A. Iliev
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ALGEBRAIC THREEFOLDS WITH TWO EXTREMAL MORPHISMS

ATANAS ILIEV

§0. Introduction

0.1. In [3] Mori gives a description of all extremal rays (extremal morphisms) arising on a smooth projective threefold with a numerically non-effective canonical bundle. Generally speaking, every smooth projective threefold V with a numerically non-effective canonical class K_V admits an extremal morphism $\pi : V \to Y$. The assumption that V admits a non-trivial pair of extremal morphisms

$$Y_1 \xleftarrow{\pi_1} V \xrightarrow{\pi_2} Y_2$$

imposes strong conditions on V. This is the essence of the Theorem 1.5 of the present work. In particular, we obtain a description of the threefolds which admit two biregular structures of conic bundles over non-singular surfaces $S_1 = Y_1$ and $S_2 = Y_2$. By the results of §3 the surfaces S_1 and S_2 must be either ruled surfaces with isomorphic basic curves, or $S_1 \simeq S_2 \simeq P^2$.

0.2. Remarks

0.2.1. In [5] E. Sato has obtained a description of the threefolds with two structures of P^1-bundles; this description corresponds to the Case A.a of Theorem 1.5. The second basic result of [5] states that *if* $\dim V \geqslant 3$ and V admits two structures of projective space bundles over projective spaces $Y_1 = P^l$ and $Y_2 = P^m$, *then*: either V is a product $V = P^l \times P^m$, or $l = m$ and $V = P(T_{P^l})$.

0.2.2. Every Fano threefold V with $\rho(V) \geqslant 2$ admits at least two extremal morphisms. However, in most of the cases V admits a ray of the type E_1. Because of that, there are too many Fano threefolds with $\rho \geqslant 2$ in the list of Mori and Mukai in [4], in contrast to the list of Theorem 1.5 in which are classified only the strongly primitive ones.

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§1. Definitions and statement of the main theorem

1.1. Everywhere in the present article, we suppose that the threefold V is a smooth projective threefold over the field of complex numbers $\mathbf{C}$.

1.2. Definitions

1.2.1. $NV = \{1\text{-cycles on } V\}/\equiv \otimes \mathbf{R}$, where $\equiv$ denotes the numerical equivalence of cycles. NV is a finite dimensional real vector space, which is dual to $NS(V) \otimes \mathbf{R}$, where $NS(V)$ is the Neron-Severi group of V.

1.2.2. (the Picard number of V) $= \rho(V) = \dim_{\mathbf{R}}(NV)$.

1.2.3. $\overline{NE}(V)$ is the closure of the convex cone $NE(V)$, generated by all the effective 1-cycles from NV (in the metrical topology of the vector space NV).

1.2.4. Let $Z \in \overline{NE}(V)$. The half-line $R = \mathbf{R}_+[z]$ is called an extremal ray, if: (a) $-K_V.z > 0$; (b) for all $Z_1, Z_2 \in \overline{NE}(V)$, the assumption $Z_1 + Z_2 \in R$ implies $Z_1 \in R$ and $Z_2 \in R$, cf. [3].

1.2.5. Let R be an extremal ray on V. Then, there exists a unique, up to an isomorphism, morphism $\pi : V \to Y$ corresponding to R, such that: (a) $\pi_*\mathcal{O}_V = \mathcal{O}_Y$; (b) if $C \subset V$ is an irreducible curve, then $[C] \in R$ if and only if $\dim \pi(C) = 0$, cf. [3]. The morphism π is called a contraction of the extremal ray R, or an extremal morphism (corresponding to R).

1.3. Description of the extremal morphisms on V (cf. [3])

Let $\pi : V \to Y$ be an extremal morphism, and let $\rho(V) \geqslant 2$. Then π can be one of the following:

1.3.1. *Type* E : $\dim Y = 3$

The morphism π is a contraction of a divisor D on V, and π corresponds to one of the types E_1, E_2, E_3, E_4, and E_5. In the case E_1 the morphism π is a contraction of a ruled surface to a smooth curve, and the threefold Y is smooth. In the case E_2 the morphism π is a contraction of a divisor $D \simeq P^2$, with a normal bundle $\mathcal{O}_D(D) \simeq \mathcal{O}_{P^2}(-1)$, to a nonsingular point on Y. In the case E_3 the morphism π is a contraction of a quadric $D \simeq P^1 \times P^1$, with a normal bundle $\mathcal{O}(-1, -1)$, to an ordinary double point on Y. Moreover, the fibers $P^1 \times t$ and $s \times P^1$ are numerically equivalent on V, for $t, s \in P^1$. In the case E_4 the morphism π is a contraction of a quadratic cone $D \subset P^3$ to a double point on Y, and $\mathcal{O}_D(D) \simeq \mathcal{O}_D \otimes \mathcal{O}_{P^3}(-1)$. In the case E_5 the morphism π is a contraction of $D \simeq P^2$ to a quadruple point on Y, and $\mathcal{O}_D(D) \simeq \mathcal{O}_{P^2}(-2)$.

1.3.2. *Type* C : $\dim Y = 2$

The variety Y is a smooth projective surface, and π corresponds to one of the types C_1 or C_2. In the case C_1 the morphism π defines a conic bundle $\pi : V \to Y$; in the case C_2 the morphism π defines a P^1-bundle $\pi : V \to Y$.

1.3.3. *Type* D : $\dim Y = 1$

The variety Y is a smooth curve, $\rho(V) = \rho(Y) + 1 = 2$, and π corresponds to one of the types D_1, D_2, and D_3. In the case D_1 the threefold V has a structure of a Del Pezzo bundle over the curve Y. In the case D_2, V is isomorphic to a $P^1 \times P^1$-bundle over the curve Y. In the case D_3 the threefold V is a P^2-bundle over Y.

