E. L. Livorni
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CLASSIFICATION OF ALGEBRAIC NON-RULED SURFACES WITH SECTIONAL GENUS LESS THAN OR EQUAL TO SIX

ELVIRA LAURA LIVORNI

Introduction

In this paper we have given a biholomorphic classification of smooth, connected, projective, non-ruled surfaces X with a smooth, connected, hyperplane section C relative to L, where L is a very ample line bundle on X, such that $g = g(C) = g(L)$ is less than or equal to six. For a similar classification of rational surfaces with the same conditions see [Li]. From the adjunction formula

$$2g - 2 = L \cdot (K_X + L)$$

where K_X denotes the canonical line bundle on X. In the cases in which $g = 0, 1$, X is rational or ruled, see [Na]. Thus we have to examine only the cases in which $g = 2, \cdots, 6$. Our classification goes far beyond the birational classification of Leonard Roth [Ro] and the one that Polkin Ionescu [Io] has given for $g \leq 4$, Our main tool is the adjunction process and the results of Andrew John Sommese which are in [So]. Our notations are as in [So] except for the following. X will denote a smooth, connected, projective, non-ruled surface and L a very ample line bundle on X. Let $\bar{L} = K_X \otimes L$. Then $h^{1,0}(X) \neq g$ [So, (1.5.2)], and $\bar{L}$ is generated by global sections [So, (1.5)], and the induced morphism ϕ has a 2-dimensional image [So, (2.0.1) and (2.1)]. Let $\hat{X} \to X$ be the Stein factorization and π: $\hat{X} \to X$ the induced morphism. Then $\hat{X}$ is a non singular projective surface, π is the blow-up at a finite number of points of $\hat{X}$, $\hat{L} = \pi_* L$ and $K_{\hat{X}} + \hat{L}$ are ample, and $K_X + L \sim \pi^*(K_{\hat{X}} + \hat{L})$ [So, (2.3)]. We call $(\hat{X}, \hat{L})$ the minimal pair of (X, L). Our main goal is to classify the pairs $(\hat{X}, \hat{L})$ and eventually the pairs (X, L). Let $d = (L^2)$, $g = g(L)$, $\hat{d} = (\hat{L}^2)$, $d' = (L'^2)$,

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g	$\hat{d}$	$\hat{c}_1^2$	$h^0(\hat{L})$	g'	d'	$h^0(K_X \otimes L)$	d	$h^0(L)$	c_1^2	$h^{1,0}(X)$	$h^{2,0}(X)$	$(\hat{X}, \hat{L})$	(X, L)
3	4	0	4	3	4	4	4	4	0	0	1	$(H_4, \mathcal{O}_{P^3}(1))$	$X = \hat{X}$
4	6	0	5	4	6	5	6	5	0	0	1	Complete intersection of a quadric and a cubic.	$X = \hat{X}$
5	8	0	6	5	8	6	7, 8	5, 6	−1, 0	0	1	One example is given by the complete intersection of three quadrics in P^4.	Blow up at most one point.
6	5	5	4	14	20	10	5	4	5	0	4	$(H_5, \mathcal{O}_{P^3}(1))$	$X = \hat{X}$
6	7	0	5	9	13	8	7	5	0	0	2		$X = \hat{X}$. $\hat{X}$ is a minimal elliptic surface.
6	8	0	5	8	12	7	8	5	−1	0	1		$X = \hat{X}$. $K3$ surface blown up at one point if it exists.
6	9	0	6	7	11	6, 5	9	6	0	0, 1	0		$X = \hat{X}$. $\hat{X}$ is a minimal elliptic surface if it exists.
6	10	0	5	6	10	5	10	5	0	2	1	$K_{\hat{X}}$ is trivial, $h^0(\hat{L}\|_{\hat{C}}) = 6$, $h^1(\hat{L}\|_{\hat{C}}) = 1$	$X = \hat{X}$. $\hat{X}$ is an abelian surface.
6	10	0	5	6	10	5	10	5	0	1	0	$K_{\hat{X}}$ is not trivial, $h^0(\hat{L}\|_{\hat{C}}) = 5$, $h^1(\hat{L}\|_{\hat{C}}) = 0$	$X = \hat{X}$. $\hat{X}$ is a hyperelliptic surface if it exists.
6	10	0	6	6	10	6	9, 10	5, 6	−1, 0	0	0	$K_{\hat{X}}$ is not trivial, $h^0(\hat{L}\|_{\hat{C}}) = 5$, $h^1(\hat{L}\|_{\hat{C}}) = 0$	Blow up at most one point. $\hat{X}$ is an Enriques surface.
6	10	0	7	6	10	7	9, 10	5, …, 7	−1, 0	0	1	$K_{\hat{X}}$ is trivial, $h^0(\hat{L}\|_{\hat{C}}) = 6$, $h^1(\hat{L}\|_{\hat{C}}) = 1$	Blow up at most one point. $\hat{X}$ is a $K3$ surface.

