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GENERATORS FOR A MAXIMALLY DIFFERENTIAL IDEAL IN POSITIVE CHARACTERISTIC

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Introduction

In this note we give the structure of maximally differential ideals in a Noetherian local ring of prime characteristic $p > 0$, in terms of their generators. More precisely, we prove the following result:

THEOREM 4. *Let A be a Noetherian local ring of prime characteristic $p > 0$ with maximal ideal $\mathfrak{m}$. Let I be a proper ideal of A. Suppose $n = \operatorname{emdim}(A)$ and $r = \operatorname{emdim}(A/I)$. If I is maximally differential under a set of derivations of A then there exists a minimal set $x_1, \ldots, x_n$ of generators of $\mathfrak{m}$ such that $I = (x_1^p, \ldots, x_r^p, x_{r+1}, \ldots, x_n)$.*

This result was proved by the author in [3, Lemma 2.2], under the additional hypothesis that A is complete and I is maximally differential under a set of k-derivations of A, where k is a coefficient field of A.

Using the methods we use to prove the above result we give a different proof for Harper's Theorem (as called by H. Matsumura, [Cf. [4, Theorem on p. 206]]). The following formulation of Harper's Theorem is due to S. Yuan [5]:

"Let A be a differentially simple ring of positive characteristic p. Then A is local. Let $\mathfrak{m}$ be the maximal ideal of A and let $n = \dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$. If $n < \infty$ then

$$A \cong k[X_1, X_2, \ldots, X_n] / (X_1^p, X_2^p, \ldots, X_n^p),$$

where k is a field and $X_1, X_2, \ldots, X_n$ are indeterminates over k."

Our proof of Harper's Theorem is very straightforward and is much simpler than the original proof by L. Harper [1] and S. Yuan's proof, both of which involve somewhat complicated computations.

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The results

By a ring we mean a commutative ring with 1.

Let A be a ring.

Let $\mathfrak{D}$ be a set of derivations of A. Then an ideal I is called a *$\mathfrak{D}$-differential ideal* if $d(I) \subset I$ for all $d \in \mathfrak{D}$. An ideal I is called a *maximally $\mathfrak{D}$-differential ideal* if it is a proper $\mathfrak{D}$-differential ideal and for every ideal J of A with $I \subsetneq J \subsetneq A$, J is not $\mathfrak{D}$-differential. An ideal I is called a *maximally differential ideal* if it is maximally $\mathfrak{D}$-differential for a set $\mathfrak{D}$ of derivations of A.

A ring is called a *differentially simple ring* if the ideal (0) is maximally differential in it.

For a derivation d of A, by d-differential we mean $\{d\}$-differential.

. LEMMA 1. *Let A be a ring of prime characteristic $p > 0$. Let δ be a derivation of A and let $x \in A$ such that $\delta(x) = 1$. Then:*

(a) *Let I be a δ-differential ideal of A. If $a_0, a_1, \ldots, a_{p-1} \in \ker(\delta)$ such that $\sum_{i=0}^{p-1} a_i x^i \in I$ then $a_i \in I$ for all $i = 0, 1, \ldots, p-1$.*

(b) *Let $E = \sum_{i=0}^{p-1} (-x)^i \delta^i / i!$. Then:*

  (i) $\delta E = -x^{p-1}\delta^p$.

  (ii) *For erery $a \in A$, $E(xa) = -x^p \delta^{p-1}(a)$.*

  (iii) *For every $a \in A$, $E^2(a) \equiv E(a) \pmod{(x^p)}$.*

Proof. (a) Since I is δ-differential, $\delta^{p-1}(\sum_{i=0}^{p-1} a_i x^i) = (p-1)! a_{p-1} \in I$. Hence $a_{p-1} \in I$. By induction, $a_{p-2}, \ldots, a_0 \in I$.

(b) Statements (i) and (ii) are straightforward from the definition of E. Statement (iii) follows from (ii). □

PROPOSITION 2. *Let A, δ, x and E be as in Lemma 1. Suppose, in addition, $x^p = 0$. Then E is a ring homomorphism. Let $A_0 = E(A)$. Then:*

  (i) $A_0 = \{a \in A \mid E(a) = a\} = \ker(\delta + x^{p-1}\delta^p)$.

  (ii) *$A^p = \{a^p \mid a \in A\} \subset A_0$ and A is a free A_0-module with basis $1, x, \ldots, x^{p-1}$.*

  (iii) *Let $\mathfrak{D}$ be a set of derivations of A such that $\delta \in \mathfrak{D}$ and let I be a maximally $\mathfrak{D}$-differential ideal of A. Then $I_0 = I \cap A_0$ is a maximally differential of A_0 and $I_0 A = I$.*

Proof. Since $x^p = 0$, by Leibnitz rule, E is a ring homomorphism. Put $\delta' = \delta + x^{p-1}\delta^p$, then δ' is a derivation of A.

