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ON TWISTING OPERATORS AND NEWFORMS OF HALF-INTEGRAL WEIGHT

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In memory of my father

Introduction

The theory of newforms is very important and useful for arithmetical study of modular forms of integral weight. It is natural to try to extend this theory into the case of modular forms of half-integral weight. Until now, several authors have attempted to find a theory of newforms of half-integral weight (cf. [She], [N], [K], [M-R-V], [She-W]). But complete results have not been obtained yet.

The purpose of this paper is to complete Kohnen's results in [K] and to establish a theory of newforms for (what is called) Kohnen space of arbitrary level (cf. §0(d)).

This paper is composed as follows: §0 is general preliminaries. §1 and §2 are preparations for the main parts of this paper. We shall deal with both the full space $S(k+1/2, N, \chi)$ and Kohnen space $S(k+1/2, N, \chi)_K$ in these sections. The main parts of this paper are §3 and §4 and the main results are Theorems (3.10-11) and Theorem (4.13). In these sections, we shall deal with only Kohnen space.

Let us explain the contents of this paper, precisely.

Let k, N be positive integers with $4 \mid N$ and χ an even character modulo N with $\chi^2 = 1$. Denote the space of cusp forms of weight $k + 1/2$, level N, and character χ by $S(k+1/2, N, \chi)$ (cf. §0(c)).

In particular, when $\mathrm{ord}_2(N) = 2$, W. Kohnen ([K]) defined a canonical subspace $S(k+1/2, N, \chi)_K$ of $S(k+1/2, N, \chi)$ which is called Kohnen space (cf. §0(d)). He also established the theory of newforms for this subspace when $N/4$ is (odd) squarefree.

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Unfortunately, when $N/4$ is not squarefree, Kohnen's theory does not work. In fact, there exists a case such that all common eigen subspace of $S(k+1/2, N, \chi)_K$ for Hecke operators have dimension ≥ 2 and hence a strong "multiplicity one theorem" does not hold good (cf. [U1, Proposition 3(3)]).

This difficulty can be resolved by decomposing $S(k+1/2, N, \chi)_K$ into eigen subspaces of twisting operators. For simplicity, we write $S = S(k+1/2, N, \chi)$ or $S(k+1/2, N, \chi)_K$. We decompose N as follows:

$$N = 2^{\mathrm{ord}_2(N)} M_1 M_{2+}, \quad M_1 = \prod_{\substack{p \mid N, p \neq 2 \\ \mathrm{ord}_p(N)=1}} p, \quad M_{2+} = \prod_{\substack{p \mid N, p \neq 2 \\ \mathrm{ord}_p(N) \geq 2}} p^{\mathrm{ord}_p(N)}.$$

Denote the set of all prime divisors of M_{2+} by Π.

In §1, we shall decompose S by twisting operators:

$$S = \Big(\bigoplus_{\kappa \in \mathrm{Map}(\Pi, \{\pm 1\})} S^{\emptyset,\kappa}\Big) \oplus \mathrm{Ker}\,(R_\Pi\,;\, S).$$

Here, $S^{\emptyset,\kappa} = \{f \in S\,;\, f \mid R_l = \kappa(l) f \text{ for all } l \in \Pi\}$ and R_Π (resp. R_l) is the twisting operator of the character $\prod_{l \in \Pi}\left(\frac{}{l}\right)$ (resp. $\left(\frac{}{l}\right)$). We define an operator B_l by $f \mid B_l(z) := f(lz)$, $z \in \mathfrak{H}$. Then $\mathrm{Ker}(R_\Pi\,;\, S) = \sum_{l \in \Pi} S\left(k+1/2, N/l, \chi\left(\frac{l}{}\right)\right) \mid B_l$ or $\sum_{l \in \Pi} S\left(k+1/2, N/l, \chi\left(\frac{l}{}\right)\right)_K \mid B_l$ (cf. Propositions (1.5) and (1.10-11)). This means that $\mathrm{Ker}(R_\Pi\,;\, S)$ consists of "oldforms".

Each $S^{\emptyset,\kappa}$ is stable under the action of all Hecke operators $\tilde{T}(n^2)((n, N) = 1)$. Moreover, there exists a case such that $S^{\emptyset,\kappa} \simeq S^{\emptyset,\kappa'}$ as Hecke modules for distinct $\kappa, \kappa' \in \mathrm{Map}(\Pi, \{\pm 1\})$ (cf. [U6]). This is the reason why Kohnen's theory does not work when $N/4$ is not squarefree ($\Leftrightarrow M_{2+} \neq 1$).

Thus, Kohnen space is not good for establishing a theory of newforms, and indeed, the space $S^{\emptyset,\kappa}$ is important for that. We shall study the space $S^{\emptyset,\kappa}$ as Hecke module in the rest of §1 and §2. In particular, we shall explicitly compute the trace relation between the traces of Hecke operators $\tilde{T}(n^2)((n, N) = 1)$ on $S^{\emptyset,\kappa}$ and the traces of Hecke operators $T(n)$ on certain subspaces of $S^0(2k, N')$, where N' varies any positive divisors of $N/2$ (cf. Proposition (2.23)). We need some assumptions about N and χ for obtaining the above trace relation.

In §3, We shall consider only Kohnen space and establish a theory of newforms on the space $S^{\emptyset,\kappa}$ (and so $S(k+1/2, N, \chi)_K$), which contains Kohnen's results in [K], by using the above trace relation formula (2.23).

The main results are as follows:

(i) There exists a canonically defined subspace $\mathfrak{S}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$ of $S^{\emptyset,\kappa}$

and a canonical decomposition:

$$S^{\emptyset,\chi} = \bigoplus_{\substack{0<e,d \\ ed\mid M_1}} \mathfrak{S}^{\emptyset,\chi}(k+1/2,\, 4dM_{2+},\, \chi)_K \mid U(e^2),$$

(cf. (3.5) and Theorem (3.10)). Here, $U(e^2)$ is the operator: $\sum_{n\geq 1} a(n)\mathbf{e}(nz) \mapsto \sum_{n\geq 1} a(e^2 n)\mathbf{e}(nz)$.

(ii) We can explicitly express the trace $\mathrm{tr}(\tilde{T}(n^2)\,;\, \mathfrak{S}^{\emptyset,\chi}(k+1/2,\, N,\, \chi)_K)$ $((n,\, N) = 1)$ with the traces $\mathrm{tr}(T(n)\,;\, \mathbf{C}f)$ for all primitive forms $f \in S(2k,\, N/4)$ and the multiplicity of each primitive form in that expression is at most one (cf. Theorem (3.10)). In particular, we have a strong "multiplicity one theorem" for $\mathfrak{S}^{\emptyset,\chi}(k+1/2,\, N,\, \chi)_K$ (cf. Theorem (3.11)).

(iii) We can define an involution $\mathbf{w}_p$ for any prime divisor p of M_1 (cf. (3.6)). This $\mathbf{w}_p$ corresponds to the Atkin-Lehner involution of integral weight (cf. Theorems (3.9-11)).

The author thinks that these results are still incomplete. Because the conductor of each primitive form of integral weight, which corresponds to $\mathfrak{S}^{\emptyset,\chi}(k+1/2,\, N,\, \chi)_K$ (cf. (ii)), is not always $N/4 = M_1 M_{2+}$ and all we can say is that it is of the form $M_1 M'$ for a certain positive divisor M' of M_{2+} (cf. Theorem (3.11)).

We shall discuss this topic in §4. We denote by $\hat{\mathfrak{N}}^{\emptyset,\chi}$ a certain subspace of $\mathfrak{S}^{\emptyset,\chi}(k+1/2,\, N,\, \chi)_K$ constructed with the spaces $\mathfrak{S}^{\emptyset,\chi'}(k+1/2,\, N',\, \chi)_{K'}$'s and the operators $U(a^2)$, R_l, where κ' is a restriction of κ and N' is a positive divisor of N with $N' < N$ (cf. the sentences before Theorem (4.13) for a precise definition). We also denote by $\mathfrak{N}^{\emptyset,\chi}$ the orthogonal complement of $\hat{\mathfrak{N}}^{\emptyset,\chi}$ in $\mathfrak{S}^{\emptyset,\chi}(k+1/2,\, N,\, \chi)_K$.

Then we have a decomposition:

$$\mathfrak{S}^{\emptyset,\chi}(k+1/2,\, N,\, \chi)_K = \mathfrak{N}^{\emptyset,\chi} \oplus \hat{\mathfrak{N}}^{\emptyset,\chi}$$

and we can characterize $\mathfrak{N}^{\emptyset,\chi}$ (resp. $\hat{\mathfrak{N}}^{\emptyset,\chi}$) as the subspace of $\mathfrak{S}^{\emptyset,\chi}(k+1/2,\, N,\, \chi)_K$ corresponding to only primitive forms of conductor $N/4$ (resp. of conductor smaller than $N/4$), under a certain assumption (4.1) on the character χ (cf. Theorem (4.13)).

Moreover, the author believes that the assumption (4.1) is not necessary (cf. the discussion at the end of §4). If it is true, the space $\mathfrak{N}^{\emptyset,\chi}$ is the true space of newforms and $S(k+1/2,\, N,\, \chi)_K$ can be constructed with $\mathfrak{N}^{\emptyset,\chi}$'s (of various κ's and various levels) and the operators $(B_l,\, U(a^2),\, R_l)$.

The author hopes that this is a final form of a theory of newforms of half-integral weight.

Finally, we have some comments. It seems likely that the results in §3 and §4 can be generalized to the full space $S(k+1/2, N, \chi)$, because their proofs depend largely on applying the results in §1 and §2 of this paper.

The proofs in §3 is to a large extent analogous to Kohnen's ([K]). We remark that Kohnen's results on the operator $w^N_{p,k+1/2,\chi}$ ([K, Proposition 4, Theorem 1]) contain a mistake. The factor $\left(\frac{N/p}{p}\right)$ falls out. Our operator $\mathbf{w}_p$ in §3 is a correction of $w^N_{p,k+1/2,\chi}$.

§0. Preliminaries

Throughout this paper, we use the following notations.

(a) General notations

Let A, B be subsets of a set X and $\{A_i\}_{i\in I}$ a family of subsets of X. If $A \cup B$ is a disjoint union, then we denote $A+B := A \cup B$ for simplicity. Similarly, if $\cup_{i\in I} A_i$ is a disjoint union, then we denote $\sum_{i\in I} A_i := \cup_{i\in I} A_i$.

We denote the set of positive integers by $\mathbf{Z}_+$ the symbol $\square$ denotes any square integer. For any $n \in \mathbf{Z}_+$, we denote by $\varphi(n)$ the order of the group $(\mathbf{Z}/n\mathbf{Z})^\times$.

For any prime p, the symbol $|\ \ |_p$ is the p-adic absolute value which is normalized with $|p|_p = p^{-1}$ and we also denote the additive valuation for any integer m by $\mathrm{ord}_p(m)$. Then $|m|_p = p^{-\mathrm{ord}_p(m)}$.

For a real number x, $[x]$ means the greatest integer m with $x \geq m$ and we denote $\mathrm{sgn}(x) = 1$ or -1, according as $x \geq 0$ or $x < 0$.

See [M, p.82] for the definition of the Kronecker symbol $\left(\frac{a}{b}\right)$ (a, b integers with $(a,b) \neq (0,0)$).

For a positive integer L and $m_1, m_2 \in (\mathbf{Z}/L\mathbf{Z})^\times$, we denote $m_1 \approx m_2$ if $m_1(\mathbf{Z}/L\mathbf{Z})^{\times 2} = m_2(\mathbf{Z}/L\mathbf{Z})^{\times 2}$. Let N be a positive integer and m an integer $\neq 0$. We write $m \mid N^\infty$ if every prime factor of m divides N.

Let k denote a non-negative integer. If $z \in \mathbf{C}$ and $x \in \mathbf{C}$, we put $z^x = \exp(x \cdot \log(z))$ with $\log(z) = \log(|z|) + \sqrt{-1}\arg(z)$, $\arg(z)$ being determined by $-\pi < \arg(z) \leq \pi$. Also we put $e(z) = \exp(2\pi\sqrt{-1}z)$.

Let $\mathfrak{H}$ be the complex upper half plane. For a complex-valued function $f(z)$ on $\mathfrak{H}$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbf{R})$, $\gamma = \begin{pmatrix} u & v \\ w & x \end{pmatrix} \in \Gamma_0(4)$ and $z \in \mathfrak{H}$, we define functions $J(\alpha, z)$, $j(\gamma, z)$ and $f|[\alpha]_k(z)$ on $\mathfrak{H}$ by: $J(\alpha, z) = cz + d$, $j(\gamma, z) = \left(\frac{-1}{x}\right)^{-1/2}\left(\frac{w}{x}\right)(wz+x)^{1/2}$ and $f|[\alpha]_k(z) = (\det\alpha)^{k/2} J(\alpha, z)^{-k} f(\alpha z)$.

For $m \in \mathbf{Z}_+$ we define an operator $U(m)$ on formal power series in $\mathbf{e}(z)$ by

$$\sum_{n\geq 0} a(n)\mathbf{e}(nz) \mid U(m) := \sum_{n\geq 0} a(mn)\mathbf{e}(nz).$$

Let χ be a Dirichlet character modulo N. Then we denote the conductor of χ by $\mathfrak{f}(\chi)$ and the p-primary component of χ by χ_p for each prime divisor p of N.

For a primitive character ψ modulo r, we put the Gauss sum for $\psi : \mathfrak{g}(\psi) = \sum_{i=1}^{r} \psi(i)\mathbf{e}(i/r)$.

