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NEW INVARIANTS AND CLASS NUMBER PROBLEM IN REAL QUADRATIC FIELDS

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In recent papers [10, 11, 12, 13, 14], we defined some new p-invariants for any rational prime p congruent to $1 \bmod 4$ and D-invariants for any positive square-free integer D such that the fundamental unit ε_D of real quadratic field $\mathbf{Q}(\sqrt{D})$ satisfies $N\varepsilon_D = -1$, and studied relationships among these new invariants and already known invariants.

One of our main purposes in this paper is to generalize these D-invariants to invariants valid for all square-free positive integers containing D with $N\varepsilon_D = 1$. Another is to provide an improvement of the theorem in [14] related closely to class number one problem of real quadratic fields. Namely, we provide, in a sense, a most appreciable estimation of the fundamental unit to be able to apply, as usual (cf. [3, 4, 5, 9, 12, 13]), Tatuzawa's lower bound of $L(1, \chi_D)$ (cf. [7]) for estimating the class number of $\mathbf{Q}(\sqrt{D})$ from below by using Dirichlet's classical class number formula.

In §1, we shall define and consider properties of new D-invariant m_D valid for all square-free positive integers D, and in §2 we shall deal with new D-invariant n_D, which is more valuable when $m_D = 0$. At that time, we shall partly alter notations of some new D-invariants related to n_D defined in [13] for avoiding any confusion of notations.

In §3, we shall consider real quadratic fields of R-D type, for which fields explicit forms of the fundamental units are well-known, and as an application of results in §1-2, we shall characterize each case of R-D type by using D-invariants studied there.

Finally, in §3 we shall provide an improvement of the theorem in [14] and Theorem 2 in [12].

Throughout this paper, we denote by $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ the set of all non-negative rational integers and by $[x]$ the greatest integer less than or equal to x. Moreover, we set

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$$\mathbf{D} = \{D > 0 : \text{square-free positive integer}\},$$

and for any D in $\mathbf{D}$, we denote by h_D and by

$$\varepsilon_D = (t_D + u_D\sqrt{D})/2 \quad (> 1)$$

the class number and the fundamental unit of the real quadratic field $\mathbf{Q}(\sqrt{D})$ respectively.

Furthermore, we set

$$\mathbf{D}_+ = \{D \in \mathbf{D} : N\varepsilon_D = 1\},$$

and

$$\mathbf{D}_- = \{D \in \mathbf{D} : N\varepsilon_D = -1\},$$

where N means the norm mapping from $\mathbf{Q}(\sqrt{D})$ to the rational number field $\mathbf{Q}$.

§1

We first prove the following:

THEOREM 1.1. *For any D in $\mathbf{D}$ greater than* 13 (*or* $t_D \geqq 5$), *it holds*

$$[u_D^2/t_D] = [t_D/D] = [\varepsilon_D/D] = [u_D/\sqrt{D}].$$

Proof. First we show that $D > 13$ implies $t_D \geqq 5$. It follows from $t_D^2 - Du_D^2 = \pm 4$ that $D \leqq Du_D^2 = t_D^2 \pm 4$. Hence,

$$D \geqq 29 \text{ implies } t_D^2 \pm 4 \geqq 29, \text{ and so } t_D \geqq 5.$$

On the other hand, for any D satisfying $13 < D < 29$ we can confirm $t_D \geqq 5$ by practical calculations.

Next, we put $m_D = [u_D^2/t_D]$. Then, for any D in $\mathbf{D}_+$ satisfying $t_D \geqq 5$ we can prove the following inequality:

$$(1) \qquad Dm_D \leqq t_D - 1 < u_D\sqrt{D} < \varepsilon_D < t_D < D(m_D + 1),$$

and for any D in $\mathbf{D}_-$ satisfying $t_D \geqq 5$ we can prove the following inequality:

$$(2) \qquad Dm_D < t_D < \varepsilon_D < u_D\sqrt{D} < t_D + 1 \leqq D(m_D + 1).$$

In the case D in $\mathbf{D}_+$ from $u_D^2 D = t_D^2 - 4$ and $t_D \geqq 5$ we know

$$t_D > u_D^2 D/t_D = t_D - 4/t_D > t_D - 1.$$

Moreover we get

$$Dm_D \leqq u_D^2 D / t_D < D(m_D + 1).$$

Hence we get

$$Dm_D \leqq t_D - 1 \quad \text{and} \quad t_D \leqq D(m_D + 1).$$

Furthermore, we know

$$u_D\sqrt{D} < \varepsilon_D < t_D$$

from Lemma 2 in [9], and

$$t_D - 1 < \sqrt{t_D^2 - 4} = u_D \sqrt{D}$$

from $t_D \geqq 5$.

In the case D in $\mathbf{D}_-$ from $u_D^2 D = t_D^2 + 4$ we know

$$t_D < u_D^2 D / t_D = t_D + 4 / t_D < t_D + 1.$$

Hence we get similarly

$$Dm_D \leqq t_D \quad \text{and} \quad t_D + 1 \leqq D(m_D + 1).$$

Moreover, we know

$$t_D < \varepsilon_D < u_D\sqrt{D}$$

from Lemma 2 in [9], and

$$u_D\sqrt{D} = \sqrt{t_D^2 + 4} < t_D + 1$$

from $t_D \geqq 5$.

Here, if we assume $t_D = Dm_D$ or $D(m_D + 1)$, then $t_D \equiv 0 \pmod D$. Hence, it follows

$$\pm 4 = t_D^2 - Du_D^2 \equiv 0 \pmod D,$$

which conflicts with $D > 13$.

Our theorem is immediately follows from (1) and (2).

