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ON A q-ANALOGUE OF THE LOG-Γ-FUNCTION

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0. Introduction

For complex numbers q and u, Carlitz defined the q-Bernoulli numbers $\{\beta_k(q)\}$ and the q-Euler numbers $\{H_k(u, q)\}$ associated to u by

$$\beta_0(q) = 1, \sum_{j=0}^{k} \binom{k}{j} q^{j+1}\beta_j(q) - \beta_k(q) = \begin{cases} 1 & (k = 1) \\ 0 & (k \geq 2) \end{cases}$$

and

$$H_0(u, q) = 1, \sum_{j=0}^{k} \binom{k}{j} q^j H_j(u, q) - uH_k(u, q) = 0 \quad (k \geq 1).$$

(See [2]). Note that if $q \to 1$, then $\beta_k(q) \to B_k$ and $H_k(u, q) \to H_k(u)$ where $\{B_k\}$ and $\{H_k(u)\}$ are the ordinary Bernoulli and Euler numbers defined by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!},$$

and

$$\frac{1-u}{e^t - u} = \sum_{k=0}^{\infty} H_k(u) \frac{t^k}{k!}.$$

In [4], Kobiltz constructed a q-analogue of the p-adic L-function $L_{p,q}(s, \chi)$ which interpolated the q-Bernoulli numbers at non positive integers, and suggested the following two problems.

(1) Are there complex analytic q-L-series which $L_{p,q}(s, \chi)$ can be viewed as interpolating, in the same way that $L_p(s, \chi)$ interpolates $L(s, \chi)$?
(2) Do Carlitz's $\beta_k(q)$ occur in the coefficients of some Stirling type series for p-adic or complex analytic q-log-Γ-functions ?

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In [5], Satoh gave an answer to the problem (1). Namely he constructed a q-L-series $L_q(s, \chi)$ which interpolated the generalized q-Bernoulli numbers defined in [4]. An answer to the problem (2) in the p-adic case was given by the author in [7]. But the problem (2) in the complex case is unsolved.

In the present paper, we give an answer to the problem (2) in the complex case. In §1, we construct a locally analytic function $g(x, u, q)$ in which the q-Euler numbers occur as the coefficients of the Stirling expansion. In §2, we construct a locally analytic function $G(x, q)$ in which the q-Bernoulli numbers occur as the coefficients of the Stirling expansion. In §3, we calculate the values of $L_q(s, \chi)$ at positive integers by using $G(x, q)$. The result is a q-analogue of the classical relation between the Dirichlet L-series and the log-Γ-function. So we can regard $G(x, q)$ as the q-log-Γ-function which the above problem (2) in the complex case requires.

1. The function $g(x, u, q)$

Let $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ be the sets of rational integers, real numbers and complex numbers. Let q be a complex number with $|q| < 1$, and u be a complex number with $|u| > 1$. For a complex number z, we use the notation $[z] = [z ; q] = (1 - q^z)/(1 - q)$. Note that

$$\lim_{n\to\infty} [n] = \frac{1}{1-q}. \tag{1.1}$$

Let

$$l(s, u, q) = \sum_{n=1}^{\infty} \frac{u^{-n}}{[n]^s}, \tag{1.2}$$

for $s \in \mathbf{C}$.

LEMMA 1 (Satoh). *$l(s, u, q)$ is analytic in the whole complex plane. For $k \in \mathbf{Z}$ with $k \geq 0$,*

$$l(-k, u, q) = \begin{cases} \dfrac{1}{u-1} & (k = 0) \\ \dfrac{u}{u-1} H_k(u, q) & (k \geq 1) \end{cases}$$

Proof. See [5].

Now we define the function

$$(1.3) \qquad g(x, u, q) = \sum_{n=0}^{\infty} u^{-n}(x + [n])\log(x + [n]).$$

By the condition $|u| > 1$, we can see that $g(x, u, q)$ is a locally analytic function.

