

***P*-HARMONIC DIMENSIONS ON ENDS**

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Dedicated to Professor Masanori Kishi on his sixtieth birthday

Consider an *end* Ω in the sense of Heins (cf. Heins [3]): Ω is a relatively non-compact subregion of an open Riemann surface such that the relative boundary $\partial\Omega$ consists of finitely many analytic Jordan closed curves, there exist no non-constant bounded harmonic functions with vanishing boundary values on $\partial\Omega$ and Ω has a single ideal boundary component. A *density* $P = P(z) dx dy$ ($z = x + iy$) is a 2-form on $\Omega \cup \partial\Omega$ with nonnegative locally Hölder continuous coefficient $P(z)$. Denote by $\mathcal{P}_P(\Omega)$ the class of nonnegative solutions of the equation

$$(1) \quad L_P u \equiv \Delta u - Pu = 0 \quad (\text{i.e. } d * du - uP = 0)$$

on Ω with vanishing boundary values on $\partial\Omega$. The *P-harmonic dimension* of Ω (or the *elliptic dimension* of P on Ω (cf. e.g. Nakai [8])), $\dim \mathcal{P}_P(\Omega)$ in notation, is defined to be the ‘dimension’ of the half module $\mathcal{P}_P(\Omega)$. The *P-harmonic dimension* $\dim \mathcal{P}_0(\Omega)$ for the particular $P \equiv 0$ is called simply the *harmonic dimension* of Ω (cf. Heins [3]).

We are particularly interested in the following result by Heins [3]:

THEOREM A. *Let $\{A_n\}$ be a sequence of mutually disjoint annuli in Ω satisfying that A_{n+1} separates A_n from the ideal boundary of Ω for every n . Suppose that the sum of moduli of A_n diverges. Then the harmonic dimension of Ω is one.*

A density P is said to be *finite* if $\int \int_{\Omega} P dx dy < \infty$. The above theorem has been generalized for finite densities P by Nakai [8] and Kawamura [4] as follows:

THEOREM B. *Let P be finite on Ω and $\{A_n\}$ be the same as in Theorem A. Then*

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the P -harmonic dimension of Ω is one.

The following is another generalization of Theorem A (cf. Segawa [9]):

THEOREM C. *Let $\{A_n\}$ be a sequence of mutually disjoint sets in Ω such that each A_n consists of at most N mutually disjoint annuli for a positive integer N and A_{n+1} separates A_n from the ideal boundary of Ω for every n . Suppose that the sum of moduli of A_n diverges. Then the harmonic dimension of Ω is at most N .*

The main purpose of this paper is to unify Theorems B and C to a form including both Theorems B and C as special cases. The main theorem is as follows:

MAIN THEOREM. *Let P be finite on Ω and $\{A_n\}$ be the same as in Theorem C. Then the P -harmonic dimension of Ω is at most N .*

We shall prove a bit more in Theorem 6. In Section 1, we prove a duality relation for P -harmonic dimensions (cf. Theorem 2), which plays a fundamental role for the proof of Theorem 6.

§1. Duality relation

1.1. A relatively noncompact subregion Ω of an open Riemann surface is referred to as a *general end* if the relative boundary $\partial\Omega$ of Ω consists of a finitely many disjoint analytic Jordan closed curves. In this section we assume that Ω is a general end. We denote by β the ideal boundary of Ω . Without loss of generality, we may assume that there exist an open Riemann surface R and its exhaustion $\{R_n\}_{n=0}^\infty$ with $\Omega = R - \bar{R}_0$. Let $e_p^{(n)}$ be the solution of the equation (1) on $\Omega \cap R_n = R_n - \bar{R}_0$ with boundary values 1 on $\partial\Omega$ and 0 on ∂R_n . Since $\{e_p^{(n)}\}$ is increasing and dominated by the constant function 1, the limit $e_p = \lim_{n \rightarrow \infty} e_p^{(n)}$ exists. Note that e_p is the solution of (1) on Ω with boundary values 1 on $\partial\Omega$ and 0 on the ideal boundary β' . The function e_p is referred to as the P -unit on Ω for P (cf. Nakai [7]). Obviously e_p does not depend on a choice of $\{R_n\}_{n=0}^\infty$. We consider the associated operator \hat{L}_P with L_P which is introduced by Nakai (cf. [7], [8]):