Let V, V' be finite-dimensional vector spaces over $\mathbf{C}$. We denote the trace of T on V for a linear operator T on V by $\mathrm{tr}(T\ ;\ V)$ and also the kernel of a linear map F from V to V' by $\mathrm{Ker}(F\ ;\ V)$.

(b) Modular forms of integral weight

Let k and N be positive integers. By $S(2k, N)$, we denote the space of all holomorphic cusp forms of weight $2k$ with the trivial character on the group $\Gamma = \Gamma_0(N)$. We also denote the subspace of $S(2k, N)$ spanned by all newforms in $S(2k, N)$ by $S^0(2k, N)$.

Let $\alpha \in GL_2^+(\mathbf{R})$. If Γ and $\alpha^{-1}\Gamma\alpha$ are commensurable, we define a linear operator $[\Gamma\alpha\Gamma]_{2k}$ on $S(2k, N)$ by: $f \,|\, [\Gamma\alpha\Gamma]_{2k} = (\det\alpha)^{k-1} \sum_{\alpha_i} f \,|\, [\alpha_i]_{2k}$, where α_i runs over a system of representatives for $\Gamma\backslash\Gamma\alpha\Gamma$. For a positive integer n with $(n, N) = 1$, we put $T_{2k,N}(n) = \sum_{ad=n} \left[\Gamma\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix}\Gamma\right]_{2k}$, where the sum is extended over all pairs of integers (a, d) such that $a, d > 0$, $a \mid d$, $ad = n$.

Let Q be a positive divisor of N such that $(Q, N/Q) = 1$. Take any element $\gamma_Q \in SL_2(\mathbf{Z})$ which satisfies the conditions:

$$\gamma_Q \equiv \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} \pmod{Q}\ ; \\ \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \pmod{N/Q}. \end{cases}$$

Put $W(Q) = \gamma_Q \begin{pmatrix} Q & 0 \\ 0 & 1\end{pmatrix}$. The following facts are well-konwn: $W(Q)$ is a normalizer of Γ ; $[W(Q)]_{2k}$ induces a $\mathbf{C}$-linear automorphism of order 2 on $S(2k, N)$ and this operator is independent of a choice of an element γ_Q. For $Q = 1$, we can take $\gamma_1 = W(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$. Hence we have $[W(1)]_{2k} = 1$. Moreover for the sake of simplicity, we use the following abbreviated notation: Let A be a subset of the set of all prime divisors of N. Then $W_A := W(\prod_{p\in A} p^{\mathrm{ord}_p(N)})$. In particular, we simply write $W_l = W_A$ if $A = \{l\}$.

Moreover, if the subscripts are obvious and any confusion does not occur, we simply write $T(n) = T_{2k,N}(n)$ and $W(Q) = [W(Q)]_{2k}$, etc..

For any $f(z) = \sum_{n=1}^{\infty} a(n)\mathbf{e}(nz) \in S(2k, N)$ and χ a primitive character modulo $\mathfrak{f} = \mathfrak{f}(\chi)$, put $f | R_\chi(z) := \sum_{n=1}^{\infty} \chi(n)a(n)\mathbf{e}(nz)$. From [Sh 3, Proposition 3.64] we have $f | R_\chi \in S(2k, N', \chi^2)$, where N' is the least common multiple of N and $\mathfrak{f}(\chi)^2$. We call this operator R_χ the twisting operator of χ.

If f and g are cusp forms of weight $2k$ on a subgroup $\tilde{\Gamma}$ of finite index in $SL_2(\mathbf{Z})$, we denote their Petersson inner product by:

$$\langle f, g\rangle = v(\tilde{\Gamma}\backslash \mathfrak{H})^{-1} \int_{\tilde{\Gamma}\backslash\mathfrak{H}} f(z)\overline{g(z)}y^{2k-2}dxdy,$$

$$v(\tilde{\Gamma}\backslash \mathfrak{H}) = \int_{\tilde{\Gamma}\backslash\mathfrak{H}} y^{-2}dxdy \quad (x = \mathrm{Re}(z),\ y = \mathrm{Im}(z)).$$

In the following sections, we shall use various properties for the operators $T_{2k,N}(n)$, $W(Q)$, and R_χ. We shall collect them in the appendix 1.

(c) Modular forms of half-integral weight

Let k denote a non-negative integer, N a positive integer divisible by 4, and χ an even character modulo N such that $\chi^2 = 1$. Put $\mu = \mathrm{ord}_2(N)$, $M = 2^{-\mu}N$ and $\Gamma = \Gamma_0(N)$. Then there is a square-free odd positive divisor M_0 of M such that $\chi = \left(\frac{M_0}{\ }\right)$ or $\left(\frac{2M_0}{\ }\right)$ (the Kronecker symbol).

Let $\mathfrak{G}(k + 1/2)$ be the group consisting of pairs (α, φ), where $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbf{R})$ and φ is a holomorphic function on $\mathfrak{H}$ satisfying $\varphi(z) = t(\det \alpha)^{-k/2-1/4}J(\alpha, z)^{k+1/2}$ with $t \in \mathbf{C}$ and $|t| = 1$. The group law is defined by: $(\alpha, \varphi(z)) \cdot (\beta, \phi(z)) = (\alpha\beta, \varphi(\beta z)\phi(z))$. For a complex-valued function f on $\mathfrak{H}$ and $(\alpha, \varphi) \in \mathfrak{G}(k + 1/2)$, we define a function $f | (\alpha, \varphi)$ on $\mathfrak{H}$ by: $f | (\alpha, \varphi)(z) = \varphi(z)^{-1}f(\alpha z)$. Moreover if there will be no confusion, we also write $\gamma^* = (\gamma, j(\gamma, z)^{2k+1})$ for all $\gamma \in \Gamma_0(4)$.

By $\Delta = \Delta_0(N, \chi) = \Delta_0(N, \chi)_{k+1/2}$, we denote the subgroup of $\mathfrak{G}(k + 1/2)$ consisting of all pairs (γ, φ), where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in \Gamma$ and $\varphi(z) = \chi(d) j(\gamma, z)^{2k+1}$ and also denote $\Delta_1 = \Delta_1(N) := \{\gamma^* \mid \gamma \in \Gamma_1(N)\}$.

We denote by $G(k + 1/2, N, \chi)$ (resp. $S(k + 1/2, N, \chi)$) the space of integral (resp. cusp) forms of weight $k + 1/2$ with the character χ on the group Γ, namely, the space of all the complex-valued holomorphic functions f on $\mathfrak{H}$ which satisfy $f \mid \xi = f$ for all $\xi \in \Delta$ and which are holomorphic (resp. are holomorphic and vanish) at all cusps of Γ. In particular, we write $S(k+1/2, N) = S(k+1/2, N, \chi)$ if χ is the trivial character. Moreover we also denote by $S(k+1/2, \Delta_1(N))$ the space of cusp forms of weight $k+1/2$ on the group $\Gamma_1(N)$ i.e., the space of all the complex-valued holomorphic functions f on $\mathfrak{H}$ which satisfy $f \mid \xi = f$ for all $\xi \in \Delta_1$ and which are holomorphic and vanish at all cusps of $\Gamma_1(N)$ ([cf. Sh 1]).

If f and g are cusp forms of weight $k+1/2$ on a subgroup $\tilde{\Gamma}$ of finite index in $\Gamma_0(4)$, we denote their Petersson inner product by:

$$\langle f, g \rangle = v(\tilde{\Gamma} \backslash \mathfrak{H})^{-1} \int_{\tilde{\Gamma} \backslash \mathfrak{H}} f(z) \overline{g(z)} y^{k-3/2} dxdy,$$

$$v(\tilde{\Gamma} \backslash \mathfrak{H}) = \int_{\tilde{\Gamma} \backslash \mathfrak{H}} y^{-2} dxdy \ (x = \mathrm{Re}(z),\ y = \mathrm{Im}(z)).$$

Now for $\nu = 0$ or 1, we denote by $\Omega^{\nu}(N, \chi)$ the set of all pairs (ρ, t), where ρ is a primitive character modulo r with $\rho(-1) = (-1)^{\nu}$ and t is a positive integer, which satisfy the following two conditions:

(0.1) $$4tr^2 \mid N.$$

(0.2) $$\chi = \left(\frac{\rho(-1)t}{\ }\right) \rho \text{ as a character modulo } N.$$

Then we consider the theta series of the following type:
$h^{\nu}(\rho\,; z) = (1/2) \sum_{m \in \mathbf{Z}} \rho(m) m^{\nu} \mathbf{e}(m^2 z)$, where $z \in \mathfrak{H}$ and $\nu = 0$ or 1.

For the case $\nu = 0$, we know that $\{h^0(\rho\,; tz) \mid (\rho, t) \in \Omega^0(N, \chi)\}$ is a $\mathbf{C}$-basis of the space $G(1/2, N, \chi)$ (cf. [S-S]). For the case $\nu = 1$, let $U(N\,; \chi)$ be the subspace of $S(3/2, N, \chi)$ generated by $\{h^1(\rho\,; tz) \mid (\rho, t) \in \Omega^1(N, \chi)\}$ over $\mathbf{C}$. By $V(N\,; \chi)$, we denote the orthogonal complement of $U(N\,; \chi)$ in $S(3/2, N, \chi)$ with respect to the Petersson inner product.

Let $\xi \in \mathfrak{G}(k+1/2)$. If Δ and $\xi^{-1}\Delta\xi$ are commensurable, we define a linear operator $[\Delta\xi\Delta]_{k+1/2}$ on $G(k+1/2, N, \chi)$ and $S(k+1/2, N, \chi)$ by: $f \mid [\Delta\xi\Delta]_{k+1/2} = \sum_{\eta} f \mid \eta$, where η runs over a system of representatives for $\Delta \backslash \Delta\xi\Delta$. Similarly, if ξ and $\xi^{-1}\Delta_1\xi$ are commensurable, we define a linear operator $[\Delta_1\xi\Delta_1]$ on $S(k+1/2, \Delta_1(N))$ by: $f \mid [\Delta_1\xi\Delta_1] = \sum_{\eta \in \Delta_1 \backslash \Delta_1\xi\Delta_1} f \mid \eta$.

Then for a positive integer n with $(n, N) = 1$, we put

$$\tilde{T}_{k+1/2,N,\chi}(n^2) = n^{k-3/2} \sum_{ad=n} a \left[\Delta \left(\begin{pmatrix} a^2 & 0 \\ 0 & d^2 \end{pmatrix}, (d/a)^{k+1/2} \right) \Delta \right]_{k+1/2},$$

where the sum is extended over all pairs of integers (a, d) such that $a, d > 0$, $a \mid d$ and $ad = n$. We simply write $\tilde{T}(n^2) := \tilde{T}_{k+1/2,N,\chi}(n^2)$ if the subscripts are obvious and any confusion does not occur. These operators $\tilde{T}(n^2)$ $((n, N) = 1)$ are hermitian and commutative with each other on $S(k + 1/2, N, \chi)$ (cf. [Sh 2, Lemma 5], [Sh 3, Proposition (3.32)], [U1, (1.9)]).

For $k = 1$, from [Sh 1, Theorem 1.7], it follows that $h^1(\rho\,; tz)$ with $(\rho, t) \in \Omega^1(N, \chi)$ is an eigen function of the Hecke operators $\tilde{T}_{3/2,N,\chi}(p^2)$ for all prime numbers $p \nmid N$. Hence, we see that $U(N\,; \chi)$ and $V(N\,; \chi)$ are invariant under the action of the Hecke operators $\tilde{T}_{3/2,N,\chi}(n^2)$ for all natural numbers n with $(n, N) = 1$ (cf. [Sh 3, Lemma 5]).

$U(N\,; \chi)$ corresponds to the space of the Eisenstein series through the Shimura correspondence and only the elements of $V(N\,; \chi)$ correspond to the cusp forms (cf. [St]). Hence, when $k = 1$, we shall be dealing with $V(N\,; \chi)$ in place of $S(3/2, N, \chi)$. Moreover, we can see the following: For any $m \in \mathbf{Z}_+$, $U(N\,; \chi) \subseteq U(Nm\,; \chi)$ and $V(N\,; \chi) \subseteq V(Nm\,; \chi)$. In fact, the first assertion follows from the definition. Next, we have $V(N\,; \chi) \subseteq S(3/2, N, \chi) \subseteq S(3/2, Nm, \chi)$ and also $V(N\,;\chi) \perp U(Nm\,; \chi)$ because $U(Nm\,; \chi)$ corresponds to the space of Eisenstein series and $V(N\,; \chi)$ corresponds to a space of cusp forms (cf. [C], [St]). Hence $V(N\,; \chi) \subseteq V(Nm\,; \chi)$. □

Let $f(z) = \sum_{n=0}^{\infty} a(n)\mathbf{e}(nz) \in G(k + 1/2, N, \chi)$ and ψ a primitive character modulo $\mathfrak{f}(\psi)$. Let N' be the least common multiple of N, $\mathfrak{f}(\psi)^2$, and $\mathfrak{f}(\psi)\mathfrak{f}(\chi)$. Then $f \mid R_\psi(z) := \sum_{n=0}^{\infty} \psi(n)a(n)\mathbf{e}(nz)$ belongs to the space $G(k + 1/2, N', \chi\psi^2)$. In particular, if f is a cusp form, so is $f \mid R_\psi$ [Sh 1, Lemma 3.6].

