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A CLASSIFICATION OF RIEMANNIAN 3-MANIFOLDS WITH CONSTANT PRINCIPAL RICCI CURVATURES $\rho_1 = \rho_2 \neq \rho_3$

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Introduction

This paper has been motivated by various problems and results in differential geometry. The main motivation is the study of curvature homogeneous Riemannian spaces initiated in 1960 by I.M. Singer (see Section 9–Appendix for the precise definitions and references). Up to recently, only sporadic classes of examples have been known of curvature homogeneous spaces which are not locally homogeneous. For instance, isoparametric hypersurfaces in space forms give nice examples of nontrivial curvature homogeneous spaces (see [FKM]). To study the topography of curvature homogeneous spaces more systematically, it is natural to start with the dimension $n = 3$. The following results and problems have been particularly inspiring.

1) K. Sekigawa [Se1] has constructed in 1975 a locally nonhomogeneous Riemann metric on R^3 with the constant principal Ricci curvatures $\rho_1 = \rho_2 = -1$, $\rho_3 = 0$. This example was extended by F. Tricerri, L. Vanhecke and the present author in [KTV1] as follows:

Let a Riemannian metric $\tilde{g}$ be given in a domain $U \subset R^3(w, x, y)$ by an orthonormal coframe of the form

$$\omega^1 = f(w, x)dw, \quad \omega^2 = dx - ydw, \quad \omega^3 = dy + xdw.$$

Then

(a) if $f(w, x) = a(w)e^{kx} + b(w)e^{-kx}$, then the corresponding principal Ricci curvatures are $\rho_1 = \rho_2 = -k^2$, $\rho_3 = 0$;

(b) if $f(w, x) = a(w)\cos kx + b(w)\sin kx$, then the corresponding principal Ricci curvatures are $\rho_1 = \rho_2 = k^2$, $\rho_3 = 0$.

Here k is a constant and $a(w)$, $b(w)$ are arbitrary functions of one variable. These metrics are always locally irreducible and *not* locally homogeneous.

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Moreover, it was proved in [KTV1] that the local isometry classes of each of the families (a) and (b) still depend on two arbitrary functions of 1 variable (modulo some constants).

The present author [K1] has given an explicit local classification of 3-dimensional Riemannian spaces with the constant index of nullity $\nu(p) = 1$ (which means that the null space of the Riemann curvature tensor has dimension 1 at all points). As a by-product, the following results follows:

THEOREM A. *Let* (M, g) *be a Riemannian manifold with the constant principal Ricci curvatures* $\rho_1 = \rho_2 \neq 0$, $\rho_3 = 0$, *which is not locally homogeneous. Then there is an open dense subset* $S \subset M$ *such that, in a neighborhood of each point* $p \in S$, (M, g) *is isometric to one of the spaces described above* (*i.e., to a space* $(U, \tilde{g})$ *of type* (a) *or* (b) *according to whether* $\rho_1 = \rho_2 = -k^2$ *or* $\rho_1 = \rho_2 = k^2$ *holds for some* $k > 0$, *respectively*).

2) The following conjecture is attributed to M. Gromov: Let M be a compact manifold. Then the set of isometry classes of metrics on M having at each point the same Riemannian curvature tensor as a given *homogeneous* Riemannian space $(\tilde{M}, \tilde{g})$ is a finite dimensional space. In order to prove or disprove this conjecture in dimension $n = 3$, one has to get an overview about all 3-dimensional local Riemannian metrics with the constant principal Ricci curvatures. (See [TV2] and [KTV1] for more details).

3) K. Tsukada [Ts] has classified all curvature homogeneous hypersurfaces isometrically immersed into space forms. The only cases which remained open were the situations $M_3 \to H^4(-k^2)$ and $M_3 \to S^4(k^2)$. The attack on the Tsukada's problem leads naturally to the study of Riemannian 3-manifolds with the costant Ricci roots $\rho_1 = \rho_2 \neq \rho_3 \neq 0$.

4) K. Yamato [Ya] has studied the criteria for the local homogeneity of 3-dimensional Riemannian manifolds with constant principal Ricci curvatures. He also gave first examples of Riemannian matrics on R^3 which are not locally homogeneous and have 3 distinct constant principal Ricci curvatures ρ_1, ρ_2, ρ_3. Such a Riemannian metric always exists if the numbers

$$A = \frac{\rho_1 + \rho_2 - \rho_3}{2},\ B = \frac{\rho_1 - \rho_3}{\rho_3 - \rho_2},\ C = -\frac{(\rho_1 + \rho_2)(\rho_3 - \rho_2)^2}{(\rho_2 - \rho_1)^2}$$

satisfy the inequalities $A > 0$, $C > 0$, $A + BC > 0$.

In the "degenerate" case $\rho_1 = \rho_2 \neq \rho_3 \neq 0$, the author gives a number of local and global criteria for the homogeneity but the existence of nonhomogeneous examples is not investigated.

5) J. Milnor [Mi] has proved that there exist some signatures of the Ricci tensor which are never reached by left invariant Riemann metrics on 3-dimensional Lie groups. It follows easily that one of these signatures is never reached by a homogeneous Riemannian 3-manifold. It is a stimulating problem to construct *explicitly* nonhomogeneous Riemannian metrics with constant principal Ricci curvatures and with the "forbidden" signature of the Ricci tensor.

6) D. De Turck (see [DeT] or [Be], Theorem 5.14) has proved the local existence of Riemannian metrics with the prescribed (nonsingular) Ricci tensor. Let us notice that the existence of Riemannian metrics with the *prescribed principal Ricci curvatures* does not follow from this theorem and a different method is needed.

The *contents* of this paper is as follows: In the first section we derive the basic system of partial differential equations for the problem in title. This is a system of nine PDE for 3 functions of 3 independent variables (in a convenient system of local coordinates). In Section 2 we partially integrate the previous system of PDE. We are left with only three PDE and a system of (many) algebraic equations for new functions which depend only on two variables. In Section 3 we calculate the covariant differential of the Ricci tensor. In Section 4 we study the geometric structure of the case $\rho_3 < 0$ ("hyperbolic case"), namely the existence of "asymptotic foliations". In Section 5 we introduce new local coordinates which are adapted to one of the asymptotic foliations. This essentially simplifies our task to resolve the given PDE system in the hyperbolic case. A quasiexplicit general solution is given as well as some explicit examples. In Section 6 we calculate the local isometry classes of the whole set of solutions. They depend on two arbitrary functions of 1 variable. Sections 7 and 8 are devoted to the (more difficult) elliptic case where $\rho_3 > 0$. Again a quasi-explicit solution is given and we prove that the local isometry classes depend *at least* on 2 arbitrary functions of 1 variable. In Section 9 we show that there are many nonhomogeneous metrics (some of them explicit) with constant principal Ricci curvatures which are never reached by the homogeneous metrics.

For attacking our problem with constant $\rho_1 = \rho_2 \neq \rho_3 \neq 0$ we are using a modification of the direct method from [K1]. The classification problem for the case $\rho_1 \neq \rho_2 \neq \rho_3 \neq \rho_1$ seems to be of a different nature and it will be studied in a subsequent paper.

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1. The basic system of PDE for the problem

Let (M, g) be a 3-dimensional Riemannian manifold whose Ricci tensor $\hat{R}_{ij}$ has constant eigenvalues $\rho_1 = \rho_2 \neq \rho_3$, $\rho_3 \neq 0$. Choose a neighborhood $\tilde{U}$ of a fixed point $m \in M$ and a smooth vector field E_3 of unit eigenvectors corresponding to the Ricci root ρ_3 in $\tilde{U}$. Let $S : D^2 \to \tilde{U}$ be a surface through m which is transversal with respect to all trajectories generated by E_3 at all cross-points and not orthogonal to such a trajectory at m. (The vector field E_3 determines an orientation of S). Then there is a normal neighborhood $U \ni m$, $U \subset \tilde{U}$, with the property that each point $p \in U$ is projected to exactly one point $\pi(p) \in S$ via some trajectory. We fix any local coordinate system (w, x) on S and then a local coordinate system (w, x, y) on U such that the values $w(p)$, $x(p)$ are defined as $w(\pi(p))$, $x(\pi(p))$ for each $p \in U$, and

$y(p) = d^+(\pi(p), p) =$ the oriented length of the trajectory joining p with $\pi(p)$.

Then $E_3 = \partial/\partial y$ can be extended in U to an orthonormal moving frame $\{E_1, E_2, E_3\}$. Let $\{\omega^1, \omega^2, \omega^3\}$ be the corresponding dual coframe. Then ω^i are of the form

$$\omega^i = a^i dw + b^i dx \ (i = 1, 2), \ \omega^3 = dy + Hdw + Gdx. \tag{1.1}$$

The Ricci tensor expressed with respect to $\{E_1, E_2, E_3\}$ has the form $\hat{R}_{ij} = \rho_i \delta_{ij}$. Because each ρ_i is expressed through the sectional curvatures K_{ij} by the formula $\rho_i = \hat{R}_{ii} = \sum_{j \neq i} K_{ij}$, there exist constants k and $\tilde{c} \neq 0$ such that

$$K_{12} = k, \quad K_{13} = K_{23} = \tilde{c}, \quad \rho_1 = \rho_2 = k + \tilde{c}, \quad \rho_3 = 2\tilde{c}. \tag{1.2}$$

Define now the connection form $\{\omega^i_j\}$ by the standard formulas

$$d\omega^i - \sum_j \omega^j \wedge \omega^i_j = 0, \ \omega^i_j + \omega^j_i = 0 \ (i, j = 1,2,3). \tag{1.3}$$

Because $R_{ijkl} = 0$ whenever at least three of the indices i, j, k, l are distinct, the formulas (1.2) are equivalent to

$$\begin{cases} dw^1_2 + \omega^1_3 \wedge \omega^3_2 = k\omega^1 \wedge \omega^2, \\ dw^1_3 + \omega^1_2 \wedge \omega^2_3 = \tilde{c}\omega^1 \wedge \omega^3, \\ dw^2_3 + \omega^2_1 \wedge \omega^1_3 = \tilde{c}\omega^2 \wedge \omega^3. \end{cases} \tag{1.5}$$

Next, differentiate the equations (1.5) and then substitute from (1.5). We obtain easily

$$\omega^1_3 \wedge \omega^1 \wedge \omega^2 = 0, \quad \omega^2_3 \wedge \omega^1 \wedge \omega^2 = 0, \tag{1.6}$$

and

(1.7) $$d(\omega^1 \wedge \omega^2) = 0.$$

The formulas (1.6) mean that ω_3^1, ω_3^2 are linear combinations of ω^1, ω^2 only, and the third equation of (1.3) then means that $d\omega^3$ is a multiple of $\omega^1 \wedge \omega^2$. From (1.1) we see that the functions G, H are independent of y.

Now, there is a local coordinate system $(\bar{w}, \bar{x}, y)$, $\bar{w} = \bar{w}(w, x)$, $\bar{x} = \bar{x}(w, x)$ (possibly in a smaller neighborhood of m) such that

(1.8) $$\omega^1 = P^1 d\bar{w} + Q^1 d\bar{x}, \quad \omega^2 = P^2 d\bar{w} + Q^2 d\bar{x}, \quad \omega^3 = dy + \bar{H}(\bar{w}, \bar{x}) d\bar{w}.$$

Indeed, because the surface S is not orthogonal to the vector field E_3 at m, the Pfaffian form $H(w, x)dw + G(w, x)dx$ from (1.1) is nonzero in a neighborhood of m in M. Then we define $\bar{w}(w, x)$ as a potential function of the Pfaffian equation $Hdw + Gdx = 0$, and the second function $\bar{x}(w, x)$ can be defined as an arbitrary smooth function which is functionally independent of $\bar{w}$. In addition, there are new Pfaffian forms $\tilde{\omega}^1$, $\tilde{\omega}^2$ such that $(\tilde{\omega}^1)^2 + (\tilde{\omega}^2)^2 = (\omega^1)^2 + (\omega^2)^2$ and $\tilde{\omega}^1$ does not involve the differential $d\bar{x}$. We can summarize:

PROPOSITION 1.1. *In a normal neighborhood of any point $m \in M$ there exists an orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$ and a local coordinate system (w, x, y) such that*

(1.9) $$\omega^1 = f dw, \quad \omega^2 = A dx + C dw, \quad \omega^3 = dy + H dw.$$

Here f, A, C, $fA \neq 0$, are smooth functions of w, x, y and $H = H(w, x)$.