1.4. DEFINITION. *The threefold V is called strongly primitive if there are no extremal rays of type E_1 on V.*

1.5. The Main Theorem

THEOREM. *Let V be a (smooth, projective) strongly primitive threefold which admits two extremal morphisms $\pi_1 : V \to Y_1$ and $\pi_2 : V \to Y_2$. Then, the following cases are possible*:

Case 1. *The morphisms π_1 and π_2 correspond to the type C. Then* $2 \leqslant \rho(V) \leqslant 3$ *and*:

(1.A) *If $\rho(V) = 3$, then*

either: (A.a). *$V \simeq S_1 \times_C S_2$, where S_1 and S_2 are ruled surfaces over a curve C,*

or: (A.b). *V is a two-sheeted covering of $S_1 \times_C S_2$, where S_1, S_2, and C are as in* (A.a).

(1.B) *If $\rho(V) = 2$, then V is a Fano threefold (see Corollary* 2.6.2).

Case 2. *The morphism π_1 corresponds to the type C, and the morphism π_2 corresponds to one of the types D or E. Then V is a Fano threefold (see Corollary* 4.2, *Corollaries* 5.3 *and* 5.4).

Case 3. *Let the extremal morphisms $\pi_1, \pi_2, \ldots$ on V be of the type E. Then the corresponding divisors $D_1, D_2, \ldots$ are mutually disjoint (see* §7).

Remark. The rest of the paper is devoted to the proof of Theorem 1.5. Especially, Case 1 is discussed in §2 and §3. It follows from the considerations in §3 that the double covering $\pi : V \to S_1 \times_C S_2$, in case (A.b), has the following properties:

Let $\mathscr{E}_i$ be a normalized locally free sheaf of rank 2, over the base curve C, such that $S_i = P_C(\mathscr{E}_i)$, $i = 1,2$ (see [1, ch. V, §2]). Let $e_i = -\deg(\det \mathscr{E}_i)$, let φ_i be the general fibre of $S_i \to C$, and let b_i be the section of S_i such that $\mathscr{L}(b_i) = \mathscr{O}_{P(\mathscr{E}_i)}(1)$, $i = 1,2$. Let $p_i : S_1 \times_C S_2 \to S_i$ be the natural projections, and let $C_i = p_i^*(b_i)$, $F_i = p_i^*(\varphi_i)$, $i = 1,2$. Then F_1 and F_2 are numerically equivalent, i.e. $F_1 \equiv F_2 \equiv F$ for some $F \in p_1^*(\mathrm{Pic}\, S_1) \cap p_2^*(\mathrm{Pic}\, S_2)$. The branch divisor $B \subset S_1 \times_C S_2$ of π is smooth, and B is numerically equivalent to $2.C_1 + 2.C_2 + 2q.F$ for some $q > 0$. Moreover, the threefold V is a standard conic bundle over S_i with a discriminant curve $\Delta_i \equiv 4.b_i + (4q - 2e_i).\ \varphi_i$, where $\{i, j\} = \{1, 2\}$.

§2. The case (C, C)

2.1. Let π_1 and π_2 be of type C. Let $\pi_1 : V \to Y_1$ and $\pi_2 : V \to Y_2$ be the corresponding extremal morphisms. In particular, $S_1 = Y_1$ and $S_2 = Y_2$ are smooth surfaces (see 1.3.2). Denote by f_k the general fiber of the morphism π_k, $k = 1,2$.

2.2. PROPOSITION. *If $\rho(V) \geqslant 3$, then $\rho(V) = 3$, and S_1 and S_2 are ruled surfaces.*

Proof.

2.2.1. Let H be a very ample divisor on S_2, and let $C \in |H|$ be a smooth curve. Then $(\pi_2^* C,\ \pi_2^* C)_V = m.f_2$, where $m = (C,\ C)_{S_2} > 0$. Therefore, $\pi_2^* C \notin \pi_1^*(\mathrm{Pic}\, S_1)$; hence, the mapping $\pi_1 : \pi_2^* C \to S_1$ is surjective. Since $\kappa(\pi_2^* C) = -\infty$, then $\kappa(S_1) = -\infty$ (here $\kappa(X)$ is the Kodaira dimension of X). Similarly $\kappa(S_2) = -\infty$. Consequently, there exist morphisms $h_k : S_k \to S_{k,o}$, where $S_{k,o}$ are ruled surfaces or P^2. As $\rho(V) \geqslant 3$, then $\rho(S_k) \geqslant 2$.

Let, for example, $S_{1,o} = P^2$. Then the surface S_1 is rational, and the morphism $h_1 : S_1 \to S_{1,o} = P^2$ is non-trivial; in the opposite case $\rho(V) = \rho(P^2) + 1 = 2$, which contradicts the assumption $\rho(V) \geqslant 3$. Consequently, there exists a morphism $h_1' : S_1 \to \mathbf{F}_1$, such that $h_1 = h_1'.\sigma$, where $\sigma : \boldsymbol{F}_1 \to P^2$ is a blowing-up of a point in P^2. Therefore, we can always assume that $S_{1,o}$ and $S_{2,o}$ are ruled surfaces (rational or non-rational).

Let $S_{k,o} = P(\mathscr{E}_k)$, let $\mathscr{L}(b_{k,o}) = \mathscr{O}_{P(\mathscr{E}_K)}(1)$, and let $\varphi_{k,o}$ be the general fiber of $S_{k,o}$, $k = 1,2$ (see the Remark after Theorem 1.5). Let

$$\mathrm{Num}\, S_k = \mathbf{Z} b_k \oplus \mathbf{Z}\varphi_k \oplus \bigoplus_{i=1}^{m_k} \mathbf{Z}_{\varepsilon_{k,i}},$$

where $\varepsilon_{k,i}$ are the exceptional curves of h_k, and b_k and φ_k are the preimages of $b_{k,o}$ and $\varphi_{k,o}$ on S_k, $k = 1,2$. Let $m = \rho(V) - 1$. Obviously $\rho(S_k) = m = m_k + 2$, $k = 1,2$.