For any D in $\mathbf{D}_+$, we put

$$A_D = \{a : 0 < a < D,\ a^2 \equiv 4 \pmod D\},$$

and

$$(A, B)_D = \{(a, b) : a \in A_D,\ a^2 - 4 = bD\}.$$

For any D in $\mathbf{D}_-$, we put

$$A_D = \{a : 0 \leqq a < D,\ a^2 \equiv -4 \pmod{D}\},$$

and

$$(A, B)_D = \{(a, b) : a \in A_D,\ a^2 + 4 = bD\}.$$

Then we can prove the following:

THEOREM 1.2. *For any D in $\mathbf{D}$, there are uniquely determined m_D in $\mathbf{N}_0$ and (a_D, b_D) in $(A, B)_D$ such that*

$$\begin{cases} t_D = D\cdot m_D + a_D, \\ u_D^2 = D\cdot m_D^2 + 2a_D\cdot m_D + b_D. \end{cases}$$

Additionally, if $D > 5$, then

$$m_D = [t_D/D] \quad \text{and} \quad 0 \leqq b_D < a_D < D,$$

and moreover

$$b_D = 0 \quad \text{if and only if} \quad a_D = 2.$$

Proof. In the case D in $\mathbf{D}_+$, we put

$$[t_D/D] = m_D \quad \text{and} \quad t_D = D \cdot m_D + a_D.$$

Then, m_D and a_D are uniquely determined, and we know

$$m_D \in \mathbf{N}_0, \quad 0 \leqq a_D < D.$$

Moreover, since

$$Du_D^2 = t_D^2 - 4 = D(Dm_D^2 + 2a_D m_D) + (a_D^2 - 4),$$

we obtain

$$a_D^2 - 4 \equiv 0 \pmod{D}.$$

Therefore, we put

$$a_D^2 - 4 = D\cdot b_D.$$

Then, b_D is also uniquely determined and we obtain

$$u_D^2 = D\cdot m_D^2 + 2a_D\cdot m_D + b_D.$$

Here, if we assume $a_D = 0$, then we get $D = 2$.
However, from $\varepsilon_2 = 1 + \sqrt{2}$ we get $N\varepsilon_2 = -1$, which contradicts with D in $\mathbf{D}_+$.
Hence, $a_D \neq 0$ holds, and so (a_D, b_D) is in $(A, B)_D$.

Next, it is clear that $b_D = 0$ if and only if $a_D = 2$.
Furthermore,

$$b_D < 0 \quad \text{if and only if} \quad a_D = 1,$$

which is equivalent to $D = 3 \quad (b_D = -1)$.
Hence, $D > 3$ implies $b_D \geqq 0$.

If we assume $b_D \geqq D$, then from $a_D^2 - 4 = Db_D \geqq D^2$ we get at once

$$-4 \geqq D^2 - a_D^2 = (D - a_D)(D + a_D) > D,$$

which is a contradiction. Hence, we get $b_D < D$.

Finally, we put $f(x) = -x^2 + Dx + 4$.
Then

$$f(0) = f(D) = 4 > 0,$$

and

$$0 < f(b_D) = (a_D + b_D)(a_D - b_D)$$

holds.
Hence, we know $a_D > b_D$ from $a_D + b_D > 0$.

In the case D in $\mathbf{D}_-$, we proved already in [13].

§2

In the case $m_D = 0$, i.e. $t_D < D$, we know by Theorem 1.1

$$u_D^2 < t_D \quad \text{provided } D > 13 \text{ or } t_D \geqq 5.$$

Therefore, in this case the invariant $n_D = [t_D / u_D^2]$ is more useful than the invariant m_D.

For any D in $\mathbf{D}_+$, we put

$$V_D = \{v : 0 \leqq v < u_D^2,\ v^2 \equiv 4 \pmod{u_D^2}\},$$

and

$$(V, W)_D = \{(v, w) : v \in V_D,\ v^2 - 4 = wu_D^2\}.$$

For any D in $\mathbf{D}_-$, we put

$$V_D = \{v : 0 \leqq v < u_D^2,\ v^2 \equiv -4 \pmod{u_D^2}\},$$

and

$$(V, W)_D = \{(v, w) : v \in V_D,\ v^2 + 4 = wu_D^2\}.$$

Then we can prove the following:

THEOREM 2.1. *For any D in $\mathbf{D}$, there are uniquely determined n_D in $\mathbf{N}_0$ and (v_D, w_D) in $(V, W)_D$ such that*

$$\begin{cases} t_D = u_D^2 \cdot n_D + v_D \\ D = u_D^2 \cdot n_D^2 + 2v_D \cdot n_D + w_D. \end{cases}$$

Additionally, if $u_D > 2$, then

$$0 \leqq w_D < v_D < u_D^2 \quad \text{and} \quad n_D = [t_D / u_D^2] = [D/t_D].$$

Proof. In the case D in $\mathbf{D}_+$, we put

$$[t_D / u_D^2] = n_D \quad \text{and} \quad t_D = u_D^2 \cdot n_D + v_D.$$

Then, n_D and v_D are uniquely determined, and

$$n_D \in \mathbf{N}_0, \quad 0 \leqq v_D < u_D^2$$

holds.

Since

$$Du_D^2 = t_D^2 - 4 = u_D^2(u_D^2 n_D^2 + 2v_D n_D) + (v_D^2 - 4),$$

we obtain

$$v_D^2 - 4 \equiv 0 \pmod{u_D^2},$$

and so v_D is in V_D.

Moreover, we put

$$v_D^2 - 4 = u_D^2 \cdot w_D.$$

Then, w_D is also uniquely determined and we obtain

$$D = u_D^2 \cdot n_D^2 + 2v_D \cdot n_D + w_D,$$

and so (v_D, w_D) is in $(V, W)_D$.