The formula (1.7) can be now written in the form

(1.10) $$(fA)'_y = 0, \text{ i.e., } fA = 1/\chi(w, x) \text{ for some function } \chi \neq 0.$$

We see immediately that, introducing a new variable $\bar{x}(w, x)$ instead of x, we obtain after such a transformation

(1.11) $$fA = 1, \text{ i.e., } \chi = 1.$$

For the connection form we obtain easily (using (1.9) and (1.10))

(1.12) $$\omega_2^1 = -A\alpha dx + R dw + \beta dy, \quad \omega_3^1 = A\beta dx + S dw, \quad \omega_3^2 = A'_y dx + T dw,$$

where

(1.13) $$\alpha = \chi(A'_w - C'_x - HA'_y), \quad \beta = \frac{\chi}{2}(H'_X + AC'_y - CA'_y),$$

and

$$R = \chi f f'_x - C\alpha + H\beta, \tag{1.14}$$

$$S = f'_y + C\beta, \tag{1.15}$$

$$T = C'_y - f\beta. \tag{1.16}$$

The curvature conditions (1.5) then give a system of nine PDE for our problem:

(A1) $(A\alpha)'_y + \beta'_x = 0,$

(A2) $R'_y - \beta'_w = 0,$

(A3) $(A\alpha)'_w + R'_x + SA'_y - A\beta T = -k\chi^{-1},$

(B1) $A''_{yy} - A\beta^2 = -\tilde{c}A,$

(B2) $-A''_{yw} + T'_x + A(\alpha S + \beta R) = \tilde{c}AH,$

(B3) $T'_y - S\beta = -\tilde{c}C,$

(C1) $(A\beta)'_y + A'_y\beta = 0,$

(C2) $S'_x - (A\beta)'_w - (A\alpha T + A'_y R) = 0,$

(C3) $S'_y + T\beta = -\tilde{c}f.$

2. The first integrals and the reduction of the basic PDE system

The aim of this section is to replace the PDE's of the series (B) and (C) by a system of algebraic equations for the new functions depending only on w and x. First of all, we eliminate the equations (B2) and (C2).

PROPOSITION 2.1. *The equation* (B2) *is a consequence of* (A1) *and* (B1).

Proof. Using (1.14)–(1.16) we obtain

$$\begin{aligned} &T'_x - A''_{yw} + A(\alpha S + \beta R) \\ &= C''_{yx} - f'_x\beta - f\beta'_x - A''_{yw} + A(\chi\beta f f'_x + \alpha f'_y + H\beta^2). \end{aligned}$$

From (1.13) we get, using also (1.10),

$$A''_{yw} = (\alpha f A + C'_x + HA'_y)'_y$$

and after the substitution we obtain, using again (1.10)

$$T'_x - A''_{yw} + A(\alpha S + \beta R) = f(-\beta'_x - (A\alpha)'_y) + H(-A''_{yy} + A\beta^2).$$

This is equal to $\tilde{c}AH$ due to (A1) and (B1).

PROPOSITION 2.2. The equation (C2) is a consequence of (A1), (A2) and (C1).

Proof. First we have, due to (1.14) and (1.10)

$$(AR)'_y = (f'_x - AC\alpha + AH\beta)'_y = f''_{xy} - (AC\alpha)'_y + H(A\beta)'_y.$$

From (C1) we get

$$(AR)'_y = f''_{xy} - (AC\alpha)'_y - H\beta A'_y. \tag{2.1}$$

Using (1.14)–(1.16) and also (A2), we obtain

$$\begin{aligned} &S'_x - (A\alpha T + A'_y R) - (A\beta)'_w \\ &= f''_{xy} + (C\beta)'_x - A\alpha(C'_y - f\beta) - (AR)'_y - \beta A'_w. \end{aligned}$$

Substituting now from (2.1) and using (1.10) we can rewrite the right-hand side in the form

$$-\beta(A'_w - C'_x - HA'_y) + \alpha\beta\chi^{-1} + C(\beta'_x + (A\alpha)'_y),$$

and this is zero due to $(1.13)_1$ and as a consequence of (A1).

PROPOSITION 2.3. *The equations* (B3) *and* (C3) *are satisfied if and only if*

$$fT - CS = \varphi_0, \tag{2.2}$$

where $\varphi_0 = \varphi_0(w, x)$ *is an arbitrary function and, moreover,*
(a) *in the* hyperbolic case $\tilde{c} = -\lambda^2$ *we have*

$$S^2 + T^2 = \lambda(\varphi_1 e^{2\lambda y} + \varphi_2 e^{-2\lambda y} - \varphi_3), \tag{2.3a}$$

$$fS + CT = \varphi_1 e^{2\lambda y} - \varphi_2 e^{-2\lambda y}, \tag{2.4a}$$

$$f^2 + C^2 = \frac{1}{\lambda}(\varphi_1 e^{2\lambda y} + \varphi_2 e^{-2\lambda y} + \varphi_3), \tag{2.5a}$$

where the functions $\varphi_i = \varphi_i(w, x)$, $i = 1, 2, 3$, *satisfy the single relation*

$$\varphi_0^2 - 4\varphi_1\varphi_2 + \varphi_3^2 = 0; \tag{2.6a}$$

(b) *in the* elliptic case $\tilde{c} = \lambda^2$ *we have*

$$S^2 + T^2 = \lambda(-\varphi_1 \sin 2\lambda y + \varphi_2 \cos 2\lambda y + \varphi_3), \tag{2.3b}$$

(2.4b) $$fS + CT = \varphi_1 \cos 2\lambda y + \varphi_2 \sin 2\lambda y,$$

(2.5b) $$f^2 + C^2 = \frac{1}{\lambda}(\varphi_1 \sin 2\lambda y - \varphi_2 \cos 2\lambda y + \varphi_3),$$

where the functions $\varphi_i = \varphi_i(w, x)$, $i = 1,2,3$, *satisfy the single relation*

(2.6b) $$\varphi_0^2 + \varphi_1^2 + \varphi_2^2 - \varphi_3^2 = 0.$$

Proof. First, using (1.15), (1.16), (B3) and (C3) to express the derivatives f'_y, C'_y, T'_y, S'_y, we obtain $(fT - CS)'_y = 0$, which is equivalent to (2.2).

Further, put $X = S^2 + T^2$, $Y = fS + CT$, $Z = f^2 + C^2$. Using the same argument as before we obtain a system of PDE

(2.7) $$\frac{\partial X}{\partial y} = -2\tilde{c}Y, \frac{\partial Y}{\partial y} = X - \tilde{c}Z, \frac{\partial Z}{\partial y} = 2Y.$$

Hence the formulas (2.3a)–(2.5a), or (2.3b)–(2.5b) follow. The last equations (2.6a), (2.6b) are consequences of the algebraic identity

$$(fT - CS)^2 + (fS + CT)^2 = (f^2 + C^2)(S^2 + T^2).$$

Hence all formulas above are consequences of (B3) and (C3).

The converse part follows easily: differentiating (2.2) and writing explicitly the equation $\partial Y/\partial y = X - \tilde{c}Z$, we get a system of two linear algebraic equations for S'_y, T'_y, which can be solved by the Cramer's rule. Hence (B3) and (C3) follow.

PROPOSITION 2.4. *There is a function* $\varphi_4(w, x)$ *such that*

(2.8) $$SA = \varphi_4(w, x).$$

Further, the equation (A3) *simplifies to the form*

(2.9) $$(A\alpha)'_w + R'_x = -(k + \tilde{c})\chi^{-1} = -\rho_1\chi^{-1}.$$

Proof. Using (C3) and (1.15), (1.16) we get

(2.10) $$(SA)'_y = SA'_y - A(T\beta + \tilde{c}f) = f'_yA'_y + (CA'_y - AC'_y) + f(A\beta^2 - \tilde{c}A).$$

Due to (B1) we obtain hence

(2.11) $$(SA)'_y = f'_yA'_y + \beta(CA'_y - AC'_y) + fA''_{yy} = (A'_yf)'_y + \beta(CA'_y - AC'_y).$$

On the other hand, using (1.15) first and (C1) later, we get

(2.12) $$(SA)'_y = [f'_y A + (A\beta) C]'_y = (f'_y A)'_y - \beta(CA'_y - AC'_y).$$

As the arithmetic mean-value of (2.11) and (2.12) we obtain

(2.13) $$(SA)'_y = \frac{1}{2}(fA)'_y = 0,$$

(using also (1.10)). Hence (2.8) follows. Then (2.13) and (2.10) imply

(2.14) $$SA'_y - AT\beta = \tilde{c}Af = \tilde{c}\chi^{-1},$$

and the equation (A3) takes on the form (2.9), q.e.d.

PROPOSITION 2.5. *The equations* (B1) *and* (C1) *are satisfied if and only if*

(2.15) $$\beta = \lambda a_0 / A^2$$

where $a_0 = a_0(w, x)$ *is an arbitrary function and, moreover,*
(a) *in the* hyperbolic case $\tilde{c} = -\lambda^2$ *we have*

(2.16a) $$A^2 = a_1 e^{2\lambda y} + a_2 e^{-2\lambda y} + a_3, \quad a_i = a_i(w, x),$$

where

(2.17a) $$a_0^2 + a_3^2 - 4a_1 a_2 = 0;$$

(b) *in the* elliptic case $\tilde{c} = \lambda^2$ *we have*

(2.16b) $$A^2 = a_1 \cos 2\lambda y + a_2 \sin 2\lambda y + a_3, \quad a_i = a_i(w, x),$$

where

(2.17b) $$a_0^2 + a_1^2 + a_2^2 - a_3^2 = 0.$$

Proof. The equation (C1) means $(A^2\beta)'_y = 0$ and hence (2.15) follows. (B1) can be written in the form

(2.18) $$2A'_y A''_{yy} = 2\lambda^2 (a_0)^2 A'_y A^{-3} - 2\tilde{c}AA'_y.$$

Integrating with respect to y we see that there is a function $\phi(w, x)$ such that

$$(A'_y)^2 = -\lambda^2 (a_0)^2 A^{-2} - \tilde{c}A^2 + \phi(w, x).$$

Hence

(2.19) $$(A^2)'_y = \pm 2\sqrt{-\tilde{c}A^4 + \phi A^2 - \lambda^2 (a_0)^2}.$$

For $\tilde{c} = -\lambda^2$ we obtain the general solution

$$A^2 = (\mu/\lambda)/\cosh t - \phi/2\lambda^2$$

where

$$\mu = \sqrt{\phi^2/4\lambda^2 + \lambda^2 {a_0}^2},\ t = \pm 2\lambda y + \nu(w, x).$$

For $\tilde{c} = \lambda^2$ we obtain the general solution

$$A^2 = (\mu/\lambda) \cos t + \phi/2\lambda^2$$

where

$$\mu = \sqrt{\phi^2/4\lambda^2 - \lambda^2 {a_0}^2},\ t = \pm 2\lambda y + \nu(w, x).$$

Here $\nu(w, x)$ is a new arbitrary function. Hence (2.16a), or (2.16b), follows with specific coefficients a_1, a_2, a_3 satisfying (2.17a), or (2.17b), respectively.

CONVENTION. *Up to the end of this section we always assume that the local coordinates* (w, x, y) *are fixed in such a way that* (1.9) *and, in addition,* (1.11) *holds, i.e.,* $\chi = 1$.

PROPOSITION 2.6. *For* $\tilde{c} = -\lambda^2$ *we have*

(2.20a) $$2\lambda a_0 AC = (\varphi_5 + 2\lambda) a_1 e^{2\lambda y} + (\varphi_5 - 2\lambda) a_2 e^{-2\lambda y} + \varphi_5 a_3,$$

and for $\tilde{c} = \lambda^2$ *we have*

(2.20b) $$2\lambda a_0 AC = (\varphi_5 a_1 + 2\lambda a_2) \cos 2\lambda y + (\varphi_5 a_2 - 2\lambda a_1) \sin 2\lambda y + \varphi_5 a_3,$$

where $\varphi_5 = \varphi_5(w, x)$ *is some function.*

In addition, in the elliptic case we always have $a_0 \neq 0$.

Proof. Subtracting the formulas (2.11) and (2.12) we get

$$(fA'_y - Af'_y)'_y + 2\beta(CA'_y - AC'_y) = 0, \text{ i.e.,}$$
$$(A^2(f/A)'_y)'_y + 2\beta A^2(C/A)'_y = 0.$$

Using (1.11) and (2.15) we get $(A^2(1/A^2)'_y)'_y + 2\lambda a_0(C/A)'_y = 0$.
By the integration we obtain

$$-(A^2)'_y A^{-2} + (2\lambda a_0 AC)A^{-2} = \varphi_5(w, x),$$

where φ_5 is an arbitrary function. This can be rewritten as

(2.21) $$2\lambda a_0 AC = \varphi_5 A^2 + (A^2)'_y$$

and then we substitute from (2.16a), or (2.16b), respectively. Hence (2.20a,b) follow.