(i) Let $a \in A_0$. Then $a = E(b)$ for some $b \in A$. Therefore by Lemma 1 $E(a) = EE(b) = E(b) = a$, as $x^p = 0$. Hence $A_0 = \{a \in A \mid E(a) = a\}$. Now we prove the other equality. Let $a \in A$ such that $a = E(a)$. Then, by Lemma 1, $\delta(a) = \delta E(a) = -x^{p-1}\delta^p(a)$. Therefore $\delta'(a) = 0$. Conversely, let $a \in \ker(\delta')$. We show by induction that $x^i\delta^i(a) = 0$ for $i = 1, \ldots, p-1$. Since $x^p = 0$, $x\delta(a) = 0$. Suppose $x^i\delta^i(a) = 0$ for $1 \leq i < p-1$. Then $0 = \delta(x^{i+1}\delta^i(a)) = x^{i+1}\delta^{i+1}(a) + (i+1)x^i\delta^i(a) = x^{i+1}\delta^{i+1}(a)$. Hence $E(a) = a$. Therefore $A_0 = \ker(\delta')$.

(ii) Since $\delta(a^p) = 0$ for all $a \in A$, $A^p \subset A_0$. Now we show that A is generated by $1, x, \ldots, x^{p-1}$ over A_0. Let $a_0 \in A$. By induction on i we construct $a_i \in A_0$ for $i = 0, 1, \ldots, p-1$ such that $a = \sum_{i=0}^{p-1} a_i x^i$. Now $a = E(a) + xb_1$ for some $b_1 \in A$. Take $a_0 = E(a)$ and $a_1 = E(b_1)$. Again $b_1 = E(b_1) + xb_2$ for some $b_2 \in A$. Take $a_2 = E(b_2)$ and so on. Since $x^p = 0$ we have $a = \sum_{i=0}^{p-1} a_i x^i$. As $A_0 = \ker(\delta')$ by (i) and $\delta'(x) = 1$, by Lemma 1, $1, x, \ldots, x^{p-1}$ are linearly independent over A_0.

(iii) Let $a \in I$. Then, by (ii), $a = \sum_{i=0}^{p-1} a_i x^i$, for some $a_0, a_1, \ldots, a_{p-1} \in A_0$. Since I is δ'-differential, $\delta'(x) = 1$ and $A_0 = \ker(\delta')$, by Lemma 1, $a_i \in I$ for all $i = 0, 1, \ldots, p-1$. Hence $I = I_0 A$.

Let $d \in \mathfrak{D}$. For $a \in A_0$, let $d_i(a)$ denote the coefficient of x^i in the expression of $d(a)$, $i = 0, 1, \ldots, p-1$. Then d_i's are derivations of A_0. (We have borrowed this construction from [2].) Let $\mathfrak{D}_0 = \{d_i \mid d \in \mathfrak{D}, i = 0, 1, \ldots, p-1\}$. We show that I_0 is maximally $\mathfrak{D}_0$-differential. First we show that I_0 is $\mathfrak{D}_0$-differential. Let $a \in I_0$ and $d \in \mathfrak{D}$. Since $a \in I$, $d(a) = \sum_{i=0}^{p-1} d_i(a)x^i \in I$. Therefore by Lemma 1 $d_i(a) \in I$. Hence $d_i(a) \in I_0$ for all $i = 0, 1, \ldots, p-1$ and $d \in \mathfrak{D}$. Therefore I_0 is $\mathfrak{D}_0$-differential. Let J be a $\mathfrak{D}_0$-differential ideal of A_0 containing I_0. Let $a \in J$ and $d \in \mathfrak{D}$. Then $d(a) = \sum_{i=0}^{p-1} d_i(a)x^i \in JA$. Therefore JA is $\mathfrak{D}$-differential. Since I is maximally $\mathfrak{D}$-differential and $I = I_0A \subset JA$ either $JA = I$ or $JA = A$. Since A is faithfully flat over A_0, $J = I_0$ or $J = A_0$ accordingly. Hence I_0 is a maximally differential ideal of A_0. □

COROLLARY 3 [Harper's Theorem, Cf. [1]]. *Let A be a differentially simple ring of positive characteristic p. Then A is local. Let $\mathfrak{m}$ be the maximal ideal of A and let $n = \dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$. If $n < \infty$ then*

$$A \cong k[X_1, \ldots, X_n]/(X_1^p, \ldots, X_n^p),$$

where k is a field and $X_1, X_2, \ldots, X_n$ are indeterminates over k.

Proof. Let $\mathfrak{D}$ be the set of all derivations of A. Then (0) is maximally $\mathfrak{D}$-differential.