Proposition 1. *For* $x \in \mathbf{C}$ *with* $|x| > 1/(1 - |q|)^2$,

$$g(x, u, q) = \frac{u}{u-1}\{x \log x + \frac{1}{u-q}(\log x + 1)\}$$
$$+ \frac{u}{u-1}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(u, q) \frac{1}{x^k}.$$

Proof. By (1.3),

$$(1.4) \qquad g(x, u, q) = \sum_{n=0}^{\infty} u^{-n}(x + [n])\log x$$
$$+ \sum_{n=0}^{\infty} u^{-n}(x + [n])\log\left(1 + \frac{[n]}{x}\right).$$

By Lemma 1, we have

$$\sum_{n=0}^{\infty} u^{-n}[n] = \frac{u}{u-1} H_1(u, q)$$
$$= \frac{u}{u-1}\frac{1}{u-q}.$$

So the first term of (1.4) can be calculated and equals

$$\frac{u}{u-1}\left\{x \log x + \frac{1}{u-q}\log x\right\}.$$

On the other hand, by using the series expansion formula

$$\log(1 + z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} z^k}{k},$$

the second term of (1.4) equals

$$(1.5) \qquad \sum_{n=0}^{\infty} u^{-n}\left\{\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\frac{[n]^k}{x^{k-1}} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\frac{[n]^{k+1}}{x^k}\right\}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} \{\sum_{n=0}^{\infty} u^{-n}[n]^{k+1}\} \frac{1}{x^k} + \frac{u}{u-1} \frac{1}{u-q}.$$

Note that $|[n]| < 1/(1-|q|)$ for $n = 1,2,3,\dots$. So the equation (1.5) holds for x with $|x| > 1/(1-|q|)^2$. By Lemma 1, we have the assertion.

COROLLARY 1. *For $x \in \mathbf{C}$ with $|x| > 1/(1-|q|)^2$,*

$$g'(x, u, q) = \frac{u}{u-1}(\log x + 1) + \frac{u}{u-1} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} H_k(u, q) \frac{1}{x^k},$$

where $g'(x, u, q)$ is the derivative $\frac{d}{dx} g(x, u, q)$.

2. The function $G(x, q)$

In [5], the q-Riemann ζ-function was defined by

$$\zeta_q(s) = \frac{2-s}{s-1}(1-q) \sum_{n=1}^{\infty} \frac{q^n}{[n]^{s-1}} + \sum_{n=1}^{\infty} \frac{q^n}{[n]^s}. \tag{2.1}$$

LEMMA 2 (Satoh). *$\zeta_q(s)$ is analytic in the whole complex plane. For $k \in \mathbf{Z}$ with $k \geq 1$,*

$$\zeta_q(1-k) = \begin{cases} q\beta_1(q) & (k=1) \\ -\dfrac{\beta_k(q)}{k} & (k \geq 2). \end{cases}$$

Proof. See [5].

By Lemma 1, Lemma 2 and (2.1), we obtain the relation

$$-\frac{\beta_k(q)}{k} = -\frac{k+1}{k} H_k(q^{-1}, q) + \frac{1}{1-q} H_{k-1}(q^{-1}, q) \tag{2.2}$$

for $k \in \mathbf{Z}$ with $k \geq 2$.

Now we define the function $G(x, q)$ by

$$G(x, q) = (qx - x - 1)g'(x, q^{-1}, q) + 2(1-q)g(x, q^{-1}, q) + \frac{1}{1+q}. \tag{2.3}$$

PROPOSITION 2. *For $x \in \mathbf{C}$ with $|x| > 1/(1-|q|)^2$,*

$$G(x, q) = \left(x - \frac{1}{1+q}\right)\log x - x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} \beta_{k+1}(q) \frac{1}{x^k}.$$

Proof. By Proposition 1 and Corollary 1, the left hand side of (2.3) equals

$$\{(q-1)x - 1\} \frac{1}{1-q} (\log x + 1) + 2\left(x \log x + \frac{q}{1-q^2} (\log x + 1)\right) + \frac{1}{1+q}$$
$$+ \{(q-1)x - 1\} \frac{1}{1-q} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} H_k(q^{-1}, q) \frac{1}{x^k}$$
$$+ 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(q^{-1}, q) \frac{1}{x^k}$$
$$= \left(x - \frac{1}{1+q}\right) \log x - x + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left\{\frac{k+2}{k+1} H_{k+1}(q^{-1}, q) - \frac{1}{1-q} H_k(q^{-1}, q)\right\} \frac{1}{x^k}.$$

By (2.2), we have the assertion.