$g' = g(K_X + L)$, $c_1^2 = (K_X^2)$, $\hat{c}_1^2 = (K_{\hat{X}}^2)$. We fix the meaning of these symbols and the assumptions above throughout this paper. We have summarized our results in the above table, where using proposition (0.1) we have computed d' and g'. As one can see we have not been able to establish in all cases if there exists a surface satisfying the invariants that we have obtained. We are very grateful to Andrew J. Sommese whose suggestions have been very helpful. We would like to thank the referee for his careful revision.

§ 0. Background material

Since most of our background material is found in [So], we will mention here, without any details, only the results we need which cannot be found there.

(0.1) PROPOSITION. *Let L be a line bundle on a smooth, connected, projective surface X. Then*:

1) $$d' = g' + g - 2$$

2) $$dd' \leq 4(g-1)^2$$

3) $$d + d' = c_1^2 + 4(g-1)\,.$$

See [Ro].

(0.2) PROPOSITION. *Let X be a smooth, connected, projective, surface embedded by a very ample line bundle $\mathscr{L}$ in $\boldsymbol{P}^4$. Then*

(0.2.1) $$\mathscr{L}\cdot\mathscr{L}(\mathscr{L}\cdot\mathscr{L} - 5) - 10(g(\mathscr{L}) - 1) + 12\chi(\mathcal{O}_X) = 2K_X\cdot K_X\,.$$

See [Ha] pg. 434.

(0.3) PROPOSITION. *Let X be a smooth, connected, projective surface and L a very ample line bundle on it. Suppose $h^0(L) = 3, 4$ then*:

1) $h^0(L) = 4$, $g = 3$, $h^{1,0}(X) = 0$, $h^{2,0}(X) = 1$, $d = 4$, $c_1^2 = 0$
$$(X, L) = (\hat{X}, \hat{L}) = (H_4, \mathcal{O}_{P^3}(1))$$

2) $h^0(L) = 4$, $g = 6$, $h^{1,0}(X) = 0$, $h^{2,0}(X) = 4$, $d = 5$, $c_1^2 = 5$
$$(X, L) = (\hat{X}, \hat{L}) = (H_5, \mathcal{O}_{P^3}(1))\,.$$

(0.4) DEFINITION. *A line bundle L on a projective variety is called spanned if $h^0(L)$ is generated by the global sections. By* [Ha, Lemma (7.8)] *this is equivalent to say that $h^0(L)$ is base-point-free.*

(0.5) PROPOSITION. *Let X be a smooth, connected, projective surface and L an ample line bundle on it. If $K_X \cdot L < 0$, then X is birational to $\boldsymbol{P}_1(C)$ cross a curve.*

Indeed since L is ample, one sees that $H^0(K_X^M) = 0$ for all $M > 0$ and that X is ruled by Castelnuovo's theorem [Bo + Hu, (3.5)].

(0.6) PROPOSITION. *With the notation and assumptions of the introduction, one has*

(i) $d \leq \hat{d} \leq 2g-2$, $h^0(L) \leq h^0(\hat{L})$,

(ii) $\hat{d} - d = (\hat{c}_1^2) - (c_1^2)^2$ *is the number of points to blow up for the morphism* $X \to \hat{X}$ *(hence in particular* $\hat{d} = d \Leftrightarrow X = \hat{X}$*),*