Let $C_1 = \pi_1^* b_1$, $C_2 = \pi_1^* \varphi_1$, $C_{i+2} = \pi_1^* \varepsilon_{1,i}$, $D_1 = \pi_2^* b_2$, $D_2 = \pi_2^* \varphi_2$, $D_{i+2} = \pi_2^* \varepsilon_{2,i}$ $(i = 1,2, \ldots m - 2)$. If $\pi_k : V \to S_k$ is a conic bundle, then $\mathrm{Pic}\, V = \pi_2^* \mathrm{Pic}\, S_k + \mathbf{Z}K_V$; if $\pi_k : V \to S_k$ is a P^1-bundle, then $\mathrm{Pic}\, V = \pi_k^* \mathrm{Pic}\, S_k + \mathbf{Z}L_k$, where L_k corresponds to a section of π_k. In both cases

$$2.\ \mathrm{Pic}\, V \subseteq \pi_k^* \mathrm{Pic}\, S_k + \mathbf{Z}K_V,$$

i.e. the divisors D_i (resp. C_i) are linear combinations, with integer or half-integer coefficients, of the divisors C_i (resp. D_i) and K_V (in the numerical sense). Therefore, there exists a system of equations of the form:

2.2.2. $$C_i + \sum_j d_{ij} D_j \equiv - d_i K_V$$
$$\sum_j c_{ij} C_j + D_i \equiv - c_i K_V,$$

where the numbers $2d_{ij}$, $2d_i$, $2c_{ij}$, and $2c_i$ are integer.

Let $D = (d_{ij})$, $C = (c_{ij})$, $d = (d_1, \ldots, d_m)^t$, $c = (c_1, \ldots, c_m)^t$, and let E be the unit matrix of rank m. By the adjunction formula $K_V.f_k = -2$, $k = 1,2$; and from 2.2.2 we conclude that $C_i.f_2 = 2d_i$, $D_i.f_1 = 2c_i$. The integers $2d_i$ and $2c_i$ are non-negative; they can be interpreted as follows:

If $d_i = 0$, then $C_i \in \pi_2^* \operatorname{Pic} S_2$;

if $d_i > 0$, then $d_i =$ the degree of the covering $\pi_2 : C_i \to S_2$;

(similarly — for c_i). Further, from 2.2.2 we derive:

$(- c_i + \sum_l c_{il} d_l).K_V \equiv D_i - \sum_{l,j} c_{il} d_{lj} D_j$, $i = 1,2, \ldots m$. Therefore, from the formula connecting Pic V and Pic S_2, we obtain that the both sides of the last equation are equal to zero, in the numerical sense. Hence, $C.D = E$, and $C d = c$. These matrix equations will be used in the proof of Proposition 2.3.

2.2.3. Let $C_i C_j = \gamma_{ij} f_1$, $D_i D_j = \delta_{ij} f_2$, $k_{ij} = K_V C_i D_j$. After multiplying the first m equations from 2.2.2 by $C_i D_j$ we obtain the following system:

2.2.4. $$R_{ijk} = 2d_i \sum_l d_{kl} \delta_{lj} + d_k k_{ij} + \gamma_{ki}.2c_j = 0.$$

By the choice of the curves b_k, φ_k, ε_{ki}, the numbers γ_{ki} and δ_{lj} satisfy the following conditions:

2.2.5. (a) $\gamma_{ii} = - p_i < 0$, $\delta_{ii} = - q_i < 0$, $i \geqslant 3$;
(b) $\gamma_{1i} = \gamma_{2i} = \delta_{1i} = \delta_{2i} = 0$, $i \geqslant 3$;
(c) $\gamma_{22} = \delta_{22} = 0$, $\gamma_{11} = - e_1$, $\delta_{11} = - e_2$,
where $- e_k = (b_k, b_k)_{S_k} = (b_{k,o}, b_{k,o})_{S_{k,o}}$, $k = 1,2$.

2.2.6. Lemma. *If $d_2 = 0$, then $d_3 = \cdots = d_m = 0$ (similarly — for c_i).*

Proof. Every $\varepsilon_{1,i}$ is a component of some degenerating fiber $\varphi_{1,i} \equiv \varphi_1$ of h_1, $\varphi_{1,i}$ being a linear combination with integer coefficients of exceptional curves and the preimage of some fiber of $S_{1,o}$. Let, for example, $\varphi_1 \equiv \sum_{n \geqslant o} \lambda_n \varepsilon_{1,n}$, where $\lambda_n \geqslant 0$ and $\varepsilon_{1,o}$ is the proper preimage of some fiber of $S_{1,o}$, over which we take blowing-ups. Then

2.2.7. $0 = 2d_2 = C_2 f_2 = \lambda_o . \pi_1^* \varepsilon_{1,o} f_2 + \sum_{n \geqslant 1} \lambda_n . 2d_{n+2}$.

Hence, $2d_{i+2} = C_{i+2} f_2 = \pi_1^* \varepsilon_{1,i} f_2 = 0$.

2.2.8. LEMMA. *If* $m = \rho(V) - 1 \geqslant 3$, *then* $\Pi_{i=3}^{m} c_i d_i = 0$.

Let us look at the equations $R_{13k} = 0$, $k \geqslant 2$, *and* $R_{23k} = 0$, $k \geqslant 2$ (*see* 2.2.4). *We shall give a proof of* 2.2.8 *on an example, which is not different from the general case.*

EXAMPLE. $m = 4$; i.e. from 2.2.5 we have $\delta_{33} \neq 0$, $\delta_{43} \neq 0$, $\delta_{13} = \delta_{23} = 0$. For definiteness, we may assume that $\delta_{33} = -2$ and $\delta_{43} = 1$; therefore $\delta_{44} = -1$. The surface S_2 is obtained from $S_{2,o}$ after blowing-up a point not lying on the base section, and a second blowing-up with a centre lying on the first exceptional divisor. The equations $R_{13k} = 0$ and $R_{23k} = 0$, $k \geqslant 2$, take the form:

$$
\begin{aligned}
\text{2.2.9.}\quad R_{132} &= -2c_3 + (-2d_{23} + d_{24}).2d_1 = -d_2 k_{13} \\
R_{13k} &= (-2d_{k3} + d_{k4}).2d_1 = -d_k k_{13},\ k = 3,4 \\
R_{23k} &= (-2d_{k3} + d_{k4}).2d_2 = -d_k k_{23},\ k = 2,3,4
\end{aligned}
$$

From 2.2.9 we easily derive that *either* $d_2 = 0$ (and hence, according to Lemma 2.2.6, $d_3 = \cdots = d_m = 0$), *or* the assumption $d_3 \neq 0$ implies $c_3 = 0$.