Remark. The formula in Proposition 2 is a q-analogue of the classical asymptotic series (see [9]).

$$\log \frac{\Gamma(x)}{\sqrt{2\pi}} \sim \left(x - \frac{1}{2}\right) \log x - x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} B_{k+1} \frac{1}{x^k},$$

where B_n is the ordinary Bernoulli number.

3. Values of $L_q(s, \chi)$

For $r \in \mathbf{Z}$ with $r \geq 1$ and a function $f(x)$, let $f^{(r)}(x)$ be the r-th derivative $\frac{d^r}{dx^r} f(x)$.

LEMMA 3. *For $r \in \mathbf{Z}$ with $r \geq 2$,*

$$\frac{(-1)^r}{(r-2)!} g^{(r)}(x, u, q) = \sum_{n=0}^{\infty} \frac{u^{-n}}{(x + [n])^{r-1}}.$$

Proof. We can see that

$$g^{(1)}(x, u, q) = \sum_{n=0}^{\infty} u^{-n} \{\log(x + [n]) + 1\},$$

and

$$g^{(2)}(x, u, q) = \sum_{n=0}^{\infty} \frac{u^{-n}}{(x + [n])}.$$

Inductively we can see that

$$g^{(r)}(x, u, q) = (-1)^{r}(r-2)! \sum_{n=0}^{\infty} \frac{u^{-n}}{(x + [n])^{r-1}}$$

for $r \geq 2$. So we have the assertion.

The q-L-series was defined by

$$L_q(s, \chi) = \frac{2-s}{s-1}(1-q) \sum_{n=1}^{\infty} \frac{q^{n}\chi(n)}{[n]^{s-1}} + \sum_{n=1}^{\infty} \frac{q^{n}\chi(n)}{[n]^{s}}, \tag{3.1}$$

where χ is a primitive Dirichlet character with conductor f. $L_q(s, \chi)$ interpolates the generalized q-Bernoulli numbers $\beta_{n,\chi}(q)$ (defined in [4]) at non positive integers. So we can regard $L_q(s, \chi)$ as the function which the Koblitz problem (1) requires (see [4],[5]).

Now we evaluate the values of $L_q(s, \chi)$ at positive integers.

LEMMA 4. *For $k \in \mathbf{Z}$ with $k \geq 2$,*

$$L_q(k, \chi) = \frac{2-k}{(k-1)!}(1-q) \frac{(-1)^{k}}{[f]^{k-1}} \sum_{a=1}^{f} q^{a(2-k)} \chi(a) g^{(k)}\left(q^{-a}\frac{[a]}{[f]}, q^{-f}, q^{f}\right)$$

$$+ \frac{1}{(k-1)!} \frac{(-1)^{k+1}}{[f]^{k}} \sum_{a=1}^{f} q^{a(1-k)} \chi(a) g^{(k+1)}\left(q^{-a}\frac{[a]}{[f]}, q^{-f}, q^{f}\right)$$