In the elliptic case, let us suppose $a_0 = 0$. Then (2.20b) implies

$$\varphi_5 a_1 + 2\lambda a_2 = 0,\ \varphi_5 a_2 - 2\lambda a_1 = 0,$$

and hence $2\lambda(a_1^2 + a_2^2) = 0$. Then (2.17b) implies $a_3 = 0$ and hence $A^2 = 0$, a contradiction with $fA \neq 0$.

The following proposition deals with a more general situation.

PROPOSITION 2.7. *For $\tilde{c} = -\lambda^2$ we have*

$$AC = b_1 e^{2\lambda y} + b_2 e^{-2\lambda y} + b_3 \tag{2.22a}$$

and for $\tilde{c} = \lambda^2$ we have

$$AC = b_1 \cos 2\lambda y + b_2 \sin 2\lambda y + b_3, \tag{2.22b}$$

where $b_i = b_i(w, y)$, $i = 1,2,3$.

Proof. For $a_0 \neq 0$ the relations (2.22a,b) follows from (2.20a,b), which is always the case for $\tilde{c} = \lambda^2$. Suppose now $\tilde{c} = -\lambda^2$ and $a_0 = 0$. Then $\beta = 0$ and from (1.16), (B3) we infer $C''_{yy} = -\tilde{c}C = \lambda^2 C$. Hence

$$C = re^{\lambda y} + se^{-\lambda y}, \tag{2.23a}$$

where r, s are functions of w, x only. On the other hand, (2.16a) and (2.17a) imply

$$A = pe^{\lambda y} + qe^{-\lambda y}, \tag{2.24a}$$

where p, q are functions of w, x only. Hence (2.22a) follows.

PROPOSITION 2.8. *Introduce the function*

$$h(w, x) = H'_x. \tag{2.25}$$

In the hyperbolic case we have

$$\begin{cases} ha_1 = 2\lambda(a_1 b_3 - a_3 b_1),\ ha_2 = 2\lambda(a_3 b_2 - a_2 b_3), \\ ha_3 = 2\lambda(a_0 + 2a_1 b_2 - 2a_2 b_1), \end{cases} \tag{2.26a}$$

and in the elliptic case we have

$$\begin{cases} ha_1 = 2\lambda(b_3 a_2 - b_2 a_3),\ ha_2 = 2\lambda(b_1 a_3 - a_1 b_3), \\ ha_3 = 2\lambda(a_0 + b_1 a_2 - a_1 b_2). \end{cases} \tag{2.26b}$$

Moreover, if $a_0 \neq 0$, *then*

$$h = 2\varepsilon\lambda a_3/a_0,\ \varepsilon = \operatorname{sgn}\tilde{c}. \tag{2.27}$$

Proof. From $(1.13)_2$ we get $h = 2\beta - (AC)'_y + 2CA'_y$. Then (2.15) implies

$$hA^2 = 2\lambda a_0 - (A^2)(AC)'_y + (AC)(A^2)'_y. \tag{2.28}$$

Now we use (2.16a,b) and (2.22a,b) to get (2.26a,b). To obtain (2.27) we use (2.26a,b), (2.20a,b), (2.17a,b) and the direct check.

Next, we shall derive additional algebraic relations. From (2.2) and (2.8) we obtain (under the condition $\chi = 1$)

$$S = \varphi_4 f,\ T = \varphi_0 A + \varphi_4 C. \tag{2.29}$$

Substitute from here into the differential equation (C3). We obtain

$$\varphi_4 f'_y + \varphi_0 A\beta + \varphi_4 C\beta = -\tilde{c}f.$$

Multiplying this equation by $2A^3$ and using (1.11), (2.15) we get

$$-2\varphi_4 AA'_y + 2a_0\varphi_0\lambda A^2 + 2\varphi_4 a_0\lambda AC = -2\tilde{c}A^2. \tag{2.30}$$

Then (2.21) implies

$$2\lambda\varphi_0 a_0 + \varphi_4\varphi_5 = -2\tilde{c}. \tag{2.31}$$

Further, from (2.29) we obtain also

$$CT + fS = \varphi_0 AC + \varphi_4(f^2 + C^2).$$

Substituting from (2.4a,b), (2.5a,b) we get in the hyperbolic case

$$\varphi_0 AC = \varphi_1(1 - \varphi_4/\lambda)e^{2\lambda y} - \varphi_2(1 + \varphi_4/\lambda)e^{-2\lambda y} - \varphi_3\varphi_4/\lambda, \tag{2.32a}$$

and in the elliptic case

$$\varphi_0 AC = (\varphi_1 + \varphi_2\varphi_4/\lambda)\cos 2\lambda y + (\varphi_2 - \varphi_1\varphi_4/\lambda)\sin 2\lambda y - \varphi_3\varphi_4/\lambda. \tag{2.32b}$$

Another consequence of (2.29) is

$$S^2 + T^2 = \varphi_4^2(f + C^2) + (2\varphi_0 AC)\varphi_4 + \varphi_0^2 A^2.$$

Using the formulas (2.3a,b), (2.5a,b), (2.16a,b) and (2.32a,b), we obtain finally

$$\varphi_0^2\lambda a_1 = \varphi_1(\varphi_4 - \lambda)^2,\ \varphi_0^2\lambda a_2 = \varphi_2(\varphi_4 + \lambda)^2,\ \varphi_0^2\lambda a_3 = \varphi_3(\varphi_4^2 - \lambda^2) \tag{2.33a}$$

in the hyperbolic case, and

(2.33b)
$$\begin{cases} \varphi_0^2\lambda a_1 = \varphi_2(\lambda^2 - \varphi_4^2) - 2\varphi_1\varphi_4\lambda, \\ \varphi_0^2\lambda a_2 = \varphi_1(\varphi_4^2 - \lambda^2) - 2\varphi_2\varphi_4\lambda,\ \varphi_0^2\lambda a_3 = \varphi_3(\lambda^2 + \varphi_4^2) \end{cases}$$

in the elliptic case.

Consider now the identity $(AC)^2 = A^2(f^2 + C^2) - 1$. Substituting from (2.5a,b), (2.16a,b) and (2.22a,b) we get a system of quadratic equations

(2.34a)
$$\begin{cases} \lambda b_1^2 = a_1\varphi_1, \\ \lambda(2b_1b_2 + b_3^2) = a_1\varphi_2 + a_2\varphi_1 + a_3\varphi_3 - \lambda, \\ \lambda b_2^2 = a_2\varphi_2, \\ 2\lambda b_1b_3 = \varphi_3 a_1 + \varphi_1 a_3, \\ 2\lambda b_2 b_3 = \varphi_3 a_2 + \varphi_2 a_3 \end{cases}$$

in the hyperbolic case, and another system of quadratic equations

(2.34b)
$$\begin{cases} \lambda(b_1^2 - b_2^2) = -(a_1\varphi_2 + a_2\varphi_1), \\ 2\lambda b_1b_2 = a_1\varphi_1 - a_2\varphi_2, \\ \lambda(b_1^2 + b_2^2 + 2b_3^2) = a_2\varphi_1 - a_1\varphi_2 + 2a_3\varphi_3 - 2\lambda, \\ 2\lambda b_1b_3 = a_1\varphi_3 - a_3\varphi_2, \\ 2\lambda b_2b_3 = a_2\varphi_3 + a_3\varphi_1 \end{cases}$$

in the elliptic case.

In the notation (2.22a,b), we can rewrite (2.20a,b) in the form

(2.35a) $\quad 2\lambda a_0 b_1 = (\varphi_5 + 2\lambda)a_1, \quad 2\lambda a_0 b_2 = (\varphi_5 - 2\lambda)a_2, \quad 2\lambda a_0 b_3 = \varphi_5 a_3,$

or

(2.35b) $\quad 2\lambda a_0 b_1 = \varphi_5 a_1 + 2\lambda a_2, \quad 2\lambda a_0 b_2 = \varphi_5 a_2 - 2\lambda a_1, \quad 2\lambda a_0 b_3 = \varphi_5 a_3,$

respectively.

Also, we can rewrite (2.32a,b) in the form

(2.36a) $\quad \lambda\varphi_0 b_1 = \varphi_1(\lambda - \varphi_4), \quad \lambda\varphi_0 b_2 = -\varphi_2(\lambda + \varphi_4), \quad \lambda\varphi_0 b_3 = -\varphi_3\varphi_4,$

or

(2.36b) $\quad \lambda\varphi_0 b_1 = \lambda\varphi_1 + \varphi_2\varphi_4, \quad \lambda\varphi_0 b_2 = \lambda\varphi_2 - \varphi_1\varphi_4, \quad \lambda\varphi_0 b_3 = -\varphi_3\varphi_4,$

respectively.

We conclude with the main results of this section.

THEOREM 2.9. *Let $\lambda \neq 0$ be a constant. Let $\varphi_0, \varphi_1, \dots, \varphi_5, a_0, a_1, a_2, a_3, b_1, b_2, b_3, h$ be functions of two variables w, x defined in some domain $V \subset R^2(w, x)$,* *satisfying eight collections of algebraic equations* (2.6), (2.17), (2.26), (2.31), (2.33), (2.34), (2.35), (2.36) (*either of the hyperbolic type, or of the elliptic type*) *with the corresponding parameter* λ, *and such that* $a_1^2 + a_2^2 + a_3^2 > 0$ *in* V.

Let A, f, C, H *be functions defined in a domain* $U \subset R^3(w, x, y)$ *where* $A \neq 0$, *by the formulas* (2.16), (1.11), (2.22) *and* (2.25) *of the corresponding type, and let the metric* g *be defined on* U *by* (1.9). *Further, let* α, β, R *be defined as in* $(1.13)_1$, (2.15), (1.14), *with* $\chi = 1$. *Then the curvature conditions* (1.5) *are satisfied for the metric* g (*with a fixed* k *and with the corresponding* $\tilde{c} = \pm \lambda^2$) *if, and only if, the system of* PDE (A1), (A2) *and* (2.9) (*with* $\chi = 1$) *is satisfied.*

Remark. The algebraic conditions mentioned above are, of course, far from being independent, but they are all useful.

The proof follows from the whole series of assertions and formulas given in this section.

3. The Riemannian invariants

Let (M, g) be given locally as in Proposition (1.1). In this short section we shall assume that the function $\chi(w, x)$ from (1.10) is *arbitrary*.

We rewrite the formulas (1.12) using the forms ω^1, ω^2, ω^3 as a basis. It follows

$$(3.1) \qquad \begin{cases} \omega_2^1 = \chi f_x' \omega^1 - \alpha\omega^2 + \beta\omega^3, \\ \omega_3^1 = f^{-1} f_y' \omega^1 + \beta\omega^2, \\ \omega_3^2 = (\beta - h\chi)\omega^1 + A^{-1}A_y'\omega^2 \quad (h = H_x'). \end{cases}$$

We shall also write, for the brevity,

$$(3.2) \qquad \omega_3^1 = a\omega^1 + b\omega^2, \ \omega_3^2 = c\omega^1 + e\omega^2,$$

where

$$(3.3) \qquad a = f^{-1}f_y', \ b = \beta, \ c = \beta - h\chi, \ e = A^{-1}A_y'.$$

Using the standard formula from [KN1]

$$(3.4) \qquad \nabla_{E_j} E_i = \sum_k \omega_i^{\ k}(E_j)E_k \quad (i, j = 1,2,3)$$

we obtain

$$\left\{\begin{array}{l} \nabla_{E_1}E_1 = -\chi f_x' E_2 - aE_3,\ \nabla_{E_1}E_2 = \chi f_x' E_1 - cE_3, \\ \nabla_{E_2}E_1 = \alpha E_2 - bE_3,\ \nabla_{E_2}E_2 = -\alpha E_1 - eE_3, \\ \nabla_{E_1}E_3 = aE_1 + cE_2,\ \nabla_{E_2}E_3 = bE_1 + eE_2, \\ \nabla_{E_3}E_1 = -bE_2,\ \nabla_{E_3}E_2 = bE_1,\ \nabla_{E_3}E_3 = 0. \end{array}\right. \tag{3.5}$$

For the Ricci tensor $\hat{R} = \hat{R}_{ij}$ we get, using the notation (1.2) and the adapted local orthonormal coframe ω^i,

$$\hat{R} = (k + \tilde{c})(\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2) + 2\tilde{c}(\omega^3 \otimes \omega^3). \tag{3.6}$$

Using (3.1) and standard formula

$$\nabla_X \omega^i = -\sum_j \omega_j^i(X)\omega^j \tag{3.7}$$

we obtain

$$\begin{aligned} \nabla \hat{R} = (\tilde{c} - k)[(a\omega^1 + b\omega^2) \otimes (\omega^1 \otimes \omega^3 + \omega^3 \otimes \omega^1) \\ + (c\omega^1 + e\omega^2) \otimes (\omega^2 \otimes \omega^3 + \omega^3 \otimes \omega^2)], \end{aligned} \tag{3.8}$$

where a, b, c, e are given by (3.3). Hence we get also

$$\| \nabla \hat{R} \|^2 = 2(\tilde{c} - k)^2(a^2 + b^2 + c^2 + e^2) = 2(\rho_1 - \rho_3)^2(a^2 + b^2 + c^2 + e^2). \tag{3.9}$$

4. The existence of asymptotic leaves in the hyperbolic case

The basic assumptions in this section are: a) (M, g) is of hyperbolic type, b) $\chi = 1$, i.e., $fA = 1$ in the given local coordinates.