Especially, since $w_D u_D^2 = v_D^2 - 4 < u_D^4 - 4$, if we assume $u_D > 2$, then we get

$$w_D < u_D^2 - (4/u_D^2) < u_D^2.$$

Furthermore, it is clear that

$$w_D < 0 \quad \text{if and only if} \quad v_D = 0 \text{ or } 1.$$

On the other hand,

$$v_D = 0 \quad \text{implies} \quad w_D u_D^2 = -4, \quad \text{and so} \quad u_D = 1 \text{ or } 2,$$

and

$$v_D = 1 \quad \text{implies} \quad w_D u_D^2 = -3, \quad \text{and so} \quad u_D = 1.$$

Hence, $u_D > 2$ implies $0 \leqq w_D < u_D^2$.

Next, we put

$$g(x) = -x^2 + u_D^2 x + 4.$$

Then we get

$$g(0) = g(u_D^2) = 4 > 0$$

and

$$0 < g(w_D) = (v_D + w_D)(v_D - w_D).$$

Hence, if $u_D > 2$, then we know $v_D > w_D$ from $v_D + w_D \geqq 0$.

Finally, in $D = n_D t_D + (n_D v_D + w_D)$,

$$u_D > 2 \quad \text{implies} \quad n_D v_D + w_D \geqq 0,$$

and

$$t_D - (n_D v_D + w_D) = (u_D^2 - v_D) \cdot n_D + (v_D - w_D) > 0.$$

Hence, we get

$$[t_D / u_D^2] = n_D = [D / t_D].$$

In the case D in $\mathbf{D}_-$, we proved already in [13].

§3

In the unique expression

$$D = k^2 + r \quad (-k < r \leqq k)$$

for any D in $\mathbf{D}$, if

$$4k \equiv 0 \pmod{r}$$

holds, then the real quadratic field $\mathbf{Q}(\sqrt{D})$ is called of Richaud-Degert type (simply R-D type).

In this section, we consider new invariants of real quadratic fields of R-D type. For real quadratic fields of R-D type, the explicit form of the fundamental unit is well-known as follows (cf. [1, 2, 6, 8]):

THEOREM 3.1 (Richaud-Degert). *Let* $\mathbf{Q}(\sqrt{D})$ $(D = k^2 + r,\ -k < r \leqq k)$ *be real quadratic fields of* R-D *type* $(4k \equiv 0 \bmod r)$.
Then, the fundamental unit ε_D *of* $\mathbf{Q}(\sqrt{D})$ *is of the following form*:

$$\begin{cases} \varepsilon_D = k + \sqrt{D} & \text{with } N\varepsilon_D = -\operatorname{sgn} r \quad \text{for } |r| = 1 \\ \qquad (\text{except for } D = 5\,;\ k = 2,\ r = 1), & \\ \varepsilon_D = (k + \sqrt{D})/2 & \text{with } N\varepsilon_D = -\operatorname{sgn} r \quad \text{for } |r| = 4, \\ \varepsilon_D = \{(2k^2 + r) + 2k\sqrt{D})\}/|r| & \text{with } N\varepsilon_D = 1 \qquad\quad \text{for } |r| \neq 1, 4. \end{cases}$$

Using this theorem, we consider each case of R-D type as follows: In the case $r = 1$, we get $D = k^2 + 1 \quad (k \geqq 1)$ and $\varepsilon_D = k + \sqrt{D} \quad (N\varepsilon_D = -1)$, and so

$$t_D = 2k, \quad u_D = 2, \quad t_D/u_D^2 = k/2,$$

which implies

$$n_D = k/2 \quad (k : \text{even}) \quad \text{or} \quad (k-1)/2 \quad (k : \text{odd}).$$

Hence,

$$\text{if } k \text{ is even, then } v_D = 0 \text{ and } w_D = 1.$$

On the other hand,

$$\text{if } k \text{ is odd, then } v_D = 2 \text{ and } w_D = 2.$$

In the case $r = -1$, we get $D = k^2 - 1 \quad (1 < k : \text{even})$ and $\varepsilon_D = k + \sqrt{D}$ $(N\varepsilon_D = 1)$, and so

$$t_D = 2k, \quad u_D = 2, \quad t_D/u_D^2 = k/2,$$

which implies

$$n_D = k/2.$$

Hence, we obtain

$$v_D = 0 \quad \text{and} \quad w_D = -1.$$

In the case $r=2$, we get $D=k^2+2 \quad (k \geqq 2)$ and $\varepsilon_D=k^2+1+k\sqrt{D}$ $(N\varepsilon_D=1)$, and so

$$t_D=2(k^2+1), \quad u_D=2k, \quad u_D^2/t_D=2k^2/(k^2+1),$$

which implies

$$m_D=1$$

from Theorem 1.1, because of $1<2k^2/(k^2+1)<2$.
Hence, we get

$$a_D=k^2, \quad b_D=k^2-2.$$

In the case $r=-2$, we get $D=k^2-2 \quad (k>2)$ and $\varepsilon_D=k^2-1+k\sqrt{D}$ $(N\varepsilon_D=1)$, and so

$$t_D=2(k^2-1), \quad u_D=2k, \quad u_D^2/t_D=2k^2/(k^2-1),$$

which implies

$$m_D=2$$

from Theorem 1.1, because of $2<2k^2/(k^2-1)<3$.
Hence, we get

$$a_D=2, \quad b_D=0.$$

In the case $r=3$, we get $D=k^2+3$ $(3 \leqq k \equiv 0 \bmod 3)$ and $\varepsilon_D=\{(2k^2+3)+2k\sqrt{D}\}/3 \quad (N\varepsilon_D=1)$, and so