Proof. By (3.1)

$$L_q(k, \chi) = \frac{2-k}{k-1}(1-q) \sum_{a=1}^{f} \sum_{j=0}^{\infty} \frac{q^{a+fj}\chi(a+fj)}{[a+fj]^{k-1}} \tag{3.2}$$

$$+ \sum_{a=1}^{f} \sum_{j=0}^{\infty} \frac{q^{a+fj}\chi(a+fj)}{[a+fj]^{k}}.$$

By using the relation

$$[a+fj]=\frac{1-q^{a+fj}}{1-q}=\frac{1-q^{a}}{1-q}+q^{a}\frac{1-q^{f}}{1-q}\frac{1-q^{fi}}{1-q^{f}}$$
$$=[a]+q^{a}[f][j\,;q^{f}],$$

the first term of (3.2) equals

$$\frac{2-k}{k-1}(1-q)\frac{1}{[f]^{k-1}}\sum_{a=1}^{f}q^{a(2-k)}\chi(a)\sum_{j=0}^{\infty}\frac{q^{fj}}{\left(q^{-a}\frac{[a]}{[f]}+[j\,;q^{f}]\right)^{k-1}},$$

and the second term of (3.2) equals

$$\frac{1}{[f]^{k}}\sum_{a=1}^{f}q^{a(1-k)}\chi(a)\sum_{j=0}^{\infty}\frac{q^{fj}}{\left(q^{-a}\frac{[a]}{[f]}+[j\,;q^{f}]\right)^{k}}.$$

By Lemma 3, we have the assertion.

PROPOSITION 3. *For $k\in\mathbf{Z}$ with $k\geq 2$,*

$$L_{q}(k,\chi)=\frac{(-1)^{k}}{(k-1)!}\frac{1}{[f]^{k}}\sum_{a=1}^{f}q^{a(2-k)}\chi(a)G^{(k)}\left(q^{-a}\frac{[a]}{[f]},q^{f}\right).$$

Proof. By (2.3), we can inductively see that

$$G^{(k)}(x,q)=(qx-x-1)g^{(k+1)}(x,q^{-1},q)+(1-q)(2-k)g^{(k)}(x,q^{-1},q) \tag{3.3}$$

for $k\in\mathbf{Z}$ with $k\geq 2$. So we obtain

$$G^{(k)}\left(q^{-a}\frac{[a]}{[f]},q^{f}\right)=\left\{(q^{f}-1)q^{-a}\frac{1-q^{a}}{1-q^{f}}-1\right\}g^{(k+1)}\left(q^{-a}\frac{[a]}{[f]},q^{-f},q^{f}\right)$$
$$+(1-q^{f})(2-k)g^{(k)}\left(q^{-a}\frac{[a]}{[f]},q^{-f},q^{f}\right)$$
$$=-q^{-a}g^{(k+1)}\left(q^{-a}\frac{[a]}{[f]},q^{-f},q^{f}\right)+[f](1-q)(2-k)g^{(k)}\left(q^{-a}\frac{[a]}{[f]},q^{-f},q^{f}\right).$$

By Lemma 4,

$$L_{q}(k,\chi)=\frac{(-1)^{k}}{(k-1)!}\frac{1}{[f]^{k}}\sum_{a=1}^{f}q^{a(2-k)}\chi(a)$$
$$\times\left\{[f](1-q)(2-k)g^{(k)}\left(q^{-a}\frac{[a]}{[f]},q^{-f},q^{f}\right)-q^{-a}g^{(k+1)}\left(q^{-a}\frac{[a]}{[f]},q^{-f},q^{f}\right)\right\}$$

So we have the assertion.

Remark. By Lemma 3 and (3.3), we can see that

$$\frac{(-1)^k}{(k-1)!} G^{(k)}(x, q) = \frac{2-k}{k-1}(1-q) \sum_{n=0}^{\infty} \frac{q^n}{(x+[n])^{k-1}} + (1+x-qx) \sum_{n=0}^{\infty} \frac{q^n}{(x+[n])^k} \tag{3.4}$$

for $k \in \mathbf{Z}$ with $k \geq 2$. This relation is a q-analogue of the classical one

$$\frac{(-1)^k}{(k-1)!} \frac{d^k}{dx^k} \log \Gamma(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^k}.$$

By considering the result in Proposition 2, Proposition 3 and (3.4), we can regard $G(x, q)$ as the q-log-Γ-function which the Koblitz problem (2) in the complex case requires.

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