \leqq \tilde{\alpha}$. Thus we see $\tilde{d}_{M'}(f'(m'_j), f'(m'_k)) \leqq (c_2\varepsilon_1/10)\tilde{N}_\varepsilon^{1/2}$ and from Lemma 3.2, $d_{M'}(m'_j, m'_k) \leqq \varepsilon_1/10$. So we can use Lemma 6.2, then,

$$\begin{aligned} d_{M'}(m'_j, m'_k) &\leqq \frac{c_6}{c_2}\tilde{N}_\varepsilon^{-1/2} d(f'(m'_j), f'(m'_k)) \\ &\leqq \frac{c_6}{c_2}\tilde{N}_\varepsilon^{-1/2}\left(\frac{4a}{r}\tilde{N}_\varepsilon^{1/2} + \varepsilon c_3 \tilde{N}_\varepsilon^{1/2} + \eta(\varepsilon) + \varepsilon c_3 \tilde{N}_\varepsilon^{1/2}\right) \\ &\leqq \frac{c_6 c_{13}}{c_2}\varepsilon\ . \end{aligned}$$

From the above, we observe,

$$\begin{aligned}&|d_M(m, m_i) - d_{M'}(F(m), m_i')|\\&\quad\leqq |d_M(m_i, m_j) - d_{M'}(m_i', m_j')| + d_M(m, m_j) + d_{M'}(F(m), m_j')\\&\quad\leqq 2ra + d_M(m, m_j) + d_{M'}(F(m), m_k') + d_{M'}(m_k', m_j')\\&\quad\leqq 2r\varepsilon + \varepsilon + \varepsilon + \frac{c_6 c_{13}}{c_2}\varepsilon := c_{11}\varepsilon\,.\end{aligned}$$

q.e.d.

(iii) Definition of the isometry I: $T_m M \to T_{F(m)}M'$.

$$\text{Put } u_i = \exp_m^{-1} m_i / |\exp_m^{-1} m_i|$$

and

$$u_i' = \exp_{F(m)}^{-1} m_i' / |\exp_{F(m)}^{-1} m_i'|\,.$$

Combining Lemma 8.1 and 8.2, we get for any $\delta'' > 0$, there exist c_{14}, $c_{15} > 0$ such that if $c_{14} \geqq d_M(m_i, m) \geqq c_{14}/10$ and $\varepsilon < c_{15}$, then

$$|\langle u_i, u_j\rangle - \langle u_i', u_j'\rangle| < \delta''\,.$$

We choose $u_{i_1}, \cdots, u_{i_d}$ satisfying $\langle u_{i_j}, u_{i_j}\rangle \geqq 1 - (1/100d^2)$ and $|\langle u_{i_j}, u_{i_k}\rangle| \leqq 1/100d^2$, $(j \neq k)$. From these, we get the orthonormal basis $\{e_i\}_{i=1}^d$ of $T_m M$ by Schmidt's orthogonalization. Namely $e_1 = u_{i_1}$,

$$e_{j+1} = \left(u_{i_{j+1}} - \sum_{k=1}^{d} \langle u_{i_{j+1}}, e_k\rangle e_k\right) \Big/ \left|u_{i_{j+1}} - \sum_{k=1}^{d} \langle u_{i_{j+1}}, e_k\rangle e_k\right|, \cdots\,.$$

We also get the orthonormal basis $\{e_i'\}_{i=1}^d$ of $T_{F(m)}M'$ from $\{u_{i_j}'\}_{i=1}^d$. Put $a_{jk} = \langle e_j, u_{i_k}\rangle$ and $a_{jk}' = \langle e_j', u_{i_k}'\rangle$. Then by inductive arguments, we see

$$|a_{jk} - a_{jk}'| \leqq (100d)^{j+k}\delta'' \leqq (100d)^{2d}\delta''\,.$$

We define the isometry I: $T_m M \to T_{F(m)}M'$ by $I(e_i) = e_i'$.

(iv) Estimate of dF.

From the definition, we know

$$df_m(\xi) = (\cdots, h'(t_i) \textstyle\sum_j a_{ij}\xi_j, \cdots)$$

for $\xi = \sum \xi_j e_j \in U_m M$ and $t_i = d_M(m, m_i)$. Put $t_i' = d_{M'}(F(m), m_i'')$.

LEMMA 8.3. *For any $\delta > 0$, there exist c_{16}, c_{17}, $c_{18} > 0$ such that if $r < c_{16}$, $\kappa < c_{17}$* (see § 2), *a, $\xi < c_{18}$, then,*

$$|dF(\xi) - I(\xi)| < \delta\,.$$

Proof. Firstly, we estimate $|df(\xi) - df'(I(\xi))|$. From the definition,

$$\begin{aligned}|df(\xi) - df'(I(\xi))|^2 &= \sum_{i=1}^{N_\varepsilon} \Big(h'(t_i) \sum_j a_{ij}\xi_j - h'(t_i') \sum_j a_{ij}'\xi_j\Big)^2\\&\leqq \sum_{t_i, t_i' \in [r/8, 7r/8]} + \sum_{\text{otherwise}}\,.\end{aligned}$$

From Lemma 8.2, there exists $c_{19} > 0$ such that if a, $\varepsilon < c_{19}$, then $|h'(t_i) - h'(t_i')| \leqq c_{17}/10d$. Thus, from $|h'(t)| \leqq 4/r$,

$$\begin{aligned}(\text{first term}) &\leqq \sum_{t_i, t_i' \in [2r/8, 6r/8]} \{h'(t_i)(\sum_j (a_{ij} - a'_{ij})\xi_j) \\ &\qquad + (h'(t_i) - h'(t_i')) \sum_j a'_{ij}\xi_j\}^2 \\ &\leqq \left(\frac{4}{r}(100d)^{2d}\delta'' d^2 + c_{17}\cdot \frac{d^2}{10d}\right)^2 \tilde{N}_\varepsilon .\end{aligned}$$

Note that if $t_i \in [0, r/8] \cup [7r/8, r]$, then $t_i' \in [0, 2r/8] \cup [6r/8, r] := J$. Since $c_{17} > \kappa > |h'(t)|$ on $t \in J$, we see

$$\begin{aligned}(\text{second term}) &\leqq (\sum_{t_i, t_i' \in J} |h'(t_i) + h'(t_i')|^2)(|\sum a_{ij}\xi_j| + |\sum a'_{ij}\xi_j|)^2 \\ &\leqq 4c_{17}^2 \cdot 4d^4 \tilde{N}_\varepsilon .\end{aligned}$$

Therefore,

$$\begin{aligned}|df(\xi) - df'(I(\xi))|^2 &\leqq \left(\left((100d)^{2d} + \frac{d}{10} + 4d^2\right)\cdot\frac{4}{r}\right)^2 (\delta'' + 2c_{17})^2 \tilde{N}_\varepsilon \\ &\leqq (100d)^{6d}\cdot r^{-2}(\delta'' + 2c_{17})^2 \tilde{N}_\varepsilon .\end{aligned}$$

Secondly, from Lemma 7.2, we find

$$|dP \circ df(\xi) - df(\xi)| \leqq 2\eta(\varepsilon) + \frac{\sin(\mu_\sigma) + \lambda}{1 + \lambda}|df(\xi)| .$$

For fixed $r > 0$, there exists $c_{20}, c_{21} > 0$ such that if a, $\varepsilon < c_{20}$, $\sigma > c_{21}$, then the righthand side of the above inequality is smaller than $(10^5 d)^{-d}(\delta/10c_3)|df(\xi)|$, by the definition of $\eta(\varepsilon)$ and μ_σ (§ 6, § 7).

Therefore since $c_2 = (10^5 d)^{-d/2} 3/2r$, (§ 3 Remark),

$$\begin{aligned}&\frac{1}{\inf_\xi |df'(\xi)|}|dP \circ df(\xi) - df' \circ I(\xi)| \\ &\quad\leqq (c_2 \tilde{N}_\varepsilon^{1/2})^{-1}\left((100d)^{3d} r^{-1}(\delta'' + 2c_{17})\tilde{N}_\varepsilon^{1/2} + (10^5 d)^{-d}\frac{\delta}{10c_3}c_3\tilde{N}_\varepsilon^{1/2}\right) \\ &\quad\leqq (10^5 d)^{5d}(\delta'' + 2c_{17}) + \frac{\delta}{10} .\end{aligned}$$

For $\delta'' > 0$ satisfying $(10^5 d)^{5d}\delta'' \leqq \delta/10$, take $c_{16} > 0$ as $c_{16} \leqq c_{14}$ and $c_{17} > 0$ as $(10^5 d)^{5d} 2c_{17} \leqq \delta/10$ and $c_{18} > 0$ as $c_{18} \leqq \min(c_{15}, c_{19}, c_{20})$.