2.2.10. LEMMA. *If* $\rho(V) \geqslant 3$, *then* $\rho(V) = 3$.

Proof. According to Lemma 2.2.8, if $m \geqslant 3$, then $\Pi_{i \geqslant 3} c_i d_i = 0$. Let, for example, $c_3 = 0$. Then $D_3 \in \pi_1^* \operatorname{Pic} S_1$. Hence $D_3 = \pi_1^* C$, where $C \in \operatorname{Pic} S_1$ and $(C, C)_{S_1} = r \in \mathbf{Z}$. Then $-q_3 f_2 = (D_3, D_3)_V = (\pi_1^* C, \pi_1^* C)_V = r f_1$, where $q_3 > 0$ (i.e. $q_3 \neq 0$) — a contradiction. Therefore $m = \rho(V) - 1 = 2$, and the Proposition 2.2 is proved.

2.3. PROPOSITION. *Let* $\rho(V) = 3$. *Then* $\pi_1^* \varphi_1 \in \pi_2^* \operatorname{Pic} S_2$ *and* $\pi_2^* \varphi_2 \in \pi_1^* \operatorname{Pic} S_1$, *where* φ_k *is the general fiber of the ruled surface* S_k, $k = 1,2$.

Proof. For convenience, we shall change the notation. As $m = 2$, the system 2.2.2 takes the form:

$$
\text{2.3.1.}\quad
\begin{array}{rrrrrcl}
-C_1 & & + & g_1 C_2 & + d_1 F_2 & \equiv & r_1 K_V \\
 & -F_1 & + & b_1 C_2 & + a_1 F_2 & \equiv & c_1 K_V \\
g_2 C_1 & + d_2 F_1 & - & C_2 & & \equiv & r_2 K_V \\
b_2 C_1 & + a_2 F_1 & & & - F_2 & \equiv & c_2 K_V,
\end{array}
$$

where $F_k = \pi_k^* \varphi_k$, $C_k = \pi_k^* b_k$, $k = 1,2$, and all the coefficients are either integers, or half-integers.

The equality $C.D = E$ (see 2.2.2) implies $g_2 = \varepsilon a_1$, $d_2 = -\varepsilon d_1$, $b_2 = -\varepsilon b_1$, and $a_2 = \varepsilon g_1$, where $\varepsilon = (g_1 a_1 - b_1 d_1)^{-1}$. From $C\,d = c$ and $D\,c = d$ (ibid.) we obtain:

2.3.2.
$$\begin{aligned} c_2 + a_2 c_1 + b_2 r_1 &= 0 \\ r_2 + d_2 c_1 + g_2 r_1 &= 0 \\ c_1 + a_1 c_2 + b_1 r_2 &= 0 \\ r_1 + d_1 c_2 + g_1 r_2 &= 0. \end{aligned}$$

After multiplying both sides of the equalities 2.3.1 by f_1 and f_2 we obtain $F_1 f_2 = 2c_1$, $F_2 f_1 = 2c_2$, $C_1 f_2 = 2r_1$, and $C_2 f_1 = 2r_2$. The system 2.2.5 for γ_{ij} and δ_{ij} takes the form:

$$C_1^{\,2} = -\,e_1 f_1,\; C_2^{\,2} = -\,e_2 f_2,\; C_1 F_1 = f_1,$$
$$C_2 F_2 = f_2,\; F_1^{\,2} = F_2^{\,2} = 0.$$

We divide the proof in several cases:

Case 1. S_1 and S_2 are rational.

CLAIM. *In Case* 1, *the equality* $c_1 c_2 r_1 r_2 = 0$ *is fulfilled.*

Proof of the Claim. Assume that $c_1 c_2 r_1 r_2 \neq 0$; and let $\varepsilon < 0$. From the equation $c_1 K_V F_1 C_2 = 2a_1 c_1 - 2b_1 e_2 c_1$, we get $K_V F_1 C_2 = 2a_1 - 2b_1 e_2$; therefore $c_2(2a_1 - 2b_1 e_2) = c_2 K_V F_1 C_2 = -\,2c_1 + 2b_2 r_2$. By 2.3.2, $2c_1 + 2a_1 c_2 = -\,2b_1 r_2$, where $b_2 = -\,\varepsilon b_1$. Hence:

2.3.3. $$(2\varepsilon - 2)\, b_1 r_2 = 2b_1 e_2 c_2.$$

From $\varepsilon < 0$, $r_2 > 0$, $c_2 > 0$, and $e_2 \geqslant 0$, we get that $b_1 = 0$; in particular $b_2 = -\,\varepsilon b_1 = 0$. Thus, from $c_1 K_V F_1 F_2 = 2b_1 c_1$ and $c_1 > 0$, we obtain that $K_V F_1 F_2 = 0$. Then, from $r_2 K_V F_1 F_2 = 0$ and $r_1 K_V F_1 F_2 = 0$, we conclude that $-\,2c_1 + 2\,g_2 c_2 = 0$ and $-\,2c_2 + 2\,g_1 c_1 = 0$. Therefore $g_1 > 0$, $g_2 > 0$, and $g_1 . g_2 = 1$.

From $K_V F_1 C_2 = 2a_1$, and from the equations of the type $R_{ijk} = 0$ for $r_2 K_V F_1 C_2$ we obtain:

2.3.4. $$(a_1 - g_2) r_2 = 2e_2 c_1,$$

where $a_1 = \varepsilon g_1$, $g_2 = \varepsilon a_1$, $g_1 > 0$, $g_2 > 0$, and $\varepsilon < 0$. In that case, the equation 2.3.4 contradicts the assumption that $c_1 > 0$ and $r_2 > 0$.

Let $\varepsilon < 0$ and $c_1 c_2 r_1 r_2 = 0$. In particular, if $r_1 = 1$ then $2e_1 c_2 = r_1 K_V F_2 C_1 = 0$. Therefore, either $c_2 = 0$, or $e_1 = 0$. If $e_1 = 0$, then $S_1 \simeq P^1 \times P^1$, and we can assume that $b_1 \subset S_1$ is a fiber (cf. 2.2.1).