(d) The Kohnen space

We keep to the notations in the subsection (c). Let k be a positive integer. Suppose that $N = 4M$ and M is an odd natural number. Then $\chi = \left(\frac{M_0}{\ }\right)$ for some positive divisor M_0 of M (cf. §0(c)). Put $\varepsilon = \left(\frac{-1}{M_0}\right) = \chi_2(-1)$, where χ_2 is the 2-primary component of χ. We define the Kohnen space $S(k + 1/2, N, \chi)_K$ as follows:

$$S\left(k + \frac{1}{2}, N, \chi\right)_K = \left\{ \begin{array}{l} S(k + \frac{1}{2}, N, \chi) \ni f(z) = \sum_{n=1}^{\infty} a(n)\mathbf{e}(nz)\,; \\ a(n) = 0 \text{ for } \varepsilon(-1)^k n \equiv 2,3 \pmod 4 \end{array} \right\}.$$

In particular, we write $S(k + 1/2, N, \chi)_K = S(k + 1/2, N)_K$ if χ is the trivial character.

Put $\xi := \xi_{k+1/2,\varepsilon} = \left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \varepsilon^{k+1/2}\mathbf{e}((2k+1)/8)\right) \in \mathfrak{G}(k+1/2)$ and $Q_{k+1/2,N,\chi} = [\Delta\xi\Delta]_{k+1/2}$, $\Delta = \Delta_0(N, \chi)_{k+1/2}$. Then we know the following from [K, Proposition 1]: $Q_{k+1/2,N,\chi}$ is a hermitian operator on $S(k+1/2, N, \chi)$; and $S(k+1/2, N, \chi)_K$ is the α-eigen subspace of $S(k+1/2, N, \chi)$ with respect to the operator $Q_{k+1/2,N,\chi}$, where $\alpha = (-1)^{[(k+1)/2]}2\sqrt{2}\varepsilon$.

For $k = 1$, from the definitions of $S(3/2, N, \chi)_K$ and $U(N\,;\chi)$, it is easily shown that $S(3/2, N, \chi)_K$ contains $U(N\,;\chi)$. We denote by $V(N\,;\chi)_K$ the orthogonal complement of $U(N\,;\chi)$ in $S(3/2, N, \chi)_K$ with respect to the Petersson inner product. Then we can see for any odd positive integer m, $V(N\,;\chi)_K \subseteq V(Nm\,;\chi)_K$ (cf. §0(c)).

From [K] §3 and §4, we know that $S(k+1/2, N, \chi)_K$ (resp. $V(N\,;\chi)_K$) is invariant under the action of the Hecke operators $\tilde{T}_{k+1/2,N,\chi}(n^2)$ (resp. $\tilde{T}_{3/2,N,\chi}(n^2)$) for all positive integers n with $(n, N) = 1$.

Moreover Kohnen introduced the following operator on $S(k+1/2, N, \chi)_K$ in [K §3]:

$$\tilde{T}_{k+1/2,N,\chi}(4) := \frac{3}{2}2^{k-3/2}\left[\Delta\left(\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, 2^{k+1/2}\right)\Delta\right]\mathbf{Pr},$$

where $\mathbf{Pr} := \dfrac{2}{3\alpha}\left(Q_{k+1/2,N,\chi} + \dfrac{\alpha}{2}\right)$ is the orthogonal projection from $S(k+1/2, N, \chi)$ onto $S(k+1/2, N, \chi)_K$ (cf. [K, Proposition 1]). We shall use this operator in §3. We collect various properties of $\tilde{T}_{k+1/2,N,\chi}(4)$ in §3.

§1. The spaces $S^{\emptyset,\chi}$, $V^{\emptyset,\chi}$ and several operators

Let k be a positive integer, N a positive integer divisible by 4, and χ an even character modulo N such that $\chi^2 = 1$. The letter z means an element of $\mathfrak{H}$. Put $\mu = \mathrm{ord}_2(N)$ and $\nu_p = \mathrm{ord}_p(N)$ for any odd prime p. We decompose N as follows: $N = 2^\mu M = 2^\mu M_1 M_{2+}$, where $M_1 := \Pi_{p|M,\nu_p=1}p$ and $M_{2+} := \Pi_{p|M,\nu_p\geq 2}p^{\nu_p}$. We also denote the set of all prime divisors of M_{2+} by $\Pi = \Pi(M_{2+}) := \{l_1, \ldots, l_r\}$. From now on to the end of this paper, these notations are fixed.

Remark. The case of $\Pi = \emptyset$ (i.e., M is squarefree) was already studied by Kohnen in [K]. We shall generalize Kohnen's results in the following sections.

For any subset I of Π, we set the following notations: $l_I := \prod_{l \in I} l$ and $\psi_I := \left(\frac{}{l_I}\right)$ (The Legendre symbol). In particular, we have $l_\emptyset = 1$ and $\psi_\emptyset = 1$. For any non-empty subset I of Π, we can define the twisting operator of ψ_I on $G(k + 1/2, N, \chi)$ and $S(k + 1/2, N, \chi)$ (cf. [U2, §1]). We denote it by $R_I := R_{\psi_I} = \prod_{l \in I} R_{\left(\frac{}{l}\right)}$. Also we put $R_\emptyset := 1$. Moreover, we write $R_I = R_l$ if $I = \{l\}$. We denote the commutative algebras gererated by $R_I (I \subseteq \Pi)$ as follows: $\mathfrak{R} := \mathbf{Z}[R_I ; I \subseteq \Pi]$ and $\mathfrak{R}_{\mathbf{C}} := \mathbf{C}[R_I ; I \subseteq \Pi]$. Then from [U2, Corollary 1.10] all R_I's $(I \subseteq \Pi)$ are hermitian with respect to the Petersson inner product. Therefore each element of $\mathfrak{R}$ is also a hermitian operator. These notations are also fixed from now on to the end of this paper.

In the following, for the sake of simplicity, we write

$$\begin{aligned} S &:= S(k + 1/2, N, \chi) \text{ or } S(k + 1/2, N, \chi)_K, \\ V &:= V(N ; \chi) \text{ or } V(N ; \chi)_K. \end{aligned} \tag{1.1}$$

From [U2 §1], it follows that the spaces S and V are fixed by the operator R_I for any $I \subseteq \Pi$. Hence both $\mathfrak{R}$ and $\mathfrak{R}_{\mathbf{C}}$ act on these spaces.

Now we shall decompose these spaces S and V with the twisting operators.

For any $I \subseteq \Pi$, put $L_I := \{a \in \mathbf{Z} ; (a, l_\Pi) = l_I\}$. Then we have for any $I \subseteq \Pi$

$$\mathbf{Z} - \Big(\bigcup_{l \in \Pi - I} \mathbf{Z} l \Big) = \sum_{J \subseteq I} L_J \text{ (disjoint union).} \tag{1.2}$$

We define the operator $\tilde{R}_I \in \mathfrak{R}$ as follows: First $\tilde{R}_\emptyset := R_\Pi{}^2$ and for any non-empty subset I of Π we define inductively $\tilde{R}_I := R_{\Pi - I}{}^2 - \sum_{\substack{I \supseteq J \\ I \neq J}} \tilde{R}_J$. Then from (1.2) and easy induction, we get the following.

PROPOSITION (1.3). *For any* $f = \sum_{n \geq 0} a(n) \mathbf{e}(nz) \in G(k + 1/2, N, \chi)$ *and any* $I \subseteq \Pi$, *we have* $f \mid \tilde{R}_I = \sum_{0 \leq n \in L_I} a(n) \mathbf{e}(nz) \in G(k + 1/2, N, \chi)$. □

From this proposition we have the following relation as operators on $G(k + 1/2, N, \chi)$: For any subsets $I, J \subseteq \Pi$ $(I \neq J)$,

$$\tilde{R}_I{}^2 = \tilde{R}_I, \ \tilde{R}_I \tilde{R}_J = \tilde{R}_J \tilde{R}_I = 0, \text{ and } \sum_{I \subseteq \Pi} \tilde{R}_I = 1. \tag{1.4}$$

Hence $\{\tilde{R}_I \mid I \subseteq \Pi\}$ is a family of projection operators on $G(k + 1/2, N, \chi)$.

Put for any subset $I \subseteq \Pi$

$$S^I := S \mid \tilde{R}_I = \{\textstyle\sum_{n \geq 1} b(n) \mathbf{e}(nz) \in S \,;\, b(n) = 0 \text{ for } n \notin L_I\}$$

and

$$V^I := V|\tilde{R}_I = \{\sum_{n\geq 1} b(n)\mathbf{e}(nz) \in V\ ;\ b(n) = 0 \text{ for } n \notin L_I\}.$$

Since $\mathcal{R}_{\mathbf{C}}$ is a commutative algebra, the space S^I and $V^I (I \subseteq \Pi)$ are stable under the action of elements of $\mathcal{R}_{\mathbf{C}}$.

PROPOSITION (1.5). *Under the above notations, we have the following.*
(1) S^I *and* S^J *are orthogonal with respect to the Petersson inner product for any* $I, J \subseteq \Pi\ (I \neq J)$.

(2) $$S = \bigoplus_{I\subseteq \Pi} S^I, \quad V = \bigoplus_{I\subseteq \Pi} V^I.$$

(3) $$\mathrm{Ker}(R_\Pi\ ; S) = \bigoplus_{\emptyset\neq I\subseteq \Pi} S^I, \quad \mathrm{Ker}(R_\Pi\ ; V) = \bigoplus_{\emptyset\neq I\subseteq \Pi} V^I.$$

Proof. Any $\tilde{R}_I \in \mathcal{R}(I \subseteq \Pi)$ is hermitian. The assertions (1) and (2) follow from this fact and (1.4).

(3) It is easy to verify that

$$\begin{aligned} f(z) = \sum_{n\geq 0} a(n)\mathbf{e}(nz) &\in \mathrm{Ker}(R_\Pi\ ; G(k+1/2, N, \chi)) \\ &\Leftrightarrow a(n) = 0 \text{ for all } n(\geq 0) \text{ such that } (n, l_\Pi) = 1 \\ &\Leftrightarrow f|\tilde{R}_\emptyset = 0 \text{ by Proposition (1.3).} \end{aligned}$$

The assertion (3) is easily deduced from the above. □

We denote the set of all maps from Π to $\{\pm 1\}$ by $\{\pm 1\}^\Pi = \mathrm{Map}(\Pi, \{\pm 1\})$. Put for any $\kappa \in \mathrm{Map}(\Pi, \{\pm 1\})$,

$$S^{\emptyset,\kappa} := \{f \in S^\emptyset\ ; f|R_l = \kappa(l)f \quad \text{for all } l \in \Pi\},$$
$$V^{\emptyset,\kappa} := \{f \in V^\emptyset\ ; f|R_l = \kappa(l)f \quad \text{for all } l \in \Pi\}.$$

If we need to specify k, N, and χ, we will denote $S^{\emptyset,\kappa} = S^{\emptyset,\kappa}(k+1/2, N, \chi)$, $S^{\emptyset,\kappa}(k+1/2, N, \chi)_K$ and $V^{\emptyset,\kappa} = V^{\emptyset,\kappa}(N, \chi)$, $V^{\emptyset,\kappa}(N, \chi)_K$.

Remark (1.6). When $\Pi = \emptyset$, we understand that the meaning of these notations is $S^{\emptyset,\kappa} = S^\emptyset = S$, $V^{\emptyset,\kappa} = V^\emptyset = V$.

Since $R_\Pi = R_I R_{\Pi - I}$ for any $I \subseteq \Pi$, we have

$$\mathrm{Ker}(R_\Pi\ ; S) \supseteq \mathrm{Ker}(R_I\ ; S) \text{ and } \mathrm{Ker}(R_\Pi\ ; V) \supseteq \mathrm{Ker}(R_I\ ; V).$$

From this fact and Proposition (1.5), it follows that for any $I \subseteq \Pi$

$$S^{\emptyset} \perp \mathrm{Ker}(R_I\,;S) \text{ and } V^{\emptyset} \perp \mathrm{Ker}(R_I\,;V).$$

Hence $S^{\emptyset}$ and $V^{\emptyset}$ are decomposed into ± 1-eigen subspaces with respect to all R_I's $(I \subseteq \Pi)$. The decompositions are as follows:

$$S^{\emptyset} = \bigoplus_{\kappa\in\{\pm 1\}^{\Pi}} S^{\emptyset,\kappa},\ V^{\emptyset} = \bigoplus_{\kappa\in\{\pm 1\}^{\Pi}} V^{\emptyset,\kappa}.$$

Moreover, for any $f \in S^{\emptyset,\kappa}$ or $V^{\emptyset,\kappa}$ we have

(1.7) $$f|R_I = \kappa_I f \text{ for all } \kappa \in \{\pm 1\}^{\Pi} \text{ and } I \subseteq \Pi.$$

Here $\kappa_I := \Pi_{l\in I}\,\kappa(l) \in \{\pm 1\}$. In particular, $\kappa_{\emptyset} = 1$.

Now we consider the Hecke operator $\tilde{T}(n^2) = \tilde{T}_{k+1/2,N,\chi}(n^2)$ for any $n \in \mathbf{Z}_+$ with $(n, N) = 1$. From [U2, Proposition (1.7)], $\tilde{T}(n^2)$ commutes with the twisting operator R_I $(I \subseteq \Pi)$. Hence it also commutes with any element of $\mathcal{R}_{\mathbf{C}}$. Therefore the spaces S^I and V^I are stable under $\tilde{T}(n^2)$ for all subsets I of Π. Similarly, $\tilde{T}(n^2)$ acts on the spaces $S^{\emptyset,\kappa}$ and $V^{\emptyset,\kappa}$.