(iii) $(2g-2-d)^2/d \geq c_1^2 \geq d - (2g-2)$,

$(2g-2-\hat{d})^2/\hat{d} \geq \hat{c}_1^2 \geq \hat{d} - (2g-2)$,

(iv) *the following are equivalent:*

(iv. 1) $c_1^2 = d - (2g-2)$,

(iv. 2) $\hat{c}_1^2 = \hat{d} - (2g-2)$,

(iv. 3) $\hat{d} = 2g-2$,

(iv. 4) $K_{\hat{X}} \approx 0$ *and* $\hat{X}$ *is a minimal model.*

(0.6.1) *Remark.* We note

$$\begin{aligned}(K_{\hat{X}} \cdot \hat{L}) &= (\hat{L} \cdot K_{\hat{X}} + \hat{L}) - (\hat{L}^2) \\ &= (L \cdot K_X + L) - (\hat{L}^2) = 2g-2-\hat{d}\end{aligned}$$

by $K_X + L = \pi^*(K_{\hat{X}} + \hat{L})$ and $\pi_* L = \hat{L}$. Similarly one has

$$(K_X \cdot L) = 2g-2-d\,.$$

Proof. By (0.6.1) and (0.5), one has $\hat{d} \leq 2g-2$ because X is not ruled. The rest for (i) and (ii) follows from [So, (2.3)].

(iii): Since the argument is similar, we prove only the second assertion. By the algebraic index theorem, one has

$$(\hat{L} \cdot K_{\hat{X}})^2 \geq (\hat{L}^2) \cdot (K_{\hat{X}}^2)\,,$$

which reduces to $(2g-2-\hat{d})^2 \geq \hat{d} \cdot \hat{c}_1^2$. Since X is not ruled, one sees that $|mK_{\hat{X}}| \neq \phi$ for some $m > 0$. Thus $(K_{\hat{X}} \cdot \hat{L} + K_{\hat{X}}) \geq 0$ by ampleness of $\hat{L} + K_{\hat{X}}$, which reduces to $\hat{c}_1^2 \geq \hat{d} - (2g-2)$.

(iv): Since $K_X + L = \pi^*(K_{\hat{X}} + \hat{L})$ and $\pi_* K_X = K_{\hat{X}}$, one has $(K_X \cdot K_X + L) = (K_{\hat{X}} \cdot K_{\hat{X}} + \hat{L})$, whence $c_1^2 + 2g - 2 - d = \hat{c}_1^2 + 2g - 2 - \hat{d}$ as above. Thus (iv. 1) $\Leftrightarrow$ (iv. 2). If (iv. 2) holds, then $(K_{\hat{X}} \cdot K_{\hat{X}} + \hat{L}) = 0$ as above. Let

$m > 0$ be such that $|mK_{\hat{X}}| \neq \phi$. Then $(mK_{\hat{X}} \cdot K_{\hat{X}} + \hat{L}) = 0$ implies $mK_{\hat{X}} \sim 0$ by ampleness of $K_{\hat{X}} + \hat{L}$. Thus (iv. 2) implies (iv. 4). (iv. 4) implies (iv. 3) because $(K_{\hat{X}} \cdot \hat{L}) = 2g - 2 - \hat{d}$ as above. If (iv. 3) holds, then one has $\hat{c}_1^2 = 0$ by (iii), whence (iv. 2). q.e.d.

(0.7) PROPOSITION. *With the notation and assumptions of the introduction, assume that* $\hat{d} = 2g - 3$. *Then* $k(X) = 1$ *and* $\hat{X}$ *is a minimal model (i.e. free from exceptional curves of the first kind).*

Proof. claim: $k(X) \leq 1$.

Assume that $k(X) = 2$, and let Y be its minimal model. Then by $\chi(\theta_Y) > 0$ and $(K_Y^2) > 0$ [$G + H_1$], one has

$$h^0(2K_X) = h^0(2K_Y) \geq \chi(2K_Y) = \chi(\theta_Y) + (2K_Y \cdot 2K_Y - K_Y)/2$$
$$= \chi(\theta_Y) + (K_Y^2) \geq 2\,.$$

Hence $\dim|2K_{\hat{X}}| \geq 1$. Since L is very ample and $L < \pi^*\hat{L}$ and $\pi_*L = \hat{L}$, the linear system $|\hat{L}|$ induces an embedding of $\hat{X}$-(finite set). Since $(2K_{\hat{X}} \cdot \hat{L}) = 2(2g - 2 - \hat{d}) = 2$ (0.6.1), the generic number of $|2K_{\hat{X}}|$ is birationally sent to a line or a conic by the rational map associated to $|\hat{L}|$. Thus $\hat{X}$ is ruled and this is a contradiction. Hence the claim is proved.

claim: $(K_{\hat{X}}^2) = 0$ and $\hat{X}$ is a minimal model.