Finally we get,

$$\begin{aligned}|dF(\xi) - I(\xi)| &= |df'^{-1} \circ dP \circ df(\xi) - I(\xi)| \\ &\leqq \frac{1}{\inf_\xi |df'(\xi)|}|dP \circ df(\xi) - df' \circ I(\xi)| < \delta . \qquad \text{q.e.d.}\end{aligned}$$

§ 9. In the case when M is noncompact

In the case when M is noncompact, let M_b be the set of all points m of M with $d_M(m, m_0) < b$ for fixed $m_0 \in M$. In the above, we get the map F_b: $M_{b-2r} \to M'_b$. Note that the estimate of constants do not depend on b, thus for fixed b_0, $F_b|_{M_{b_0}} = F_{b'}|_{M_{b_0}}$ for b, $b' \gg b_0$. Let F: $M \to M'$ be the inductive limit of F_b.

We see that F is a diffeomorphism. The injectivity and immersivity follows from those of F_b. Surjectivity follows from Lemma 8.3 and the implicit function theorem. q.e.d.

§ 10. Proof of Theorem 2

From the result of Heintze-Karcher [8] or Maeda [11], we get the estimate of the injectivity radius i_M in terms of d, Δ, ρ, v, namely,

$$i_M \geqq \min\left(\pi/\Delta^{1/2}, \frac{\pi v}{\omega_d}\cdot\exp\left(-(d-1)\rho\Delta^{1/2}\right)\right).$$

Therefore we can use Theorem 1. Take a, $\varepsilon > 0$ which satisfy the assumption of Theorem 1. Let M_{N_1} be the set of elements in $M(d, \Delta, \rho, v)$, which have a minimal ε-dense subset $\{m_i\}_{i=1}^{N_1}$. From the volume comparison theorem, we see $N_1 \leqq b_{-\Delta}(\rho)/b_\Delta(\varepsilon/2) := N_0$. Therefore it suffices to estimate the number of the diffeomorphism classes in M_{N_1} for $N_1 \leqq N_0$.

Now, take a function

$$\Phi\colon M_{N_1} \longrightarrow Q = \prod_{k=1}^{N_1(N_1-1)} [\log(\varepsilon/2), \log(\rho)]$$

defined by

$$\Phi(M) = \{\log(d_M(m_i^k, m_j^k))\}_{k=1}^{N_1(N_1-1)},$$

where Q is the direct product of the intervals $[\log(\varepsilon/2), \log(\rho)]$ and k is a loxicographic order of (i, j). We define the distance d_Q on Q by,

$$d_Q(x, y) = \max_{1\leqq k\leqq N_1(N_1-1)} |x_k - y_k|,$$

where $x = \{x_i\}$, $y = \{y_i\}$.

Then, Theorem 1 says that if $d_Q(\Phi(M), \Phi(M')) \leqq -\log(1-a) := b_1$, then M and M' are diffeomorphic. Therefore it is sufficient to estimate the cardinality of maximal set P_{N_1} in Q, of which elements α, β ($\alpha \neq \beta$) satisfy $d_Q(\alpha, \beta) > b_1$,

$$\sharp P_{N_1} \leqq \left(\frac{2b_2}{b_1}\right)^{N_1(N_1+1)} \leqq \left(\frac{2b_2}{b_1}\right)^{N_0(N_0+1)},$$

where $b_2 = \log(\rho) - \log(\varepsilon/2)$. After all we can estimate the number of the diffeomorphism classes of $M(d, \Delta, \rho, v)$, which is smaller than $N_0(2b_2/b_1)^{N_0(N_0+1)}$.

q.e.d.

§11. Outline of the proof of Theorem 3

Let M be a compact d-dimensional Riemannian manifold with $|K_M| \leqq \Delta$ and $\mathrm{Ric}_M \geqq d-1$. Let $m, n, m_1, m_2, \cdots$, be the points of M and $p, q, p_1, p_2, \cdots$, be the points of S^d. We denote by $TD(m)$ the interior of the tangential cut locus i.e., $TD(m) =$ the interior of $\{v \in T_mM \mid d_M(m, \exp_m v) = |v|\}$. For the linear isometry I: $T_pS^d \to T_mM$, we define the map $F = \exp_m \circ I \circ \exp_p^{-1}$: $B_\pi(p) \to M$. Put $D' = \exp_p(I^{-1}(TD(m)))$. From the theorem of Myers, we see $D' \subset B_\pi(p)$. Moreover if the closure of D' is not contained in $B_\pi(p)$, then $\mathrm{diam}_M = \pi$, so M is isometric to S^d by Cheng's Theorem [2]. We may argue the case when the closure of D' is contained in $B_\pi(p)$.

We give an outline of the proof of Theorem 3. From $|K_M| \leqq \Delta$, $|dF|$ can be estimated in D'. We see that $\mathrm{vol}(S^d - D')$ is small and $|dF|$ is close to 1 on much part in D' —this is "good" part—, using the fact $\mathrm{vol}(M) \geqq \mathrm{vol}(S^d) - \delta$. Since the volume of the "bad" part is small, we can choose $\varepsilon/2$-dense, $\varepsilon/4$-discrete subset $\{p_i\}$ of S^d in D' such that the geodesic connecting the points of $\{p_i\}$ intersects small "bad" part. So we see that $d_{S^d}(p_i, q_j)$ is not much smaller than $d_M(m_i, m_j)$, where $m_i = F(p_i)$. Therefore, if we see that

(1) $\{m_i\}$ is ε-dense, $\varepsilon/10$-discrete in M.

(2) $d_{S^d}(p_i, p_j)$ is not much larger than $d_M(m_i, m_j)$,

then, from Theorem 1, we find that M is diffeomorphic to S^d. We show (1) by the following arguments. If not, then there exists a point $n \in M$ such that $\min d_M(n, m_i)$ is larger than $3\varepsilon/2$. Since F does not much expand on "good" part and so $B_{\varepsilon/4}(n)$ is intersect only "bad" part. But since "bad" part is very small, it cannot cover $B_{\varepsilon/4}(n)$. This contradicts the fact F is surjection. Assume that (2) does not hold, namely there exist p_i, p_j such that $d_{S^d}(p_i, p_j)$ is much larger than $d_M(m_i, m_j)$. Let B_1, B_2 be the ball with the center p_i, p_j, of which radius is a half of $d_{S^d}(p_i, p_j)$. From the assumption, we see that $\mathrm{vol}(B_1 \cup B_2)$ is much larger than $\mathrm{vol}(F(B_1 \cup B_2))$. It contradicts the fact $\mathrm{vol}(M) > \mathrm{vol}(S^d) - \delta$.

§ 12. Estimate of *dF*

LEMMA 12.1. i) $|\det F| \leqq 1$ *on* D'.

ii) *For any* $\delta_3 > 0$, *there exists* $L = L(d, \Delta\colon \delta_3) > 0$ *such that*

$$|dF| \leqq L \qquad \text{on } B_{\pi-\delta}(p)\,.$$

Proof. From $\mathrm{Ric}_M \geqq d-1$, i) follows from the volume comparison theorem (cf. [7] or [13]). For ii), we quote from [1] 6.4.1, that is $|(d\exp_m)_{rv}w| \leqq |w|(s_{-\Delta}(t)/r)$ on M, where $|v| = 1$, $v \perp w$ and this inequality holds as long as $s_{(1/2)(-\Delta+\Delta)}(r) = r$ is positive. Since $|(d\exp_p)_{rv}w| = |w|(\sin(r)/r)$ on S^d, we may put $L = s_{-\Delta}(\pi-\delta_3)/\sin(\pi-\delta_3)$. q.e.d.

Put $\overline{A}[\delta_4] = \{q \in D' \,|\, |dF_q| > 1+\delta_4\}$ and $\overline{B}[\delta_4] = \{q \in D' \,|\, |\det dF_q| < 1-\delta_4\}$. Notice that $\overline{A}$ does not mean the closure of A here.

LEMMA 12.2. *For any* δ_4, $\delta_5 > 0$, *there exists* $\delta_6 = \delta_6(d, \Delta; \delta_5) > 0$ *such that if* $\mathrm{vol}(M) \geqq \mathrm{vol}(S^d) - \delta_6$, *then* $\mathrm{vol}(\overline{A}[\delta_4]) < \delta_5$, $\mathrm{vol}(\overline{B}[\delta_4]) < \delta_5$ *and* $\mathrm{vol}(S^d - D') < \delta_5$.

Since the proof of this lemma is elementary but complicated, so we only give here an outline and the detailed proof is left over to Section 14. It seems to be able to prove more easily.

From Lemma 12.1, F is volume decreasing. With $F(D') = M$ and $\mathrm{vol}(M) \geqq \mathrm{vol}(S^d) - \delta$, we see that the $\mathrm{vol}(\overline{B}[\delta_4]) < \delta_5$ and $\mathrm{vol}(S^d - D') < \delta_5$. To show the first inequality, we observe that the arguments of the equality case of the volume comparison theorem in [8] can be modified to the near-equality case. So we find K_M is close to 1 on much part. From this, using Rauch's comparison theorem, we see $|dF|$ is close to 1 on much part.

§ 13. Proof of Theorem 3

(i) Construction of ε-dense set $\{p_i\}$ on S^d.