We shall start with some additional algebraic formulas.

PROPOSITION 4.1. *We have*

$$\varphi_5 = 2\varphi_4, \tag{4.1}$$

$$\varphi_0 A^2 - \lambda a_0(f^2 + C^2) + \varphi_5 AC + h = 0. \tag{4.2}$$

Proof. Let first $a_0\varphi_0 \neq 0$. Then $a_1 a_2 \neq 0$ holds due to (2.17a), and $\varphi_1\varphi_2 \neq 0$ holds due to (2.6a). From the formulas $(2.34\text{a})_{1,3}$ we see that $b_1 b_2 \neq 0$. Using (2.35a) and (2.36a) we obtain

$$\frac{b_1}{b_2} = \frac{(\varphi_5 + 2\lambda)}{(\varphi_5 - 2\lambda)}\frac{a_1}{a_2} = \frac{\varphi_1(\varphi_4 - \lambda)}{\varphi_2(\varphi_4 + \lambda)}.$$

Substituting for a_1/a_2 from (2.33a) we get

$$(\varphi_5 + 2\lambda)/(\varphi_5 - 2\lambda) = (\varphi_4 + \lambda)/(\varphi_4 - \lambda),$$

and hence (4.1) follows.

Suppose now that $\varphi_0 = 0$. Then (2.31) implies (because $\tilde{c} = -\lambda^2$)

$$\varphi_4\varphi_5 = 2\lambda^2 \neq 0. \tag{4.3}$$

From (2.36a) it follows

$$\varphi_1(\varphi_4 - \lambda) = 0, \quad \varphi_2(\varphi_4 + \lambda) = 0, \quad \varphi_3 = 0. \tag{4.4}$$

Because $\lambda \neq 0$ and $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 > 0$, we get two cases:

(a) $\varphi_1 \neq 0$, $\varphi_2 = 0$. Then $\varphi_4 = \lambda$ and (4.3) implies $\varphi_5 = 2\lambda = 2\varphi_4$;

(b) $\varphi_1 = 0$, $\varphi_2 \neq 0$. Then $\varphi_4 = -\lambda$ and (4.3) implies $\varphi_5 = -2\lambda = 2\varphi_4$.

Hence (4.1) follows once again. For $a_0 = 0$ we use (2.35a), and the proof is similar.

To derive (4.2), let us suppose first $a_0 \neq 0$. Then (2.35a) and (4.1) imply

$$b_1 = \frac{(\varphi_4 + \lambda)a_1}{\lambda a_0}, \; b_2 = \frac{(\varphi_4 - \lambda)a_2}{\lambda a_0}, \; b_3 = \frac{\varphi_4 a_3}{\lambda a_0}, \tag{4.5}$$

and (2.31) can be rewritten, due to (4.1), in the form

$$\lambda\varphi_0 a_0 = \lambda^2 - \varphi_4^2. \tag{4.6}$$

Now we first substitute for A^2, $f^2 + C^2$, AC, φ_5 and h into (4.2) from (2.16a), (2.5a), (2.22a) with (4.5), (4.1) and (2.27), respectively.

If $\varphi_0 \neq 0$, we express a_1, a_2, a_3 from (2.33a) and use (4.6) to eliminate φ_0. Then the identity (4.2) follows.

If $a_0 \neq 0$, $\varphi_0 = 0$, we calculate φ_1, φ_2 explicitly from $(2.34\text{a})_{1,3}$. From (4.5) and (4.6) we get $b_1 b_2 = 0$, hence $\varphi_1\varphi_2 = 0$, and (2.6a) implies $\varphi_3 = 0$. Substituting for φ_i into (4.2), we easily check this identity.

Suppose finally $a_0 = 0$. Then (2.21) and (2.28) imply $h = -\varphi_5(AC) - (AC)'_y$, and (4.2) is equivalent to $\varphi_0 A^2 - (AC)'_y = 0$. This is easily checked by using (4.6), (2.33a), (2.35a) and (2.36a) (for $\varphi_0 \neq 0$), or $(2.34\text{a})_{1,3}$ (for $\varphi_0 = 0$).

Now, we shall introduce a useful geometric concept (cf. also [K1]):

DEFINITION 4.2. An *asymptotic leaf* of (M, g) is a smooth surface $N \subset M$ such that a) N is tangent to the field ξ of principal Ricci directions corresponding to the principal Ricci curvature ρ_3, b) the family $\{T_pN\}_{p\in N}$ of tangent planes of N

is parallel along each trajectory of ξ contained in N (with respect to the Riemannian connection of (M, g)).

Now, we are going to prove

THEOREM 4.3. *Let (M, g) be of hyperbolic type. Then for each point $m \in M$ there is a neighborhood $U \ni m$ and an adapted local coordinate system (w, x, y) (i.e., one satisfying* (1.9) *and* (1.11)) *with the following property: a surface $N \subset U$ is an asymptotic leaf if and only if its tangent planes satisfy the quadratic equation*

$$\lambda a_0 dx^2 + 2\varphi_4 dxdw - \varphi_0 dw^2 = 0 \tag{4.7}$$

along N. As a consequence, there are two distinct asymptotic leaves through any point $p \in U$.

Proof. Choose an adapted normal coordinate neighborhood $U(w, x, y)$ of m. Let now $N \subset U$ be an asymptotic leaf. Then the tangent planes along N contain the vector field E_3 and can be described by a formula

$$\sin\varphi \cdot \omega^1 + \cos\varphi \cdot \omega^2 = 0, \tag{4.8}$$

where φ is a smooth function on N. This means

$$T_pN = \operatorname{span}\{\cos\varphi \cdot E_1 - \sin\varphi \cdot E_2,\ E_2\}_p,\ p \in N. \tag{4.9}$$

Now, the integrability condition

$$[\cos\varphi \cdot E_1 - \sin\varphi \cdot E_2,\ E_3] \in \operatorname{span}\{\cos\varphi \cdot E_1 - \sin\varphi \cdot E_2,\ E_3\} \tag{4.10}$$

and the parallelism condition

$$\nabla_{E_3}(\cos\varphi \cdot E_1 - \sin\varphi \cdot E_2) \in \operatorname{span}\{\cos\varphi \cdot E_1 - \sin\varphi \cdot E_2,\ E_3\} \tag{4.11}$$

must be satisfied along N. Hence it follows that also the condition

$$\nabla_{\cos\varphi \cdot E_1 - \sin\varphi \cdot E_2} E_3 \in \operatorname{span}\{\cos\varphi \cdot E_1 - \sin\varphi \cdot E_2,\ E_3\} \tag{4.12}$$

holds along N. From the formulas (3.5) we obtain that (4.12) is equivalent to

$$\sin^2\varphi \cdot b + \sin\varphi\cos\varphi(e - a) - \cos^2\varphi \cdot c = 0. \tag{4.13}$$

Using (4.8) as a proportion formula, we see that the tangent distribution of N satisfies the equation

$$c(\omega^1)^2 + (e - a)\omega^1\omega^2 - b(\omega^2)^2 = 0. \tag{4.14}$$

Substituting for ω^1, ω^2 from (1.8) and expressing a, b, c, e in the form (3.3) we get hence (using also (1.11) and the derivative of this relation)

$$(\lambda a_0 A^2)dx^2 + (2\lambda a_0 AC - (A^2)'_y)dxdw + (h - \lambda a_0 f^2 + \lambda a_0 C^2 - 2CA'_y)dw^2 = 0.$$

Finally we substitute from (2.21), (4.1), (4.2) to obtain (4.7).

It remains to prove the converse implication and the last statement of Theorem 4.3. Let us observe first that, due to (4.6), the discriminant of the quadratic equation (4.7) is equal to $\lambda^2 > 0$. Hence in the given neighborhood $U \subset M$, (4.7) determines two different smooth 2-dimensional tangent distributions. We see that the equation (4.7) decomposes in two ways as

$$\varphi_0 dw - (\varphi_4 + \varepsilon\lambda)dx = 0, \ (\varphi_4 + \varepsilon\lambda)dw + \lambda a_0 dx = 0, \tag{4.15}$$

where either $\varepsilon = 1$, or $\varepsilon = -1$. At the initial point $m \in M$, either the equations with $\varepsilon = 1$ are linearly independent, or the equations with $\varepsilon = -1$ are linearly independent. Hence the corresponding equations (4.15) are linearly independent in a normal neighborhood $U' \subset U$ of m. We see that both 2-dimensional distributions are integrable and any smooth surface $N \subset U'$ satisfying (4.7) is an integral manifold of one of these distributions (in fact a level surface of some potential function of (4.15)). Also, the previous calculations show that the condition (4.11) is satisfied along each surface N satisfying (4.7); i.e., the constructed integral manifolds are asymptotic leaves (forming two distinct "asymptotic foliations"). This concludes the proof of Theorem 4.3.

5. The quasiexplicit classification in the hyperbolic case and examples

The aim of the previous section was to prepare the following

COROLLARY 5.1. *Let (M, g) be of hyperbolic type. Then, in a normal neighborhood $U \subset M$ of any point m, there are adapted local coordinates $\bar{w}, \bar{x}, \bar{y}$ such that* (1.9) *and* (1.11) *hold and, moreover,* $\bar{a}_0 = 0$.

Proof. Let ε be a sign for which $\varphi_4 + \varepsilon\lambda \neq 0$ in a normal neighborhood $U \ni m$. Then choose a new variable $\bar{w}$ as a potential function $P(w, x)$ of the equation $(\varphi_4 + \varepsilon\lambda)dw + \lambda a_0 dx = 0$. Using the new local coordinates $\bar{w}, x, y$, in U, we obtain, for a new orthonormal frame $\{\bar{E}_1, \bar{E}_2, E_3\}$,

$$\bar{\omega}^1 = \bar{f}d\bar{w}, \quad \bar{\omega}^2 = \bar{A}dx + \bar{C}d\bar{w}, \quad \omega^3 = dy + Hdw. \tag{5.1}$$

Now, we shall look for a substitution $\bar{y} = y + \phi(w, x)$ such that

(5.2) $$dy + Hdw = d\bar{y} + \bar{H}d\bar{w},$$

where $\bar{H} = \bar{H}(w, x)$ is another unknown function.

The corresponding conditions can be written in the form

(5.3) $$\phi'_w + \bar{H}\bar{w}'_w = H, \quad \phi'_x + \bar{H}\bar{w}'_x = 0.$$

The (local) integrability condition of (5.3) is

(5.4) $$\bar{H}'_x\bar{w}'_w - \bar{H}'_w\bar{w}'_x = H_x,$$

which is a linear PDE for $\bar{H}$.

Let us fix one solution $\bar{H}(w, x)$ of (5.4). Then the function ϕ is determined by (5.3) up to an additive constant. We see that the formula (1.9) is satisfied with respect to the new variables $\bar{w}, x, \bar{y}$. Finally, the condition (1.11) can be also satisfied by introducing a new variable $\bar{x}$.

Having fixed the new local coordinates $\bar{w}, \bar{x}, \bar{y}$ (possibly in a smaller neighborhood $U' \subset U$), the equation of the asymptotic leaves has the form (4.7). But now one of the asymptotic foliations is given by the level surfaces $\bar{w} =$ const. Hence the corresponding equation (4.7) cannot involve the square $d\bar{x}^2$ and thus $\bar{a}_0 = 0$ holds in the whole neighborhood, q.e.d.