Since $k(X) \leq 1$, the minimal model Y of X satisfies $(K_Y^2) = 0$, whence $(K_{\hat{X}}^2) \leq (K_Y^2) = 0$. By (0.6), (iii), one sees $(K_{\hat{X}}^2) = 0, -1$. By (0.6), (iv), one has $(K_{\hat{X}}^2) \neq -1$ because (iv. 3) does not hold. Hence $(K_{\hat{X}}^2) = 0$ and the claim is proved.

It now remains to prove $K_{\hat{X}} \not\approx 0$. Indeed this follows from $(K_{\hat{X}} \cdot \hat{L}) = 2g - 2 - \hat{d} = 1$ (0.6.1). q.e.d.

§ 1. Classification

We use the notation and assumptions of the introduction. By (0.3), one may assume $h^0(L) \geq 5$. By (0.6), one has $d \leq \hat{d} \leq 2g-2$, $5 \leq h^0(L) \leq h^0(\hat{L})$, and $g \leq 6$ by our original assumption.

We treat 2 cases $\hat{d} = 2g - 2$ and $\hat{d} < 2g - 2$, we can give comments on the existence later.

(1.1) THEOREM. *If* $\hat{d} = 2g - 2$, *then one of the following holds*:

(1.1.1) $g = 6$, $d = \hat{d} = 10$, $h^0(L) = 5$, $c_1^2 = 0$, $X = \hat{X}$ *is an abelian surface or a hyperelliptic surface,* (1.1.2) $g=6$, $\hat{d}=10$, $h^0(\hat{L})=6$, $d=9, 10$, $h^0(L)=d-4$, $\hat{c}_1^2=0$, *$\hat{X}$ is an Enriques surface and $\hat{L}$ is very ample,*

(1.1.3) $g=4, 5, 6$, $h^0(\hat{L})=g+1$, $h^0(L)=g+1-(\hat{d}-d)$, $(\hat{c}_1^2)=0$. *$\hat{X}$ is a $K3$ surface and $\hat{L}$ is very ample. One has $d=\hat{d}$ and $X=\hat{X}$ is a complete intersection of a quadric and a cubic of P^4 if $g=4$, see* [Sa$_1$], *and $d=\hat{d}$ or $\hat{d}-1$ if $g=5, 6$.*

Proof. By (0.6), (iv), one sees that $K_{\hat{X}}\approx 0$ and $\hat{X}$ is a minimal model, and one has $\chi(\theta_{\hat{X}})=0, 1, 2$ [Bo + Hu]. Since $K_{\hat{X}}\approx 0$, one has $H^i(\hat{L})=0$ $(i>0)$ by Kodaira Vanishing Theorem. Hence one has

$$(*)\qquad 5\leq h^0(L)\leq h^0(\hat{L})=\chi(\theta_X)+\frac{\hat{d}}{2}\leq \chi(\theta_X)+g-1\,.$$

Case i). $\chi(\theta_X)=0$, X is an abelian surface or a hyperelliptic surface. In this case, one has $g=6$ and $h^0(L)=5$, $(K_X^2)=d-10$. Thus by (0.2), one has $d(d-5)-50=2(d-10)$, whence $d=10$ and $X=\hat{X}$. This is (1.1.1).

Case ii). $\chi(\theta_X)=1$, X is an Enriques surface.

One has $g\geq 5$ by (*), and $h^0(\hat{L})=g$. If $g=5$, then $h^0(\hat{L})=5$ and $(K_X^2)=\mathrm{d}-8$, and one has $d(d-5)-40+12=2(d-8)$ by (0.2), which has no solution. Thus this is a contradiction and one obtains $g=6$. Since X is an Enriques surface, one has $h^{1,0}(X)=0$, whence $\hat{L}$ is very ample by [So, (2.4)]. Thus if $d\leq 8$, then $H^0(L)$ is the subspace of $H^0(\hat{L})$ with at least 2 base points [So, (2.3.2)] and $h^0(L)\leq h^0(\hat{L})-2=4$, which is a contradiction. Hence $d=9$ or 10. This is (1.1.2).

Case iii). $\chi(\theta_X)=2$, X is a $K3$ surface.