The main purpose of §1 and §2 is to compute the trace of the Hecke operator $\tilde{T}(n^2)$ on the spaces $S^{\emptyset,\kappa}$ and $V^{\emptyset,\kappa}$. Before computation, we prepare several notations and propositions.

For any $m \in \mathbf{Z}_+$, put $\delta_m := \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$ and $\tilde{\delta}_m := (\delta_m, m^{-k/2-1/4}) \in \mathfrak{G}(k+1/2)$. Then we can easily verify $\Delta_0(Nm, \chi')_{k+1/2} \subseteq \tilde{\delta}_m^{-1}\Delta_0(N, \chi)_{k+1/2}\tilde{\delta}_m$, where $\chi' = \chi\left(\frac{m}{}\right)$. Hence for any $f \in G(k+1/2, N, \chi)$, $f\,|\,\tilde{\delta}_m$ is stable under any elements of $\Delta_0(Nm, \chi')_{k+1/2}$. Therefore from the standard argument, it follows that the mapping $f \mapsto f\,|\,\tilde{\delta}_m$ maps $G(k+1/2, N, \chi)$ into $G(k+1/2, Nm, \chi')$. We also have the similar results for the spaces S and V. We need the following proposition for that proof.

Proposition (1.8). *Let m be a positive integer and put $\chi' = \chi\left(\frac{m}{}\right)$ as a character modulo Nm. We have the following relations for all $f \in G(k+1/2, N, \chi)$ and $n \in \mathbf{Z}_+$ such that $(n, Nm) = 1$:*

$$f\,|\,\tilde{T}_{k+1/2,N,\chi}(n^2) = f\,|\,\tilde{T}_{k+1/2,Nm,\chi}(n^2),$$
$$f\,|\,\tilde{T}_{k+1/2,N,\chi}(n^2)\tilde{\delta}_m = f\,|\,\tilde{\delta}_m\tilde{T}_{k+1/2,Nm,\chi'}(n^2).$$

Proof. We get these relations from straightforward computation. □

Now we can prove the following.

PROPOSITION (1.9). *Under the above notations, we have the following. Let m be a positive integer and put $\chi' = \chi\left(\frac{m}{}\right)$ as a character modulo Nm. Then the mapping $f \mapsto f \mid \tilde{\delta}_m$ maps the spaces $G(k+1/2, N, \chi)$, $S(k+1/2, N, \chi)$, $V(N;\chi)$ into the spaces $G(k+1/2, Nm, \chi')$, $S(k+1/2, Nm, \chi')$, $V(Nm;\chi')$ respectively. Moreover, if m is an odd positive integer, the mapping $f \mapsto f \mid \tilde{\delta}_m$ maps the spaces $S(k+1/2, N, \chi)_K$ and $V(N;\chi)_K$ into $S(k+1/2, Nm, \chi')_K$ and $V(Nm;\chi')_K$, respectively.*

Proof. We already proved the case of $G(k+1/2, N, \chi)$ and the proof for $S(k+1/2, N, \chi)$ is exactly similar to those for $G(k+1/2, N, \chi)$.

Next, we consider the case of $V(N;\chi)$. $V(N;\chi)$ has a $\mathbf{C}$-basis consisting of common eigen forms for $\tilde{T}(n^2)((n, N) = 1)$. Take any element f of such a basis. Then the system of eigen values of f corresponds to a certain primitive cusp form of weight 2 through Shimura correspondence (cf. [C], [St]).

By Proposition (1.8), $f \mid \tilde{\delta}_m \tilde{T}(n^2) = f \mid \tilde{T}(n^2)\tilde{\delta}_m = \lambda(n) f \mid \tilde{\delta}_m$ for any $n \in \mathbf{Z}_+$ such that $(n, Nm) = 1$. Here, $\lambda(n)$ is the eigen value of f with respect to $\tilde{T}(n^2)$. Hence $f \mid \tilde{\delta}_m \in S(3/2, Nm, \chi')$ is a common eigen form of $\tilde{T}(n^2)((n, Nm) = 1)$ and those system of eigen values also corresponds to a primitive cusp form.

Therefore $f \mid \tilde{\delta}_m$ is orthogonal to the space of theta series $U(Nm;\chi')$, i.e., $f \mid \tilde{\delta}_m \in V(Nm;\chi')$ (cf. [C], [St]).

For the case of $S(k+1/2, N, \chi)_K$, it is sufficiently to check the condition of vanishing of Fourier coefficients. That is an easy computation.

Finally, since $V(N;\chi)_K = V(N;\chi) \cap S(3/2, N, \chi)_K$, the assertion for $V(N;\chi)_K$ follows from the results for $V(N;\chi)$ and $S(3/2, N, \chi)_K$. □

PROPOSITION (1.10). *The notations are the same as above. Let m be an odd positive divisor of N. Suppose that a $\mathbf{C}$-valued function f on $\mathfrak{H}$ satisfies the following two conditions* : (i) *$f(z+1) = f(z)$ for all $z \in \mathfrak{H}$* ; (ii) *$f(mz) \in G(k+1/2, N, \chi)$ (resp. $S(k+1/2, N, \chi)$).*

Then we get the following.

(1) *If $m\mathfrak{f}(\chi') \mid N$, $f \in G(k+1/2, N/m, \chi')$ (resp. $S(k+1/2, N/m, \chi')$).*
(2) *If $m\mathfrak{f}(\chi') \nmid N$, $f = 0$.*

Here, $\chi' = \chi\left(\frac{m}{\cdot}\right)$ *and* $\mathfrak{f}(\chi')$ *is the conductor of* χ'.

Proof. See [S-S, p.45, Lemma 7]. □

This assertion also holds good for the Kohnen space and $V(N;\chi)$.

COROLLARY (1.11). *The notations* (m, χ', *etc.,*) *are the same as in Proposition* (1.10). *Suppose that a* $\mathbf{C}$*-valued function* f *on* $\mathfrak{H}$ *satisfies the following two conditions*: (i) $f(z+1)=f(z)$ *for all* $z \in \mathfrak{H}$; (ii) $f(mz) \in S(k+1/2, N, \chi)_K$ (*resp.* $V(N;\chi)$, $V(N;\chi)_K$). *Then we get the following.*
(1) *If* $m\mathfrak{f}(\chi') \mid N$, $f \in S(k+1/2, N/m, \chi')_K$ (*resp.* $V(N/m;\chi')$, $V(N/m;\chi')_K$).
(2) *If* $m\mathfrak{f}(\chi') \nmid N$, $f=0$.

Proof. Since $S(k+1/2, N, \chi)_K \subseteq S(k+1/2, N, \chi)$ and $V(N;\chi)_K \subseteq V(N;\chi) \subseteq S(3/2, N, \chi)$, the above assertion (2) follows from the assertion (2) of the Proposition (1.10).

Now we shall consider the case of $m\mathfrak{f}(\chi') \mid N$.

First assume $f(mz) \in S(k+1/2, N, \chi)_K$. Then from Proposition (1.10) we know $f \in S(k+1/2, N/m, \chi')$. Put $f=\sum_{n\geq 1} a(n)\mathbf{e}(nz)$. Since $f(mz) \in S(k+1/2, N, \chi)_K$, we have that if $a(n) \neq 0$, $\chi_2(-1)(-1)^k nm \equiv 0,1 \pmod 4$. Moreover, we also have $\chi_2'(-1)(-1)^k n \equiv \chi_2(-1)(-1)^k nm \pmod 4$.

Hence $a(n) \neq 0 \Rightarrow \chi_2'(-1)(-1)^k n \equiv 0,1 \pmod 4$. Therefore from the definition of the Kohnen space, $f \in S(k+1/2, N/m, \chi')_K$.

Next assume $f(mz) \in V(N;\chi) (\subseteq S(3/2, N, \chi))$. By Proposition (1.10), we know that $f \in S(3/2, N/m, \chi')$. It is sufficient to show that f is orthogonal to $U(N/m;\chi')$.

$U(N/m;\chi')$ has a $\mathbf{C}$-basis $\{h^1(\rho;tz)\ ;\ (\rho,t) \in \Omega^1(N/m,\chi')\}$. Then we can easily verify that $h^1(\rho;tmz) = m^{-k/2-1/4} h^1(\rho;tz) \mid \tilde{\delta}_m \in U(N;\chi)$. Hence, for any $(\rho, t) \in \Omega^1(N/m, \chi')$,

$$\begin{aligned}\langle f, h^1(\rho;tz)\rangle &= \langle f, m^{k/2+1/4} h^1(\rho;tmz) \mid \tilde{\delta}_m^{-1}\rangle \\ &= m^{k/2+1/4}\langle f \mid \tilde{\delta}_m, h^1(\rho;tmz)\rangle \quad \text{(cf. [U2, Lemma (1.9)])} \\ &= m^{k+1/2}\langle f(mz), h^1(\rho;tmz)\rangle = 0.\end{aligned}$$

Therefore we get $f \perp U(N/m;\chi')$ i.e., $f \in V(N/m;\chi')$.

Finally, the assertion for $V(N;\chi)_K$ is an easy consequence of the above results, because $V(N;\chi)_K = S(3/2, N, \chi)_K \cap V(N;\chi)$. □

Let I be any subset of Π. Put $\chi^{(I)} := \chi\left(\frac{l_I}{\,}\right)$. This $\chi^{(I)}$ is considered as a Dirichlet character modulo N/l_I. We also set

$$S_I := \begin{cases} S(k+1/2, N/l_I, \chi^{(I)}), & \text{if } S = S(k+1/2, N, \chi)\,; \\ S(k+1/2, N/l_I, \chi^{(I)})_K, & \text{if } S = S(k+1/2, N, \chi)_K\,; \end{cases}$$

and

$$V_I := \begin{cases} V(N/l_I\,; \chi^{(I)}), & \text{if } V = V(N\,; \chi)\,; \\ V(N/l_I\,; \chi^{(I)})_K, & \text{if } V = V(N\,; \chi)_K. \end{cases}$$

Let A be a $\mathbf{C}$-vector space consisting of $\mathbf{C}$-valued functions on $\mathfrak{H}$. Define for any positive integer m:

$$A^{(m)} := A \mid \tilde{\delta}_m = \{a(mz) \mid a(z) \in A\}.$$

Then we get the following relations (1.12)–(1.14).

(1.12) For any subset I of Π,

$$S_I^{(l_I)} = \bigoplus_{I \subseteq J \subseteq \Pi} S^J, \quad V_I^{(l_I)} = \bigoplus_{I \subseteq J \subseteq \Pi} V^J.$$

Proof. Take any $f = \sum_{n \geq 1} a(n)\mathbf{e}(nz) \in S_I^{(l_I)}$. Then $a(n) = 0$ if $l_I \nmid n$. Since $\mathbf{Z}l_I = \sum_{I \subseteq J \subseteq \Pi} L_J$, $f = \sum_{I \subseteq J \subseteq \Pi} \sum_{1 \leq n \in L_J} a(n)\mathbf{e}(nz) = \sum_{I \subseteq J \subseteq \Pi} (f \mid \tilde{R}_J) \in \bigoplus_{I \subseteq J \subseteq \Pi} S^J$.

Next, take any J such that $I \subseteq J \subseteq \Pi$ and any $f = \sum_{1 \leq n \in L_J} a(n)\mathbf{e}(nz) \in S^J$. Put $g(z) := f(z/l_I)\,(z \in \mathfrak{H})$. g satisfies the two conditions in Proposition (1.10) and Corollary (1.11). We also have $\mathfrak{f}(\chi^{(I)}) \mid (N/l_I)$. Hence $g \in S_I$ and $f \in S_I^{(l_I)}$. □

(1.13) For any $I, J \subseteq \Pi$ with $I \cap J \neq \emptyset$, we have

$$S^J \mid R_I = \{0\}, \; V^J \mid R_I = \{0\}.$$

Proof. This assertion follows from Proposition (1.3). □

(1.14) Let I and J be any subset of Π with $I \cap J = \emptyset$. We have the following equation as the mapping from S_J (resp. V_J) into S (resp. V):

$$\left(\frac{l_J}{l_I}\right) R_I \tilde{\delta}_{l_J} = \tilde{\delta}_{l_J} R_I.$$

Proof. Since $l_I^2 \mid (N/l_J)$, we can define the twisting operator R_I on S_J and V_J. Then the equation follows from easy calculation. □

Now we can compute the trace of $\tilde{T}(n^2)$ on the spaces $S^{\emptyset,\chi}$ and $V^{\emptyset,\chi}$. Since the computation for $V^{\emptyset,\chi}$ is exactly the same as those for $S^{\emptyset,\chi}$, we shall consider only the space $S^{\emptyset,\chi}$.