LEMMA 13.1. *For any* δ_7, $\delta_8 > 0$, *there exists* $\delta_9 = \delta_9(d, \Delta; \delta_7, \delta_8) > 0$ *and a* δ_7*-dense subset* $\{p_i\}$ *of* S^d *in* $B_{\pi-\delta_7/10}(p)$ *such that if* $\mathrm{vol}(M) \geqq \mathrm{vol}(S^d) - \delta_9$, *then*

$$\frac{d_M(F(p_i), F(p_j))}{d_{S^d}(p_i, p_j)} \leqq 1 + \delta_8 \qquad \text{for } d_{S^d}(p_i, p_j) < \frac{\pi}{20}\,.$$

Proof. We may assume $0 < \delta_8 < \delta_7 < 1$. Take *a* $\delta_7/2$-dense, $\delta_7/2$-discrete subset $\{q_i\}_{i=1}^N$ of S^d in $B_{\pi-\delta_7/20}(p)$. Put $N=\#\{q_i\}$ and $B_i=B_{\delta_7/100}(q_i)$. Note that $B_i\subset B_{\pi-\delta_7/20}(p)$. Take

$$\delta_{10}\leqq\left(\frac{d}{20N}\cdot\frac{10}{\pi}\cdot b_1\left(\frac{\delta_8}{100}\right)\cdot\left(\frac{\delta_8}{1000}\right)^{d-1}\right)^{1/(d-1)}.$$

We define

$$\Lambda[\delta_8]=\{q\in B_{\pi-\delta_{10}}(p)\,|\,|dF_q|\leqq 1+\delta_8/2\}=B_{\pi-\delta_{10}}(p)-\bar{\Lambda}[\delta_8/2]\,.$$

From Lemma 12.2, there exists $\delta_9>0$ such that if $\mathrm{vol}\,(M)\geqq\mathrm{vol}\,(S^d)-\delta_9$, then

$$\mathrm{vol}\,(B_{\pi-\delta_{10}}(p)-\Lambda[\delta_8])<\frac{1}{20N}\cdot b_1\left(\frac{\delta_8}{100}\right)\cdot\frac{2}{d}\cdot\left(\frac{\alpha}{4}\right)^d\sin^{1-d}\left(\frac{\pi}{10}\right),$$

where $\alpha=\delta_7\delta_8/200L$ and $L=L(\delta_{10})=s_{-\Delta}(\pi-\delta_{10})/\sin(\pi-\delta_{10})$ in Lemma 12.1.

Hereafter we denote by $\gamma_{p,q}$ the minimal geodesic from p to q. Then, we observe that for $q_i'\in B_i$, $q_j'\in B_j$, if $\gamma_{q_i',q_j'}\subset B_{\pi-\delta_{10}}(p)$, then

$$\begin{aligned}d_M(F(q_i'),F(q_j'))&\leqq\int_{\gamma_{q_i',q_j'}}|dF|\,dt\\&=\int_{\Lambda[\delta_8]\cap\gamma_{q_i',q_j'}}|dF|\,dt+\int_{\gamma_{q_i',q_j'}-\Lambda[\delta_8]}|dF|\,dt\\&\leqq(1+\delta_8/2)d_{S^d}(q_i',q_j')+L\cdot m(\gamma_{q_i',q_j'}-\Lambda[\delta_8]):=A_1\end{aligned}$$

where $m(\cdot)$ is the canonical measure on $\gamma_{q_i',q_j'}$.

If $m(\gamma_{q_i',q_j'}-\Lambda[\delta_8])\leqq\alpha$, then

$$(*)\qquad\begin{aligned}\frac{d_M(F(q_i'),F(q_j'))}{d_{S^d}(q_i',q_j')}&\leqq\frac{A_1}{d_{S^d}(q_i',q_j')}\\&\leqq 1+\frac{\delta_8}{2}+\frac{L\alpha}{d_{S^d}(q_i',q_q')}\\&\leqq 1+\frac{\delta_8}{2}+\frac{\delta_8}{2}=1+\delta_8\,.\end{aligned}$$

In the following, we prove that p_i can be taken in $\Lambda[\delta_8]\cap B_i$. For the existence of $p_1\in B_1\cap\Lambda[\delta_8]$, we only note the inequality $\mathrm{vol}\,(B_{\pi-\delta_{10}}-\Lambda[\delta_8])<\mathrm{vol}\,(B_1)$.

Nextly, suppose that there exist points $p_1,p_2,\cdots,p_k$ $(p_i\in B_i)$ such that

$$\frac{d_M(F(p_i),F(p_j))}{d_{S^d}(p_i,p_j)}\leqq 1+\delta_8\qquad\text{for }d_{S^d}(p_i,p_j)\leqq\frac{\pi}{20}.\quad(1\leqq i,j\leqq k)$$

Then, we show that there exists $p_{k+1} \in B_{k+1}$ which satisfies

$$\frac{d_M(F(p_{k+1}), F(p_i))}{d_{S^d}(p_{k+1}, p_i)} \leqq 1 + \delta_8 \qquad \text{for } i \leqq k .$$

In fact, if not, then for any $q \in B_{k+1}$, there exists $p_i \in B_i$ such that

$$\frac{d_M(F(q), F(p_i))}{d_{S^d}(q, p_i)} > 1 + \delta_8 .$$

Then from (*), $m(\gamma_{q,p_i} - \Lambda[\delta_8]) > \alpha$ or $\gamma_{q,p_i} \cap B_{\delta_{10}}(\tilde{p}) \neq \phi$, where $\tilde{p}$ is the antipodal point of p. Let S_i^1 be the set of $q \in B_{k+1}$ such that $m(\gamma_{q,p_1} - \Lambda[\delta_8]) > \alpha$ and S_i^2 be the set of $q \in B_{k+1}$ such that $\gamma_{p_1,q} \cap B_{\delta_{10}}(\tilde{p}) \neq \phi$ and $S_i = S_i^1 \cup S_i^2$. Since, by the assumption, $B_{k+1} \subset \bigcup_i S_i$, we may assume that

$$(**) \qquad \operatorname{vol}(S_1) = \max_i \operatorname{vol}(S_i) \geqq \frac{1}{2N} \cdot \operatorname{vol}(B_{k+1}) .$$

Let C^i be the cone consisting of the points of $\gamma_{p_1,q}(q \in S_1^i)$ and $\tilde{C}^i = \exp_{p_1}^{-1}(C^1)$. Put $E_t^i = C^i \cap B_t(p_1)$. Since $m(\gamma_{q,p_1} - \Lambda[\delta_8]) > \alpha$, for $q \in S_1^1$, from the Fubini's theorem, we observe

$$\begin{aligned} &\operatorname{vol}(B_{\pi-\delta_{10}}(p) - \Lambda[\delta_8]) \\ &\quad \geqq \int_{U_{p_1}S^d \cap \tilde{C}_1 \ni v} \left(\int_0^{\pi-\delta_{10}} \chi_{\gamma_v - \Lambda[\delta_8]}(t) \cdot \sin^{d-1}(t)\, dt \right) dv_{U_{p_1}S^d} . \end{aligned}$$

where γ_v is the geodesic emanating from p_1 with initial vector v, $\chi_A(t)$ is the characteristic function of the set A and $dv_{U_pS^d}$ is the canonical measure on U_pS^d induced from Lebesgue measure on T_pS^d.

$$\begin{aligned} &\geqq \int_{U_{p_1}S^d \cap \tilde{C}_1} \left(\left(\int_0^{\alpha/2} + \int_{\pi-\delta_{10}-\alpha/2}^{\pi-\delta_{10}} \right) \sin^{d-1}(t)\, dt \right) dv_{U_{p_1}S^d} \\ &\geqq \int_{U_{p_1}S^d \cap \tilde{C}_1} \left(\int_0^{\alpha} \left(\frac{t}{2}\right)^{d-1} dt \right) dv_{U_{p_1}S^d} \\ &= \int_{U_{p_1}S^d \cap \tilde{C}_1} \frac{2}{d} \left(\frac{\alpha}{2}\right)^d dv_{U_{p_1}S^d} . \end{aligned}$$

Namely,

$$\int_{U_{p_1}S^d \cap \tilde{C}_1} dv_{U_{p_1}S^d} \leqq \operatorname{vol}(B_{\pi-\delta_{10}} - \Lambda[\delta_8]) \cdot \frac{d}{2} \cdot \left(\frac{2}{\alpha}\right)^{\alpha} .$$

On the other hand, since $d_{S^d}(p_1, B_{\delta_{10}}(\tilde{p})) > \delta_8/100$, we see

$$\operatorname{vol}(E_{\pi/10}^2) \leqq \operatorname{vol}(E) ,$$

where E is the cone in S^d, which contains $B_{\delta_{10}}(\tilde{p})$ far from its summit with distance $\delta_8/100$ and the length of generating line is smaller than $\pi/10$. From the spherical trigonometry, we calculate

$$\mathrm{vol}\,(E) \leqq \frac{1}{d}\cdot\frac{\pi}{10}\cdot\left(\frac{1000\delta_{10}}{\pi\delta_8}\right)^{d-1}.$$