One has $5\leq h^0(L)\leq h^0(\hat{L})=g+1\leq 7$ by (*). Since $h^{1,0}(X)=0$, $\hat{L}$ is very ample [So, (2.4)]. One has $d=\hat{d}$ and X is a complete intersection of a quadric and a cubic of P^4 if $g=4$ See [Sa$_1$]. Suppose now that $d\leq\hat{d}-2$. Then as in case ii), one has $5\leq h^0(L)\leq h^0(\hat{L})-2=5$ and $(K_X^2)=d-10$. By (0.2), one has $d(d-5)-50+24=2(d-10)$. Thus $d^2-7d-6=0$ which has no solution. Hence one has $d\geq\hat{d}-1$. q.e.d.

(1.2) THEOREM. *If $\hat{d}\leq 2g-3$, then one of the following holds*:

(1.2.1) $g=6$, $d=\hat{d}=7$, $h^0(L)=5$, $c_1^2=0$, $h^{2,0}(X)=2$, $h^{1,0}(X)=0$, *$|K_X|$ is base-point-free and induces the elliptic fibration of X,*

(1.2.2) $g=6$, $d=\hat{d}=8$, $h^0(L)=5$, $c_1^2=-1$, *$|K_X|$ consists of one conic*

D and D is an exceptional curve of the first kind, its contraction is a minimal $K3$ surface,

(1.2.3) $g = 6,\ d = \hat{d} = 9,\ h^0(L) = 6,\ c_1^2 = 0,\ h^{2,0}(X) = 0,\ h^{1,0}(X) = 0, 1.$ *X is a minimal elliptic surface.*

Proof. Let C be a general member of $|L|$. We start with

claim: $|L|_C|$ is a special linear system.

If otherwise, one has $4 \leq h^0(L) - 1 \leq h^0(L|_C) = 1 - g + d \leq g - 2 \leq 4$. Thus $h^0(L) = 5$, $g = 6$, $d = \hat{d} = 9$, and $X = \hat{X}$ is an elliptic surface (0, 7). Hence by (0.2), one has $9 \cdot 4 - 50 + 12\chi(\mathcal{O}_X) = 0$, that is $12\chi(\mathcal{O}_X) = 14$, which is a contradiction. Thus our claim is proved.

Since $L|_C$ is very ample and its degree d satisfies $0 < d < \deg K_C$, one has

$$(**) \qquad 3 \leq h^0(L) - 2 \leq \dim |L|_C| < d/2 < g - 1$$

by Clifford's theorem [Sa$_2$, (1.4)]. Hence $d \geq 7$, $g \geq 5$. We claim that $g = 6$. Indeed if $g = 5$, then $d = \hat{d} = 7$ and $h^0(L) = 5$ by (**), and one has $(K_X^2) = 0$ by $1/7 \geq (K_X^2) > -1$ by (0.6), (iii), (iv). Applying (0.2), one has $7.2 - 40 + 12\chi(\mathcal{O}_X) = 0$, i.e. $12\chi(\mathcal{O}_X) = 26$, which is a contradiction. Thus our claim $g = 6$ is proved, and we treat 2 cases $h^0(L) \geq 6$ and $h^0(L) = 5$.

Case i). $h^0(L) \geq 6$.

In this case one has $h^0(L) = 6$, $d = \hat{d} = 9$ by (**). Thus $X = \hat{X}$ is a minimal elliptic surface by (0.7). Since $(K_X \cdot L) = 1$ (0.6.1), $h^{2,0}(X) = 0$. Because otherwise $|K_X|$ contains a line which is a fiber of the elliptic fibration, which is a contradiction. One has $h^{1,0}(X) = 0, 1$ by $\chi(\mathcal{O}_X) = 1 - h^{1,0}(X) \geq 0$.

Case ii). $h^0(L) = 5$.

One has $7 \leq d \leq \hat{d} \leq 9$, $(10 - d)^2/d \geq (K_X^2) > d - 10$ by (0.6), (iii), (iv), $d(d - 5) - 50 + 12\chi(\theta_X) = 2(K_X^2)$ by (0.2). If $d = 9$, then $(K_X^2) = 0$ and $36 - 50 + 12\chi(\theta_X) = 0$ which is a contradiction. Thus $d = 8$ or 9.

Case ii.1) $d = 8$.