$$t_D=2(2k^2+3)/3, \quad u_D=4k/3, \quad u_D^2/t_D=8k^2/(6k^2+9),$$

which implies

$$m_D=1$$

from Theorem 1.1, because of $1<8k^2/(6k^2+9)<2$.
Hence, we get

$$a_D=(k^2-3)/3, \quad b_D=(k^2-9)/9.$$

In the case $r=-3$, we get $D=k^2-3 \quad (3<k \equiv 0 \bmod 3)$ and $\varepsilon_D=\{(2k^2-3)+2k\sqrt{D}\}/3 \quad (N\varepsilon_D=1)$, and so

$$t_D=2(2k^2-3)/3, \quad u_D=4k/3, \quad u_D^2/t_D=8k^2/(6k^2-9),$$

which implies

$$m_D = 1$$

from Theorem 1.1, because of $1 < 8k^2/(6k^2 - 9) < 2$.
Hence, we get

$$a_D = (k^2 + 3)/3, \quad b_D = (k^2 + 9)/9.$$

In the case $|r| = 4$, we get $D = k^2 \pm 4$ $(k \geqq 4)$ and $\varepsilon_D = (k + \sqrt{D})/2$ $(N\varepsilon_D = -1)$, and so

$$t_D = k, \quad u_D = 1, \quad t_D/u_D^2 = k,$$

which implies $n_D = k$.
Hence, we obtain

$$v_D = 0 \quad \text{and} \quad w_D = \pm 4.$$

In the case $|r| \geqq 5$, we get $t_D = 2(2k^2 + r)/|r|$, $u_D = 4k/|r|$, which implies

$$t_D/u_D^2 = (|r|/4) + (\operatorname{sgn} r)(r^2/8k^2).$$

Here, since $0 < r^2/8k^2 \leqq 1/8$, we obtain

$$n_D = [|r|/4] - 1 \quad (\text{for } 0 > r \equiv 0 \bmod 4),$$

or

$$n_D = [|r|/4] \qquad (\text{for other cases}).$$

From these considerations, we obtain the following theorem and table:

THEOREM 3.2. *For real quadratic fields* $\mathbf{Q}(\sqrt{D})$ *of* R-D *type, if* $D > 5$, *then*

$$m_D = \begin{cases} 2 & \text{for } r = -2, \\ 1 & \text{for } r = 2, \pm 3, \\ 0 & \text{for } |r| \neq 2, 3. \end{cases}$$

$D = k^2 + r$	r	t_D	u_D	m_D	a_D	b_D
$D \geqq 6$	2	$2k^2 + 2$	$2k$	1	k^2	$k^2 - 2$
$D \geqq 7$	-2	$2k^2 - 2$	$2k$	2	2	0
$D \geqq 39$	3	$2(2k^2 + 3)/3$	$4k/3$	1	$(k^2 - 3)/3$	$(k^2 - 9)/9$
$D \geqq 33$	-3	$2(2k^2 - 3)/3$	$4k/3$	1	$(k^2 + 3)/3$	$(k^2 + 9)/9$

$D=k^2+r$	r	t_D	u_D	n_D	v_D	w_D	
$D \geqq 17$	1	$2k$	2	$k/2$	0	1	k : even
$D \geqq 10$	1	$2k$	2	$(k-1)/2$	2	2	k : odd
$D \geqq 15$	-1	$2k$	2	$k/2$	0	-1	k : even only
$D \geqq 29$	4	k	1	k	0	4	
$D \geqq 21$	-4	k	1	k	0	-4	

Furthermore, we can prove the following three propositions, which characterize each case of R-D type:

PROPOSITION 3.1. (1) $\mathbf{Q}(\sqrt{D})$ *is of* R-D *type with* $|r|=1$, *if and only if* $u_D=2$.
(2) $Q(\sqrt{D})$ *is of* R-D *type with* $|r|=4$, *if and only if* $u_D=1$.

Proof. From $t_D^2-Du_D^2=\pm 4$ we can obtain the following:
In the case $u_D=1$, we get directly $D=t_D^2\pm 4$.
In the case $u_D=2$, we know first that t_D is even, and so we can put $t_D=2k$ with a positive integer k. Hence we get $D=k^2\pm 1$.

The converse is clear from the above table.

PROPOSITION 3.2. (1) $\mathbf{Q}(\sqrt{D})$ *is of* R-D *type with* $r=\pm 1$ (k: *even*) *if and only if* $(v_D, w_D)=(0, \pm 1)$.
(2) $\mathbf{Q}(\sqrt{D})$ *is of* R-D *type with* $r=\pm 2$, *if and only if* $u_D^2=4D\mp 8$.
(3) $\mathbf{Q}(\sqrt{D})$ *is of* R-D *type with* $r=\pm 3$, *if and only if* $9u_D^2=16D\mp 48$.
(4) $\mathbf{Q}(\sqrt{D})$ *is of* R-D *type with* $r=\pm 4$, *if and only if* $(v_D, w_D)=(0, \pm 4)$.

Proof. (1), (4) In the case $v_D=0$, by Theorem 2.1 we obtain immediately $D=u_D^2n_D^2+w_D$. Hence, if additionally we assume $w_D=\pm 1$, or ± 4 respectively, then $\mathbf{Q}(\sqrt{D})$ is of R-D type and $r=w_D$.

The converse is clear from the above table.