Thus we estimate, from $m(\gamma_v \cap B_1) < \delta_8/50$.

$$\begin{aligned}\mathrm{vol}\,(S_1) &\leqq \mathrm{vol}\,(E^1_{\pi/10}\cap B_1) + \mathrm{vol}\,(E^2_{\pi/10})\\ &= \int_{U_{p_1}S^d\cap \tilde{C}_1 \ni v}\left(\int_\gamma \chi_{(\gamma_v\cap B_1)}(t)\sin^{d-1}(t)\,dt\right)dv_{U_{p_1}S^d} + \mathrm{vol}\,(E^2_{\pi/10})\\ &\leqq \int_{U_{p_1}S^d\cap\tilde{C}_1}\sin^{d-1}\left(\frac{\pi}{10}\right)\cdot\frac{\delta_8}{50}\,dv_{U_pS^d} + \mathrm{vol}\,(E^2_{\pi/10})\\ &\leqq \mathrm{vol}\,(B_{\pi-\delta_{10}}(p) - \Lambda[\delta_8])\cdot\frac{d}{2}\cdot\left(\frac{2}{\alpha}\right)^d\cdot\sin^{d-1}\left(\frac{\pi}{10}\right)\cdot\frac{\delta_8}{50}\\ &\quad + \frac{1}{d}\cdot\frac{\pi}{10}\cdot\left(\frac{1000\delta_{10}}{\pi\delta_8}\right)^{d-1}\\ &\leqq \frac{1}{10N}\cdot b_1\left(\frac{\delta_8}{100}\right) < \frac{1}{2N}\mathrm{vol}\,(B_{k+1})\,,\end{aligned}$$

namely,

$$\mathrm{vol}\,(S_1) < \frac{1}{2N}\cdot\mathrm{vol}\,(B_{k+1})\,,$$

It contradicts (**). q.e.d.

(ii) Proof of Theorem 3.

We take a, $\varepsilon > 0$, which satisfy the assumption of Theorem 1. For $\delta_7 = \varepsilon/2$, take $\delta_8 > 0$ satisfying $\delta_8 \leqq \min\left((1/2)b_\Delta(\delta_7/10)\omega_d^{-1},\ a/10\right)$. Let $\{p_i\}$ and $\alpha > 0$ be the same as in Lemma 13.1. From Theorem 1, it suffices to prove that there exists $\delta > 0$ such that, if $\mathrm{vol}\,(M) \geqq \mathrm{vol}\,(S^d) - \delta$, then $\{F(p_i)\}$ is an ε-dense, $\varepsilon/10$-discrete in M and it satisfies

$$\frac{d_M(F(p_i), F(p_j))}{d_{S^d}(p_i, p_j)} \geqq 1 - a\,, \qquad \text{for } 0 < d_{S^d}(p_i, p_j) < \frac{\pi}{20}\,.$$

CLAIM 1: *$\{F(p_i)\}$ is $2\delta_7$ $(=\varepsilon)$-dense in M.*

Proof of Claim 1. If not, then there exists $n \in M$ such that

$$B_{\delta_7/10}(n) \cap \left(\bigcup_i B_{3\delta_7/2}(F(p_i))\right) = \phi\,.$$

Put

$$B'_i = \{q \in B_{\delta_7}(p_i) \mid q \in \gamma_{q',p_i},\ q' \in \partial B_{\delta_7}(p_i),$$
$$m(\gamma_{q',p_i} - \Lambda[\delta_8]) > \alpha \text{ or } \gamma_{q',p_i} \cap B_{\delta_{10}}(\bar p) \neq \phi\}$$

and $\tilde B_i = B_{\delta_7}(p_i) - B'_i$. From (*) in the proof of Lemma 13.1, we see $F(\bigcup_i \tilde B_i) \subset (\bigcup B_{3\delta_7/2}(F(p_i)))$.

From the similar argument to Lemma 13.1, we see

$$\mathrm{vol}\,(B'_i) \leqq \frac{d}{2}\cdot\left(\frac{4}{\alpha}\right)^d \mathrm{vol}\,(B_{\pi-\delta_{10}}(p) - \Lambda[\delta_8]) + \frac{1}{d}\left(\frac{\pi}{10}\right)\left(\frac{1000\delta_{10}}{\pi\delta_8}\right)^{d-1} := A_1$$
$$\mathrm{vol}\,(\tilde B_i) \geqq \mathrm{vol}\,(B_{\delta_7}(p_i)) - A_1\,.$$

Note that

$$\mathrm{vol}\,(F(\tilde B_i \cap D' - \bar B[\delta_8]) \geqq (1-\delta_8)\,\mathrm{vol}\,(\tilde B_i \cap D' - \bar B[\delta_8]),$$

where $\bar B[\delta_8]$ appears in Lemma 12.2.

From this, we have

$$\begin{aligned}\mathrm{vol}\,(M) &\geqq \mathrm{vol}\,(B_{\delta_7/10}(n)) + \mathrm{vol}\,(\bigcup_i F(\tilde B_i))\\ &\geqq \mathrm{vol}\,(B_{\delta_7/10}(n)) + (1-\delta_8)(\,\mathrm{vol}\,(\bigcup (B_i \cap D' - \bar B[\delta_8]))\\ &\geqq \mathrm{vol}\,(B_{\delta_7/10}(n)) + (1-\delta_8)(\mathrm{vol}\,(\bigcup_i \tilde B_i)) - \mathrm{vol}\,(S^d - D') - \mathrm{vol}\,(\bar B[\delta_8])\\ &\geqq \mathrm{vol}\,(B_{\delta_7/10}(n)) + (1-\delta_8)(\mathrm{vol}\,(\bigcup_i B_{\delta_7}(p_i))) - NA_1 - \mathrm{vol}\,(S^d - D')\\ &\quad - \mathrm{vol}\,(\bar B[\delta_8])\\ &\geqq \mathrm{vol}\,(B_{\delta_7/10}(n)) + \mathrm{vol}\,(S^d) - \delta_8\,\mathrm{vol}\,(S^d) - NA_1 - \mathrm{vol}\,(S^d - D')\\ &\quad - \mathrm{vol}\,(\bar B[\delta_8])\end{aligned}$$

where $N = \#\{p_i\}$.

From Lemma 12.2, there exists $\delta_{11} > 0$ such that if $\mathrm{vol}\,(M) \geqq \mathrm{vol}\,(S^d) - \delta_{11}$, then

$$\delta_8\,\mathrm{vol}\,(S^d) + NA_1 + \mathrm{vol}\,(S^d - D') + \mathrm{vol}\,(\bar B[\delta_8]) < b_\Delta(\delta_7/10)$$
$$\leqq \mathrm{vol}\,(B_{\delta_7/10}(n)).$$

(The constants are determined in following order, $\delta_7 \to \delta_8 \to \delta_{10} \to L \to \alpha \to \delta_{11}$.)

Therefore, we see,

$$\mathrm{vol}\,(M) > \mathrm{vol}\,(S^d) + \mathrm{vol}\,(B_{\delta_7/10}(n)) - b_\Delta(\delta_7/10) \geqq \mathrm{vol}\,(S^d) \geqq \mathrm{vol}\,(M)\,.$$

It is a contradiction.

CLAIM 2: $\dfrac{d_M(F(p_i), F(p_j))}{d_{S^d}(p_i, p_j)} \geqq 1 - \delta_8 > 1 - a\,.$

Proof of Claim 2. If not, then we may assume $d_M(F(p_1), F(p_2)) < (1-\delta_8)d_{S^d}(p_1, p_2)$, Put $d' = d_{S^d}(p_1, p_2)$ and $d'' = d_M(F(p_1), F(p_2))$. There exists $\delta_{12} > 0$ such that

$$b_1\left(\frac{d'}{2}\right) - b_1\left(\frac{d'}{2} - \delta_{12}\right) < \frac{1}{10} \cdot b_\Delta\left(\frac{d'}{2} - \frac{d''}{2}\right).$$

For this δ_{12}, similarly as Lemma 13.1, there exists $\eta > 0$ such that if $d_M(F(q'), F(p_i)) > d'/2$ for $q' \in \partial B_{d'/2-\delta_{12}}(p_i)$, then

$$m(\gamma_{q',p_i} - \Lambda[\delta_8]) > \eta \quad \text{or} \quad \gamma_{q',p_i} \cap B_{\delta_{10}}(\tilde{p}) \neq \phi .$$

Put

$$B = \bigcup_{i=1}^{2} (B_{d'/2-\delta_{12}}(p_i) - \{q \in B_{d'/2-\delta_{12}}(p_i) \mid q \in \gamma_{q',p_i},$$
$$q' \in \partial B_{d'/2}(p_i),\ m(\gamma_{q',p_i} - \Lambda[\delta_8]) > \eta \quad \text{or} \quad \gamma_{q',p_i} \cap B_{\delta_{10}}(\tilde{p}) \neq \phi\}) .$$

and

$$A_2 = \frac{d}{2}\left(\frac{4}{\eta}\right)^d \mathrm{vol}\,(B_{\pi-\delta_{10}}(p) - \Lambda[\delta_8]) + \frac{1}{d}\left(\frac{\pi}{10}\right)\left(\frac{1000\delta_{10}}{\pi\delta_8}\right).$$

Then we observe $F(B) \subset (B_{d'/2}(F(p_1)) \cup B_{d'/2}(F(p_2)))$ and

$$\begin{aligned}
&\mathrm{vol}\,(F(B_{d'/2-\delta_{12}}(p_1) \cup B_{d'/2-\delta_{12}}(p_2))) - A_2\\
&\quad \leqq \mathrm{vol}\,(F(B)) \leqq \mathrm{vol}\,(B_{d'/2}(F(p_1)) \cup B_{d'/2}(F(p_2)))\\
&\quad \leqq \mathrm{vol}\,(B_{d'/2}(F(p_1))) + \mathrm{vol}\,(B_{d'/2}(F(p_2)))\\
&\qquad - \mathrm{vol}\,(B_{d'/2-d''/2}(z)) ,
\end{aligned}$$

where z is the mid point of the minimal geodesic from $F(p_1)$ to $F(p_2)$.