$$(2) \quad \hat{L}_P u \equiv \Delta u + 2\nabla(\log e_p) \cdot \nabla u$$

where e_p is the P -unit on Ω . Denote by $B_P(\Omega)$ the class of *bounded* solutions of the equation

$$(3) \quad \hat{L}_p u = 0$$

on Ω with continuous boundary values on $\partial\Omega$. Note that $1 \in B_p(\Omega)$. To begin with we show the following

LEMMA 1. *Suppose that u belongs to $B_p(\Omega)$. Then u satisfies the following inequalities:*

$$\min_{p \in \partial\Omega} u(p) \leq \inf_{p \in \Omega} u(p) \leq \sup_{p \in \Omega} u(p) \leq \max_{p \in \partial\Omega} u(p).$$

Proof. We have only to show the last inequality since $-u$ also belongs to $B_p(\Omega)$. By adding a constant we may assume $u \geq 0$. Set $M = \max_{p \in \partial\Omega} u(p)$. Let v_n be the solution of (1) on $\Omega \cap R_n$ with boundary values u on $\partial\Omega$ and 0 on ∂R_n . Since $\{v_n\}$ is increasing and $v_n \leq M e_p$ ($n = 1, 2, \dots$), $v = \lim_{n \rightarrow \infty} v_n$ exists and is a solution of (1). It is clear that $v/e_p \leq M$ on Ω . Thus we complete the proof if we show that $u = v/e_p$, or $u e_p = v$.

Note that $L_p(u e_p) = 0$ and $0 \leq v \leq u e_p$ on Ω . There exists a constant $C > 0$ such that $0 \leq u \leq C$, i.e. $0 \leq u e_p \leq C e_p$. Let w_n be the solution of (1) on $\Omega \cap R_n$ with boundary values 0 on $\partial\Omega$ and e_p on ∂R_n . Then $w_n = e_p - e_p^{(n)}$. By the minimum principle, $0 \leq u e_p - v \leq C w_n$ on $\Omega \cap R_n$. Since $\lim_{n \rightarrow \infty} e_p^{(n)} = e_p$, $\lim_{n \rightarrow \infty} w_n = 0$ and therefore $u e_p = v$. \square

1.2. Let $B_p^0(\Omega)$ be the subspace of $B_p(\Omega)$ which consists of functions with the limit 0 at β :

$$B_p^0(\Omega) = \{u \in B_p(\Omega) : \lim_{p \rightarrow \beta} u(p) = 0\}.$$

Next consider the quotient space

$$\mathfrak{B}_p(\Omega) = B_p(\Omega) / B_p^0(\Omega)$$

and denote by $\dim \mathfrak{B}_p(\Omega)$ the dimension of the linear space $\mathfrak{B}_p(\Omega)$. Our first achievement of this paper is the following duality relation for $\mathcal{P}_p(\Omega)$ and $\mathfrak{B}_p(\Omega)$ (cf. Segawa [9]):

THEOREM 2. *If either $\mathcal{P}_p(\Omega)$ or $\mathfrak{B}_p(\Omega)$ is of finite dimension, then the P-harmonic dimension $\dim \mathcal{P}_p(\Omega)$ coincides with $\dim \mathfrak{B}_p(\Omega)$:*

$$\dim \mathcal{P}_p(\Omega) = \dim \mathfrak{B}_p(\Omega).$$

The proof of the above theorem is given in no.1.4. By the definition of

$\mathcal{B}_p(\Omega)$ and the fact $1 \in \mathcal{B}_p(\Omega)$, $\dim \mathcal{B}_p(\Omega) = 1$ is equivalent that $\lim_{p \rightarrow \beta} u(p)$ exists for every u in $B_p(\Omega)$. Therefore Theorem 2 implies the following, which was originally obtained by Hayashi [3] (cf. Nakai [7]):