(2) If $\mathbf{Q}(\sqrt{D})$ is of R-D type and $r=\pm 2$, then $u_D=2k$, and hence

$$u_D^2=4k^2=4(k^2\pm 2)\mp 8=4D\mp 8.$$

Conversely, if $u_D^2=4D\mp 8$, then $u_D^2\equiv 0 \pmod 4$, and so $u_D\equiv 0 \pmod 2$. Hence, we can put $u_D=2k$ with a suitable natural number k, and obtain immediately $D=k^2\pm 2$, which shows that $\mathbf{Q}(\sqrt{D})$ is of R-D type and $r=\pm 2$.

(3) If $\mathbf{Q}(\sqrt{D})$ is of R-D type and $r=\pm 3$, then $u_D=4k/3$, and hence

$$9u_D^2 = 16k^2 = 16(k^2 \pm 3) \mp 48 = 16D \mp 48.$$

Conversely, if $9u_D^2 = 16D \mp 48$, then $9u_D^2 \equiv 0 \pmod{4^2}$, and so $u_D \equiv 0 \pmod 4$. Hence, similarly we can put $u_D = 4k$ with a suitable natural number k, and obtain $D = 9k^2 \pm 3$, which shows that $\mathbf{Q}(\sqrt{D})$ is of R-D type and $r = \pm 3$.

PROPOSITION 3.3. (1) *$\mathbf{Q}(\sqrt{D})$ is of* R-D *type with $r = 1$ (k: odd), if and only if $u_D = 2$ and $(v_D, w_D) = (2, 2)$.*

(2) *$\mathbf{Q}(\sqrt{D})$ is of* R-D *type with $r = -2$, if and only if $m_D = 2$ and $(a_D, b_D) = (2, 0)$.*

Proof. (1) If we assume $u_D = 2$ and $(v_D, w_D) = (2, 2)$, then from Theorem 2.1 we get immediately

$$D = 4n_D^2 + 4n_D + 2 = (2n_D + 1)^2 + 1,$$

which shows that $\mathbf{Q}(\sqrt{D})$ is of R-D type and $r = 1$.

The converse is clear from the above table.

(2) If we assume $m_D = 2$ and $(a_D, b_D) = (2, 0)$, then from Theorem 1.2, we get similarly $u_D^2 = 4(D + 2)$. Hence, there exists a natural number k satisfying $D + 2 = k^2$ i.e. $D = k^2 - 2$. Therefore, $\mathbf{Q}(\sqrt{D})$ is of R-D type and $r = -2$.

The converse is clear from the above table.

§4

In this section, in connection with class number problem, we consider finiteness properties and estimations from below for the class number of real quadratic fields.

We first prove the following theorem related to class number one problem for real quadratic fields:

THEOREM 4.1. *For arbitrarily chosen and fixed natural number h_0 and real number c greater than 2, there exists only a finite number of real quadratic fields $\mathbf{Q}(\sqrt{D})$ ($D \in \mathbf{D}$) such that*

$$\varepsilon_D < D \cdot e^{\frac{1}{D^c}} \quad \text{and} \quad h_D \leqq h_0.$$

Proof. We first define a symbol δ_D depend on D in $\mathbf{D}$ by

$$\delta_D = 0 \quad \text{for} \quad D \equiv 1 \pmod 4,$$

and

$$\delta_D = 1 \quad \text{for} \quad D \equiv 2,\ 3 \pmod 4.$$

Then, we get $d = 4^{\delta_D} \cdot D$ for the discriminant d of real quadratic field $\mathbf{Q}(\sqrt{D})$.

Moreover, by applying Tatuzawa's result (cf. [7])

$$L(1,\ \chi_d) > 0.655/(sd^{1/s})$$

(for any $s \geqq 11.2$, $d \geqq e^s$ and with one possible exception of d) to Dirichlet's

classical class number formula

$$h_d = (2 \log \varepsilon_D)^{-1} \sqrt{d} \cdot L(1,\ \chi_d),$$

we obtain

$$h_D > 4^{\delta_D (s-2)/2s} \cdot 0.3275 s^{-1} \cdot D^{(s-2)/2s} / \log \varepsilon_D$$

for any $s \geqq 11{,}2$ and $D \geqq e^s$ in D.
Here, if we assume $\varepsilon_D < D \cdot e^{D^{\frac{1}{c}}}$, and put

$$\alpha = (s-2)/(2s), \quad \beta = 1/c,$$

then

$$\alpha > \beta \quad \text{if and only if} \quad s > 2c/(c-2).$$

On the other hand, if we put moreover

$$f_s(D) = D^{\alpha}/(D^{\frac{1}{c}} + \log D),$$

then we get for any $s > 2c/(c-2)$

$$h_D > 4^{\delta_D (s-2)/2s} \cdot 0.3275 s^{-1} \cdot f_s(D),$$

and under the assumption $\alpha > \beta > 0$, $f_s(D)$ tends to infinity as D tends to infinity. Therefore, if we choose any s satisfying

$$s > \max\{11.2,\ 2c/(c-2)\},$$

then there exists a positive number D_0 such that

$$h_D > h_0 \quad \text{holds for any } D \text{ in } \mathbf{D} \text{ with } D > D_0,$$

in other words,

$$h_D \leqq h_0 \quad \text{implies} \quad D \leqq D_0.$$

From this theorem, we can obtain immediately the following two corollaries:

COROLLARY 4.1. *For arbitrarily chosen and fixed real number c greater than 2, there exists only a finite number of real quadratic fields $\mathbf{Q}(\sqrt{D})$ $(D \in \mathbf{D})$ such that*

$$\varepsilon_D < D \cdot e^{D^{\frac{1}{c}}} \quad \textit{and} \quad h_D = 1.$$

COROLLARY 4.2. *There exist infinitely many real quadratic fields of class number one if and only if there exist infinitely many real quadratic fields $\mathbf{Q}(\sqrt{D})$ $(D \in \mathbf{D})$ satisfying*

$$\varepsilon_D > D \cdot e^{D^{\frac{1}{c}}} \quad \textit{and} \quad h_D = 1$$

for any fixed number c greater than 2.