These inequalities imply that

$$\begin{aligned}
\mathrm{vol}\,(M) &\leqq \mathrm{vol}\,(F(D' - (B_{d'/2}(p_1) \cup B_{d'/2}(p_2))))\\
&\quad + \mathrm{vol}\,(F(D' \cap (B_{d'/2}(p_1) \cup B_{d'/2}(p))))\\
&\leqq \mathrm{vol}\,(S^d - (B_{d'/2}(p_1) \cap B_{d'/2}(p_2)))\\
&\quad + \mathrm{vol}\,(B_{d'/2}(p_1) - B_{d'/2-\delta_{12}}(p_1))\\
&\quad + \mathrm{vol}\,(B_{d'/2}(p_2) - B_{d'/2-\delta_{12}}(p_2))\\
&\quad + \mathrm{vol}\,(F(B_{d'/2-\delta_{12}}(p_1) \cup B_{d'/2-\delta_{12}}(p_2))\\
&\leqq \mathrm{vol}\,(S^d - (B_{d'/2}(p_1) \cup B_{d'/2}(p_2)))\\
&\quad + 2\left(b_1\left(\frac{d'}{2}\right) - b_1\left(\frac{d'}{2} - \delta_{12}\right)\right) + A_2\\
&\quad + \mathrm{vol}\,(B_{d'/2}(F(p_1)) + \mathrm{vol}\,(B_{d'/2}(F(p_2))\\
&\quad - \mathrm{vol}\,(B_{d'/2-d''/2}(z))\\
&\leqq \mathrm{vol}\,(S^d - (B_{d'/2}(p_1) \cup B_{d'/2}(p)))\\
&\quad + \mathrm{vol}\,(B_{d'/2}(p_1)) + \mathrm{vol}\,(B_{d'/2}(p_2)) + A_2\\
&\quad - \frac{1}{2} b_\Delta\left(\frac{d'}{2} - \frac{d''}{2}\right)\\
&\leqq \mathrm{vol}\,(S^d) + A_2 - \frac{1}{2} b_\Delta\left(\frac{d'}{2} - \frac{d''}{2}\right).
\end{aligned}$$

Note that the second term of A_2 can be small if we take sufficiently small δ_{10}. From Lemma 12.2, there exists $\delta_{13} > 0$ such that if $\mathrm{vol}(M) \geqq \mathrm{vol}(S^d) - \delta_{13}$, then

$$A_2 < \frac{1}{4}\cdot b_\Delta\left(\frac{d'}{2} - \frac{d''}{2}\right).$$

We take

$$\delta_0 = \min\left(\delta_{13}, \frac{1}{5}\cdot b_\Delta\left(\frac{d'}{2} - \frac{d''}{2}\right)\right)$$

in Theorem 3. Then if $\mathrm{vol}(M) \geqq \mathrm{vol}(S^d) - \delta_0$, then

$$\mathrm{vol}(M) < \mathrm{vol}(S^d) - \frac{1}{4}\cdot b_\Delta\left(\frac{d'}{2} - \frac{d''}{2}\right) < \mathrm{vol}(S_d) - \delta_0 .$$

It is a contradiction. $\varepsilon/10$-discreteness of $\{F(p_i)\}$ follows immediately from Claim 2 with $a \leqq \varepsilon^2/10$. q.e.d.

Corollary follows from the above and the following two theorems.

THEOREM A. (C. B. Croke [5], Theorem B.) *Let M be a compact d-dimensional Riemannian manifold with $\mathrm{diam}(M) \leqq D < \pi$ and $\mathrm{Ric}_M \geqq d-1$. Then there exists $C(d, D) > 1$ such that $\lambda_1(M) \geqq C(d, D)\cdot d$.*

THEOREM B. (A. Kasue [10], Theorem 4.1.) *Given $d, \Delta, v_0 > 0$ with $\Delta > 1$, $v_0 < \omega_d$, for any $V \in (v_0, \omega_d)$, there exists a constant $\rho = \rho(d, \Delta, v_0; V) > 0$ with $\rho < \pi$ such that if d-dimensional Riemannian manifold M has the property that $\mathrm{Ric}_M \geqq d-1$, $|K_M| \leqq \Delta$, $\mathrm{vol}(M) \geqq v_0$ and $\mathrm{diam}(M) \geqq \rho$, then $\mathrm{vol}(M) \geqq V$.*

§14. Proof of Lemma 12.2.

We firstly take constants which satisfy the following.

$$\delta_{22} < \log(1+\delta_4), \quad \delta_3 < \frac{\delta_5}{4\,\mathrm{vol}(S^{d-1})} \quad \text{and} \quad \log\left(\frac{s_{-\Delta}(\delta_3)}{\sin(\delta_3)}\right) < \frac{\delta_{22}}{3},$$

$$\delta_{21} < \frac{\delta_{22}}{2}, \quad \delta_{20} < \min\left(\delta_3, \frac{\delta_3^2\delta_{21}}{36\pi(\Delta+1)}\right), \quad \delta_{19} < \frac{\sin(\delta_3)\delta_{21}}{3(16d^2+1)},$$

$$\delta_{18} < \frac{\delta_{19}}{2}, \quad \delta_{17} < \left(\frac{\delta_3\delta_{18}}{6}\right)^2,$$

$$\delta_{16} < \min\left(\frac{4\Delta}{\delta_3^2}, \frac{\delta_{17}\delta_{20}}{10d^3}, \frac{\delta_{18}}{3}, \frac{\delta_{18}^2}{12K_3}, \frac{\delta_{19}^2}{12K_1^2K_4}, \frac{\delta_{21}}{3}\right), \quad \delta_{15} < \frac{\delta_{20}}{10},$$

$$\delta_{14} < \min\left(\delta_4, 1 - \exp\left(-\frac{\delta_{20}\delta_{16}}{10}\right)\right),$$

$$K_4 > \frac{d^3 K_3}{\delta_{21}}, \quad K_3 > \frac{3}{2}\pi d\Delta\left(\frac{s_{-\Delta}(\pi)}{s_\Delta(\delta_3)}\right)^2 + \frac{K_2 s_{-\Delta}(\pi)}{s_\Delta(\delta_3)},$$

$$K_2 > \left(\frac{s_{-\Delta}(\pi)}{s_\Delta(\delta_3)} + s_{-\Delta}(\pi/2\Delta^{1/2})\right)(s_\Delta(\pi/2\Delta^{1/2}))^{-1},$$

$$K_1 > \frac{s_{-\Delta}(\pi)^{d-1}}{\sin^{d-1}(\delta_3)s_\Delta(\delta_3)^{d-1}(1-\delta_7)}.$$

Then we can conclude by putting

$$\delta_6 = \min\left(\frac{\delta_5\delta_{14}\delta_{15}}{6\pi}\sin^{d-1}\left(\frac{\delta_5}{3}\right), \ \delta_5\delta_{14}\right).$$

Put

$$\bar{C}[\delta_3, \delta_{14}, \delta_{15}] = \{v \in U_pS^d \mid \gamma(t) = \exp_p tv,$$
$$m(\gamma([0, \pi - \delta_3]) \cap (\bar{B}[\delta_{14}] \cup S^d - D')) > \delta_{15}\},$$
$$\bar{D}[\delta_3, \delta_{14}, \delta_{15}] = U_pS^d - \bar{C}[\delta_3, \delta_{14}, \delta_{15}],$$

and

$$D[\delta_3, \delta_{14}, \delta_{15}] = I(\bar{D}[\delta_3, \delta_{14}, \delta_{15}]).$$

CLAIM 1: *If* $\mathrm{vol}(M) \geqq \mathrm{vol}(S^d) - \delta_6$, *then*

$$\mathrm{vol}(\bar{B}[\delta_4]) \leqq \mathrm{vol}(\bar{B}[\delta_{14}]) \leqq \frac{\delta_6}{\delta_{14}} < \delta_5, \qquad \mathrm{vol}(S^d - D') < \delta_6$$

$$\mathrm{vol}_{(U_pS^d)}(\bar{C}[\delta_3, \delta_{14}, \delta_{15}]) \leqq \frac{3\delta_6}{\delta_{14}\delta_{15}\sin^{d-1}(\delta_{15}/3)} < \frac{\delta_5}{2\pi}.$$

where $\mathrm{vol}_{(U_pS^d)}$ *means the canonical measure on* U_pS^d.