Let I and J be subsets of Π such that $I \cap J = \emptyset$ and n a positive integer with $(n, N) = 1$. Then the mapping $f \mapsto f|\tilde{\delta}_{l_J}$ is an isomorphism from S_J onto $S_J^{(l_J)}$. Hence by using (1.8) and (1.12-14),

$$\left(\frac{l_J}{l_I}\right) \mathrm{tr}(R_I\tilde{T}(n^2)\ ;\ S_J) = \mathrm{tr}(R_I\tilde{T}(n^2)\ ;\ S_J^{(l_J)})$$

$$= \mathrm{tr}\Big(R_I\tilde{T}(n^2)\ ;\ \bigoplus_{J\subseteq K\subseteq \Pi} S^K\Big) = \mathrm{tr}\Big(R_I\tilde{T}(n^2)\ ;\ \bigoplus_{J\subseteq K\subseteq \Pi - I} S^K\Big).$$

Therefore, we have the following by summing them up with respect to J:

$$\sum_{I\cap J=\emptyset} (-1)^{\#J}\left(\frac{l_J}{l_I}\right) \mathrm{tr}(R_I\tilde{T}(n^2)\ ;\ S_J) = \sum_{I\cap J=\emptyset} (-1)^{\#J}\, \mathrm{tr}\Big(R_I\tilde{T}(n^2)\ ;\ \bigoplus_{J\subseteq K\subseteq \Pi-I} S^K\Big)$$

$$= \sum_{I\cap J=\emptyset}\ \sum_{J\subseteq K\subseteq \Pi-I} (-1)^{\#J}\, \mathrm{tr}(R_I\tilde{T}(n^2)\ ;\ S^K)$$

$$= \sum_{K\subseteq \Pi-I} \Big(\sum_{J\subseteq K} (-1)^{\#J}\Big) \mathrm{tr}(R_I\tilde{T}(n^2)\ ;\ S^K)$$

$$= \mathrm{tr}(R_I\tilde{T}(n^2)\ ;\ S^{\emptyset}) = \mathrm{tr}\Big(R_I\tilde{T}(n^2)\ ;\ \bigoplus_{\chi\in\{\pm1\}^{\Pi}} S^{\emptyset,\chi}\Big) = \sum_{\chi\in\{\pm1\}^{\Pi}} \mathrm{tr}(R_I\tilde{T}(n^2)\ ;\ S^{\emptyset,\chi})$$

$$= \sum_{\chi\in\{\pm1\}^{\Pi}} \kappa_I \mathrm{tr}(\tilde{T}(n^2)\ ;\ S^{\emptyset,\chi}).$$

Here, $\sum_{I\cap J=\emptyset}$ is the sum extended over all subsets J of Π with $I \cap J = \emptyset$. Moreover, for any κ, $\kappa_0 \in \{\pm 1\}^{\Pi}$ we have

$$\sum_{I\subseteq \Pi} \kappa_I \kappa_{0,I} = \prod_{l\in\Pi} (1 + \kappa(l)\kappa_0(l)) = \begin{cases} 2^{\#\Pi}, & \text{if } \kappa = \kappa_0\,; \\ 0, & \text{otherwise.} \end{cases}$$

Thus we obtain

PROPOSITION (1.15). *Let n be a positive integer such that $(n, N) = 1$ and κ any element of $\{\pm 1\}^{\Pi}$. We have*

$$\mathrm{tr}(\tilde{T}(n^2)\ ;\ S^{\emptyset,\kappa}) = 2^{-\#\Pi} \sum_{I\subseteq\Pi} \kappa_I \Big\{\sum_{I\cap J=\emptyset} (-1)^{\#J}\left(\frac{l_J}{l_I}\right) \mathrm{tr}(R_I\tilde{T}(n^2)\ ;\ S_J)\Big\},$$

$$\mathrm{tr}(\tilde{T}(n^2)\ ;\ V^{\emptyset,\kappa}) = 2^{-\#\Pi} \sum_{I\subseteq\Pi} \kappa_I \Big\{\sum_{I\cap J=\emptyset} (-1)^{\#J}\left(\frac{l_J}{l_I}\right) \mathrm{tr}(R_I\tilde{T}(n^2)\ ;\ V_J)\Big\},$$

where $\sum_{I\cap J=\emptyset}$ *is the sum extended over all subsets* J *of* Π *with* $I \cap J = \emptyset$. □

Now we introduce new operators $\tilde{W}(Q)$, Y_p and study their properties.

Let Q be an odd positive divisor of N such that $(Q, N/Q) = 1$. Take any element $\gamma_Q \in SL_2(\mathbf{Z})$ satisfying the conditions:

$$\gamma_Q \equiv \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & (\mathrm{mod}\ Q)\ ; \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & (\mathrm{mod}\ N/Q). \end{cases}$$

Then $\gamma_Q \in \Gamma_0(N/Q) \subseteq \Gamma_0(4)$. Put $\gamma_Q^* := (\gamma_Q, j(\gamma_Q, z)^{2k+1})$ and $\tilde{W}(Q) := \gamma_Q^* \tilde{\delta}_Q \in \mathfrak{G}(k+1/2)$.

Remark (1.16). For any $f \in G(k+1/2, N, \chi)$, $f \mid \tilde{W}(Q)$ is independent of a choice of an element γ_Q, because a gap of two elements is at most an element of $\Gamma(N)$. So we can choose a convenient element for calculations.

The following facts are easily verified by straightforward calculations:

(1.17) For all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,

$$\tilde{W}(Q)\gamma^*\tilde{W}(Q)^{-1} = (W(Q)\gamma W(Q)^{-1})^*\left(1, \left(\frac{Q}{d}\right)\right).$$

Hence $\tilde{W}(Q)\Delta_0(N, \chi)\tilde{W}(Q)^{-1} = \Delta_0\left(N, \chi\left(\frac{Q}{\cdot}\right)\right)$ and $\tilde{W}(Q)$ is a normalizer of $\Delta_1(N)$.

From these facts, we can see that the mapping $f \mapsto f \mid \tilde{W}(Q)$ gives an isomorphism from $S(k+1/2, N, \chi)$ onto $S\left(k+1/2, N, \chi\left(\frac{Q}{\cdot}\right)\right)$. Moreover we have the following properties for $\tilde{W}(Q)$.

PROPOSITION (1.18) *Let* Q *and* Q' *be odd positive divisors of* N *such that* $(Q, N/Q) = (Q', N/Q') = (Q, Q') = 1$. *Then for any* $f \in S(k+1/2, N, \chi)$,

$$f \mid \tilde{W}(Q)\tilde{W}(Q) = \left(\frac{-1}{Q}\right)^{-k-1/2} \chi_Q(-1)\chi_{N/Q}(Q) f,$$
$$f \mid \tilde{W}(Q)\tilde{W}(Q') = \chi_{Q'}(Q) f \mid \tilde{W}(QQ').$$

Here, χ_Q, $\chi_{N/Q}$, *and* $\chi_{Q'}$ *are the* Q, N/Q, *and* Q'*-primary components of* χ *respectively.*

Proof. We get these relations from similar straightforward computation to the case of integral weight. □

Next we investigate the relations between $\tilde{W}(Q)$ and the other operators $U(p)$, $\tilde{T}(n^2)$, and R_l. For simplicity, we write $\Gamma = \Gamma_0(N)$, $\Gamma_1 = \Gamma_1(N)$, $\Delta = \Delta_0(N, \chi)$, and $\Delta_1 = \Delta_1(N)$.

We remark that for each odd prime divisor p of N, the mapping $f \mapsto f \mid U(p)$ gives a linear map from $S(k+1/2, N, \chi)$ to $S\left(k+1/2, N, \chi\left(\frac{p}{\cdot}\right)\right)$, and we also have the following identity: for any $f \in S(k+1/2, \Delta_1)$,

$$f \mid U(p) = p^{k/2-3/4} f \mid \left[\Delta_1 \left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, p^{k/2+1/4}\right) \Delta_1\right] \tag{1.19}$$

$$= p^{k/2-3/4} \sum_{i \in \mathbf{Z}/p\mathbf{Z}} f \mid \left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, p^{k/2+1/4}\right) \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}^*$$

(cf. [Sh 1, Proposition 1.5]).

PROPOSITION (1.20). *Let p, l be any odd prime divisors of N with $p \neq l$ and put $Q = p^{\mathrm{ord}_p(N)}$. Then for any $f \in S(k+1/2, N, \chi)$, we have the following:*

(1)
$$f \mid U(l) \tilde{W}(Q) = \chi_p(l) f \mid \tilde{W}(Q) U(l).$$

(2) *Suppose $l \in \Pi$ ($\Leftrightarrow \mathrm{ord}_l(N) \geq 2$). Then*

$$f \mid R_l U(p) = \left(\frac{p}{l}\right) f \mid U(p) R_l, \quad f \mid R_l \tilde{W}(Q) = \left(\frac{Q}{l}\right) f \mid \tilde{W}(Q) R_l.$$

(3) *If n is a positive integer prime to N, then*

$$f \mid U(p) \tilde{T}_{k+1/2, N, \chi\left(\frac{p}{\cdot}\right)}(n^2) = f \mid \tilde{T}_{k+1/2,N,\chi}(n^2) U(p),$$

$$f \mid \tilde{W}(Q) \tilde{T}_{k+1/2, N, \chi\left(\frac{Q}{\cdot}\right)}(n^2) = f \mid \tilde{T}_{k+1/2,N,\chi}(n^2) \tilde{W}(Q).$$

Proof. (1) From (1.17) and the fact $\Delta_0(N) \triangleright \Delta_1$, we have the identity:

$$\Delta_1 \xi \Delta_1 \tilde{W}(Q) = \gamma^* \tilde{W}(Q) \Delta_1 \xi \Delta_1,$$

where $\xi = \left(\begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix}, l^{k/2+1/4}\right)$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that $d \equiv 1 \pmod{N/Q}$ and $d \equiv l \pmod{Q}$. The relation (1) is verified by this fact.

(2) The first relation of (2) is easily verified by checking the coincidence of

the Fourier coefficients of the both sides.

Next, from [U2, Proposition (1.5)], for any $f \in S(k+1/2, N, \chi)$, we have

$$f \mid R_l = \mathrm{g}\left(\left(\frac{}{l}\right)\right)^{-1} \sum_{m \in H} \left(\frac{m}{l}\right) f \mid [\Delta \xi(m) \Delta],$$

where $H = (\mathbf{Z}/l\mathbf{Z})^{\times} / (\mathbf{Z}/l\mathbf{Z})^{\times 2}$ and $\xi(m) = \left(\begin{pmatrix} l & m \\ 0 & l \end{pmatrix}, 1\right)$.

We also have the following identity from (1.17):

$$\Delta \xi(m) \Delta \tilde{W}(Q) = \tilde{W}(Q) \Delta' \xi(m') \Delta',$$

where $\Delta' = \Delta_0\left(N, \chi\left(\frac{Q}{}\right)\right)$ and $m' \in H$ such that $Qm' \equiv m \pmod{l}$. The second relation of (2) is verified by this fact.

(3) From the definition of $\tilde{T}(n^2)$ (cf. §0(c)), it is sufficient to study the relations between $\tilde{W}(Q)$, $U(p)$ and $[\Delta \tau(n) \Delta]$. Here, for simplicity, we write $\tau(n) = \left(\begin{pmatrix} 1 & 0 \\ 0 & n^2 \end{pmatrix}, n^{k+1/2}\right)$. It is easily shown that for any $f \in S(k+1/2, N, \chi)$,

$$f \mid [\Delta \tau(n) \Delta] = f \mid [\Delta_1 \tau(n) \Delta_1] \quad \text{(cf. [Sh 1, p. 450])}.$$

For each Δ_1-double coset $\Delta_1 g \Delta_1$, we put $\deg(\Delta_1 g \Delta_1)$ = the degree of $\Delta_1 g \Delta_1 :=$ the number of left Δ_1-cosets contained in $\Delta_1 g \Delta_1$. Then we have

$$\deg(\Delta_1 \tau(n) \Delta_1) \deg(\Delta_1 \xi \Delta_1) = \deg(\Delta_1 \xi \tau(n) \Delta_1)$$

and $\xi \tau(n) = \tau(n) \xi$. Hence we have the identities of elements in the abstract Hecke algebra:

$$\Delta_1 \tau(n) \Delta_1 \cdot \Delta_1 \xi \Delta_1 = \Delta_1 \tau(n) \xi \Delta_1 = \Delta_1 \xi \Delta_1 \cdot \Delta_1 \tau(n) \Delta_1.$$

The first relation of (3) is shown by these identities.

Finally, we can show the second relation of (3) in a similar method to (1) and (2). □

From [U3], we know that the operators $\tilde{W}(Q)$ and $U(p)$ map $V(N; \chi)$ to $V(N; \psi)$ for some character ψ, if $k = 1$.

Proposition (1.21). *Let p be an odd prime divisor of N and put $Q = p^{\mathrm{ord}_p(N)}$. Then for any $f \in V(N; \chi)$, we have $f \mid U(p) \in V\left(N; \chi\left(\frac{p}{}\right)\right)$ and $f \mid \tilde{W}(Q) \in V\left(N; \chi\left(\frac{Q}{}\right)\right)$. In particular, $\tilde{W}(Q)$ gives an isomorphism from $V(N; \chi)$ onto*

$V\left(N\,;\chi\left(\frac{Q}{}\right)\right)$.

Proof. Put $\xi=\left(\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}, p^{3/4}\right)$. For any $v\in V(N\,;\chi)$ and any $u'\in U\left(N\,;\chi\left(\frac{p}{}\right)\right)$,

$$\langle v\mid U(p),\,u'\rangle=p^{-1/4}\sum_{i\in\mathbf{Z}/p\mathbf{Z}}\left\langle v\mid\xi\begin{pmatrix}1 & i\\ 0 & 1\end{pmatrix}^{*},\,u'\right\rangle$$

$$=p^{-1/4}\sum_{i\in\mathbf{Z}/p\mathbf{Z}}\left\langle v,\,u'\mid\begin{pmatrix}1 & -i\\ 0 & 1\end{pmatrix}^{*}\xi^{-1}\right\rangle=p^{3/2}\langle v,\,u'(pz)\rangle,$$

where $z\in\mathfrak{H}$ (cf. [U2, Lemma (1.9)]).