COROLLARY 3. *The P -harmonic dimension $\dim \mathcal{P}_p(\Omega)$ is one if and only if there exists $\lim_{p \rightarrow \beta} u(p)$ for every u in $B_p(\Omega)$.*

1.3. Consider the linear space \mathcal{E} generated by $\mathcal{P}_p(\Omega)$, i.e.

$$\mathcal{E} = \{h_1 - h_2 : h_1, h_2 \in \mathcal{P}_p(\Omega)\},$$

and the bilinear functional

$$(u, h) \mapsto \langle u, h \rangle = - \int_{\partial\Omega} u * dh = \int_{\partial\Omega} u \frac{\partial h}{\partial n} ds$$

defined on $B_p(\Omega) \times \mathcal{E}$ where $\partial/\partial n$ is the inner normal derivative. Let $g_n(\cdot, p)$ be the Green's function of (1) on $\Omega \cap R_n$ with pole at p for each $n \in \mathbf{N}$, the set of positive integers. Note that $g_n(\cdot, p)$ converges to the Green's function $g(\cdot, p)$ of (1) on Ω with pole at p uniformly on each compact subset in $\Omega \cup \partial\Omega$. Set

$$\mathcal{Q} = \{h \in \mathcal{P}_p(\Omega) : \langle 1, h \rangle = 1\}.$$

We maintain

LEMMA 4. *If $u \in B_p(\Omega)$, then*

$$\limsup_{p \rightarrow \beta} u(p) = \sup \langle u, \mathcal{Q} \rangle$$

and

$$\liminf_{p \rightarrow \beta} u(p) = \inf \langle u, \mathcal{Q} \rangle$$

where $\langle u, \mathcal{Q} \rangle = \{\langle u, h \rangle : h \in \mathcal{Q}\}$.

Proof. We first show that

$$(4) \quad u(p)e_p(p) = - \frac{1}{2\pi} \int_{\partial\Omega} u * dg(\cdot, p) \quad (p \in \Omega)$$

for every $u \in B_p(\Omega)$. Suppose that $p \in \Omega \cap R_n$. Let u_n be the solution of (3) on $\Omega \cap R_n$ with boundary values u on $\partial\Omega$ and 0 on ∂R_n . By Lemma 1, u_n converges to u uniformly on each compact subset in $\Omega \cup \partial\Omega$. Observe that $u_n e_p$ is the solu-

tion of (1) on $\Omega \cap R_n$ with boundary values u on $\partial\Omega$ and 0 on ∂R_n . Hence the Green's formula yields that $u_n(p)e_p(p) = - (1/2\pi) \int_{\partial\Omega} u * dg_n(\cdot, p)$. By letting $n \rightarrow \infty$, we have (4).

Take an arbitrary cluster value a of u at β and a sequence $\{p_n\}$ with $\lim_{n \rightarrow \infty} p_n = \beta$ and $\lim_{n \rightarrow \infty} u(p_n) = a$. Applying (4) to $1 \in B_p(\Omega)$, we see that $e_p(p_n) = - (1/2\pi) \int_{\partial\Omega} u * dg(\cdot, p_n)$, i.e.

$$- \int_{\partial\Omega} * d \frac{g(\cdot, p_n)}{2\pi e_p(p_n)} = 1.$$

From this it follows that a suitable subsequence of $\{(1/2\pi) g(\cdot, p_n)/e_p(p_n)\}$ converges to a function G , which belongs to Q , uniformly on each compact subset of $\Omega \cup \partial\Omega$. By (4) we also have

$$u(p_n) = - \int_{\partial\Omega} u * d \frac{g(\cdot, p_n)}{2\pi e_p(p_n)}.$$

Therefore we conclude that

$$a = - \int_{\partial\Omega} u * dG,$$

i.e. $a \in \langle u, Q \rangle$, which implies

$$\inf_{p \rightarrow \beta} \langle u, Q \rangle \leq \liminf_{p \rightarrow \beta} u(p) \leq \limsup_{p \rightarrow \beta} u(p) \leq \sup \langle u, Q \rangle.$$