Proof of Claim 1. Since

$$\begin{aligned}\mathrm{vol}(S^d) - \delta_6 \leqq \mathrm{vol}(M) &= \int_M dv_M = \int_{D'} |\det dF|\, dv_{S^d} \\ &\leqq \int_{\bar{B}[\delta_{14}]} (1-\delta_{14}) dv_{S^d} + \int_{D' - \bar{B}[\delta_{14}]} dv_{S^d} \\ &= \mathrm{vol}(S^d) - \mathrm{vol}(S^d - D') - \delta_{14}\,\mathrm{vol}(\bar{B}[\delta_{14}]),\end{aligned}$$

we see

$$\mathrm{vol}(\bar{B}[\delta_{14}]) < \frac{\delta_6}{\delta_{14}} \quad \text{and} \quad \mathrm{vol}(S^d - D') < \delta_6.$$

From the Fubini's theorem,

$$\begin{aligned}
&\operatorname{vol}(\bar{B}[\delta_{14}] \cup S^d - D') \\
&= \int_{U_p S^d \ni v} \left(\int \chi_{(\gamma_v(t) \cap \bar{B}[\delta_{14}] \cup (S^d - D'))}(t) \sin^{d-1} t \, dt \right) dv_{U_p S^d} \\
&\leqq \int_{\bar{C}[\delta_3, \delta_{14}, \delta_{15}]} \frac{\delta_{15}}{3} \cdot \sin^{d-1}\left(\frac{\delta_{15}}{3}\right) dv_{U_p S^d} ,
\end{aligned}$$

namely

$$\begin{aligned}
\operatorname{vol}_{(U_p S^d)}(\bar{C}[\delta_3, \delta_{14}, \delta_{15}]) &\leqq \frac{3 \operatorname{vol}(\bar{B}[\delta_{14}] \cup S^d - D')}{\delta_{15} \sin^{d-1}(\delta_{15}/3)} \\
&\leqq \frac{3\delta_6}{\delta_{14}\delta_{15} \sin^{d-1}(\delta_{15}/3)} < \frac{\delta_5}{2\pi} .
\end{aligned}$$

q.e.d.: Claim 1.

For $v \in D[\delta_3, \delta_{14}, \delta_{15}]$, put $\gamma(t) = \exp_m tv$ and $\bar{\gamma}(t) = \exp_p tI^{-1}(v)$. Let $U_i(t)$ (resp. $\bar{U}_i(t)$) ($1 \leqq i \leqq d-1$) be the linearly independent parallel vector fields along tv (resp. $tI^{-1}(v)$) which is perpendicular to v (resp. $I^{-1}(v)$). Put $Y_i(t) = d\exp_m(tU_i(t))$, $\bar{Y}_i(t) = d\exp_p(t\bar{U}_i(t))$ and $W_i(t) = P_t \circ I \circ P_{-t}\bar{Y}_i(t)$, where P_t and P_{-t} are the parallel translations along $\gamma(t)$ and $\bar{\gamma}(t)$ respectively. For $\bar{\gamma}(s_0) \in D' - \bar{B}[\delta_7]$, we put

$$\begin{aligned}
E^{\gamma}_{s_0}[\delta_{16}] &= E^{\gamma}_{s_0}[\delta_3, \delta_{14}, \delta_{15}, \delta_{16}] \\
&= \{\gamma(s) \mid s \in [0, s_0],\ (\log|\bar{Y}_1(s) \wedge \cdots \wedge \bar{Y}_{d-1}(s)|)' \\
&\qquad \leqq (\log|Y_1(s) \wedge \cdots \wedge Y_{d-1}(s)|)' + \delta_{16}\} .
\end{aligned}$$

CLAIM 2: $m(\gamma([0, s_0]) - E^{\gamma}_{s_0}[\delta_{16}]) \leqq \dfrac{-\log(1-\delta_{14})}{\gamma_{16}} < \dfrac{\delta_{20}}{10}$.

Proof of Claim 2. It is an easy consequence of the following two inequalities,

$$\begin{aligned}
&(\log|Y_1(s) \wedge \cdots \wedge Y_{d-1}(s)|)' \leqq (\log|\bar{Y}_1(s) \wedge \cdots \wedge \bar{Y}_{d-1}(s)|)' , \\
&\log|\bar{Y}_1(s_0) \wedge \cdots \wedge \bar{Y}_{d-1}(s)| \leqq (\log|Y_1(s_0) \wedge \cdots \wedge Y_{d-1}(s_0) - \log(1-\delta_{14}) ,
\end{aligned}$$

q.e.d.: Claim 2

In the following, we fix $s_1 \in E^{\gamma}_{s_0}[\delta_{16}]$. We may assume $s_1 \geqq \pi - \delta_3 - \delta_{20}/10 - \delta_{15} > \pi/2$. Since the value

$$(\log|\bar{Y}_1(s) \wedge \cdots \wedge \bar{Y}_{d-1}(s)|)' - (\log|Y_1(s) \wedge \cdots \wedge Y_{d-1}(s)|)'$$

does not change when we replace Y_i and $\bar{Y}_i$ by linear combination, so we may assume that $\{Y_i(s_i)\}$ and $\{\bar{Y}_i(s_1)\}$ are orthonormal.

We denote by $I_{s_1}(Y_i, Y_i)$ the index form of Y_i along $\gamma|_{[0,s_1]}$.

CLAIM 3: *If* $\gamma(s_1) \in \gamma[0, s_0]) - E^{\gamma}_{s_0}[\delta_{16}]$, *then*

$$I_{s_1}(W_i, W_i) \leqq I_{s_1}(Y_i, Y_i) + \delta_{16} .$$

Proof of Claim 3. From the argument of Heintze-Karcher [8], we see

$$\begin{aligned}
&(\log |Y_1(s_1) \wedge \cdots \wedge Y_{d-1}(s_1)|)' \\
&= \sum_{i=1}^{d-1} I_{s_1}(Y_i, Y_i) \qquad (\{Y_i(s_1)\} \text{ are orthonormal.}) \\
&\leqq \sum_{i=1}^{d-1} I_{s_1}(W_i, W_i) \qquad (\text{the index lemma.}) \\
&\leqq \sum_{i=1}^{d-1} I_{s_1}(\bar{Y}_i, \bar{Y}_i) = (\log |\bar{Y}_1(s_1) \wedge \cdots \wedge \bar{Y}_{d-1}(s_1)|)' \\
&= (\log |Y_1(s_1) \wedge \cdots \wedge Y_{d-1}(s_1)|)' + \delta_{16} \\
&= \sum_{i=1}^{d-1} I_{s_1}(Y_i, Y_i) + \delta_{16} .
\end{aligned}$$

Thus with the index lemma, $I_{s_1}(Y_i, Y_i) \leqq I_{s_1}(W_i, W_i)$, we get

$$I_{s_1}(W_i, W_i) \leqq I_{s_1}(Y_i, Y_i) + \delta_{16} \qquad \text{for each } i .$$

q.e.d.: Claim 3

Since $\{Y_j(s)\}$ is a basis of $T_{\gamma(s)}M$, we may put $W_i(s) = \sum_{j=1}^{d} f_{ij}(s) Y_j(s)$. For fixed i, we define

$$\begin{aligned}
F^i_{s_1}[\delta_{17}] &= F^i_{s_1}[\delta_3, \varepsilon_{14}, \delta_{15}, \delta_{16}, \delta_{17}] \\
&= \left\{\gamma(s) \,|\, s \in [0, s_1],\ \left|\sum_{j=1}^{d} f'_{ij}(s) Y_j(s)\right|^2 < \delta_{17}\right\} .
\end{aligned}$$

CLAIM 4: (i) $m(\gamma([\delta_3, s_1]) - F^i_{s_1}[\delta_{17}]) < \dfrac{\delta_{16}}{\delta_{17}}$.

(ii) *If* $\gamma(s) \in F^i_{s_1}[\delta_{17}]$, *then,*

$$\left|\int_0^s \sum_{j=1}^{d} f''_{ij} f_{ik} g(Y_j, Y_k)' dt\right| < \delta_{18} .$$

Proof of Claim 4. From the arguments of Cheeger-Ebin [3] (Chap 1, §8, 1.21), we have

$$I_{s_1}(W_i, W_i) = I_{s_1}(Y_i, Y_i) + \int_0^{s_1} \left|\sum_{j=1}^{d} f'_{ij} Y_j\right|^2 dt ,$$

therefore,

$$\int_0^s \left|\sum_{j=1}^{d} f'_{ij} Y_j\right|^2 dt \leqq \delta_{16} \qquad \text{for } s \leqq s_1 .$$

This implies (i).