Obviously, we have $u'(pz)\in U(Np\,;\chi)$. Observing $V(N\,;\chi)\subseteq V(Np\,;\chi)$ (cf. §0(c)), we get the first assertion. See [U3] for the proof of the second assertion. □

When $\mathrm{ord}_2(N)=2$, these operators $\tilde{W}(Q)$ and $U(p)$ also fix the Kohnen space.

Proposition (1.22). *Suppose that $N=4M$ with an odd positive integer M. Let p be an odd prime divisor of N and $Q=p^{\mathrm{ord}_p(N)}$. Then*

$$S(k+1/2,\,N,\,\chi)_K\mid U(p)\subseteq S\left(k+1/2,\,N,\,\chi\left(\frac{p}{}\right)\right)_K$$

and the operator $\tilde{W}(Q)$ gives an isomorphism from $S(k+1/2,\,N,\,\chi)_K$ onto $S\left(k+1/2,\,N,\,\chi\left(\frac{Q}{}\right)\right)_K$.

Proof. We can easily check the condition (cf. §0(d)) for Fourier coefficients. So the first assertion is obvious.

Next, $S(k+1/2,\,N,\,\chi)_K$ is an eigen subspace of $S(k+1/2,\,N,\,\chi)$ with respect to the operator $[\varDelta\xi_{k+1/2,\chi_2(-1)}\varDelta]$ (cf. §0(d) and [K, Proposition 1]). Hence it is sufficient for the proof of the second assertion to study the relation between $\tilde{W}(Q)$ and $\varDelta\xi_{k+1/2,\chi_2(-1)}\varDelta$.

Observing (1.16), we can take γ_Q as follows:

$$SL_2(\mathbf{Z}) \ni \gamma_Q \equiv \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & (\mathrm{mod}\ Q), \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & (\mathrm{mod}\ M/Q), \\ \begin{pmatrix} 1 & 0 \\ 4(1-Q) & 1 \end{pmatrix} & (\mathrm{mod}\ 16). \end{cases}$$

Put $\Delta' = \Delta_0\left(N, \chi\left(\frac{Q}{\cdot}\right)\right)$ and

$$\xi_0 = \left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, 1\right) = \xi_{k+1/2, \chi_2(-1)}\,(1, \chi_2(-1)^{-k-1/2}\mathbf{e}(-(2k+1)/8)).$$

Then we can show the following identity:

$$\tilde{W}(Q)\Delta'\xi_0\Delta' = \Delta\xi_0\Delta\left(1, \chi_2(Q)\left(\frac{-1}{Q}\right)^{k+1/2}\right)\tilde{W}(Q) \quad \text{(cf. (1.17))}.$$

From this identity and Proposition (1.18), the second assertion is easily verified. □

COROLLARY (1.23). *Under the same notation and assumption as in* (1.22), *we have* $V(N\,;\chi)_K \mid U(p) \subseteq V\left(N\,;\chi\left(\frac{p}{\cdot}\right)\right)_K$ *and* $\tilde{W}(Q)$ *gives an isomorphism from* $V(N\,;\chi)_K$ *onto* $V\left(N\,;\chi\left(\frac{Q}{\cdot}\right)\right)_K$.

Proof. These assertions easily follow from (1.21), (1.22), and $V(N\,;\chi)_K = V(N\,;\chi) \cap S(3/2, N, \chi)_K$. □

For any odd prime divisor p of N with $\mathrm{ord}_p(N) = 1 (\Leftrightarrow p \mid M_1)$, we define the operator Y_p on $S(k+1/2, N, \chi)$ by the following:

$$f \mid Y_p := p^{-k/2+3/4} f \mid U(p)\tilde{W}(p), \quad (f \in S(k+1/2, N, \chi)).$$

From the above Proposition (1.20), we have the following.

PROPOSITION (1.24). *Let p and q be any distinct two odd prime divisors of N such that* $\mathrm{ord}_p(N) = \mathrm{ord}_q(N) = 1$. *Then for any* $f \in S(k+1/2, N, \chi)$, *we have the following*:

(1) $$f \mid Y_pY_q = f \mid Y_qY_p,\ f \mid Y_pU(q) = \chi_p(q)\left(\frac{q}{p}\right) f \mid U(q)Y_p.$$

(2) *Suppose* $l \in \Pi$. *Then*

$$f \mid Y_p R_l = f \mid R_l Y_p.$$

(3) If n is a positive integer prime to N, then

$$f \mid Y_p \tilde{T}_{k+1/2,N,\chi}(n^2) = f \mid \tilde{T}_{k+1/2,N,\chi}(n^2) Y_p.$$

□

Let us find the relation that is satisfied by this operator Y_p.

Let p be any odd prime divisor of N with $\mathrm{ord}_p(N) = 1$. Take an element $\gamma_p \in SL_2(\mathbf{Z})$ satisfying the condition:

$$\gamma_p := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & (\mathrm{mod}\, p)\,; \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & (\mathrm{mod}\, N/p). \end{cases}$$

Then by using (1.19), for each $f \in S(k+1/2, N, \chi)$,

$$\begin{aligned} f \mid Y_p &= \sum_{i \in \mathbf{Z}/p\mathbf{Z}} f \mid \left(\begin{pmatrix} 1 & i \\ 0 & p \end{pmatrix}, p^{k/2+1/4} \right) \gamma_p{}^* \tilde{\delta}_p \\ &= \sum_{i \in \mathbf{Z}/p\mathbf{Z}} f \mid \left(\begin{pmatrix} p(a+ic) & b+id \\ p^2 c & pd \end{pmatrix}, j(\gamma_p, pz)^{2k+1} \right). \end{aligned}$$

For $i = 0$, we have the following identity from straightforward computation and Proposition (1.18):

$$(1.25) \quad \begin{aligned} f \mid \left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, p^{k/2+1/4} \right) \tilde{W}(p) &= \left(\frac{-1}{p} \right)^{-k-1/2} \chi_{N/p}(p) f \mid \tilde{W}(p) \tilde{\delta}_p \\ &= \left(\frac{-1}{p} \right) \chi_p(-1) f \mid \tilde{W}(p)^{-1} \tilde{\delta}_p. \end{aligned}$$

Here, $\chi_{N/p}$ is the N/p-primary component of χ.

If $i \not\equiv 0\ (\mathrm{mod}\, p)$, $a + ic$ and pc are relatively prime and so there exist $u, v \in \mathbf{Z}$ such that $u(a+ic) + vpc = 1$. Hence $\begin{pmatrix} u & v \\ -pc & a+ic \end{pmatrix} \in \Gamma_0(N)$ and

$$\begin{pmatrix} u & v \\ -pc & a+ic \end{pmatrix}^* \left(\begin{pmatrix} p(a+ic) & b+id \\ p^2 c & pd \end{pmatrix}, j(\gamma_p, pz)^{2k+1} \right) = \left(\begin{pmatrix} p & x \\ 0 & p \end{pmatrix}, \left(\frac{i}{p} \right) \right),$$

where $x = u(b+id) + vdp$.

Since $x \equiv -u\ (\mathrm{mod}\, p)$, $ui \equiv 1\ (\mathrm{mod}\, p)$, and $\chi^2 = 1$, we have

$$f \mid \left(\begin{pmatrix} p(a+ic) & b+id \\ p^2c & pd \end{pmatrix}, j(\gamma_p, pz)^{2k+1}\right) = \chi(a+ic)\, f \mid \left(\begin{pmatrix} p & -u \\ 0 & p \end{pmatrix}, \left(\frac{i}{p}\right)\right)$$
$$= \chi_p(u) \left(\frac{u}{p}\right) f \mid \left(\begin{pmatrix} p & -u \\ 0 & p \end{pmatrix}, 1\right).$$

From these results, we have the following identity:

(1.26) Let p be any odd prime divisor of N with $\mathrm{ord}_p(N) = 1$. For each $f = \sum_{n\geq 1} a(n)\mathbf{e}(nz) \in S(k+1/2, N, \chi)$,

$$f \mid Y_p = \left(\frac{-1}{p}\right) \chi_p(-1)\, f \mid \tilde{W}(p)^{-1}\tilde{\delta}_p + \sum_{u \in (\mathbf{Z}/p\mathbf{Z})^\times} \chi_p(u) \left(\frac{u}{p}\right) f \mid \left(\begin{pmatrix} p & -u \\ 0 & p \end{pmatrix}, 1\right)$$
$$= \left(\frac{-1}{p}\right) \chi_p(-1)\, f \mid \tilde{W}(p)^{-1}\tilde{\delta}_p$$
$$+ \begin{cases} \left(\frac{-1}{p}\right)^{-1/2} p^{1/2} \sum_{n\geq 1} a(n) \left(\frac{n}{p}\right)\mathbf{e}(nz), & \text{if } \chi_p = 1, \\ -\sum_{n\geq 1} a(n)\mathbf{e}(nz) + p\sum_{n\geq 1} a(pn)\mathbf{e}(pnz), & \text{if } \chi_p = \left(\frac{}{p}\right). \end{cases}$$

Let us apply $p^{-k/2+3/4}U(p)$ to the both sides of the identity in (1.26).

Put $g := f \mid \tilde{W}(p)^{-1} = \sum_{n\geq 1} b(n)\mathbf{e}(nz)$. Then

$$g \mid \tilde{\delta}_p p^{-k/2+3/4}\, U(p) = p \sum_{n\geq 1} b(n)\mathbf{e}(pnz) \mid U(p) = pg.$$

Therefore we get

$$p^{-k/2+3/4} f \mid Y_p U(p) = \left(\frac{-1}{p}\right) \chi_p(-1) pf \mid \tilde{W}(p)^{-1}$$
$$+ \begin{cases} 0, & \text{if } \chi_p = 1, \\ p^{-k/2+3/4}(p-1)\, f \mid U(p), & \text{if } \chi_p = \left(\frac{}{p}\right). \end{cases}$$

Thus, we obtain the following.

Proposition (1.27). *Let p be any odd prime divisor of N with* $\mathrm{ord}_p(N) = 1$. *For each $f \in S(k+1/2, N, \chi)$,*

$$f \mid Y_p^2 = \begin{cases} \left(\frac{-1}{p}\right) pf, & \text{if } \chi_p = 1, \\ (p-1)\, f \mid Y_p + pf, & \text{if } \chi_p = \left(\frac{}{p}\right). \end{cases}$$

Hence Y_p is a semi-simple operator on $S(k+1/2, N, \chi)$. Moreover the adjoint operator of Y_p on $S(k+1/2, N, \chi)$ is given by $Y_p' = \left(\frac{-1}{p}\right)\chi_p(-1)Y_p$.

Proof. From the identity (1.26) and [U2, Lemma (1.9)], the adjoint operator Y_p' on $S(k+1/2, N, \chi)$ is given by:

$$g \mid Y_p' = \left(\frac{-1}{p}\right)\chi_p(-1)\, g \mid \tilde{\delta}_p^{-1}\tilde{W}(p) + \sum_{u\in(\mathbf{Z}/p\mathbf{Z})^\times}\left(\frac{u}{p}\right)\chi_p(u) g \mid \left(\begin{pmatrix} p & u \\ 0 & p\end{pmatrix}, 1\right)$$

$(g \in S(k+1/2, N, \chi))$.

Observing $g \mid \tilde{\delta}_p^{-1} = g \mid \left(\begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix}, p^{k/2+1/4}\right)$ and (1.25),

$$\begin{aligned} g \mid Y_p' &= g \mid \tilde{W}(p)^{-1}\tilde{\delta}_p + \sum_{u\in(\mathbf{Z}/p\mathbf{Z})^\times}\left(\frac{u}{p}\right)\chi_p(u) g \mid \left(\begin{pmatrix} p & u \\ 0 & p\end{pmatrix}, 1\right) \\ &= \left(\frac{-1}{p}\right)\chi_p(-1) g \mid Y_p. \end{aligned}$$

□

From this proposition, we can see that Y_p is a linear automorphism of $S(k+1/2, N, \chi)$. Since $\tilde{W}(p)$ gives an isomorphism from $S\left(k+1/2, N, \chi\left(\frac{p}{}\right)\right)$ onto $S(k+1/2, N, \chi)$, $U(p)$ is also a linear isomorphism from $S(k+1/2, N, \chi)$ onto $S\left(k+1/2, N, \chi\left(\frac{p}{}\right)\right)$. We can also see that S and V are stable under Y_p for each prime divisor p of M_1 by using (1.21–23). Here, the letters S and V are the same meaning as in (1.1).

Finally, we can know the behavior of the spaces $S^{\emptyset,\chi}$ and $V^{\emptyset,\chi}$ under the operators $\tilde{W}(p)$, $U(p)$, and Y_p from Propositions (1.20) and (1.24). Thus we get the following.