Let $c_1 = 0$, but $c_2 > 0$. Then 2.3.2 implies that $c_2 + b_2 r_1 = 0$, i.e. $b_2 < 0$. But, from the equations $0 = c_1K_VF_2C_1 = -2c_2 + 2b_1r_1$ and $b_2 = -\varepsilon b_1$, $\varepsilon < 0$, we obtain that $b_2 > 0$, which is impossible.

COROLLARY. *If $\varepsilon < 0$, and S_1 and S_2 are rational, then $c_1 = c_2 = 0$.*

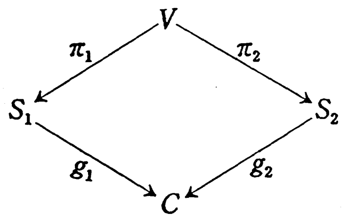
If $\varepsilon > 0$, we come to a contradiction in the same way. The Claim is proved. Proceeding in a similar way, from the above Claim and from 2.3.2, we obtain the following

COROLLARY. *In the Case 1 we have $c_1 = c_2 = 0$.*
Then, from 2.3.1, we obtain that $F_1 \in \pi_2^* \operatorname{Pic} S_2$ and $F_2 \in \pi_1^* \operatorname{Pic} S_1$.

Case 2. S_1 or S_2 is non-rational.

Let, for example, S_1 be an irregular ruled surface and let $g_1 : S_1 \to C$, $g_2 : S_2 \to C'$ be the corresponding representations of the surfaces S_1 and S_2 as P^1-bundles over the curves C and C', where $g(C) = g \geqslant 1$. Then the general fibers of $|L_1| = g_1\pi_1 : V \to C$ and $|L_2| = g_2\pi_2 : V \to C'$ are rational surfaces.

Let $|L_1| \neq |L_2|$. Then $f = L_2|_{L_1}$ is a curve on L_1 and $(f, f)_{L_1} = L_2L_2L_1 = 0$. Hence, the restriction $|L_2||_{L_1} : L_1 \to C$ defines, on the rational surface L_1, a structure of bundle with rational curves as fibers and with a non-rational base C, which is impossible. Therefore $C' \simeq C$, and the diagram



where $g(C) = g \geqslant 1$, is commutative. Evidently, in this case $\pi_1^* \varphi_1 \in \pi_2^* \operatorname{Pic} S_2$ and $\pi_2^* \varphi_2 \in \pi_1^* \operatorname{Pic} S_1$. The Proposition 2.3 is proved.

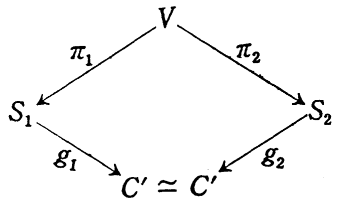
2.4. COROLLARY. *If $\rho(V) = 3$, then the equation 2.3.2 take the form:*

$$rK_V \equiv -C_1 - C_2 + dF,$$

where $F_1 \equiv F_2 \equiv F \in \pi_1^* \operatorname{Pic} S_1 \cap \pi_2^* \operatorname{Pic} S_2$, the numbers $2r$ and $2d$ are integer, and $r > 0$.

The Corollary is a direct consequence from Proposition 2.3, and from the first and the third equations of 2.3.1. Note that from the two other equations of 2.3.1 we obtain that $b_1 = b_2 = 0$ and $a_1 = a_2 = 1$. Then, the former two equations give $\varepsilon = -1$.

2.5. COROLLARY. *If $\rho(V) = 3$, then there exists a curve C such that the diagram*



is commutative.

Proof. For S_1 and S_2 — non-rational, the Corollary is proved in 2.3, Case 2.

Let S_1 and S_2 be rational ruled surfaces. By Proposition 2.3, we have $\pi_1^* \varphi_1 \in \pi_2^* \operatorname{Pic} S_2$ and $\pi_2^* \varphi_2 \in \pi_1^* \operatorname{Pic} S_1$. Consequently, there are correctly defined morphisms $\lambda = g_1 \pi_1 \pi_2^{-1} g_2^{-1} : C' \to C$ and $\lambda^{-1} = g_2 \pi_2 \pi_1^{-1} g_1^{-1} : C \to C'$, where $g_1 : S_1 \to C \simeq P^1$ and $g_2 : S_2 \to C' \simeq P^1$ define structures of ruled surfaces on S_1 and S_2. Therefore $\lambda \in \operatorname{Aut} P^1$; and if we replace g_2 by $\lambda . g_2$, we shall obtain the commutative diagram from above.

2.6. *Case* $\rho(V) = 2$

Let us consider the case $\rho(V) < 3$. Then $\rho(V) = 2$, and there are on V two extremal rays R_1 and R_2 of type C. As $\rho(V) = \dim_{\mathbf{R}}(NV) = 2$, then R_1 and R_2 form a base of the two-dimensional real vector space NV. Let $R_1 = \mathbf{R}_+[l_1]$ and $R_2 = \mathbf{R}_+[l_2]$. Since R_1 and R_2 are extremal rays in the two-dimensional cone $\overline{NE}(V) \subset NV$, and since $K_V . l_1 < 0$, $K_V . l_2 < 0$, then $K_V . Z < 0$ for any $Z \in \overline{NE}(V)$. By the Kleiman's criterion we derive that $-K_V$ is ample, i.e. V is a Fano threefold.