Next we show that

$$(5) \quad \liminf_{p \rightarrow \beta} u(p) \leq \inf \langle u, Q \rangle \leq \sup \langle u, Q \rangle \leq \limsup_{p \rightarrow \beta} u(p).$$

Suppose that $h \in Q$ and $u \in B_p(\Omega)$. Let h_{nm} be the solution of (1) on $R_m - \bar{R}_n$ ($m > n$) with boundary values h on ∂R_n and 0 on ∂R_m . The Green's formula yields that

$$\begin{aligned} \langle u, h \rangle &= - \int_{\partial\Omega} u * dh = - \int_{\partial\Omega} u_m e_p * dh = \int_{\partial R_n} u_m e_p * dh - h * d(u_m e_p) \\ &= \int_{\partial R_n} u_m e_p * dh - h_{nm} * d(u_m e_p) \end{aligned}$$

and

$$\int_{\partial R_n} u_m e_p * dh_{nm} - h_{nm} * d(u_m e_p) = \int_{\partial R_m} u_m e_p * dh_{nm} - h_{nm} * d(u_m e_p) = 0$$

where u_m is defined at the beginning of the proof. Therefore, by letting $m \rightarrow \infty$, we have

$$(6) \quad \langle u, h \rangle = \int_{\partial R_n} u e_p * d(h - h_n)$$

where $h_n = \lim_{m \rightarrow \infty} h_{nm}$. Applying (6) to $u = 1$, we also have

$$(7) \quad \int_{\partial R_n} e_p * d(h - h_n) = \langle 1, h \rangle = - \int_{\partial \Omega} * dh = 1.$$

Hence (6) and (7) imply that

$$\inf_{p \in \partial R_n} u(p) \leq \inf \langle u, Q \rangle \leq \sup \langle u, Q \rangle \leq \sup_{p \in \partial R_n} u(p).$$

Thus (5) follows from the above.

The proof is herewith complete. \square

1.4. Proof of Theorem 2. By definition, the dimension $\dim \mathcal{E}$ of the linear space \mathcal{E} coincides with $\dim \mathcal{P}_p(\Omega)$.

Consider the \mathcal{E} -kernel ($B_p(\Omega)$ -kernel resp.)

$$K_1 = \bigcap_{h \in \mathcal{E}} \{u \in B_p(\Omega) : \langle u, h \rangle = 0\}$$

$$(K_2 = \bigcap_{u \in B_p(\Omega)} \{h \in \mathcal{E} : \langle u, h \rangle = 0\} \text{ resp.})$$

of the bilinear functional $(u, h) \mapsto \langle u, h \rangle$. By virtue of Lemma 4, it is easily seen that $K_1 = B_p^0(\Omega)$, and hence $\mathcal{B}_p(\Omega) = B_p(\Omega)/K_1$. Since $\{u|_{\partial \Omega} : u \in B_p(\Omega)\} = C(\partial \Omega)$, it follows from $h \in K_2$ that $\partial h / \partial n \equiv 0$ on $\partial \Omega$. Combining this with the fact $h \equiv 0$ on $\partial \Omega$, we have $K_2 = \{0\}$ (cf. e.g. Miranda [6]). Therefore we can consider $\mathcal{B}_p(\Omega) = B_p(\Omega)/K_1$ ($\mathcal{E} = \mathcal{E}/K_2$ resp.) to be a subspace of \mathcal{E}^* ($\mathcal{B}_p(\Omega)^*$ resp.) where we denote by X^* the conjugate space of a linear space X . In particular we have

$$\dim \mathcal{B}_p(\Omega) \leq \dim \mathcal{E}^*$$

and

$$\dim \mathcal{E} \leq \dim \mathcal{B}_p(\Omega)^*.$$

Hence we have

$$\dim \mathcal{B}_p(\Omega) = \dim \mathcal{E} = \dim \mathcal{P}_p(\Omega),$$

since linear spaces of finite dimension are isomorphic to their conjugate spaces. \square