Let $K = \{a \mid d(a) = 0 \text{ for all } a \in \mathfrak{D}\}$. By differential simplicity of A it follows that K is a field. Hence p is prime.

If $a \in A$ is a nonunit then $a^p = 0$ as the ideal (a^p) is $\mathfrak{D}$-differential. Hence A is local of Krull dimension zero. We prove the result by induction on n. If $n = 0$ then $\mathfrak{m} = \mathfrak{m}^2$. Hence $d(\mathfrak{m}) = d(\mathfrak{m}^2) \subset \mathfrak{m}$ for all $d \in \mathfrak{D}$. Therefore $\mathfrak{m}$ is $\mathfrak{D}$-differential. Hence $m = (0)$, i.e., A is a field. Suppose $n \geq 1$. Then there exist $d \in \mathfrak{D}$ and $x \in \mathfrak{m}$ such that $d(x) \notin \mathfrak{m}$. By replacing d by $d(x)^{-1}d$, we may assume that $d(x) = 1$. Since $x^p = 0$, by Proposition 2 there exists a local subring A_0 of A such that $A^p \subset A_0$, A is a free A_0-module with basis $1, x, \ldots, x^{p-1}$ and (0) is maximally differential in A_0. Then $A \cong A_0[X]/(X^p)$ where X is an indeterminate over A_0 and $A_0 \cong A/(x)$. Let $\mathfrak{m}_0$ be the maximal ideal of A_0. Now, $\dim_{A_0/\mathfrak{m}_0}(\mathfrak{m}_0/\mathfrak{m}_0^2) = \dim_{A/\mathfrak{m}}(\mathfrak{m}/(x) + \mathfrak{m}^2) = n - 1$ as $x \notin \mathfrak{m}^2$. Therefore, by induction, we are through. □

THEOREM 4. *Let A be a Noetherian local ring of prime characteristic $p > 0$ with maximal ideal $\mathfrak{m}$. Let I be a proper ideal of A. Suppose $n = \operatorname{emdim}(A)$ and $r = \operatorname{emdim}(A/I)$. If I is maximally differential under a set of derivations of A then there exists a minimal set $x_1, \ldots, x_n$ of generators of $\mathfrak{m}$ such that $I = (x_1^p, \ldots, x_r^p, x_{r+1}, \ldots, x_n)$.*

Proof. Let $\mathfrak{D}$ denote the set of all derivations d of A such that $d(I) \subset I$. Then I is maximally $\mathfrak{D}$-differential.

If $a \in \mathfrak{m}$ then (a^p) is $\mathfrak{D}$-differential and hence $a^p \in I$.

We prove the result by induction on r. If $r = 0$ then there is nothing to prove. Let $r \geq 1$. Then there exist $\delta \in \mathfrak{D}$ and $x \in \mathfrak{m}$ such that $\delta(x) \notin \mathfrak{m}$. By replacing δ by $(\delta(x))^{-1}\delta$ we may assume that $\delta(x) = 1$. Let $B = A/(x^p)$, $\mathfrak{n} = \mathfrak{m}/(x^p)$ and $J = I/(x^p)$. Let y be the image of x in B.

For $d \in \mathfrak{D}$, let d' denote the derivation on B induced by d and let $\mathfrak{D}' = \{d' \mid d \in \mathfrak{D}\}$. Then J is maximally $\mathfrak{D}'$-differential in B, $\delta'(y) = 1$ and $y^p = 0$. Therefore by Proposition 2 there exists a local subring B_0 of B such that $B^p \subset B_0$, B is a free B_0-module with basis $1, y, \ldots, y^{p-1}$, $J_0 = J \cap B_0$ is maximally differential in B_0 and $J = J_0B$. It is immediate from above data that $B_0 \cong B/(y) \cong A/(x)$ and $B_0/J_0 \cong B/J + (y) \cong A/I + (x)$. Since $x \notin I + \mathfrak{m}^2$ it follows that $\operatorname{emdim}(B_0) = n - 1$ and $\operatorname{emdim}(B_0/J_0) = r - 1$. Hence by induction $J_0 = (y_1^p, \ldots, y_{r-1}^p, y_r, \ldots, y_{n-1})$ for a minimal set $y_1, y_2, \ldots, y_{n-1}$ of generators of the maximal ideal $\mathfrak{n}_0$ of B_0. Therefore $J = (y_1^p, \ldots, y_{r-1}^p, y_r, \ldots, y_{n-1})$ and

$\mathfrak{n} = (y, y_1, \ldots, y_{n-1})$. Let x_i be a lift of y_i in A for $i = 1,2,\ldots,n-1$. Then $I = (x^p, x_1^p, \ldots, x_{r-1}^p, x_r, \ldots, x_{n-1})$ and $\mathfrak{m} = (x, x_1, \ldots, x_{n-1})$. □

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