By the integration by parts, we observe,

$$\int_0^s \left|\sum_{j=1}^d f'_{ij} Y_j\right|^2 dt = \left[\sum_{j,k=1}^d f'_{ij} f_{ik} g(Y_j, Y_k)\right]_0^s - \int_0^s \sum_{j,k=1}^d f''_{ij} f_{ik} g(Y_j, Y_k) dt - \int_0^s \sum_{j,k=1}^d f'_{ij} f_{ik} (g(Y'_j, Y_k) + g(Y_j, Y'_k)) dt .$$

For the estimate of

$$\left|\int_0^s \sum_{j,k=1}^d f''_{ij} f_{ik} g(Y_j, Y_k) dt\right| ,$$

firstly we see $g(Y'_j, Y_k) = g(Y_j, Y'_k)$ by taking the derivation of the both sides. (cf. [3] p. 25 (**))

Nextly, from R.C.T., we have

$$|\bar{Y}_i(s)| = |\bar{Y}'_i(0)| \sin(s) = |\bar{Y}_i(s_i)| \cdot \frac{\sin(s)}{\sin(s_1)} = \frac{\sin(s)}{\sin(s_1)} < \frac{2}{\delta_3} ,$$

$$|Y_k(s)| \leqq |Y'_k(0)| s_{-\Delta}(s) \leqq |Y_k(s_1)| \cdot \frac{s_{-\Delta}(s)}{s_\Delta(s_1)} = \frac{s_{-\Delta}(s)}{s_\Delta(s_1)} \leqq \frac{s_{-\Delta}(\pi)}{s_\Delta(\delta_3)}$$

Thirdly, we estimate $|f_{ik}|$. Put $Y_i(s) = \sum a_{ik} e_k$, $W_i(s) = \sum b_{ik} e_k$, where $\{e_i\}_{i=1}^d$ is the orthonormal basis of $T_{\gamma(s)}M$. From $W_i = \sum f_{ij} Y_j$, we get $b_{ik} = \sum f_{ij} a_{jk}$. Let B^j_ℓ be the matrix such that the ℓ-th column of $A = (a_{jk})$ is replaced by $b_{j\ell}$. By Cramer's formula, $f_{ik} = \det B^i_k / \det A$. Note that $\det A = |Y_1 \wedge \cdots \wedge Y_d|$ and

$$\max_{i,k} |\det B^i_k| \leqq \max_{i,k} (|W_i| \prod_{j \neq k} |Y_j|) \leqq \frac{\sin(s)}{\sin(s_1)} \left(\frac{s_{-\Delta}(s)}{s_\Delta(\delta_3)}\right)^{d-1} .$$

Since $\bar{\gamma}(s) \in D' - \bar{B}[\delta_7]$,

$$|Y_1 \wedge \cdots \wedge Y_d| \leqq |Y_1 \wedge \cdots \wedge Y_d| (1 - \delta_7) \leqq \sin^d(s)(1 - \delta_7) .$$

It implies

$$|f_{ik}| \leqq \max |\det B^i_k / \det A| \leqq \frac{1}{\sin^d(s)(1 - \delta_7)} \cdot \frac{\sin(s)}{\sin(s_1)} \left(\frac{s_{-\Delta}(s)}{s_\Delta(s_1)}\right)^{d-1} \leqq K .$$

Fourthly we have

$$|Y'_k(s_1)| \leqq K_2$$

by the following arguments.

We may assume $\Delta > 1$. Decompose $Y_k(s)$ as $Y_k(s) = Z_1(s) + Z_2(s)$, where

$Z_i(s)$ are Jacobi fields with $Z_1(s_1)=Z_2'(s_1)=0$, $Z_2(s)=Y_k(s_1)=1$ and $Z_1'(s_1)=Y_k'(s_1)$. From the Berger's comparison theorem ([2] 1.29),

$$|Z_2(s_1-\pi/2\Delta^{1/2})|\leqq c_{-\Delta}(\pi/2\Delta^{1/2}) .$$

Then,

$$\begin{aligned}|Z_1(s_1-\pi/2\Delta^{1/2})| &\leqq |Y_k(s_1-\pi/2\Delta^{1/2})|+|Z_2(s_1-\pi/2\Delta^{1/2})|\\ &\leqq \frac{s_{-\Delta}(\pi)}{s_\Delta(\delta_3)}+c_{-\Delta}(\pi/2\Delta^{1/2}) .\end{aligned}$$

Thus, we get

$$|Y_k'(s_1)|=|Z_1'(s_1)|\leqq\left(\frac{s_{-\Delta}(\pi)}{s_\Delta(\delta_3)}+c_{-\Delta}(\pi/2\Delta^{1/2})\right)(s_\Delta(\pi/\Delta^{1/2}))^{-1}\leqq K_2$$

Fifthly we have

$$(*)\quad\begin{aligned}\int_0^s\sum_{k=1}^d|Y_k'|^2dt &\leqq \int_0^{s_1}\sum_{k=1}^d|Y_k'|^2dt\\ &=\sum_{k=1}^d\int_0^{s_1}g(R(Y_k,\dot\gamma)\dot\gamma,\ Y_k)dt+g(Y_k'(s_1),\ Y_k(s_1))dt\\ &\leqq d\cdot\int_0^{s_1}(3/2)\Delta|Y_k|^2dt+|Y_k'(s_1)||Y_k(s_1)|dt\\ &\leqq\frac{3}{2}\pi d\Delta\left(\frac{s_{-\Delta}(\pi)}{s_\Delta(\delta_3)}\right)^2+\frac{K_2s_{-\Delta}(\pi)}{s_\Delta(\delta_3)}\leqq K_3 .\end{aligned}$$

Therefore, we get, from $W_i=\sum f_{ij}Y_j$,

$$\begin{aligned}&\left|\int_0^s\sum_{j,k=1}^d f_{ij}''f_{ik}g(Y_j,\ Y_k)\,dt\right|\\ &\leqq\int_0^s\left|\sum_{j=0}^d f_{ij}'Y_j\right|^2dt+\left|\sum_{i=1}^d f_{ij}'(s)g(Y_j(s),\ W_i(s))\right|\\ &\quad+2\left|\int_0^s\sum_{j=1}^d f_{ij}'f_{ik}g(Y_j,\ Y_k')\,dt\right|\\ &\leqq\delta_{16}+\left|\sum_{j=1}^d f_{ij}'(s)Y_j(s)\right||W_i(s)|\\ &\quad+2\left(\int_0^s\left|\sum_{j=1}^d f_{ij}'Y_j\right|^2dt\right)^{1/2}\left(\int_0^s\left|\sum_{j=1}^d f_{ik}Y_k'\right|^2dt\right)^{1/2}\\ &\leqq\delta_{16}+\frac{2\delta_{17}^{1/2}}{\delta_3}+2(\delta_{16}K_3)^{1/2}<\delta_{13} .\end{aligned}$$

q.e.d.: Claim 4

We put

$$G_{s_1}^i[K_4]=\left\{\gamma(s)\in F_{s_1}^i[\delta_{17}]\left|\sum_{j=1}^d|Y_j'(s)|^2\leqq K_4\right.\right\} .$$

Then, from Claim 4 and $(*)$, we see,

$$m(G^i_{s_1}[K_4]) \geqq s_1 - \frac{\delta_{16}}{\delta_{17}} - \frac{K_3}{K_4} .$$

CLAIM 5: *If $\gamma(s) \in G^i_{s_1}[K_4]$, then*

$$\left|\int_0^s g(\bar{R}(\bar{Y}_i, \dot{\bar{\gamma}})\dot{\bar{\gamma}}, \bar{Y}_i) - g(R(Y_i, \dot{\gamma})\dot{\gamma}, Y_i)dt\right| \leqq \delta_{19} .$$