In this case, $0 \geq (K_X^2) \geq -1$ and $24 - 50 + 12\chi(\mathcal{O}_X) = 2(K_X^2)$. Thus $(K_X^2) = -1$, $\chi(\mathcal{O}_X) = 2$. Hence $h^{2,0}(X) \geq 1$ and $|K_X|$ has an effective member D. Since $(D \cdot L) = 2$ (0.6.1), D is a conic. Since X is not ruled, one sees $\dim |D| = 0$ and $h^{2,0}(X) = 1$ and $h^{1,0}(X) = 0$ by $\chi(\mathcal{O}_X) = 2$. We claim that D is smooth and irreducible. Indeed by $(D^2) = -1$, D is reduced because it is a conic. If D is reducible, then $D = D_1 + D_2$, where D_1 and D_2 are lines ($D_1 \neq D_2$) and $(D_1 \cdot D_2) = 0, 1$. Since $D_1 \simeq \boldsymbol{P}^1$, one has $-2 = 2p_a(D) - 2 = (D^1 \cdot 2D_1 + D_2)$, whence $(D_1 \cdot D_2) = 0$ and $(D_1^2) = -1$. One has $(D_2^2) = -1$ in the same way. This contradicts $(D^2) = -1$. Thus D is smooth and irreducible. Since $(D \cdot L + K_X) = 1 > 0$, there are no exceptional curve of the first kind collapsed by π and one has $X = \hat{X}$. Since D is an exceptional curve of the first kind, one can contract D by $f\colon X \to Y$. Then Y is a minimal $K3$ surface by $K_Y \sim f_* K_X \sim 0$, $h^{1,0}(Y) = 0$. This is case (1.2.2).

Case ii.2). $d = 7$.

In this case one has $1 \geq (K_X^2) \geq -2$ and $14 - 50 + 12\chi(\mathcal{O}_X) = 2(K_X^2)$. Thus $(K_X^2) = 0$, $\chi(\mathcal{O}_X) = 3$. Hence $h^0(K_X) \geq 2$, and $|K_X|$ contains a pencil of cubic curves by $(K_X \cdot L) = 3$ (0.6.1). Since X is not ruled, the general member of $|K_X|$ must be a smooth elliptic curve. Thus one sees that $|K_X|$ is a pencil free from base points by $(K_X^2) = 0$. Hence $h^0(K_X) = 2$ and $h^{1,0}(X) = 0$ by $\chi(\mathcal{O}_X) = 3$. Since X is a minimal elliptic surface ($k(X) = 1$), one has $X = \hat{X}$. This is case (1.2.1).

(1.3) *Remark.* An example of an abelian surface in (1.1.1) is given in [Ho + Mu]. An example of (1.1.2) with $d = 10$ is given in [G + H$_1$, p. 749] ($V_1(w)$ given there is an Enriques surface of degree 10 in $\boldsymbol{P}^5$). An example of (1.1.3) with $d = \hat{d}$ and $g = 5$ is given by a complete intersection of 3 quadrics of $\boldsymbol{P}^5$. To construct an example of a surface in (1.1.3) with $d = \hat{d}$ and $g = 6$, consider the quartic surface X_4 in $\boldsymbol{P}^3$. Let ℓ be a line on it and let E be the line bundle $\mathcal{O}(2 - l)$, which is very ample. Since $l \cdot l = -2$, $\ell = -\ell \cdot \mathcal{O}(2) = 2$, we have

$$E \cdot E = 4 \cdot 4 - 4 - 2 = 10\,.$$

Moreover, because $K_{X_4} = \mathcal{O}_{X_4}$, it follows that

$$2g(E) - 2 = E \cdot (K_{H_4} + E) = 0 + 10\,.$$

Hence $g(E) = 6$. Furthermore, by the long cohomology sequence of the short exact sequence

$$0 \to \mathcal{O}_{X_4} \to E \to E|_C \to 0$$

where $C \in |E|$, we have that $h^0(E) = 7$, $h^1(E) = 0 = h^2(E)$. To construct an example of (1.2.1), let $f\colon Y \to \boldsymbol{P}^1$ be the $\boldsymbol{P}^2$-bundle $\boldsymbol{P}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1))$, with the tautological line bundle L and let $M = 3L + f^*\mathcal{O}(1)$. Then M is base-point-free and its generic member X is smooth and $K_X \sim (f|_X)^*\mathcal{O}(1)$. One can see that L is base-point-free and the induced map $Y \to \boldsymbol{P}^4$ induced an embedding $X \subset \boldsymbol{P}^4$ in (1.2.1). We do not know if there are examples of hyperelliptic surfaces in (1.1.1), or examples of (1.1.2), (1.2.2) and (1.2.3).

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Istituto di Matematica
Universita' Degli Studi Dell'Aquila Degli Abruzzi
via Roma, 33
67100 L'AQUILA
Italy