PROPOSITION (1.28). *Let p be any odd prime divisor of N with* $\mathrm{ord}_p(N) = 1$. *Put* $\chi' = \chi\left(\frac{p}{}\right)$. *Then we have*

(1) *The mapping $f \mapsto f \mid Y_p$ gives the automorphisms of $S(k+1/2, N, \chi)$, $S(k+1/2, N, \chi)_K$, $V(N;\chi)$, and $V(N;\chi)_K$.*

(2) *The mapping $f \mapsto f \mid U(p)$ gives the isomorphism from $S(k+1/2, N, \chi)$ (resp. $S(k+1/2, N, \chi)_K$, $V(N;\chi)$, $V(N;\chi)_K$) onto $S(k+1/2, N, \chi')$ (resp. $S(k+1/2, N, \chi')_K$, $V(N;\chi')$, $V(N;\chi')_K$).*

(3) *Let* $\kappa \in \mathrm{Map}(\Pi, \{\pm 1\})$ *and put* $\kappa' := \kappa \cdot \left(\frac{p}{}\right) \in \mathrm{Map}(\Pi, \{\pm 1\})$. *Then the operator* Y_p *gives automorphisms of* $S^{\emptyset,\kappa}$ *and* $V^{\emptyset,\kappa}$. *Hence the both operators* $\tilde{W}(p)$ *and* $U(p)$ *give isomorphisms from* $S^{\emptyset,\kappa}(k+1/2, N, \chi)$, $S^{\emptyset,\kappa}(k+1/2, N, \chi)_K$, $V^{\emptyset,\kappa}(N, \chi)$, *and* $V^{\emptyset,\kappa}(N, \chi)_K$ *onto* $S^{\emptyset,\kappa'}(k+1/2, N, \chi')$, $S^{\emptyset,\kappa'}(k+1/2, N, \chi')_K$, $V^{\emptyset,\kappa'}(N, \chi')$, *and* $V^{\emptyset,\kappa'}(N, \chi')_K$, *respectively.* □

If $\chi_p = 1$, $Z_p := \left(\frac{-1}{p}\right)^{1/2} p^{-1/2} Y_p$ is a hermitian involution on $S(k+1/2, N, \chi)$ (cf. Proposition (1.27)).

Let us characterize the eigen subspace for this operator Z_p in terms of Fourier coefficients.

PROPOSITION (1.29). *Let* p *be any odd prime divisor of* N *with* $\mathrm{ord}_p(N) = 1$. *Suppose that* $\chi_p = 1$. *Then* $Z_p := \left(\frac{-1}{p}\right)^{1/2} p^{-1/2} Y_p$ *is a hermitian involution on* $S(k+1/2, N, \chi)$ *and for each* $\lambda \in \{\pm 1\}$, *the* λ-*eigen subspace of* $S(k+1/2, N, \chi)$ *on* Z_p *coincides with the following subspace:*

$$\hat{S}_\lambda := \left\{ f = \sum_{n \geq 1} a(n)\mathbf{e}(nz) \in S(k+1/2, N, \chi)\ ;\ a(n) = 0 \text{ if } \left(\frac{n}{p}\right) = -\lambda \right\}.$$

Proof. Take any $g = \sum_{n\geq 1} b(n)\mathbf{e}(nz) \in \hat{S}_1 \cap \hat{S}_{-1}$. Then $b(n) = 0$ if $(p, n) = 1$. Put $h = g \mid U(p) \in S\left(k+1/2, N, \chi\left(\frac{p}{}\right)\right)$.

Since $h(pz) = g(z)$ and $p\mathfrak{f}\left(\chi\left(\frac{p}{}\right)\right) \nmid N$, we have $h = 0$ from (1.10). Hence $g = 0$. Therefore $\hat{S}_1 \cap \hat{S}_{-1} = \{0\}$.

For the proof of the assertion, it is sufficient to show that $\hat{S}_\lambda$ contains the λ-eigen subspace of $S(k+1/2, N, \chi)$ on Z_p for each $\lambda \in \{\pm 1\}$.

Let $\lambda \in \{\pm 1\}$ and take any $f = \sum_{n\geq 1} a(n)\mathbf{e}(nz) \in S(k+1/2, N, \chi)$ such that $f \mid Z_p = \lambda f$. Then from (1.26),

$$f \mid Z_p = \left(\frac{-1}{p}\right)^{1/2} p^{-1/2} f \mid Y_p = \left(\frac{-1}{p}\right)^{-1/2} p^{-1/2} f \mid \tilde{W}(p)^{-1}\tilde{\delta}_p + \sum_{n\geq 1} a(n) \left(\frac{n}{p}\right) e(nz).$$

Moreover, since $\lambda f = \left(\frac{-1}{p}\right)^{1/2} p^{-k/2+1/4} f \mid U(p)\tilde{W}(p)$, we have

$$f \mid \tilde{W}(p)^{-1} = \lambda \left(\frac{-1}{p}\right)^{1/2} p^{-k/2+1/4} f \mid U(p).$$

Hence

$$\begin{aligned}\lambda f = f \mid Z_p &= \lambda p^{-k/2-1/4} f \mid U(p)\tilde{\delta}_p + \sum_{n\geq 1} a(n)\left(\frac{n}{p}\right)\mathbf{e}(nz)\\ &= \lambda p^{-k/2-1/4}\sum_{n\geq 1} a(pn)\mathbf{e}(nz)\mid \tilde{\delta}_p + \sum_{n\geq 1} a(n)\left(\frac{n}{p}\right)\mathbf{e}(nz)\\ &= \lambda \sum_{n\geq 1} a(pn)\mathbf{e}(pnz) + \sum_{n\geq 1} a(n)\left(\frac{n}{p}\right)\mathbf{e}(nz).\end{aligned}$$

Comparing the n-th Fourier coefficient of the both sides, we have

$$\lambda a(n) = \begin{cases} \lambda a(n), & \text{if } p \mid n, \\ \left(\frac{n}{p}\right) a(n), & \text{if } (p, n) = 1. \end{cases}$$

Therefore, if $\left(\frac{n}{p}\right) = -\lambda$, $a(n) = 0$. Thus the proof is completed. □

§2. Computations of the traces

We keep to the notations in §1.

In this section, we shall compute traces by using the trace relations ([U1, Theorem], [U2, Theorem (4.1)]. Some assumption is necessary to use these trace relations. Hence, from now on, we assume the following additional conditions for the level N and the character χ:

Assumption (2.1). $2 \leq \mu = \mathrm{ord}_2(N) \leq 4$ and also $\mathfrak{f}(\chi_2) = 8$ if $\mu = 4$.

Moreover we use the following notations:

$$\omega := \begin{cases} \mu - 1, & \text{if } S = S(k + 1/2, N, \chi) \text{ or if } V = V(N\,;\chi), \\ 0, & \text{if } S = S(k + 1/2, N, \chi)_K \text{ or if } V = V(N\,;\chi)_K. \end{cases}$$

For any odd prime p and any subset A of Π, put

$$A(p) := \mathrm{ord}_p(l_A) = \begin{cases} 1, & \text{if } p \in A\,; \\ 0, & \text{if } p \notin A. \end{cases}$$

In this section, we shall compute the trace in the statement of Proposition (1.15), i.e., $\mathrm{tr}(\tilde{T}(n^2)\,;S^{\emptyset,\chi})$ and $\mathrm{tr}(\tilde{T}(n^2)\,;V^{\emptyset,\chi})$ with four steps. See Proposition (2.23) for the final results of this section.

(I) Before the computation, we prepare several notations.

Let I be any subset of Π. Put $N=2^{u}L_0^IL_2^I$, $L_0^I(>0)$ is the l_I-primary part of N, and $M=L_0^IL_2^I$. $N_0=\Pi_{l\in I}\, l^{2[(\nu_l-1)/2]+1}$.

For an odd prime p and integers a, b, α, β (α, $\beta\geq 0$ and $0\leq a\leq\beta/2$), put

$$\lambda_\alpha^\beta(p,b;a)=\begin{cases}1, & \text{if } a=0;\\ 1+\left(\dfrac{-b}{p}\right), & 1\leq a\leq[(\beta-1)/2];\\ \left(\dfrac{-b}{p}\right)^\alpha, & \beta \text{ is even and } a=\beta/2.\end{cases}$$

From [U1, Theorem], [U2, Theorem (4.1)], and Assumption (2.1), we can express the trace of $R_I\tilde{T}(n^2)$ by the trace of Hecke operators on the spaces of the cusp forms of weight $2k$. The precise assertions are as follows.

Let I and $J\subseteq\Pi$ such that $I\cap J=\emptyset$ and n a positive integer such that $(n,N)=1$. We have the following formula.

$$(2.2)\quad \begin{Bmatrix}\operatorname{tr}(R_I\tilde{T}(n^2);S_J) & \text{if } k\geq 2\\ \operatorname{tr}(R_I\tilde{T}(n^2);V_J) & \text{if } k=1\end{Bmatrix}$$

$$=\left(\frac{l_J}{l_I}\right)\left(\frac{-1}{l_I}\right)^k\prod_{l\in I}\chi_l(n)\prod_{(p,l_I)=1}\chi_p(-l_I)$$

$$\times\sum_{N_1}\prod_{p\mid(M/l_J)}\lambda_{\tilde{\alpha}_p}^{\tilde{\nu}_p}(p,l_In;\operatorname{ord}_p(N_1)/2)\ \operatorname{tr}([W(N_0N_1)]_{2k}T(n);S(2k,2^{\omega}N_0N_1N_2)),$$

where the notations are as follows: For any odd prime p, $\tilde{\nu}_p=\operatorname{ord}_p(N/l_J)=\nu_p-J(p)$ and $\chi^{(J)}{}_p=\left(\frac{}{p}\right)^{\tilde{\alpha}_p}$ ($\tilde{\alpha}_p=0,1$). Here $\chi^{(J)}{}_p$ is the p-primary component of $\chi^{(J)}$. Put $\tilde{L}_2^I=L_2^I/l_J$. Then N_1 runs over all square divisors of $\tilde{L}_2^I$, and $N_2:=\tilde{L}_2^I\,\Pi_{p\mid N_1}\mid\tilde{L}_2^I\mid_p$.

Let us simplify the formula (2.2). First, we have $\tilde{L}_2^I=M_1\,\Pi_{l\in\Pi-I}\,l^{\nu_l-J(l)}$. Since $\operatorname{ord}_p(N_1)$ is even and $\operatorname{ord}_p(N_1)\leq 1$ for a prime divisor p of M_1, we have $\operatorname{ord}_p(N_1)=0$. Hence, $N_1=\Pi_{l\in\Pi-I}l^{2e_l}$, where e_l ($l\in\Pi-I$) runs over all integers such that $0\leq 2e_l\leq\nu_l-J(l)$. Moreover we have

$$N_2=M_1\prod_{l\in\Pi-I}\begin{cases}1, & \text{if } e_l>0,\\ l^{\nu_l-J(l)}, & \text{if } e_l=0.\end{cases}$$

Next, we have $\chi^{(J)}{}_p=\chi_p\!\left(\frac{l_J}{}\right)_p=\chi_p\left(\frac{}{p}\right)^{J(p)}$ for any odd prime p. Put $\chi_p=\left(\frac{}{p}\right)^{\alpha_p}$,

$\alpha_p = 0,1$. Hence $\tilde{\alpha}_p \equiv \alpha_p + J(p) \pmod 2$. From $p \mid M \Leftrightarrow p \mid (M/l_J)$, $(l_I n, L_2^I) = 1$, and the above, we have $\lambda_{\tilde{\alpha}_p}^{\tilde{\nu}_p}(p, l_I n\,;\, \mathrm{ord}_p(N_1)/2) = \lambda_{\alpha_p + J(p)}^{\nu_p - J(p)}(p, l_I n\,;\, \mathrm{ord}_p(N_1)/2)$ for any odd prime p.

Thus we obtain the following.

Let I be any subset of Π and n a positive integer such that $(n, N) = 1$. For simplicity, we put $C_1 = \left(\frac{-1}{l_I}\right)^k \Pi_{l \in I} \chi_l(n) \Pi_{(p, l_I)=1} \chi_p(-l_I)$ and for $l \in \Pi - I$, $J \subseteq \Pi - I$, and a non-negative integer a,

$$\eta_l(J, a) = \begin{cases} l^{2a}, & \text{if } a > 0, \\ l^{\nu_l - J(l)}, & \text{if } a = 0. \end{cases}$$

Then we have

$$(2.3) \quad \sum_{I \cap J = \emptyset} (-1)^{\# J} \left(\frac{l_J}{l_I}\right) \begin{Bmatrix} \mathrm{tr}(R_I \tilde{T}(n^2)\,;\, S_J) & \text{if } k \geq 2 \\ \mathrm{tr}(R_I \tilde{T}(n^2)\,;\, V_J) & \text{if } k = 1 \end{Bmatrix}$$

$$= C_1 \sum_{I \cap J = \emptyset} (-1)^{\# J} \sum_{\substack{(e_l)_{l \in \Pi - I} \\ 0 \leq 2e_l \leq \nu_l - J(l)}} \prod_{l \in \Pi - I} \lambda_{\alpha_l + J(l)}^{\nu_l - J(l)}(l, l_I n\,;\, e_l)$$

$$\times \mathrm{tr}\Big([W_\Pi]_{2k} T(n)\,;\, S\Big(2k, 2^\omega N_0 M_1 \prod_{l \in \Pi - I} \eta_l(J, e_l)\Big)\Big)$$

$$= C_1 \sum_{\substack{(e_l)_{l \in \Pi - I} \\ 0 \leq 2e_l \leq \nu_l}} \sum_{\substack{I \cap J = \emptyset \\ 0 \leq J(l) \leq \nu_l - 2e_l \\ (l \in \Pi - I)}} (-1)^{\# J} \prod_{l \in \Pi - I} \lambda_{\alpha_l + J(l)}^{\nu_l - J(l)}(l, l_I n\,;\, e_l)$$

$$\times \mathrm{tr}\Big([W_\Pi]_{2k} T(n)\,;\, S\Big(2k, 2^\omega N_0 M_1 \prod_{l \in \Pi - I} \eta_l(J, e_l)\Big)\Big),$$

where $\sum_{I \cap J = \emptyset}$ is the sum extended over all subsets J of Π with $I \cap J = \emptyset$ and $W_\Pi = W(N_0 \Pi_{l \in \Pi - I} l^{2e_l})$ (cf. §0(b)).