§2. Proof of Main Theorem

2.1. In this section, we give a proof of Main Theorem in terms of extremal length.

Hereafter we assume that Ω is a *parabolic* end: i.e. there exist no non-constant bounded harmonic functions on Ω with vanishing boundary values on $\partial\Omega$. The following was proved by Nakai [8] essentially:

PROPOSITION 5. *If Ω is parabolic and P is finite on Ω , then every bounded solution of (3) on Ω has finite Dirichlet integral on $\Omega - R_1$.*

For the proof we refer to Nakai [8] and Kawamura [4].

2.2. We denote by $\lambda(\Gamma)$ the extremal length of a curve family Γ in Ω . For the definition and details of extremal length we refer to e.g. Ahlfors and Sario [1]. For every positive integer n , let $\Gamma_n(\Omega)$ be the totality of 1-cycles γ in Ω such that γ consists of at most n closed curves and separates $\partial\Omega$ from the ideal boundary β . The following is the main achievement of this paper (cf. Shiga [10]):

THEOREM 6. *Suppose that P is finite on Ω . If the extremal length $\lambda(\Gamma_N(\Omega))$ is zero for an $N \in \mathbf{N}$, then the P -harmonic dimension $\dim \mathcal{P}_p(\Omega)$ is at most N .*

Proof. Set $\Gamma_1 = \Gamma_1(\Omega)$ and $\Gamma_n = \Gamma_n(\Omega) - \Gamma_{n-1}(\Omega)$ ($n = 2, 3, \dots$). Since $\Gamma_N(\Omega) = \cup_{n=1}^N \Gamma_n$, there exists a $\nu \in \mathbf{N}$ such that $\nu \leq N$ and $\lambda(\Gamma_\nu) = 0$. We shall show that $\dim \mathcal{P}_p(\Omega)$ is at most ν . Take arbitrary $\nu + 1$ functions $u_1, \dots, u_{\nu+1}$ in $\mathcal{B}_p(\Omega)$. By virtue of Theorem 2, we have only to show that a nonzero linear combination $c_1 u_1 + \dots + c_{\nu+1} u_{\nu+1}$ of $u_1, \dots, u_{\nu+1}$ belongs to $\mathcal{B}_p^0(\Omega)$.

Consider the ‘density’ ρ on Ω such that

$$\rho |dz| = \begin{cases} \sum_{i=1}^{\nu+1} |\nabla u_i| |dz| & \text{on } \Omega - R_1 \\ 0 & \text{on } \Omega \cap R_1. \end{cases}$$

By Schwarz’ inequality and Proposition 5, we have

$$(8) \quad \int \int_{\Omega} \rho^2 dx dy \leq (\nu + 1) \sum_{i=1}^{\nu+1} D_{\Omega-R_1}(u_i) < \infty,$$

where $D_{\Omega-R_1}(u_i) = \int \int_{\Omega-R_1} |\nabla u_i|^2 dx dy$. Set $\Gamma_\nu^m = \{\gamma \in \Gamma_\nu : \gamma \cap R_m = \emptyset\}$. By means of $\lambda(\Gamma_\nu) = 0$, we have $\lambda(\Gamma_\nu^m) = 0$ for every $m \in \mathbf{N}$ (cf. Kusunoki [5]). Therefore, because of (8), we can find a sequence $\{\gamma_n\}$ in Γ_ν such that γ_n converges to the ideal boundary β and $\lim_{n \rightarrow \infty} \int_{\gamma_n} \rho |dz| = 0$. In particular, we obtain

$$(9) \quad \lim_{n \rightarrow \infty} \int_{\gamma_n} |\nabla u_i| |dz| = 0 \quad (i = 1, \dots, \nu + 1).$$