Remark. In the elliptic case, the asymptotic leaves can be also defined, but they are imaginary. On the other hand, we always have $a_0 \neq 0$ according to the last part of Proposition 2.6.

Now, we shall proceed with the classification in the hyperbolic case under the hypothesis $a_0 = 0$.

PROPOSITION 5.2. *In the adapted coordinates from Corollary* 5.1 *we always have*

(5.5) $$A = pe^{\lambda y}, \quad f = \frac{1}{p}e^{-\lambda y}, \quad C = re^{\lambda y} + Se^{-\lambda y}, \quad h = 2\lambda ps,$$

where p, r, s *are some functions of* w, x ; $p \neq 0$.

Proof. First we have $\beta = 0$ due to (2.15). From the equations (1.15) and (C3) we obtain $f = te^{\lambda y} + ue^{-\lambda y}$. Now, (2.24a) means that $A = pe^{\lambda y} + qe^{-\lambda y}$ and the relation $Af = 1$ implies that either $t = q = 0$, $up = 1$, or $u = p = 0$, $tq = 1$. Replacing possibly λ by $-\lambda$, we can always assume the first case. The expression for C is just (2.23a). From $(1.13)_2$ we see that $h = CA'_y - AC'_y = 2\lambda ps$.

PROPOSITION 5.3. *If the formulas* (5.5) *hold with arbitrary functions* $p(w, x)$, $r(w, x)$, $s(w, x)$, *then all algebraic equations from Theorem* 2.9 (*of hyperbolic type*) *are satisfied.*

The proof is an easy direct check. In particular, we get here $\varphi_0 = 2\lambda r p^{-1}$, $\varphi_4 = -\lambda$, $\varphi_5 = -2\lambda$.

Thus, we are left with the differential equations (A1), (A2) and (2.9). Obviously, we can write down this system of PDE in the simple form

$$(A\alpha)'_y = 0,\ R'_y = 0,\ (A\alpha)'_w + R'_x = -(k+\tilde{c}) = -\rho_1. \tag{5.6}$$

PROPOSITION 5.4. *The system of PDE* (5.6) *together with the condition* $h = 2\lambda ps$ *is equivalent to the following system of PDE in two independent variables*:

(D1) $\left[\frac{1}{p^2} + s^2\right]'_x = 0,$

(D2) $H'_x = 2\lambda ps,$

(D3) $p'_w - r'_x - \lambda pH = 0,$

(D4) $-(ps'_x)'_w + (rs'_x)'_x = -\rho_1.$

Proof. We see easily by the definition of α and R that

$$A\alpha = p(p'_w - r'_x - \lambda pH)e^{2\lambda y} - ps'_x,\ R = \frac{1}{2}\left[\frac{1}{p^2} + s^2\right]'_x e^{-2\lambda y} + rs'_x.$$

Hence the result follows.

PROPOSITION 5.5. *In the real analytic case, the general solution of the PDE system* (D1)–(D4) *depends* (*locally*) *on* 5 *arbitrary functions of the variable* w.

Proof. From (D1) one gets $s = p^{-1}\sqrt{p^2\varphi^2 - 1}$, where $\varphi = \varphi(w)$ is an arbitrary function. Then (D2) and (D3) are equivalent to a unique equation

$$\left(\frac{p'_w - r'_x}{\lambda p}\right)'_x = 2\lambda\sqrt{p^2\varphi^2 - 1} \tag{5.7}$$

and (D4) can be written in the form

$$\left(\frac{-p'_x}{p\sqrt{p^2\varphi^2 - 1}}\right)'_w + \left(\frac{rp'_x}{p^2\sqrt{p^2\varphi^2 - 1}}\right)'_x = -\rho_1. \tag{5.8}$$

Now, (5.7) and (5.8) form a system of two 2nd order PDE for r and p which can be expressed with respect to r''_{xx} and p''_{xx} respectively. Hence the Cauchy-Kowalewski theorem can be applied and the solution of (D1)–(D4) depends on 4 additional arbitrary functions of w.

PROPOSITION 5.6. *For any (local) hyperbolic metric g determined by Proposition 5.5 we have*

$$\| \nabla \hat{R} \|^2 = 2(\rho_1 - \rho_3)^2(2\lambda^2 + h^2). \tag{5.9}$$

Consequently, if $h \neq$ const., then the metric g is not locally homogeneous.

Proof. The formula (5.9) follows from (3.3), (3.9), (5.5) and the identity $\beta = 0$.
From Proposition 5.6, the proof of Proposition 5.5 and formula (D2) we get

COROLLARY 5.7. *The hyperbolic metrics constructed in Proposition 5.5 are not locally homogeneous, in general.*

We conclude with some *explicit* examples, which can be checked easily.

EXAMPLE 5.8. Put $p = p(w)$, $s = s(w)$, $H = 2\lambda p(w)s(w)x + \varphi(w)$,

$$r = -\lambda^2 p^2(w)s(w)x^2 + [p'(w) - \lambda p(w)\varphi(w)]x + \psi(w),$$

where $p(w)$, $s(w)$, $\varphi(w)$, $\psi(w)$ are arbitrary functions. The corresponding Ricci roots are $\rho_1 = \rho_2 = 0$, $\rho_3 = -2\lambda^2$.

EXAMPLE 5.9. Put $p = \sqrt{1 + x^2}$, $s = \dfrac{x}{\sqrt{1 + x^2}}$, $H = \lambda\left(x^2 + \dfrac{1}{4}\right)$.

$$r = -\frac{\lambda^2}{4}x(1 + x^2)^{3/2}$$

The corresponding Ricci roots are $\rho_1 = \rho_2 = \dfrac{1}{4}\lambda^2$, $\rho_3 = -2\lambda^2$.

EXAMPLE 5.10. Put $p = \sqrt{1 + x^4}$, $s = \dfrac{x^2}{\sqrt{1 + x^4}}$, $H = \dfrac{2}{3}\lambda x^3$, $r = -\dfrac{1}{9}\lambda^2 \times (1 + x^4)^{3/2}$. The corresponding Ricci roots are $\rho_1 = \rho_2 = \dfrac{2}{9}\lambda^2$, $\rho_3 = -2\lambda^2$.

We see from Proposition 5.6 that all these examples are nontrivial, i.e., the corresponding metrics are not locally homogeneous. The question if there exist non-trivial *explicit* examples for other Ricci roots $\rho_1 = \rho_2 = \text{const.}$, $\rho_3 = -2\lambda^2$, remains open.

6. The geometric existence theorem for the hyperbolic case

In this section we shall calculate the "number" of locally non-isometric spaces corresponding to any prescribed $\rho_1 = \rho_2 \neq \rho_3$, $\rho_3 = -2\lambda^2 < 0$. To avoid technical difficulties, we shall express our family of metrics in other local coordinates. For this purpose we shall assume that the function $\chi = \chi(w, x)$ from (1.10) can be *arbitrary*. We obtain first

PROPOSITION 6.1. *Every Riemannian space (M, g) with the given constant Ricci roots $\rho_1 = \rho_2 \neq \rho_3$, $\rho_3 = -2\lambda^2$, can be expressed locally in the form $(U, \tilde{g})$, where*

$U \subset R^3(w, x, y)$, $\tilde{g} = \sum_{j=1}^{3} (\omega^i)^2$, *and*

$$\omega^1 = te^{-\lambda y}dw, \quad \omega^2 = pe^{\lambda y}dx + se^{-\lambda y}dw, \quad \omega^3 = dy + Hdw. \tag{6.1}$$

Here t, p, s, H are functions of two variables w, x satisfying the following system of PDE:

(E1) $(t^2 + s^2)'_x = 0$,

(E2) $H'_x = 2\lambda ps$,

(E3) $p'_w - \lambda pH = 0$,

(E4) $(s'_x t^{-1})'_w = \rho_1 pt$.

Proof. We obtain the expression (6.1) from (5.1), (5.5) just fixing a new variable $\tilde{x} = \tilde{x}(w, x)$ as a potential function of the equation $pdx + rdw = 0$. The equation (E2) follows again from $(1.13)_2$ where $\beta = 0$. From (6.1) and (1.10) we see

$$Af = pt, \ \chi = (pt)^{-1}. \tag{6.2}$$

Using the expressions for f, A, C in (6.1) and formulas (1.13), (1.14), we obtain

$$\begin{cases} A\alpha = t^{-1}(p'_w - \lambda pH)e^{2\lambda y} - t^{-1}s'_x, \\ R = (tt'_x + ss'_x)(pt)^{-1}e^{-2\lambda y} - (pt)^{-1}s(p'_w - \lambda pH). \end{cases} \tag{6.3}$$

Then the equation (A1), (A2) mean that $A\alpha$, R are independent of y; hence (E3) and (E1) follow. At the same time, we obtain

$$A\alpha = -t^{-1}s'_x, \ R = 0. \tag{6.4}$$

The equation (2.9) (which is equivalent with (A3)) and (6.2) imply (E4), q.e.d.

Due to Proposition 5.5 (or by the direct check) we see that *the solutions* (t, p, s, H) *of* (E1)–(E4) *depend, in the real analytic case, on* 5 *arbitrary functions of* 1 *variable.*

Suppose now that $(\bar{M}, \bar{g})$ be another Riemannian manifold with the same constant Ricci roots as (M, g); then we have locally $\bar{g} = \sum(\bar{\omega}^i)^2$, where

(6.5) $$\bar{\omega}^1 = \bar{t}e^{-\lambda\bar{y}}d\bar{w},\ \bar{\omega}^2 = \bar{p}e^{\lambda\bar{y}}d\bar{x} + \bar{s}e^{-\lambda\bar{y}}d\bar{w},\ ,\ \bar{\omega}^3 = d\bar{y} + \bar{H}d\bar{w}.$$

Let $F : U \to \bar{U}$ be a local isometry between (M, g) and $(\bar{M}, \bar{g})$ with the coordinate expression

(6.6) $$\bar{w} = \bar{w}(w, x, y),\ \bar{x} = \bar{x}(w, x, y),\ \bar{y} = \bar{y}(w, x, y).$$

We want to determine a specific form of the equations (6.6) and the specific relations between the basic coefficients of (6.1) and (6.5), respectively.

From the geometrical meaning we see that the vector field $E_3 = \partial/\partial y$ must be mapped by the tangent mapping F_* into the vector field $\varepsilon'\bar{E}_3$, where $\varepsilon' = \pm 1$. Hence the vector fields E_1, E_2 are mapped into the vector fields $\cos\varphi \cdot \bar{E}_1 - \sin\varphi\cdot\bar{E}_2$, $\varepsilon(\sin\varphi\cdot\bar{E}_1 + \cos\varphi\cdot\bar{E}_2)$ respectively, where $\varepsilon = \pm 1$, and φ is a function on U. Hence we obtain (denoting the induced forms $F_*\bar{\omega}^i$ simply by $\bar{\omega}^i$)

(6.7) $$\bar{\omega}^1 = \cos\varphi\cdot\omega^1 + \varepsilon\sin\varphi\cdot\omega^2,\ \bar{\omega}^2 = -\sin\varphi\cdot\omega^1 + \varepsilon\cos\varphi\cdot\omega^2,\ \bar{\omega}^3 = \varepsilon'\omega^3.$$

Now, we can compare the expression (3.8) for the tensor $\nabla\hat{R}$ with the analogous expression for the tensor $\bar{\nabla}\hat{\bar{R}}$:

(6.8) $$\bar{\nabla}\hat{\bar{R}} = (\bar{c} - k)[(\bar{a}\bar{\omega}^1 + \bar{b}\bar{\omega}^2)\otimes(\bar{\omega}^1\otimes\bar{\omega}^3 + \bar{\omega}^3\otimes\bar{\omega}^1) + (\bar{c}\bar{\omega}^1 + \bar{e}\bar{\omega}^2)\otimes(\bar{\omega}^2\otimes\bar{\omega}^3 + \bar{\omega}^3\otimes\bar{\omega}^2)].$$

Here we calculate

(6.9) $$\bar{a} = a = -\lambda,\quad \bar{b} = b = 0,\quad \bar{c} = -\bar{h}(\bar{p}\bar{t})^{-1},\quad c = -h(pt)^{-1},\quad \bar{e} = e = \lambda.$$

Let us substitute (6.7) into (6.8) and then evaluate the equality $\bar{\nabla}\hat{\bar{R}} = \nabla\hat{R}$ as a system of equalities between the coefficients of the corresponding tensor monomials. By a lengthy but routine calculation we obtain two cases:

(6.10A) $$\varepsilon' = 1 \text{ and } \sin\varphi = 0,$$

(6.10B) $$\varepsilon' = -1 \text{ and } 2\lambda pt\cos\varphi + \varepsilon\, h\sin\varphi = 0.$$

If (6.10A) holds, then (6.7) implies

(6.11) $$\bar{\omega}^1 = \varepsilon_1\omega^1,\ \bar{\omega}^2 = \varepsilon_2\omega^2,\ \bar{\omega}^3 = \omega^3.$$

Let us suppose, e.g., $\varepsilon_1 = \varepsilon_2 = 1$ (the other subcases are treated similarly). Then we get from (6.1) and (6.5)

$$\text{(6.12)} \qquad \begin{cases} \bar{t}e^{\lambda-\bar{y}}d\bar{w} = te^{-\lambda y}\,dw, \\ \bar{p}e^{\lambda\bar{y}}d\bar{x} + \bar{s}e^{-\lambda\bar{y}}d\bar{w} = pe^{\lambda y}dx + se^{-\lambda y}dw, \\ d\bar{y} + \bar{H}d\bar{w} = dy + Hdw. \end{cases}$$

Here, in addition, the function t, p, s, H satisfy the PDE system (E1)–(E4) and the functions $\bar{t}, \bar{p}, \bar{s}, \bar{H}$ have to satisfy the analogous PDE system $(\overline{\text{E1}})-(\overline{\text{E4}})$.