Proof of Claim 5. From $g(\bar{Y}''_i, \bar{Y}_i) = g(W''_i, W_i)$ and

$$W''_i = \left(\sum_{j=1}^d f_{ij} Y_j\right)'' = \sum_{j=1}^d (f''_{ij} Y_j + 2f'_{ij} Y'_j + f_{ij} Y''_j) ,$$

we find if $\gamma(s) \in G^i_{s_1}[K_4]$, then,

$$\begin{aligned}
&\left|\int_0^s g(\bar{R}(\bar{Y}_i, \dot{\bar{\gamma}})\dot{\bar{\gamma}}, \bar{Y}_i) - g(R(W_i, \dot{\gamma})\dot{\gamma}, W_i)dt\right| \\
&\quad = \left|\int_0^s g(\bar{Y}''_i, \bar{Y}_i) - \sum_{j=1}^d f_{ij} g(Y''_j, W_i)dt\right| \\
&\quad = \left|\int_0^s \sum_{j=1}^d f''_{ij} g(Y_j, W_i) + 2 \sum_{j,k=1}^d f'_{ij} f_{ik} g(Y_j, Y'_k)dt\right| \\
&\quad \leqq \left|\int_0^s \sum_{j=1}^d f''_{ij} g(Y_j, W_i)dt\right| + 2\left|\int_0^s \sum_{j=1}^d f'_{ij} Y_j dt\right| \max(|f_{ik}||Y'_k|) \\
&\quad = \delta_{18} + 2\delta_{17} K_1 K_4^{1/2} < \delta_{19} .
\end{aligned}$$

q.e.d.: Claim 5

We put $G^{\{Y\}} = \bigcap_{i=1}^d G^i_{s_1}[K_4]$. We take another orthonormal basis $\{X^{ij}_k\}$ at $\gamma(s_1)$ with $X^{ij}_1 = (X_i + Y_j)/(|Y_i + Y_j|)$ and repeat the above arguments for each (i, j). Put $G^\gamma = \bigcap_k G^{\{X^{ij}_k\}}$ and

$$G = \bigcap_{\dot{\gamma}(0) \in D[\delta_3, \delta_{14}, \delta_{15}]} G^\gamma .$$

Then, since $s_1 \geqq \pi - \delta_3 - \delta_{20}/10 - \delta_{15}$, we see

$$(**) \qquad m(G^\gamma) \geqq \pi - \delta_3 - \frac{\delta_{20}}{10} - \delta_{15} - d^3\left(\frac{\delta_{16}}{\delta_{17}} + \frac{K_3}{K_2}\right) > \pi - \delta_3 - \delta_{20} .$$

On the other hand, we find if $\gamma(s) \in G^\gamma$, then for any $\bar{X} \in T_{\bar{\gamma}(s)} S^d$,

$$\left|\int_0^s g(R(W_X, \dot{\gamma})\dot{\gamma}, W_X) - g(\bar{R}(\bar{X}, \dot{\bar{\gamma}})\dot{\bar{\gamma}}, \bar{X})\, dt\right| \leqq \pi(16d^2 + 1)|X|^2 \delta_{19} ,$$

where $W_X = P_s \circ I \circ P_{-s}(\bar{X})$. It is easily derived from the following inequality,

$$\left|\int_0^s K\left(\sum_{i=1}^d \lambda_i Y_i, \sum_{i=1}^d \lambda_i Y_i\right)dt\right| \leqq \sum_{i=1}^d \lambda_i^2 \left|\int_0^s K(Y, Y)\, dt\right|$$

$$+ 2\sum_{i=1}^{d}|\lambda_i\lambda_j|\left(\left|\int_0^s K(Y_i + Y_j,\ Y_i + Y_j)\,dt\right|\right.$$
$$\left. + \left|\int_0^s K(Y_i,\ Y_i)\,dt\right| + \left|\int_0^s K(Y_j,\ Y_j)\,dt\right|\right),$$

where $K(Z_1, Z_2) := g(\bar{R}(Z_1, \dot{\bar{\gamma}})\dot{\bar{\gamma}}, Z_2) - g(R(W_{Z_1}, \dot{\gamma})\dot{\gamma}, W_{Z_2})$ and $\sum_{i=1}^{d}\lambda_i^2 = 1$.

CLAIM 6: *For* $\gamma(t) \in G_\gamma$, $|dF_{\bar{\gamma}(t)}| < 1 + \delta_4$.

Proof of Claim 6. Similarly as above, put $Y(t) = d\exp_m(tU(t))$ and $\bar{Y}(t) = d\exp_p(tI^{-1}(U(t)))$. Take $\gamma(t_1) \in G^\gamma$ with $t_1 \geqq \delta_3$ and put

$$V(t) = \frac{Y(t)}{|Y(t_1)|}\bar{V}(t) - \frac{\bar{Y}(t)}{|\bar{Y}(t_1)|} \quad \text{and} \quad W(t) = P_t \circ I \circ P_{-t}\bar{V}(t) = \sum_{i=1}^{d} f_i V_1\,,$$

where $\{V_i\}$ are the linearly independent Jacobi fields such that $\{V_i(t_1)\}$ are orthonormal. For fixed

$$s_1 \geqq \pi - \delta_3 + \frac{\log(1 - \delta_{14})}{\varepsilon_{16}} - \delta_{15},$$

put

$$\bar{V}_1(t) = \frac{\bar{Y}(t)}{|Y(s_1)|} = \bar{V}(t)\cdot\frac{|\bar{Y}(t_1)|}{|\bar{Y}(s_1)|} \quad \text{and} \quad W_1(t) = P_t \circ I \circ P_{-t}\bar{V}_1(t).$$

Then, similarly as above, we see

$$|I_{t_1}(V, V) - I_{t_1}(W, W)| \leqq \delta_{16}$$

and therefore

$$\begin{aligned}
&|I_{t_1}(V, V) - I_{t_1}(\bar{V}, \bar{V})| \\
&\quad \leqq \delta_{16} + \left|\int_0^{t_1}(g(R(W, \dot{\gamma})\dot{\gamma}, W) - g(R(\bar{V}, \dot{\bar{\gamma}})\dot{\bar{\gamma}}, \bar{V}))\,dt\right| \\
&\quad \leqq \delta_{16} + \left|\int_0^{t_1}(g(R(W_1, \dot{\gamma})\dot{\gamma}, W_1) - g(R(\bar{V}_1, \dot{\bar{\gamma}})\dot{\bar{\gamma}}, \bar{V}_1))\,dt\right|\cdot\frac{|\bar{Y}(s_1)|}{|\bar{Y}(t_1)|} \\
&\quad \leqq \delta_{16} + (16d^2 + 1)\delta_{19}\left(\frac{\sin(s_1)}{\sin(\delta_3)}\right) < \delta_{21}\,.
\end{aligned}$$

Namely,

$$|(\log|Y(t_1)|)' - (\log|\bar{Y}(t_1)|)'| \leqq \delta_{21}\,.$$

For $\gamma(t) \notin G^\gamma$, since the value $(\log|Y(t)|)'$ does not change when $Y(t)$ replace by constant multiple of $Y(t)$, for $t \leqq t_1$, we see

$$
\begin{aligned}
(\log|Y(t)|)' &= \frac{Y(t)}{Y(t_1)} I_t(V, V) \\
&\leqq \int_0^{t_1} \left|\sum_{i=1}^{d} f_i' V_i\right|^2 dt + \int_0^{t_1} g(R(W, \dot\gamma)\dot\gamma, W)\, dt \\
&\leqq \delta_{16} + 2\Delta\left(\frac{2}{\delta_3}\right)^2 < 3\Delta\pi\left(\frac{2}{\delta_3}\right)^2,
\end{aligned}
$$

and similarly,

$$
(\log|\bar{Y}(t)|)' \leqq 3\pi\left(\frac{2}{\delta_3}\right)^2 .
$$

Integrating these, we get

$$
\log|Y(t)| - \log|\bar{Y}(t)| \leqq \log\left(\frac{s_{-\Delta}(\delta_3)}{\sin(\delta_3)}\right) + \delta_{21}\pi + \left(\frac{2}{\delta_3}\right)^2 3(\Delta + 1)\pi\delta_{20} \leqq \delta_{22} .
$$

Therefore we see

$$
|dF_{\bar\gamma(t)}| = \frac{|Y(t)|}{|\bar{Y}(t)|} \leqq \exp(\delta_{22}) < 1 + \delta_4 . \qquad \text{q.e.d.: Claim 6}
$$

Note that $\bar{A}[\delta_4] \subset D' - F^{-1}(G) := \tilde{A}[\delta_4]$.

CLAIM 7: $\operatorname{vol}(\bar{A}[\delta_4]) \leqq \operatorname{vol}(\tilde{A}[\delta_4]) < \delta_5$.

Proof of Claim 7. Since $m(F^{-1}(G^\gamma)) = m(G^\gamma)$, we have, from Claim 1 and $(**)$,

$$
\begin{aligned}
\operatorname{vol}(\bar{A}[\delta_4]) &\leqq \int_{\bar{C}[\delta_3, \delta_{14}, \delta_{15}]} \left(\int_0^\pi \sin^{d-1} t\, dt\right) dv_{U_pS^d} \\
&\quad + \int_{\bar{D}[\delta_3, \delta_{14}, \delta_{15}]} \left(\int_{(\gamma([0,\pi]) - G^\gamma)} \sin^{d-1} t\, dt\right) dv_{U_pS^d} \\
&\leqq \operatorname{vol}_{(U_pS^d)}(\bar{C}[\delta_3, \delta_{14}, \delta_{15}])\pi \\
&\quad + \max m(\gamma([0, \pi]) - G^\gamma) \operatorname{vol}(S^{d-1}) \\
&\leqq \frac{\delta_5}{2\pi}\pi + (\delta_{25} + \delta_3)\operatorname{vol}(S^{d-1}) \leqq \delta_5 . \qquad \text{q.e.d.: Claim 7}
\end{aligned}
$$

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