By definition, every γ_n consists of exactly ν closed curves $\gamma_{n1}, \dots, \gamma_{n\nu}$. Accordingly (9) implies that there exist a subsequence $\{\gamma_{nk}\}$ of $\{\gamma_n\}$ and vectors $\mathbf{v}_i = (a_{i1}, \dots, a_{i\nu}) \in \mathbf{R}^\nu$ ($i = 1, \dots, \nu + 1$) such that

$$(10) \quad \lim_{k \rightarrow \infty} \max_{p \in \gamma_{nkj}} |u_i(p) - a_{ij}| = 0 \quad (j = 1, \dots, \nu).$$

Evidently we can find $(c_1, \dots, c_{\nu+1}) \in \mathbf{R}^{\nu+1} - \{(0, \dots, 0)\}$ such that $\sum_{i=1}^{\nu+1} c_i \mathbf{v}_i = (0, \dots, 0)$. Therefore, (10) yields that

$$(11) \quad \lim_{k \rightarrow \infty} \max_{p \in \gamma_{nk}} \left| \sum_{i=1}^{\nu+1} c_i u_i(p) \right| = 0.$$

Since each γ_n separates $\partial\Omega$ from the ideal boundary β , it follows from Lemma 1 and (11) that $\lim_{p \rightarrow \beta} \sum_{i=1}^{\nu+1} c_i u_i = 0$. This completes the proof. \square

2.3. Proof of Main Theorem. Main Theorem is easily verified from Theorem 6 as follows. Assume that $\{A_n\}$ is the same as in Theorem C. Set $A_n = \cup_{j=1}^{\nu(n)} A_{nj}$ where A_{nj} 's are mutually disjoint annuli and $\nu(n) \leq N$. Let Λ_n be the totality of 1-cycles γ in A_n such that $\gamma = \cup_{j=1}^{\nu(n)} \gamma_{nj}$ where each γ_{nj} is a closed curve in A_{nj} and separates two boundary components of A_{nj} . Set $\Gamma = \cup_{n=1}^{\infty} \Lambda_n$. Note that $\Gamma \subset \Gamma_N(\Omega)$. By virtue of Theorem 6, we have only to show that $\lambda(\Gamma) = 0$.

It is well-known that $\lambda(A_n) = 2\pi / \text{mod } A_n$, where $\text{mod } A_n$ is the modulus of A_n (cf. Ahlfors and Sario [1]). Since A_n 's are mutually disjoint, we see $\lambda(\Gamma)^{-1} \geq \sum_{n=1}^{\infty} \lambda(A_n)^{-1}$. Hence, from the assumption $\sum_{n=1}^{\infty} \text{mod } A_n = \infty$ it follows that $\lambda(\Gamma) = 0$. \square

REFERENCES

- [1] L. Ahlfors and L. Sario, Riemann Surfaces, Princeton, 1960.
- [2] K. Hayashi, Les solutions positives de l'equation $\Delta u = Pu$ sur une surface de Riemann, Kōdai Math. Sem. Rep., **13** (1961), 20–24.
- [3] M. Heins, Riemann surfaces of infinite genus, Ann. of Math., **55**(1952), 296–317.
- [4] M. Kawamura, Picard principle for finite densities on Riemann surfaces, Nagoya Math. J., **67** (1977), 35–40.
- [5] Y. Kusunoki, On Riemann's period relations on open Riemann surfaces, Mem. Coll. Sci. Kyoto Univ. Ser. A Math., **30** (1956), 1–22.
- [6] C. Miranda, Partial Differential Equations of Elliptic Type, Springer, 1970.
- [7] M. Nakai, Picard principle and Riemann theorem, Tôhoku Math. J., **28** (1976), 277–292.
- [8] M. Nakai, Picard principle for finite densities, Nagoya Math. J., **70** (1978), 7–24.
- [9] S. Segawa, A duality relation for harmonic dimensions and its applications, Kodai Math., J., **4** (1981), 508–514.
- [10] H. Shiga, On harmonic dimensions and bilinear relations on open Riemann surfaces, J. Math. Kyoto Univ., **21** (1981), 861–879.

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