From the first and the last equation (6.12) we get

$$\text{(6.13)} \qquad \bar{w} = \varphi(w),\ \bar{y} = y + \mu(w),$$

and subsequently

$$\text{(6.14)} \qquad \bar{t} = te^{\lambda\mu}/\varphi'(w),\ \bar{H} = (H - \mu'(w))/\varphi'(w),$$

where φ, μ are some functions. Substituting (6.13) into the middle equation of (6.12), we get formulas involving a new arbitrary function:

$$\text{(6.15)} \qquad \bar{x} = \psi(x), \quad \bar{p} = pe^{-\lambda\mu}/\psi'(x), \quad \bar{s} = se^{\lambda\mu}/\varphi'(w), \quad \bar{h} = h/(\psi'(x)\varphi'(w))$$

Now we also see that each of the equations $(\overline{\text{E1}})-(\overline{\text{E4}})$ is a consequence of the corresponding equations (E1)–(E4). We conclude:

In the first case, (6.10a), *the functions* $\bar{w}, \bar{x}, \bar{y}, \bar{t}, \bar{p}, \bar{s}, \bar{H}$ *can be obtained from* w, x, y, t, p, s, H *by formulas involving* 3 *arbitrary functions of* 1 *variable.*

If (6.10B) holds, then

$$\text{(6.16)} \qquad \bar{\omega}^1 = \frac{\varepsilon_1}{q}(h\omega^1 - 2\lambda pt\omega^2), \quad \bar{\omega}^2 = \frac{\varepsilon_2}{q}(2\lambda pt\omega^1 + h\omega^2), \quad \bar{\omega}^3 = -\omega^3,$$

where

$$\text{(6.17)} \qquad q^2 = 4\lambda^2p^2t^2 + h^2.$$

From (E1) and (E2) we get

$$\text{(6.18)} \qquad q = 2\lambda p\alpha(w), \text{ where } \alpha(w) = \sqrt{s^2 + t^2}.$$

The explicit form of (6.16) is, for $\varepsilon_1 = -1$, $\varepsilon_2 = 1$ (the other combinations of signs are treated similarly)

$$
\begin{cases} \bar{t}e^{-\lambda\bar{y}}d\bar{w} = [\alpha(w)]^{-1}pte^{\lambda y}dx, \\ \bar{p}e^{\lambda\bar{y}}d\bar{x} + \bar{s}e^{-\lambda y}d\bar{w} = \alpha(w)e^{-\lambda y}dw + [\alpha(w)^{-1}pse^{\lambda y}dx, \\ d\bar{y} + \bar{H}d\bar{w} = -dy - Hdw. \end{cases} \tag{6.19}
$$

From the last equation we get

$$
\bar{y} = -y - \phi(w, x), \quad d\phi = Hdw + \bar{H}d\bar{w}. \tag{6.20}
$$

Substituting from here in the first and the second equation (6.19) we obtain

$$
\bar{w} = \varphi(x), \quad \bar{x} = \psi(w), \tag{6.21}
$$

$$
\begin{cases} \bar{t} = pte^{-\lambda\phi}/(\alpha(w)\varphi'(x)), \; \bar{p} = \alpha(w)e^{\lambda\phi}/\psi'(w), \\ \bar{s} = pse^{-\lambda\phi}/(\alpha(w)\varphi'(x)), \; \bar{h} = h/(\varphi'(w)\psi'(x)). \end{cases} \tag{6.22}
$$

Now, $\overline{(\mathrm{E}1)}$ together with (6.21) means that $\bar{s}^2 + \bar{t}^2 = [\bar{\alpha}(x)]^2$, where $\bar{\alpha}(x)$ is a new function of one variable, and (6.22) implies

$$
pe^{-\lambda\phi} = \varphi'(x)\bar{\alpha}(x). \tag{6.23}
$$

We see that if $\varphi(x)$, $\psi(w)$, $\bar{\alpha}(x)$ are arbitrary but fixed, then all functions $\bar{w}$, $\bar{x}$, $\bar{y}$, $\bar{t}$, $\bar{p}$, $\bar{s}$, $\bar{H}$ are determined by w, x, y, t, p, s, H. We conclude:

In the second case, (6.10B), *the functions* $\bar{w}, \bar{x}, \ldots, \bar{H}$ *can be obtained from* $w, x, \ldots, H$ *by formulas involving at most* 3 *arbitrary functions of* 1 *variable.*

We can summarize:

THEOREM 6.2. *For any constant Ricci roots* $\rho_1 = \rho_2 \neq \rho_3$, $\rho_3 = -2\lambda^2$, *the isometry classes of germs of the corresponding (real analytic) hyperbolic metrics are parametrized by the pairs of germs of arbitrary functions of one variable.*

Proof. Using the local form (6.1) for our metrics, we see that the germs of these metrics depend on the *quintuplets* of germs of arbitrary functions of one variable. At the same time, a *triplet* of germs of arbitrary functions (plus some combination of signs) is needed to generate any fixed isometry class of germs of the metrics.

7. The quasiexplicit classification in the elliptic case

In the elliptic case, the calculations become more complicated. We assume again $\chi = 1$ in this section.

PROPOSITION 7.1. *The basic coefficient functions from Theorem 2.9 are determined, in general, by the following formulas*:

$$(7.1)\quad a_1 = \frac{\varphi_2(\lambda^2 - \varphi_4^2) - 2\varphi_1\varphi_4}{\lambda\varphi_0^2},\ a_2 = \frac{\varphi_1(\varphi_4^2 - \lambda^2) - 2\varphi_2\varphi_4}{\lambda\varphi_0^2},$$

$$(7.2)\quad a_3 = \frac{\varphi_3(\lambda^2 + \varphi_4^2)}{\lambda\varphi_0^2},$$

$$(7.3)\quad b_1 = \frac{\varphi_2\varphi_4 + \varphi_1\lambda}{\lambda\varphi_0},\ b_2 = \frac{\lambda\varphi_2 - \varphi_1\varphi_4}{\lambda\varphi_0},\ b_3 = \frac{-\varphi_3\varphi_4}{\lambda\varphi_0},$$

$$(7.4)\quad a_0\varphi_0 = -\frac{\lambda^2 + \varphi_4^2}{\lambda},\ \varphi_0 = \sqrt{\varphi_3^2 - \varphi_1^2 - \varphi_2^2},$$

$$(7.5)\quad \varphi_5 = 2\varphi_4,\ h = -2\lambda\varphi_3/\varphi_0,$$

and $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ *are arbitrary functions of* w, x.

Proof. First, the equations (2.5b), (2.6b) imply $\varphi_3 \neq 0$ and then $(2.33\text{b})_3$ implies $\varphi_0 \neq 0$. Then (7.1), (7.2) follow from (2.33b) and (7.3) follows from (2.36b). From (2.17b) we also have $a_3 \neq 0$.

Now, if $b_3 = 0$, then $(2.35\text{b})_3$, $(2.36\text{b})_3$ imply $\varphi_4 = \varphi_5 = 0$ and hence $\varphi_5 = 2\varphi_4$. The formula (2.31) implies $a_0\varphi_0 = -\lambda$, i.e., (7.4). Finally, from (2.27), (7.2) and (7.4) we get $h = 2\lambda a_3/a_0 = -2\lambda\varphi_3/\varphi_0$.

Assume now $b_3 \neq 0$. Then (2.35b), (2.36b) imply $\varphi_4\varphi_5 \neq 0$ and

$$(7.6)\quad \frac{b_1}{b_3} = \frac{\varphi_5 a_1 + 2\lambda a_2}{\varphi_5 a_3} = \frac{\lambda\varphi_1 + \varphi_2\varphi_4}{-\varphi_3\varphi_4},$$

$$(7.7)\quad \frac{b_2}{b_3} = \frac{\varphi_5 a_2 - 2\lambda a_1}{\varphi_5 a_3} = \frac{\lambda\varphi_2 - \varphi_1\varphi_4}{-\varphi_3\varphi_4}.$$

Hence we get, by a routine calculation

$$(7.8)\qquad (\varphi_1^2 + \varphi_2^2)(\varphi_5 - 2\varphi_4) = 0.$$

Now, $\varphi_1^2 + \varphi_2^2 \neq 0$ implies $\varphi_5 = 2\varphi_4$; (2.31) then implies (7.4) and hence

$$h = \frac{2\lambda a_3}{a_0} = \frac{2\lambda\varphi_3(\lambda^2 + \varphi_4^2)}{\lambda\varphi_0^2 a_0} = -\frac{2\lambda\varphi_3}{\varphi_0},$$

which concludes the proof.

Suppose finally $\varphi_1 = \varphi_2 = 0$. Then $a_1 = a_2 = b_1 = b_2 = 0$ and $a_3 = \varepsilon a_0$,

$\varphi_3 = \varepsilon'\varphi_0$. $(2.35b)_3$ and $(2.36b)_3$ imply

$$\varphi_5 = 2\lambda\varepsilon b_3, \quad \varphi_4 = -\lambda\varepsilon' b_3. \tag{7.9}$$

Further, $(2.34b)_3$ implies

$$\lambda b_3^2 = a_3\varphi_3 - \lambda = \varepsilon\varepsilon' a_0\varphi_0 - \lambda, \tag{7.10}$$

and (2.31) together with (7.9) gives

$$\varphi_0 a_0 - \lambda\varepsilon\varepsilon' b_3^2 = -\lambda. \tag{7.11}$$

Substituting (7.10) into (7.11) we get $(1 + \varepsilon\varepsilon')\lambda = 0$, i.e. $\varepsilon' = -\varepsilon$. Then (7.9) implies $\varphi_5 = 2\varphi_4$, and the rest is the same as in the case $\varphi_1^2 + \varphi_2^2 \neq 0$.

By a direct check we see that all algebraic formulas (of the elliptic type) from Theorem 2.9 are satisfied due to (7.1)–(7.5) for arbitrary functions φ_1, φ_2, φ_3, φ_4.

We shall now express the system of PDE (A1), (A2) and (2.9) as a system of PDE in two independent variables. This leads to a routine but rather lengthy calculation, for which we only give some hints.

First we summarize the formulas (2.16b), (2.22b), (2.5b) and $(7.5)_2$ in the form

$$\begin{cases} A^2 = a_1 \cos 2\lambda y + a_2 \sin 2\lambda y + a_3, \\ AC = b_1 \cos 2\lambda y + b_2 \sin 2\lambda y + b_3, \\ f^2 + C^2 = (1/\lambda)(\varphi_1 \sin 2\lambda y - \varphi_2 \cos 2\lambda y + \varphi_3), \\ h = -2\varphi_3\lambda/\varphi_0, \ \varphi_0 = \sqrt{\varphi_3^2 - \varphi_1^3 - \varphi_2^2}, \end{cases} \tag{7.12}$$

where a_i, b_i are specified by (7.1)–(7.3).