2.6.1. COROLLARY. *If $\rho(V) = 2$ and (π_1, π_2) is of type (C, C), then V is a Fano threefold.*

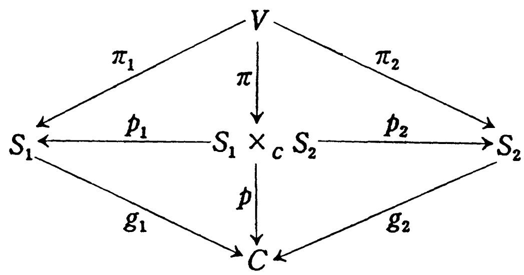
2.6.2. COROLLARY (*see* [4]). *In the conditions of* 2.6.1, *the threefold V is one of the following*:

(1) *a divisor of bidegree* (2,2) *in* $P^2 \times P^2$;
(2) *a divisor of bidegree* (2,1) *in* $P^2 \times P^2$;
(3) *a divisor of bidegree* (1,1) *in* $P^2 \times P^2$;
(4) *a two-sheeted covering of a divisor* D *of bidegree* (1,1) *in* $P^2 \times P^2$, *with a branch divisor* $B \in | - K_D |$.

§3. Construction of threefolds of type (C, C) with $\rho = 3$,

Let V be of type (C, C), and let $\pi_1 : V \to S_1$, $\pi_2 : V \to S_2$, C, etc., be as in §2. It follows from the considerations in §2 that there exists a commutative diagram of natural morphisms:

3.1.



where p_1 and p_2 are the natural projections, and $\deg \pi = 1$ or $\deg \pi = 2$ (see 2.5).

We shall consider the case $\deg \pi = 2$. In this setting, we shall obtain numerical formulae for the branch divisor of the double covering π.

3.2. Let f_k be the (general) fiber of π_k, and let σ_k be the (general) fiber of R_k, $k = 1,2$. Evidently $\sigma_k \simeq P^1$ for any σ_k, $k = 1,2$. Let $\mathscr{E}_k$, b_k, φ_k, C_k, $k = 1,2$, and $F_1 \equiv F_2 \equiv F$ be as in the Remark after Theorem 1.5. Let $\varepsilon_k = \det(\mathscr{E}_k)$, $e_k = -\deg(\varepsilon_k)$; and let $C_{1V} = \pi^* C_1$, $C_{2V} = \pi^* C_2$, $F_{1V} \equiv F_{2V} \equiv F_V$ be the divisors on V, in the sense indicated in Corollary 2.4, i.e. $F_V \equiv \pi^* p^*(x) \equiv \pi^* F$ (where $F \equiv p^*(x)$, $x \in C$). It is easy to see that:

3.3. $K_{S_1 \times_C S_2} = -2C_1 - 2C_2 + (\varepsilon_1 + \varepsilon_2 + k)F$, where $k = K_C$ is the canonical divisor of C. The branch divisor B of π has the form:

3.4. $B = 2q_1 C_1 + 2q_2 C_2 + 2\mathfrak{q}F$, where q_1 and q_2 are integers, and $2\mathfrak{q}F$ is used in the sense that $2\mathfrak{q}F = p^*(\mathfrak{q})$ for the divisor $\mathfrak{q}$ on C.

We claim that $q_1 = q_2 = 1$.

In fact, as $p_1 : S_1 \times_C S_2 \to S_1$ is a P^1-bundle, then $\mathrm{Pic}(S_1 \times_C S_2) = p_1^* \operatorname{Pic} S_1 \oplus \mathbf{Z}C_2$ (since C_2 is an 1-section of p_1). Therefore $\mathrm{Pic}(S_1 \times_C S_2) = \mathbf{Z}C_1 \oplus g_1^*$

(Pic C) $\oplus \mathbf{Z}C_2$, in sense that $g_1^*(\text{Pic } C).F = p_1^* g_1^*(\text{Pic } C) = p^*(\text{Pic } C)$. As $\pi : f_1 \to \sigma_1$ is a two-sheeted covering for the general $f_1 \simeq P^1$ and $\pi(f_1) = \sigma_1 \simeq P^1$, then it has two branch points. Therefore $\deg(B|_{\sigma_1}) = \deg(B|_{\sigma_2}) = 2$, i.e. $2 = \deg(B|_{\sigma_1}) = (2q_1C_1 + 2q_2C_2 + 2qF).\sigma_1 = 2q_2C_2\sigma_1 = 2q_2$; similarly – for q_1. As V is smooth, then B is smooth, and we derive:

COROLLARY. *The (smooth) branch divisor B of π has the form*

$$B = 2C_1 + 2C_2 + 2\mathfrak{q}F$$

for some divisor $\mathfrak{q}$ on C, where $2\mathfrak{q}F = p^*(2\mathfrak{q})$.

3.5. In the context of the situation, we shall derive some necessary numerical conditions for B.

The general surface $C_{1V} = \pi^*C_1$ is smooth, and it is a two-sheeted covering of C_1 with a branch divisor $B|_{C_1} = (2C_1 + 2C_2 + 2qF)|_{C_1} = 2C_{12} + 2(\varepsilon + \mathfrak{q})\sigma_1$, where $C_{12} = C_1.C_2$, and $(C_{12}, C_{12})_{C_1} = C_2C_2C_1 = -e_2\sigma_2C_1 = -e_2$. Therefore, for the existence of a (smooth) effective divisor $C_{12} \subset C_1$, one must have $2(q - e_1) \geqslant 2e_2$ if $e_2 \geqslant 0$, or $2(q - e_1) \geqslant \frac{1}{2}.2e_2 = e_2$ if $e_2 < 0$ (see [1, Ch. V, §2]); here $q = \deg(\mathfrak{q})$. The same restrictions are available for C_2 and e_1, and we derive:

COROLLARY. *Let B, e_1, and e_2 be as in* 3.2–3.4. *Then for $q = \deg(\mathfrak{q})$ we have*:

(a) $q \geqslant e_1 + e_2$ *if* $e_1 \geqslant 0, e_2 \geqslant 0$;
(b) $2q \geqslant 2e_1 + e_2$ *if* $e_1 \geqslant 0, e_2 < 0$;
(c) $2q \geqslant e_1 + 2e_2$ *if* $e_1 < 0, e_2 \geqslant 0$;
(d) $2q \geqslant \max\{2e_1 + e_2, e_1 + 2e_2\}$ *if* $e_1 < 0, e_2 < 0$.