Furthermore, we can provide the following lower bounds for h_D :

PROPOSITION 4.1. *For any $s \geqq 11.2$ and $D \geqq e^s$ in D,*
(1) *if $m_D \neq 0$, then*

$$h_D > 0.3275 \cdot 4^{\delta_D (s-2)/2s} \cdot s^{-1} \cdot D^{(s-2)/2s} / \{\log\,(m_D + 1) D\}$$

holds with one possible exception of D.
(2) *if $m_D = 0$ (i.e. $n_D \neq 0$), then*

$$h_D > 0.3275 \cdot 4^{\delta_D (s-2)/2s} \cdot s^{-1} \cdot D^{(s-2)/2s} / \{\log\,(D/n_D) + 1)\}$$

holds with one possible exception of D,
(3) *if $\mathbf{Q}(\sqrt{D})$ is a real quadratic field of* R-D *type, then*

$$h_D > 0.3275 \cdot 4^{\delta_D (s-2)/2s} \cdot s^{-1} \cdot D^{(s-2)/2s} / \log 3D.$$

holds with one possible exception of D.

Proof. In case of $m_D \neq 0$, from Theorem 1.1 we know first

$$\varepsilon_D < D(m_D + 1).$$

In case of $m_D = 0$, we know $n_D \neq 0$ from Theorem 1.1, and so from Theorem 1.3 in [13] we get $\varepsilon_D < (D/n_D) + 1$ for D in $\mathbf{D}_-$, and also get similarly $\varepsilon_D < D/n_D$ for D in $\mathbf{D}_+$.

In case of real quadratic field of R-D type, from Theorem 3.2 we get $\varepsilon_D < 3D$.

Hence, in each case, by applying these upper bounds for ε_D to the formula

$$h_D > 4^{\delta_D(s-2)/2s} \cdot 0.3275 s^{-1} \cdot D^{(s-2)/2s} / \log \varepsilon_D$$

obtained in the proof of Theorem 4.1, we can prove Proposition 4.1.

$$t_D = Dm_D + a_D \qquad t_D = u_D^2 n_D + v_D$$
$$u_D^2 = Dm_D^2 + 2a_D m_D + b_D \qquad D = u_D^2 n_D^2 + 2v_D n_D^2 + 2v_D n_D + w_D$$
$$a_D^2 \pm 4 = b_D D \qquad v_D^2 \pm 4 = w_D u_D^2$$
$$m_D = [t_D / D] \qquad n_D = [t_D / u_D^2]$$

D	r	t_D	u_D	h_D	m_D	a_D	b_D	n_D	v_D	w_D
# 2	1	2	2	−1	1	0	2			
# 3	−1	4	2	1	1	1	−1			
# 5		1	1	−1				1	0	4
6	2	10	4	1	1	4	2			
# 7	−2	16	6	1	2	2	0			
10	1	6	2	−2				1	2	2
# 11	2	20	6	1	1	9	7			
# 13		3	1	−1				3	0	4
14	−2	30	8	1	2	2	0			
15	−1	8	2	2				2	0	−1
# 17	1	8	2	−1				2	0	1
# 19		340	78	1	17	17	15			
21	−4	5	1	1				5	0	−4
22		394	84	1	17	20	18			
# 23	−2	48	10	1	2	2	0			
26	1	10	2	−2				2	2	2
# 29	4	5	1	−1				5	0	4
30	5	22	4	2				1	6	2
# 31		3040	546	1	98	2	0			
33	−3	46	8	1	1	13	5			
34	−2	70	12	2	2	2	0			
35	−1	12	2	2				3	0	−1
# 37	1	12	2	−1				3	0	1
38	2	74	12	1	1	36	34			
39	3	50	8	2	1	11	3			
# 41		64	10	−1	1	23	13			
42	6	26	4	2				1	10	6
# 43		6964	1062	1	161	41	39			
46		48670	7176	1	1058	2	0			

D	r	t_D	u_D	h_D	m_D	a_D	b_D	n_D	v_D	w_D
# 47	−2	96	14	1	2	2	0			
51	2	100	14	2	1	49	47			
# 53	4	7	1	−1				7	0	4
55		178	24	2	3	13	3			
57		302	40	1	5	17	5			
58		198	26	−2	3	24	10			
# 59		1060	138	1	17	57	55			
# 61		39	5	−1				1	14	8
62	−2	126	16	1	2	2	0			
65	1	16	2	−2				4	0	1
66	2	130	16	2	1	64	62			
# 67		97684	11934	1	1457	65	63			
69		25	3	1				2	7	5
70		502	60	2	7	12	2			
# 71		6960	826	1	98	2	0			
# 73		2136	250	−1	29	19	5			
74		86	10	−2	1	12	2			
77	−4	9	1	1				9	0	−4
78	−3	106	12	2	1	28	10			
# 79	−2	160	18	3	2	2	0			
82	1	18	2	−4				4	2	2
# 83	2	164	18	1	1	81	79			
85	4	9	1	−2				9	0	4
86		20810	2244	1	241	84	82			
87	6	56	6	2				1	20	11
# 89		1000	106	−1	11	21	5			
91		3148	330	2	34	54	32			
93		29	3	1				3	2	0
94		4286590	442128	1	45602	2	0			
95	−5	78	8	2				1	14	3
# 97		11208	1138	−1	115	53	29			
# 101	1	20	2	−1				5	0	1
102	2	202	20	2	1	100	98			
# 103		455056	44838	1	4418	2	0			
105	5	82	8	2				1	18	5
106		8010	778	−2	75	60	34			
# 107		1924	186	1	17	105	103			
# 109		261	25	−1	2	43	17			
110	10	42	4	2				2	10	6
111		590	56	2	5	35	11			
# 113		1552	146	−1	13	83	61			