Substitute into (A1) the function $A\alpha$ in the form

$$A\alpha = \frac{1}{2}[(A^2)'_w - 2(AC)'_x + (AC)A^{-2}(A^2)'_x - H(A^2)'_y] \tag{7.13}$$

and the function β in the form $\beta = \lambda a_0/A^2$, using (7.12) everywhere and taking the common denominator A^4 for all terms. Then the nominator of the left-hand side of the equation (A1) is a linear combination of c^3, c^2s, c^2, cs, c, s, 1 respectively, where we put $c = \cos 2\lambda y$, $s = \sin 2\lambda y$. Hence we get 7 (formally independent) partial differential equations which are linear with respect to a'_{0x}, a'_{1x}, a'_{2x}, a'_{3x} and V_1, V_2, where

$$V_1 = a'_{1w} - 2b'_{1x} - 2\lambda H a_2, \quad V_2 = a'_{2w} - 2b'_{2x} + 2\lambda H a_1. \tag{7.14}$$

Here the third and the fourth equation read

$$a_2V_1 + a_1V_2 + b_2a'_{1x} + b_1a'_{2x} = 0, \tag{7.15}$$

$$-a_1V_1 + a_2V_2 - b_1a'_{1x} + b_2a'_{2x} = 0. \tag{7.16}$$

Using the formula (2.17b) in the form

$$a_0 = \sqrt{a_3^{\,2} - a_1^{\,2} - a_2^{\,2}} \tag{7.17}$$

and its derivative

$$a_0a'_{0x} = a_3a'_{3x} - a_1a'_{1x} - a_2a'_{2x}, \tag{7.18}$$

we can eliminate the derivative a'_{0x} in all remaining equations. Now, using the relation (2.26b), and also (7.1)–(7.3) when it is necessary, we can see that *all the seven PDE are algebraic consequences of* (7.15), (7.16).

Next, substitute in (A2) the function R in the form

$$R = \frac{1}{2}[(f^2 + C^2)'_x + H(h + (AC)'_y) - (AC)A^{-2}(A^2)'_w], \tag{7.19}$$

and β in the form $\beta = \lambda a_0/A^2$, using (7.12) everywhere, and taking the common denominator A^4 for all terms, again. By the same argument as in the previous case, we obtain once more 7 partial differential equations, which are now linear with respect to a'_{0w}, a'_{1w}, a'_{2w}, a'_{3w} and W_1, W_2, where

$$W_1 = \varphi'_{1x} - 2b_1\lambda^2H,\ W_2 = \varphi'_{2x} - 2b_2\lambda^2H. \tag{7.20}$$

The third and the fourth equation have the form

$$\varphi'_{1x} - 2b_1\lambda^2H + \frac{\lambda}{a_0}a'_{1w} - \frac{\varphi_4}{a_0}a'_{2w} = 0, \tag{7.21}$$

$$\varphi'_{2x} - 2b_2\lambda^2H + \frac{\varphi_4}{a_0}a'_{1w} + \frac{\lambda}{a_0}a'_{2w} = 0. \tag{7.22}$$

We can eliminate a'_{0w} in the other equations using the analogue of (7.18).

It is again a routine to check that *the remaining PDE are algebraic consequences of* (7.21) and (7.22). Moreover, we can even check that the systems (7.15), (7.16) and (7.21), (7.22) are *algebraically equivalent!* Hence the PDE system (A1), (A2) in *three* variables in equivalent to the system (7.21), (7.22) in *two* variables. (This is in accordance with the hyperbolic case, in which the equations (A1), (A2) have been also reduced to two equations with *two* independent variables, namely to (D3) and (D1).

The last equation of (7.12) can be written in the form

$$H'_x = -2\varphi_3\lambda/\varphi_0, \quad \varphi_0 = \sqrt{\varphi_3^2 - \varphi_1^2 - \varphi_2^2} \tag{7.23}$$

Finally, (A1) and (A2) imply $\frac{\partial}{\partial y}[(A\alpha)'_w + R'_x] = 0$, and hence the equation (2.9) can be written in the form $[(A\alpha)'_w + R'_x]_{y=0} = -\rho_1$, i.e., in the form

$$[(A\alpha)_{y=0}]'_w + [(R)_{y=0}]'_x = -\rho_1. \tag{7.24}$$

We see easily from (7.13) and (7.19) that

$$(A\alpha)_{y=0} = \frac{1}{2}\left[(a_1 + a_3)'_w - 2(b_1 + b_3)'_x - 2\lambda a_2 H + \frac{(b_1 + b_3)(a_1 + a_3)'_x}{a_1 + a_3}\right], \tag{7.25}$$

$$(R)_{y=0} = \frac{1}{2}\left[\frac{(\varphi_3 - \varphi_2)'_x}{\lambda} + H(h + 2\lambda b_2) - \frac{(b_1 + b_3)(a_1 + a_3)'_w}{a_1 + a_3}\right], \tag{7.26}$$

and hence (7.24) takes on the form

$$\varphi''_{3xx} - \varphi''_{2xx} = F(\varphi_i, H, \varphi'_{iw}, \varphi'_{ix}, H'_w, \varphi''_{iwx}, \varphi''_{iww}), \tag{7.27}$$

where F is a real analytic function of its variables.

Let $\varphi_4(w, x)$ be fixed as an arbitrary real analytic function, and consider the PDE system formed by equations (7.21), (7.22), (7.23) and (7.27). We can apply to this system an easy modification of the Cauchy-Kowalewski Theorem (see e.g. [K1], Section 9). It follows that the general solution of this system (with fixed φ_4) depends locally on five arbitrary functions of the variable w, namely $\varphi_1(w, x_0)$, $\varphi_2(w, x_0)$, $\varphi_3(w, x_0)$, $\varphi'_{3x}(w, x_0)$, $H(w, x_0)$ around a point $(w_0, x_0) \in R^2(w, x)$.

This general solution determines locally all metrics of the elliptic type with the prescribed constant Ricci roots $\rho_1 = \rho_2 \neq \rho_3 = 2\lambda^2$.

Next, we have

PROPOSITION 7.2. *The norm of the covariant differential of the Ricci tensor is given, in the elliptic case, by the formula*

$$\| \nabla\hat{R} \|^2 = 2(\rho_1 - \rho_3)^2(h^2 - 2\lambda^2). \tag{7.28}$$

Proof. We use (3.9), (3.3) for $\chi = 1$, (2.15), (2.16b), the identity $a + e = 0$ and the elementary trigonometric identities. By a routine check we get $a^2 + b^2 + c^2 + e^2 = h^2 - 2\lambda^2$. (Cf. formula (5.9)).

Hence we see that the function h^2 is a Riemannian invariant.

Now, we have

PROPOSITION 7.3. *A 3-dimensional Riemannian space* (M, g) *with the prescribed constant Ricci roots* $\rho_1 = \rho_2$, $\rho_3 = 2\lambda^2$ *is locally homogeneous if, and only if,* $h = \text{const}$.

Proof. The "only if" part is a consequence of Proposition 7.2. Suppose now $h = \text{const}$. Using identities $a + e = 0$, $b - c = h$, $a^2 + b^2 + c^2 + e^2 = h^2 - 2\lambda^2$ and formula (3.8), we can prove easily that there is an adapted orthonormal coframe $(\bar{\omega}^1, \bar{\omega}^2, \omega^3)$ on a neighborhood of any point $p \in M$ for which all components of the tensor $\nabla \hat{R}$ are constant (which means that all the new functions $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{e}$ are constant). Because our space is 3-dimensional, we deduce that, with respect to the corresponding orthonormal frame, all components of the curvature tensor R, and those of its covariant derivative ∇R, are constant. According to [Se2] (see also [K2]), the space (M, g) is locally homogeneous. (Let us mention that the same proposition holds in the hyperbolic case, too).

Hence we obtain

COROLLARY 7.4. *The 3-dimensional Riemannian spaces from Proposition 7.3 are not locally homogeneous, in general.*

Proof. Consider a neighborhood U of a point $p \in M$ with an adapted local coordinate system (w, x, y). Then we have, let us say, $p = (w_0, x_0, 0)$. According to the argument prior to Proposition 7.1, the function $\varphi_3(w, x_0)/\varphi_0(w, x_0)$ can be chosen as arbitrary. According to (7.23), the function $h(w, x_0)$ can be chosen as an arbitrary real analytic function.

Remark 7.5. The author was unable to find an explicit and not locally homogeneous example in the elliptic case.

8. The geometric existence theorem for the elliptic case

The complete solution of the local isometry problem in the elliptic case seems to be tiresome. Nevertheless, the following theorem gives a satisfactory counterpart to Theorem 6.2:

THEOREM 8.1. *For any constant Ricci roots $\rho_1 = \rho_2 \neq \rho_3$, $\rho_3 = 2\lambda^2$, there exists a family $\tilde{f}$ of elliptic metrics such that the isometry classes of germs of $\tilde{f}$ are parametrized by the pairs of germs of arbitrary (real analytic) functions of one variable.*

Proof. We shall limit ourselves to the local metrics which, in some adapted local coordinates, satisfy $\varphi_4 = 0$. According to the previous section, the germs of these metrics are parametrized by germs of five arbitrary functions of 1 variable.

Suppose that (M, g), $(\bar{M}, \bar{g})$ are two Riemannian 3-manifolds with the same constant Ricci roots $\rho_1 = \rho_2 \neq \rho_3 = 2\lambda^2$, and let the local expression for (M, g) be given by (1.9), (7.12) and Proposition 7.1 with $\varphi_4 = 0$. Suppose that the local expression for $(\bar{M}, \bar{g})$ is given by the analogous formulas with $\bar{\varphi}_4 = 0$. The only basic functions are φ_i, H and $\bar{\varphi}_i$, $\bar{H}$, respectively, $i = 1,2,3$.

Having any local isometry $F : U \to \bar{U}$ between (M, g) and $(\bar{M}, \bar{g})$ given by formulas (6.6) we see that (6.7) holds once again and hence

$$(\bar{\omega}^1)^2 + (\bar{\omega}^2)^2 = (\omega^1)^2 + (\omega^2)^2, \quad \bar{\omega}^3 = \varepsilon' \omega^3, \quad \varepsilon' = \pm 1. \tag{8.1}$$

We can assume $\varepsilon' = 1$, the opposite case is treated similarly. We obtain then from (6.7), $(1.9)_3$

$$\bar{w} = \bar{w}(w, x), \quad \bar{x} = \bar{x}(w, x), \tag{8.2}$$

$$\bar{y} = y + \phi(w, x), \quad d\phi = -\bar{H}d\bar{w} + Hdw. \tag{8.3}$$

We shall now substitute into the first equation of (8.1). We get first

$$(\omega^1)^2 + (\omega^2)^2 = \lambda^{-1}(\varphi_1 \sin 2\lambda y - \varphi_2 \cos 2\lambda y + \varphi_3)dw^2 \tag{8.4}$$
$$+ 2(b_1 \cos 2\lambda y + b_2 \sin 2\lambda y + b_3)dwdx + (a_1 \cos 2\lambda y + a_2 \sin 2\lambda y + a_3)dx^2$$

$$(\bar{\omega}^1)^2 + (\bar{\omega}^2)^2 = \lambda^{-1}(\bar{\varphi}_1 \sin 2\lambda\bar{y} - \bar{\varphi}_2 \cos 2\lambda\bar{y} + \bar{\varphi}_3)d\bar{w}^2 \tag{8.5}$$
$$+ 2(\bar{b}_1 \cos 2\lambda\bar{y} + \bar{b}_2 \sin 2\lambda\bar{y} + \bar{b}_3)d\bar{w}d\bar{x} + (\bar{a}_1 \cos 2\lambda\bar{y} + \bar{a}_2 \sin 2\lambda\bar{y} + a_3)d\bar{x}^2.$$

In (8.4) we put $y = \bar{y} - \phi$ and use the standard trigonometric formulas for developing the sine and cosine of a difference of arguments; in (8.5) we substitute

$$d\bar{w} = \bar{w}_w dw + \bar{w}_x dx, \quad d\bar{x} = \bar{x}_w dw + \bar{x}_x dx. \tag{8.6}$$

We also notice that $b_3 = \bar{b}_3 = 0$ due to (7.3). Then the equality of the right-hand sides of (8.4) and (8.5) means the equalities between 3 pairs of quadratic forms in dw, dx which are coefficients of 1, $\sin 2\lambda\bar{y}$, $\cos 2\lambda\bar{y}$, respectively. For each pair of quadratic forms we compare the coefficients of dx^2, $dxdw$ and dw^2, respectively. As a result, we obtain the following system of 9 PDE for the functions