3.6. The canonical divisor K_V, and the surfaces C_{1V} and C_{2V}

3.6.1. It follows from the preceding that

$K_V = -C_{1V} - C_{2V} + (\varepsilon_1 + \varepsilon_2 + \mathfrak{k} + \mathfrak{q})F_V$. Therefore, by the adjunction formula

$K_{C_{1V}} = -C_{12V} - (\varepsilon_1 + \varepsilon_2 + \mathfrak{k} + \mathfrak{q})f_1$, where $C_{12V} = C_{1V}.C_{2V}$. Evidently, the self-intersection number of C_{12V} in C_{1V} is equal to $-2e_2$, and $C_{12V}.f_1 = 2$. Therefore $K_{C_{1V}}.K_{C_{1V}} = (8 - 8g) - (4q - 4e_1 - 2e_2)$, where $q = \deg(\mathfrak{q})$ and $g = g(C)$.

3.6.2. From the Corollary in 3.5, we obtain $4q - 2e_1 - 2e_2 \geqslant 0$; similarly for C_{2V}. From $K_{C_{1V}} \equiv -C_{2V}|_{C_{1V}} + (2g - 2 - e_1 - e_2 + q).f_1$ we conclude that

$h_1 : C_{1V} \to (C_{1V})_{\min}$ is a composition of σ-processes with centers lying on the curve $h_1(C_{12V})$; here $(C_{1V})_{\min}$ is some (relatively) minimal model of C_{1V}.

3.6.3. For $F_V = \pi^* p^*(x)$, $x \in C$, we have similarly: $K_{F_V} = (-C_{1V} - C_{2V}).F_V$ and $K_{F_V}.K_{F_V} = 4$. The surface F_V is obtained from P^2 after blowing-up of five points.

3.7. Examples of Fano threefolds of type (C, C)

3.7.1. Let $\deg \pi = 2$. Then V is of type (C_1, C_1), and $K_V = -C_{1V} - C_{2V} + (2g - 2 + q - e_1 - e_2).F_V$. Let, moreover, V be a Fano threefold. Then $K_V C_{1V} C_{2V} = 4g + 2q - 4 < 0$; in particular $g = 0$ and $q \leqslant 1$. Therefore (see the Corollary in 3.5) $1 \geqslant q \geqslant e_1 + e_2$ (since $e_1 \geqslant 0$, $e_2 \geqslant 0$).

Let $e_1 \geqslant e_2$, and let $e_1 = 1$, $e_2 = 0$. Then $B|_{C_1} = 2C_2|_{C_1}$ is not a reducible divisor, which is impossible. Therefore $e_1 = e_2 = 0$ and $K_V \equiv -C_{1V} - C_{2V} - F_V$. The manifold V is a two-sheeted covering of $S_1 \times_C S_2 = (P^1 \times P^1_o) \times_{P^1_o} (P^1_o \times P^1) \simeq P^1 \times P^1 \times P^1$ with a branch divisor $B = 2C_1 + 2C_2 + 2F$ of multidegree (2,2,2).

3.7.2. Let $\deg \pi = 1$. Then $V = S_1 \times_C S_2$, and $K_V \equiv -2C_1 - 2C_2 + (2g - 2 - e_1 - e_2).F$, where $K_V C_1 C_2 = 2g - 2 + e_1 + e_2$ and $K_V{}^3 = 24.(2g - 2)$. If V is a Fano threefold, then $K_V{}^3 < 0$, hence $g = 0$. Therefore $K_V C_1 C_2 = e_1 + e_2 - 2 < 0$, where $e_1 \geqslant 0$, $e_2 \geqslant 0$.

Let $e_1 \geqslant e_2$, and let, for example, $e_1 = 1$, $e_2 = 0$. Then $V = \mathbf{F}_1 \times_{P^1} (P^1 \times P^1)$, and $K_V \equiv -2C_1 - 2C_2 - 3F$, $K_V C_1 C_2 = -1$, $K_V{}^3 = -48$, $K_V \sigma_1 = K_V \sigma_2 = -2$. On the other hand, $-K_V|_F = 2\sigma_1 + 2\sigma_2$, $-K_V|_{C_1} = 2C_2|_{C_1} + \varphi_1$, and $-K_V|_{C_2} = 2C_1|_{C_2} + 3\varphi_2$ are ample divisors on the surfaces $F \simeq P^1 \times P^1$, $C_1 \simeq P^1 \times P^1$, and $C_2 \simeq \mathbf{F}_1$. Since $K_V.C_{12} = -1$, and $C_{12} = C_1 \cap C_2$ is a rational curve, then there exists an extremal ray $R_3 = \mathbf{R}_+[C_{12}]$ (see [3]). As $(C_2|_{C_1}, C_2|_{C_1}) = 0$, then $C_2|_{C_1}$ moves in C_1 as one of the rulings of the quadric $D = C_1 \simeq P^1 \times P^1$. The restriction of the normal bundle $N_{C_1|V}$ on C_{12} has a degree -1. In fact $N_{C_1|V}|_{C_{12}} \simeq \mathcal{O}_{C_1}(C_1) \otimes \mathcal{O}_{C_{12}} \simeq \mathcal{O}_{C_1}(-\sigma_1) \otimes \mathcal{O}_{C_{12}} \simeq \mathcal{O}_{C_{12}}(-1)$, where $C_{12} = P^1$. Therefore, we can contract C_1 along C_{12} (see [2, p.1020], or [3]); that is, there is an extremal ray of type E_1 on V, i.e. V is not strongly primitive (see 1.4 and 0.2.2).

Let $e_1 = e_2 = 0$. Then $V \simeq (P^1 \times P^1_o) \times_{P^1_o} (P^1_o \times P^1) \simeq P^1 \times P^1 \times P^1$, and $-K_V = 2C_1 + 2C_2 + 2F$ is a divisor of multidegree (2,2,2) in $P^1 \times P^1 \times P^1$.

3.8. The discriminant curves for π_1 and π_2

Let Δ_1 and Δ_2 be the discriminant curves of π_1 and π_2. Then $\Delta_1 \equiv 4b_1 + (4q - 2e_2)\varphi_1$ on S_1, and $\Delta_2 \equiv 4b_2 + (4q - 2e_1)\varphi_2$ on S_2. These numerical equalities follow immediately from the formula $-4.K_S \equiv \pi_*(-K_V)^2 + \Delta$, connecting the discriminant curve Δ of a conic bundle $\pi : V \to S$ with the canonical bundles of V and S.