D	r	t_D	u_D	h_D	m_D	a_D	b_D	n_D	v_D	w_D
114		2050	192	2	17	112	110			
115		2252	210	2	19	67	39			
118		613834	56508	1	5201	116	114			
119	− 2	240	22	2	2	2	0			
122	1	22	2	− 2				5	2	2
123	2	244	22	2	1	121	119			
# 127		9461248	839550	1	74498	2	0			
129		33710	2968	1	261	41	13			
130		114	10	− 4				1	14	2
# 131		21220	1854	1	161	129	127			
133		173	15	1	1	40	12			
134		291850	25212	1	2177	132	130			
# 137		3488	298	− 1	25	63	29			
138	− 6	94	8	2				1	30	14
# 139		155126500	13157658	1	1116017	137	135			
141	− 3	190	16	1	1	49	17			
142	− 2	286	24	3	2	2	0			
143	− 1	24	2	2				6	0	− 1
145	1	24	2	− 4				6	0	1
146	2	290	24	2	1	144	142			
# 149		61	5	− 1				2	11	5
# 151		3456296080	281269386	1	22889378	2	0			
154		42590	3432	2	276	86	48			
155		498	40	2	3	33	7			
# 157		213	17	− 1	1	56	20			
158		15486	1232	1	98	2	0			
159		2648	210	2	16	104	68			
161		23550	1856	1	146	44	12			
# 163		128160052	10038270	1	786257	161	159			
165	− 4	13	1	2				13	0	− 4
166	− 3	*		1						
# 167	− 2	336	26	1	2	2	0			
170	1	26	2	− 4				6	2	2
# 173	4	13	1	− 1				13	0	4
174		2902	220	2	16	118	80			
177		124846	9384	1	705	61	21			
178		3202	240	2	17	176	174			
# 179		8380420	626382	1	46817	177	175			
# 181		1305	97	− 1	7	38	8			
182	13	54	4	2				3	6	2
183		974	72	2	5	59	19			

D	r	t_D	u_D	h_D	m_D	a_D	b_D	n_D	v_D	w_D
185		136	10	−2				1	36	13
186		15002	1100	2	80	122	80			
187		3364	246	2	17	185	183			
190		104042	7548	2	547	112	66			
# 191		17988000	1301566	1	94178	2	0			
# 193		3528264	253970	−1	18281	31	5			
194	−2	390	28	2	2	2	0			
195	−1	28	2	4				7	0	−1
# 197	1	28	2	−1				7	0	1
# 199		32532393040	2306160198	1	*					
201		1030190	72664	1	5125	65	21			
202		6282	442	−2	31	20	2			
203	7	114	8	2				1	50	39
205		43	3	2				4	7	5
206		119070	8296	1	578	2	0			
209		93102	6440	1	445	97	45			
210	14	58	4	4				3	10	6
# 211		556708747300	*	1	*	209	207			
213	−12	73	5	1				2	23	21
214		*		1						
215	−10	88	6	2				2	16	7
217		7688126	521904	1	35429	33	5			
218		502	34	−2	2	66	20			
219	−6	148	10	4				1	48	23
221	−4	15	1	2				15	0	−4
222	−3	298	20	2	1	76	26			
# 223	−2	448	30	3	2	2	0			
226	1	30	2	−8				7	2	2
# 227	2	452	30	1	1	225	223			
# 229	4	15	1	−3				15	0	4
230	5	182	12	2				1	38	10
231	6	152	10	4				1	52	27
# 233		46312	3034	−1	198	178	136			
235	10	92	6	6				2	20	11
237	12	77	5	1				3	2	0
238		23326	1512	2	98	2	0			
# 239		12390240	801458	1	51842	2	0			
# 241		142022136	9148450	−1	589303	113	53			
246		177610	11324	2	721	244	242			
247		170584	10854	2	690	154	96			
# 251		7349780	463914	2	29281	249	247			

D	r	t_D	u_D	h_D	m_D	a_D	b_D	n_D	v_D	w_D
253		1861	117	1	7	90	32			
254	−2	510	32	3	2	2	0			
255	−1	32	2	4				8	0	−1
# 257	1	32	2	−3				8	0	1
258	2	514	32	2	1	256	254			
259		1694450	105288	2	6542	72	20			
262		209961034	12971436	1	801377	260	258			
# 263		278256	17158	1	1058	2	0			
265		12144	746	−2	45	219	181			
266		1370	84	2	5	40	6			
267		4804	294	2	17	265	263			
# 269		164	10	−1				1	64	41
# 271		*		1						
273		1454	88	2	5	89	29			
274		2814	170	−4	10	74	20			
# 277		2613	157	−1	9	120	52			
278		5002	300	1	17	276	274			
# 281		2127064	126890	−1	7569	175	109			
282		4702	280	2	16	190	128			
# 283		276548164	16439082	1	977201	281	279			
285	−4	17	1	2				17	0	−4
286		1123670	66444	2	3928	262	240			
287	−2	576	34	2	2	2	0			
290	1	34	2	−4				8	2	2
291	2	580	34	4	1	289	287			
# 293	4	17	1	−1				17	0	4
295		4049998	235800	2	13728	238	192			
298		819114	47450	−2	2748	210	148			
299		830	48	2	2	232	180			
301		22745	1311	1	75	170	96			
302		8553246	492184	1	28322	2	0			
303		5048	290	2	16	200	132			
305		978	56	2	3	63	13			
# 307		177058564	10105266	1	576737	305	303			
309		5045	287	1	16	101	33			
310		1697438	96408	2	5475	188	114			
# 311		33767760	1914794	1	108578	2	0			
# 313		253724736	14341370	−1	810622	50	8			
314		886	50	−2	2	258	212			
# 317		89	5	−1				3	14	8
318	−6	214	12	2				1	70	34