$\bar{x}(w, x)$, $\bar{w}(w, x)$:

$$(8.7)\quad \begin{cases} \lambda^{-1}\bar{\varphi}_3\bar{w}_x^{\ 2} + \bar{a}_3\bar{x}_x^{\ 2} = a_3, \\ \lambda^{-1}\bar{\varphi}_3\bar{w}_x\bar{w}_w + \bar{a}_3\bar{x}_x\bar{x}_w = 0, \\ \lambda^{-1}\bar{\varphi}_3\bar{w}_w^{\ 2} + \bar{a}_3\bar{x}_w^{\ 2} = \lambda^{-1}\varphi_3, \end{cases}$$

$$(8.8)\quad \begin{cases} \lambda^{-1}\bar{\varphi}_1\bar{w}_x^{\ 2} + 2\bar{b}_2\bar{w}_x\bar{x}_x + \bar{a}_2\bar{x}_x^{\ 2} = a_1 \sin 2\lambda\phi + a_2 \cos 2\lambda\phi, \\ \lambda^{-1}\bar{\varphi}_1\bar{w}_x\bar{w}_w + \bar{b}_2(\bar{w}_x\bar{x}_w + \bar{x}_x\bar{x}_w) + \bar{a}_2\bar{x}_x\bar{x}_w = b_1 \sin 2\lambda\phi + b_2 \cos 2\lambda\phi, \\ \lambda^{-1}\bar{\varphi}_1\bar{w}_w + 2\bar{b}_2\bar{w}_w\bar{x}_w + \bar{a}_2\bar{x}_w^{\ 2} = \lambda^{-1}(\varphi_1 \cos 2\lambda\phi - \varphi_2 \sin 2\lambda\phi), \end{cases}$$

$$(8.9)\quad \begin{cases} \lambda^{-1}\bar{\varphi}_1\bar{w}_x^{\ 2} - 2\bar{b}_1\bar{w}_x\bar{x}_x - \bar{a}_1\bar{x}_x^{\ 2} = -a_1 \cos 2\lambda\phi + a_2 \sin 2\lambda\phi, \\ \lambda^{-1}\bar{\varphi}_2\bar{w}_x\bar{w}_w - \bar{b}_1(\bar{w}_x\bar{x}_w + \bar{x}_x\bar{x}_w) - \bar{a}_1\bar{x}_x\bar{x}_w = -b_1 \cos 2\lambda\phi + b_2 \sin 2\lambda\phi, \\ \lambda^{-1}\bar{\varphi}_2\bar{w}_w^{\ 2} - 2\bar{b}_1\bar{w}_w\bar{x}_w - \bar{a}_1\bar{x}_w^{\ 2} = \lambda^{-1}(\varphi_1 \sin 2\lambda\phi + \varphi_2 \cos 2\lambda\phi). \end{cases}$$

PROPOSITION 8.2. *The functions $\bar{w}$, $\bar{x}$, $\bar{y}$, $\bar{\varphi}_i$, $\bar{H}$ can be obtained from w, x, y, φ_i, H by formulas involving 3 arbitrary functions of one variable.*

Proof. We shall use the formulas (7.1)–(7.3) simplified by $\varphi_4 = 0$, i.e.,

$$(8.10)\qquad a_1 = \varphi_2\lambda/\varphi_0^{\ 2}, \quad a_2 = -\varphi_1\lambda/\varphi_0^{\ 2}, \quad a_3 = \varphi_3\lambda/\varphi_0^{\ 2},$$

$$(8.11)\qquad b_1 = \varphi_1/\varphi_0, \quad b_2 = \varphi_2/\varphi_0, \quad b_3 = 0,$$

and the corresponding formulas $\overline{(8.10)}$, $\overline{(8.11)}$.

Suppose first that, at a fixed point $p \in U$, all the derivatives $\bar{w}_w$, $\bar{w}_x$, $\bar{x}_w$, $\bar{x}_x$ are nonzero. They remain nonzero in a neighborhood $U_p \subset U$. From $(8.7)_2$ we get

$$(8.12)\qquad \frac{\lambda^2}{\bar{\varphi}_0^{\ 2}} = \frac{\lambda\bar{a}_3}{\bar{\varphi}_3} = -\frac{\bar{w}_x\bar{w}_w}{\bar{x}_x\bar{x}_w}.$$

Now, substitute for $\bar{a}_3$ from (8.12) into $(8.7)_1$ and $(8.7)_3$, and then divide $(8.7)_1$ by $(8.7)_3$. We get easily

$$(8.13)\qquad \frac{\lambda^2}{\varphi_0^{\ 2}} = -\frac{\bar{w}_x\bar{x}_x}{\bar{w}_w\bar{x}_w}.$$

Hence

$$(8.14)\qquad \frac{\bar{\varphi}_0}{\varphi_0} = \varepsilon\,\frac{\bar{x}_x}{\bar{w}_w}, \quad \varepsilon = \pm 1.$$

Substituting first for $\bar{a}_3$ and a_3 from (8.10), $\overline{(8.10)}$ into $(8.7)_1$ and then using (8.12), (8.13), (8.14) we get

$$\frac{\bar{\varphi}_3}{\varphi_3} = \frac{\bar{x}_x}{\bar{w}_w} \cdot \frac{1}{\Delta} = \frac{\varepsilon\,\bar{\varphi}_0}{\Delta\,\varphi_0}, \tag{8.15}$$

where

$$\Delta = \bar{w}_w \bar{x}_x - \bar{w}_x \bar{x}_w \neq 0 \tag{8.16}$$

is the Jacobian of (8.2).

Now, substitute (8.12) into the middle equations of (8.8) and (8.9). We obtain

$$\begin{cases} 2\lambda^{-1}\bar{\varphi}_1 \bar{w}_x \bar{w}_w + \bar{b}_2(\bar{x}_x \bar{w}_w + \bar{x}_w \bar{w}_x) = b_1 \sin 2\lambda\phi + b_2 \cos 2\lambda\phi, \\ 2\lambda^{-1}\bar{\varphi}_2 \bar{w}_x \bar{w}_w - \bar{b}_1(\bar{x}_x \bar{w}_w + \bar{x}_w \bar{w}_x) = -\, b_1 \cos 2\lambda\phi + b_2 \sin 2\lambda\phi. \end{cases} \tag{8.17}$$

The sum of the squares of the last equations gives, with regard to (8.12),

$$\frac{\bar{\varphi}_1^{\,2} + \bar{\varphi}_2^{\,2}}{\bar{\varphi}_0^{\,2}}\,\Delta^2 = \frac{\varphi_1^{\,2} + \varphi_2^{\,2}}{\varphi_0^{\,2}}. \tag{8.18}$$

Hence and from (2.6b) we obtain

$$\left(\frac{\bar{\varphi}_3^{\,2}}{\bar{\varphi}_0^{\,2}} - 1\right)\Delta^2 = \frac{\varphi_3^{\,2}}{\varphi_0^{\,2}} - 1, \tag{8.19}$$

and using (8.15) we obtain

$$\Delta^2 = 1, \text{ i.e., } \Delta = \pm 1. \tag{8.20}$$

(The same follows from $\bar{\omega}^1 \wedge \bar{\omega}^2 = \varepsilon\omega^1 \wedge \omega^2$ and $\bar{A}\bar{f} = Af = 1$).

Now, (8.13) and (8.20) (with fixed φ_0!) give a system of two 1st order PDE for $\bar{x}(w, x)$, $\bar{w}(w, x)$, which can be written in the form satisfying the conditions of the Cauchy-Kowalewski theorem. Hence the functions $\bar{x}$, $\bar{w}$ depend on two arbitrary functions of 1 variable (supposing that the coefficient functions of (M, g) are fixed). The equalities (8.14), (8.15) then show that $\bar{\varphi}_0$ and $\bar{\varphi}_3$ are well-defined (up to a sign) by $\bar{x}(w, x)$ and $\bar{w}(w, x)$. Because h^2 is a Riemannian invariant (Proposition 7.2), we get

$$\bar{H}'_{\bar{x}} = \pm H'_x, \tag{8.21}$$

and hence $\bar{H}$ is determined (for known $\bar{x}$, $\bar{w}$) up to an arbitrary function of one variable. Then the second equation (8.3) means that $\phi(w, x)$ is uniquely determined by H, $\bar{H}$, $\bar{w}$ up to an additive constant. Finally, the equations (8.17) determine $\bar{\varphi}_1$, $\bar{\varphi}_2$ from the already known functions. The function $\bar{y}$ is also determined by $(8.3)_1$. This concludes the proof of Proposition 8.2 in the general case.

If some of the derivatives $\bar{w}_w$, $\bar{w}_x$, $\bar{x}_w$, $\bar{x}_x$ vanishes at p, then introducing new independent variables as linear combinations of w, x we see that the Cauchy-Kowalewski theorem can be applied to the system (8.13), (8.20), once more. Hence $\bar{w}$, $\bar{x}$ again depend on two arbitrary functions of 1 variable. The rest of the proof is a minor modification of the previous argument.

Now, Theorem 8.1 follows as a consequence of Proposition 8.2.

9. Appendix: Curvature homogeneous spaces

A Riemannian manifold (M, g) is said to be *curvature homogeneous* if, for each pair of points p and q, there exists a (linear) isometry F between the tangent spaces T_pM and T_qM such that $F^*(R_q) = R_p$.

Further, let $(\bar{M}, \bar{g})$ be a homogeneous Riemannian space, i.e., a connected Riemannian manifold such that the isometry group $I(\bar{M}, \bar{g})$ acts transitively on $\bar{M}$. We say that a Riemannian manifold (M, g) has *the same curvature tensor* as the space $(\bar{M}, \bar{g})$ if, each point $p \in M$, there exists a (linear) isometry $F : T_pM \rightarrow T_o\bar{M}$ such that $F^*(\bar{R}_o) = R_p$. Here o is a fixed point of $\bar{M}$, and $\bar{R}$, R denote the corresponding Riemannian curvature tensors of $(\bar{M}, \bar{g})$, (M, g), respectively. We also say briefly that (M, g) *admits* $(\bar{M}, \bar{g})$ *as a homogeneous model.* In such a case, (M, g) is obviously curvature homogeneous.

The curvature homogeneous spaces have introduced by I.M. Singer [Si], who put the question whether there exist Riemannian manifolds which are curvature homogeneous but not locally homogeneous. The first examples of this kind were constructed by K. Sekigawa [Se1] and H. Takagi [Ta]. A systematic study of this topic has been done by F. Tricerri, L. Vanhecke and the present author in [KTV1–3] and [TV1–2], where complete references can be found. Obviously, all spaces studied in the previous sections are also curvature homogeneous and not locally homogeneous, in general.

Up to recently, an open problem was to decide whether each curvature homogeneous space admits some homogeneous model. In [KTV3] the authors have proved that a four-dimensional example by K. Tsukada [Ts] is curvature homogeneous *without* any homogeneous model. They have also proved that all 3-dimensional examples by K. Yamato (see Introduction) possess homogeneous models.

As one consequence of the present paper we are now able to give explicit examples in dimension 3 which are curvature homogeneous and do not abmit homogeneous models.

PROPOSITION 9.1. *The examples* 5.9 *and* 5.10 *from Section* 5 *do not admit any homogeneous model.*

Proof. According to K. Sekigawa [Se2], each 3-dimensional homogeneous Riemannian space is locally isometric either to a Riemannian symmetric space, or a Lie group with a left invariant metric. According to J. Milnor [Mi], p. 310, *no* 3-dimensional Lie group admits a left invariant Riemann metric whose principal Ricci curvatures have the signs $(+, +, -)$. We conclude hence easily that a 3-dimensional homogeneous Riemannian space *cannot* admit (constant) Ricci roots with the signs $(+, +, -)$. But the corresponding triplets of the Ricci roots for the examples 5.9 and 5.10 are $\left(\frac{1}{4}\lambda^2, \frac{1}{4}\lambda^2, -2\lambda^2\right)$ and $\left(\frac{2}{9}\lambda^2, \frac{2}{9}\lambda^2, -2\lambda^2\right)$, respectively.

Moreover, we have also proved the following:

PROPOSITION 9.2. Let $\rho_1 = \rho_2 > 0$, $\rho_3 < 0$ be real numbers. Then the local isometry classes of Riemannian metrics with the prescribed constant principal Ricci curvatures ρ_1, ρ_2, ρ_3 depend on 2 arbitrary functions of one variable. On the other hand, there is no homogeneous Riemannian 3-space with the principal Ricci curvatures as above.

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