§4. The Case (C, D)

4.1. Let the extremal morphism π_1 be of the type C, and let π_2 be of the type D. In particular $\rho(V) = 2$ (see 1.3.3). In just the same way as in 2.6 we obtain that V is a Fano threefold.

4.2. COROLLARY. *Let the pair (π_1, π_2) be of the type (C, D). Then V is one of the following (see* [4]*):*

(1) $V = P^2 \times P^1$;

(2) *a two-sheeted covering* $\pi : V \to P^2 \times P^1$ *with a branch divisor* $B \subset P^2 \times P^1$ *of bidegree* (4,2);

(3) *a two-sheeted covering* $\pi : V \to P^2 \times P^1$ *with a branch divisor* $B \subset P^2 \times P^1$ *of bidegree* (2,2).

§5. The Case (C, E)

Let π_1 be of type C, and π_2 be of type E. We have to prove that if V is strongly primitive, then V must be a Fano threefold. We shall consider separately the cases E_2, E_3, E_4, and E_5 (see 1.3.1 and 1.4).

5.1. *The cases (C, E_2) and (C, E_5)*

Let π_2 belongs to one of the types E_2 or E_5. In particular, the morphism π_2 is a contraction of a divisor $D \simeq P^2$ in V to a point (see 1.3.1). The morphism π_1 maps $D \simeq P^2$ onto S_1. Actually, in the opposite case π_1 contracts D, because π_2 ($=$ a contracting of D) is an extremal morphism. But π_1 is also an extremal morphism, hence π_1 coincides with π_2 — a cootradiction. Therefore $\pi_2(D) = S_1$ and $S_1 \simeq P^2$.

5.2. *The cases (C, E_3) and (C, E_4)*

Let π_2 belongs to one of the types E_3 or E_4. Just as above, the fact that π_1 and π_2 are different extremal morphisms, implies that the morphism π_1 maps the quadric $D \subset V$, corresponding to π_2 (see 1.3.1), onto the surface S_1. As S_1 is smooth, it must be either P^2 (in the cases E_3 and E_4) or $P^1 \times P^1$ (in the case E_3).

Let $S_1 \simeq P^1 \times P^1$. Let ψ_1 and ψ_2 be the rulings of S_1, and let $\varphi_1 = s \times P^1 \equiv P^1 \times t = \varphi_2$ be the rulings of $D \simeq P^1 \times P^1$. Since $(\pi_1|_D^* \psi_i, \pi_1|_D^* \psi_i)_D = 0$, $i = 1,2$, then $\pi_1|_D^* \psi_i = m_i\varphi_i$, where m_i is a positive integer. Therefore $(\pi_1|_D^* \psi_1, \pi_1|_D^* \psi_2)_D = (m_1\varphi_1, m_2\varphi_2)_D = m_1m_2$. On the other hand, the last equals to $m = \deg(\pi|_D)$. But φ_1 and φ_2 are numerically equivalent on V; therefore $0 = \varphi_1.\pi_1^* \psi_1$

$= \varphi_2.\pi_1^* \psi_1 = m_2$. In particular, $\deg \pi = m = m_1 m_2 = 0$, which is impossible. Consequently, in the cases (C, E_3) and (C, E_4) the surface S_1 is isomorphic to P^2.

5.3. Corollary. *Let π_1 be of the type C and π_2 be of the type E (E_2, E_3, E_4, or E_5). Then V is a Fano threefold.*

Proof. In fact, we obtained that in all cases $S_1 \simeq P^2$ (see 5.1 and 5.2). Therefore $\rho(V) = \rho(P^2) + 1 = 2$, and V admits two different extremal morphisms. It follows that V is a Fano threefold (see 2.6).

5.4. Corollary (*see* [4]). *Let V, π_1, π_2, etc., be as in* 5.3. *Then V is one of the following*:

(1) $V = P(\mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(1))$, *in the case* (C, E_2) ;
(2) $V = P(\mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(2))$, *in the case* (C, E_5) ;
(3) *V is a two-sheeted covering of $Y = P(\mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(1))$ with a branch divisor $B \in |-K_Y|$, in the cases (C, E_3) and (C, E_4).*

§6. The case (D, D)

6.1. Let π_1 and π_2 be both of the type D (see 1.3.3). Let $S_1 = \pi_1^*(x)$, $x \in C_1$, and $S_2 = \pi_2^*(x)$, $x \in C_2$, where $\pi_k : V \to C_k$, $k = 1,2$, are the corresponding extremal morphisms. As $\rho(V) = \rho(C_k) + 1 = 2$, then S_1 is represented in the form $S_1 \equiv a.S_2 + b.K_V$, for some rational a, b. In particular, $K_{S_1} = (K_V + S_1)|_{S_1} = (1/b).(-a.S_2 + (b+1).S_1)|_{S_1}$. Hence

$$K_{S_1}.K_{S_1} = (1/b^2).((b+1).S_1 - a.S_2)^2.S_1 = 0,$$

since $S_1.S_1 = S_2.S_2 = 0$. On the other hand, the divisor $-K_{S_1}$ must be ample, since S_1 is a Del Pezzo surface, P^2, or $P^1 \times P^1$ (see 1.3.3). We come to a contradiction.

6.2. Corollary. *There are no manifolds for which π_1 and π_2 are both of type D.*

§7. The case $(E, E, \dots, E)$

Let V admits morphisms $\pi_1, \pi_2, \dots, \pi_n$ of the type E, and let V be strongly primitive. Let $D_1, D_2, \dots, D_n$ be the corresponding divisors on V, which $\pi_1, \pi_2, \dots, \pi_n$

contract (see 1.3.1). Then, by [4, p. 124 (8.1)], the divisors D_i are mutually disjoint. Consequently, the contractions π_i carry out independently.

Theorem 1.5 is proved.

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Institute of Mathematics
Bulgarian Academy of Sciences
ul. Acad. G. Bonchev, bl.8
1113 Sofia, Bulgaria