D	r	t_D	u_D	h_D	m_D	a_D	b_D	n_D	v_D	w_D
319		25803560	1444722	2	80888	288	260			
321	−3	430	24	3	1	109	37			
322	−2	646	36	4	2	2	0			
323	−1	36	2	4				9	0	−1
326	2	650	36	3	1	324	322			
327	3	434	24	2	1	107	35			
329		4752830	262032	1	14446	96	28			
330	6	218	12	4				1	74	38
# 331		*		1						
334		*		1						
335		1208	66	2	3	203	123			
# 337		2031654672	70671282	−1	*					
339		195940	10642	2	577	337	335			
341		277	15	1				1	52	12
345		13522	728	2	39	67	13			
346		186	10	−6				1	86	74
# 347		1283204	68886	1	3697	345	343			
# 349		18420	986	−1	52	272	212			
# 353		142528	7586	−1	403	269	205			
354		516130	27432	2	1457	352	350			
355		1909618	101352	2	5379	73	15			
357	−4	19	1	2				19	0	−4
358		*		1						
# 359	−2	720	38	3	2	2	0			
362	1	38	2	−2				9	2	2
365	4	19	1	−2				19	0	4
366		1815850	94916	2	4961	124	42			
# 367		*		1						
370		654	34	−4	1	284	218			
371		3390	176	2	9	51	7			
# 373		10236	530	−1	27	165	73			
374		6730	348	2	17	372	370			
377		466	24	2	1	89	21			
# 379		*		1						
381		2030	104	1	5	125	41			
382		*		1						
# 383		37536	1918	1	98	2	0			
385		191662	9768	2	497	317	261			
386		223110	11356	2	578	2	0			
# 389		2564	130	−1	6	230	136			
390	−10	158	8	4				2	30	14

D	r	t_D	u_D	h_D	m_D	a_D	b_D	n_D	v_D	w_D
391		14677360	14677360	2	37538	2	0			
393		92874286	4684888	1	236321	133	45			
394		790046070	39801946	− 2	2005193	28	2			
395	− 5	318	16	2				1	62	15
# 397		3447	173	− 1	8	271	185			
398	− 2	798	40	1	2	2	0			
399	− 1	40	2	8				10	0	− 1
# 401	1	40	2	− 5				10	0	1
402	2	802	40	2	1	400	398			
403		1339756	66738	2	3324	184	84			
406		118936190	5902704	2	292946	114	32			
407		5326	264	2	13	35	3			
# 409		*		− 1						
410		162	8	4				2	34	18
411		99460	4906	2	241	409	407			
413		61	3	1				6	7	5
415		36825608	1807698	2	88736	168	68			
417		170645294	8356536	1	409221	137	45			
418		67714	3312	2	161	416	414			
# 419		540349940	26397822	1	1289617	417	415			
# 421		444939	21685	− 1	1056	363	313			
422		14045002	683700	1	33281	420	418			
426		177502	8600	2	416	286	192			
427	− 14	124	6	6				3	16	7
429	− 12	145	7	2				2	47	45
430		5724502	276060	2	13312	342	272			
# 431		303121440	14600846	1	703298	2	0			
# 433		*		− 1						
434	− 7	250	12	4				1	106	78
435	− 6	292	14	4				1	96	47
437	− 4	21	1	1				21	0	− 4
438	− 3	586	28	4	1	148	50			
# 439	− 2	880	42	5	2	2	0			
442	1	42	2	− 8				10	2	2
# 443	2	884	42	3	1	441	439			
445	4	21	1	− 4				21	0	4
446		220332030	10433024	1	494018	2	0			
447	6	296	14	2				1	100	51
# 449		378942664	17883410	− 1	843970	134	40			
451		92942980	4376514	2	206081	449	447			
453	12	149	7	1				3	2	0

D	r	t_D	u_D	h_D	m_D	a_D	b_D	n_D	v_D	w_D
454		*		1						
455	14	128	6	4				3	20	11
# 457		*		−1						
458		214	10	−2				2	14	2
# 461		365	17	−1				1	76	20
462	21	86	4	4				5	6	2
# 463		*		1						
465		31742	1472	2	68	122	32			
466		1876638850	86933616	2	*					
# 467		3251252	450450	1	6961	465	463			
469		65	3	3				7	2	0
470		3382	156	2	7	92	18			
471		15677390	722376	2	33285	155	51			
473	−11	174	8	3				2	46	33
474		387098	17780	2	816	314	208			
478		*		1						
# 479		5978880	273182	1	12482	2	0			
481		1928280	87922	−2	4008	432	388			
482	−2	966	44	2	2	2	0			
483	−1	44	2	4				11	0	−1
485	1	44	2	−2				11	0	1
# 487		*		1						
489		*		1						
# 491		*		1						
493		111	5	−2				4	11	5
494		146070	6572	2	295	340	234			
497		2403774	197824	1	4836	282	160			
498		359554	16112	2	721	496	494			
# 499		8980	402	5	17	497	495			

# indicates prime number.

$h_D = -n$ means that $N\varepsilon_D = -1$ and $h_D = n$.

r representes the integer such that $D = k^2 + r$, $-k < r \leqq k$ and $4k \equiv 0 \pmod r$ for real quadratic field $\mathbf{Q}(\sqrt{D})$ of R-D type.

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