For a while, we fix I and constants $e_l (l \in \Pi - I)$. Put $(\Pi - I)' := \{l \in \Pi - I \mid e_l = 0\}$. By using this, we decompose any subset J of $\Pi - I$ as follows:

$$J = J' + J'',\ J' = J \cap (\Pi - I)',\ J'' = J - J'.$$

Then we have $e_l = 0$ and $\nu_l = \mathrm{ord}_l(N) \geq 2$ for any $l \in J'$. Since $J'(l) = 0$ or 1, the condition $0 \leq J'(l) \leq \nu_l - 2e_l$ is always satisfied for any $l \in (\Pi - I)'$. Hence we have the following bijection:

$$\{J \subseteq \Pi - I\,;\, 0 \leq J(l) \leq \nu_l - 2e_l \text{ for all } l \in \Pi - I\}$$
$$\leftrightarrow \{J' \subseteq (\Pi - I)'\} \times \{J'' \subseteq (\Pi - I)''\,;\, 0 \leq J''(l) \leq \nu_l - 2e_l \text{ for all } l \in (\Pi - I)''\}.$$

Moreover we have for any $J_1, J_2 \subseteq \Pi - I$,

$$\prod_{l\in\Pi-I}\eta_l(J_1,e_l)=\prod_{l\in\Pi-I}\eta_l(J_2,e_l)\Leftrightarrow J_1'=J_2'.$$

Hence the value of trace in the right hand side of the formula (2.3) does not depend on J''. Therefore

$$\begin{aligned}(2.4)\quad &\sum_{\substack{I\cap J=\emptyset\\0\le J(l)\le\nu_l-2e_l\\(l\in\Pi-I)}}(-1)^{\#J}\prod_{l\in\Pi-I}\lambda_{\alpha_l+J(l)}^{\nu_l-J(l)}(l,l_In;e_l)\\&\times\operatorname{tr}\Big([W_\Pi]_{2k}T(n);S\Big(2k,2^\omega N_0M_1\prod_{l\in\Pi-I}\eta_l(J,e_l)\Big)\Big)\\=&\sum_{J'\subseteq(\Pi-I)'}(-1)^{\#J'}\operatorname{tr}\Big([W_{I+(\Pi-I)''}]_{2k}T(n);S\Big(2k,2^\omega N_0M_1\prod_{l\in\Pi-I}\eta_l(J',e_l)\Big)\Big)\\&\times\sum_{\substack{J''\subseteq(\Pi-I)''\\0\le J''(l)\le\nu_l-2e_l\\(l\in(\Pi-I)'')}}(-1)^{\#J''}\prod_{l\in(\Pi-I)''}\lambda_{\alpha_l+J''(l)}^{\nu_l-J''(l)}(l,l_In;e_l).\end{aligned}$$

We can easily compute the sum for J''. The result is as follows:

$$\begin{aligned}&\sum_{\substack{J''\subseteq(\Pi-I)''\\0\le J''(l)\le\nu_l-2e_l\\(l\in(\Pi-I)'')}}(-1)^{\#J''}\prod_{l\in(\Pi-I)''}\lambda_{\alpha_l+J''(l)}^{\nu_l-J''(l)}(l,l_In;e_l)\\&=\prod_{l\in(\Pi-I)''}\Big(\sum_{i=0}^{\min(1,\nu_l-2e_l)}(-1)^i\lambda_{\alpha_l+i}^{\nu_l-i}(l,l_In;e_l)\Big)\\&=\prod_{l\in(\Pi-I)''}\begin{cases}\chi_l(-l_In), & \text{if } e_l=[\nu_l/2];\\0, & \text{otherwise.}\end{cases}\end{aligned}$$

Hence, if there is $l\in\Pi-I$ such that $e_l\neq0,[\nu_l/2]$, then the value of the formula (2.4) is equal to zero. Therefore we may assume that $e_l=0,[\nu_l/2]$ for all $l\in\Pi-I$.

Replacing the notations: $(\Pi-I)'$ by J and J' by K, for any subset I of Π and positive integer n with $(n,N)=1$, we have the following.

$$\begin{aligned}(2.5)\quad &\sum_{I\cap J=\emptyset}(-1)^{\#J}\Big(\frac{l_J}{l_I}\Big)\begin{Bmatrix}\operatorname{tr}(R_I\tilde T(n^2);S_J) & \text{if } k\ge2\\\operatorname{tr}(R_I\tilde T(n^2);V_J) & \text{if } k=1\end{Bmatrix}\\&=C_2\sum_{J\subseteq\Pi-I}\sum_{K\subseteq J}(-1)^{\#K}\prod_{l\in J}\chi_l(-l_In)\\&\quad\times\operatorname{tr}\Big([W_{\Pi-J}]_{2k}T(n);S\Big(2k,2^\omega N_0M_1\prod_{l\in\Pi-I-J}l^{2[\nu_l/2]}\prod_{l\in J}l^{\nu_l-K(l)}\Big)\Big),\end{aligned}$$

where $C_2:=C_1\prod_{l\in\Pi-I}\chi_l(-l_In)$. Then $C_2=\Big(\frac{-1}{l_I}\Big)^k\prod_{l\in\Pi}\chi_l(n)\prod_{p\mid(2M_1)}\chi_p(-l_I)$ by using $\chi^2=1$.

(II) Since $[W_{\Pi-J}]_{2k}$ and $T(n)$ fix the space of newforms, we can decompose the right hand side of (2.5) into a linear combination of traces on spaces of newforms.

Now, we introduce some notations. Let I, J, K be subsets of Π such that $J \subseteq \Pi - I$ and $K \subseteq J$. The letters s_l $(l \in I)$, t_l $(l \in \Pi - I - J)$, u_l $(l \in J)$, β, and $w_p(p \mid M_1)$ mean integers satisfying the following conditions:

$$\begin{cases} 0 \leq s_l \leq [(\nu_l - 1)/2], & \text{for all } l \in I; \\ 0 \leq t_l \leq [\nu_l/2], & \text{for all } l \in \Pi - I - J; \\ 0 \leq u_l \leq \nu_l - K(l), & \text{for all } l \in J; \\ 0 \leq \beta \leq \omega; & \\ 0 \leq w_p \leq 1, & \text{for all } p \mid M_1. \end{cases}$$

Moreover for a positive integer k, we put

$$tr_k((s_l), (t_l) ; (u_l), \beta, (w_p)) = tr_k((s_l)_{l\in I}, (t_l)_{l\in\Pi-I-J} ; (u_l)_{l\in J}, \beta, (w_p)_{p|M_1})$$
$$:= \mathrm{tr}\Big([W_{\Pi-J}]_{2k}\, T(n) ; S^0\Big(2k, 2^\beta \prod_{p|M_1} p^{w_p} \prod_{l\in I} l^{2([(\nu_l-1)/2]-s_l)+1} \times \prod_{l\in\Pi-I-J} l^{2([\nu_l/2]-t_l)} \prod_{l\in J} l^{u_l}\Big)\Big).$$

Applying Proposition (A.1) to the right hand side of (2.5), we have

$$\sum_{I\cap J=\emptyset} (-1)^{\#J}\Big(\frac{l_J}{l_I}\Big) \begin{Bmatrix} \mathrm{tr}(R_I\tilde{T}(n^2) ; S_J) & \text{if } k \geq 2 \\ \mathrm{tr}(R_I\tilde{T}(n^2) ; V_J) & \text{if } k = 1 \end{Bmatrix}$$
$$= C_2 \sum_{J\subseteq\Pi-I} \prod_{l\in J} \chi_l(-l_I n) \sum_{K\subseteq J} (-1)^{\#K} \sum_{(s_l)_{l\in I}} \sum_{(t_l)_{l\in\Pi-I-J}}$$
$$\times \sum_{(u_l)_{l\in J}} \prod_{l\in J} (\nu_l - K(l) - u_l + 1) \sum_\beta (\omega - \beta + 1) \sum_{(w_p)_{p|M_1}} \prod_{p|M_1} (2 - w_p)$$
$$\times\, tr_k((s_l), (t_l) ; (u_l), \beta, (w_p))$$
$$= C_2 \sum_{J\subseteq\Pi-I} \prod_{l\in J} \chi_l(-l_I n) \sum_{(s_l)} \sum_{(t_l)} \sum_\beta (\omega - \beta + 1) \sum_{(w_p)} \prod_{p|M_1} (2 - w_p)$$
$$\times \sum_{K\subseteq J} \sum_{(u_l)} (-1)^{\#K} \prod_{l\in J} (\nu_l - K(l) - u_l + 1)\, tr_k((s_l), (t_l) ; (u_l), \beta, (w_p)).$$

We compute the above sum extended over K and (u_l). The value of $tr_k((s_l), (t_l) ; (u_l), \beta, (w_p))$, is determined by $I, J, (s_l), (t_l), (u_l), \beta$, and (w_p). In particular, it does not depend on K. Hence we have

$$\sum_{K\subseteq J} \sum_{\substack{(u_l)_{l\in J} \\ 0\leq u_l\leq\nu_l-K(l)}} (-1)^{\#K} \prod_{l\in J} (\nu_l - K(l) - u_l + 1)\, tr_k((s_l), (t_l) ; (u_l), \beta, (w_p))$$

$$= \sum_{\substack{(u_l)_{l\in J} \\ 0\leq u_l\leq \nu_l}} tr_k((s_l),\ (t_l)\ ;\ (u_l),\ \beta,\ (w_p)) \sum_{\substack{K\subseteq J \\ 0\leq K(l)\leq \nu_l-u_l \\ (l\in J)}} (-1)^{\# K} \prod_{l\in J} (\nu_l - K(l) - u_l + 1)$$

$$= \sum_{\substack{(u_l)_{l\in J} \\ 0\leq u_l\leq \nu_l}} tr_k((s_l),\ (t_l)\ ;\ (u_l),\ \beta,\ (w_p)).$$

Combining these results and Proposition (1.15), we can express the trace of $\tilde{T}(n^2)$ on $S^{\emptyset,\chi}$ and $V^{\emptyset,\chi}$ ($\kappa \in \{\pm 1\}^{\Pi}$) as follows:

$$\begin{Bmatrix} \mathrm{tr}(\tilde{T}(n^2)\ ;\ S^{\emptyset,\chi}) & \text{if } k\geq 2 \\ \mathrm{tr}(\tilde{T}(n^2)\ ;\ V^{\emptyset,\chi}) & \text{if } k=1 \end{Bmatrix}$$

$$(2.6) \qquad = 2^{-\#\Pi} \sum_{I\subseteq \Pi} \kappa_I C_2 \sum_{J\subseteq \Pi - I} \prod_{l\in J} \chi_l(-l_I n) \sum_{\beta=0}^{\omega} (\omega-\beta+1) \sum_{(w_p)_{p|M_1}} \prod_{p|M_1} (2-w_p)$$

$$\times \sum_{(s_l)_{l\in I}} \sum_{(t_l)_{l\in \Pi-I-J}} \sum_{(u_l)_{l\in J}} tr_k((s_l),\ (t_l)\ ;\ (u_l),\ \beta,\ (w_p)),$$

where (s_l), (t_l), (u_l), and (w_p) run over integers such that

$$\begin{cases} 0\leq s_l\leq [(\nu_l-1)/2], & \text{for all } l\in I\,; \\ 0\leq t_l\leq [\nu_l/2], & \text{for all } l\in \Pi - I - J\,; \\ 0\leq u_l\leq \nu_l, & \text{for all } l\in J\,; \\ 0\leq w_p\leq 1, & \text{for all } p\mid M_1. \end{cases}$$

We consider the level of the space in the trace $tr_k((s_l),\ (t_l)\ ;\ (u_l),\ \beta,\ (w_p))$. If s_l ($l\in I$) runs over the set $\{x\in \mathbf{Z}\mid 0\leq x\leq [(\nu_l-1)/2]\}$, then $2([(\nu_l-1)/2] - s_l)+1$ runs over all odd integers x such that $0\leq x\leq \nu_l$. Similarly, if t_l ($l\in \Pi - I - J$) runs over the set $\{x\in \mathbf{Z}\mid 0\leq x\leq [\nu_l/2]\}$, then $2([\nu_l/2] - t_l)$ runs over all even integers x such that $0\leq x\leq \nu_l$.

For simplicity, we put $K := \Pi - I - J$. Moreover for a given system of integers $(\rho_l)_{l\in \Pi}$, $0\leq \rho_l\leq \nu_l$ ($l\in \Pi$), we set the following notations:

$$\Pi(\rho)_{\mathrm{odd}} := \{l\in \Pi \mid \rho_l \text{ is odd}\}, \quad \Pi(\rho)_{\mathrm{even}} := \{l\in \Pi \mid \rho_l \text{ is even}\}.$$

Then we have

$$\sum_{(s_l)_{l\in I}} \sum_{(t_l)_{l\in K}} \sum_{(u_l)_{l\in J}} tr_k((s_l),\ (t_l)\ ;\ (u_l),\ \beta,\ (w_p))$$

$$= \sum_{\substack{(\rho_l)_{l\in \Pi}, 0\leq \rho_l\leq \nu_l \\ \rho_l:\mathrm{odd}(l\in I) \\ \rho_l:\mathrm{even}(l\in K)}} \mathrm{tr}\left([W_{I+K}]_{2k} T(n)\ ;\ S^0\left(2k,\ 2^{\beta} \prod_{p|M_1} p^{w_p} \prod_{l\in \Pi} l^{\rho_l}\right)\right)$$

and we also have

$$\sum_{\Pi=I+J+K} \sum_{\substack{(\rho_l)_{l\in\Pi}, 0\le\rho_l\le\nu_l \\ \rho_l:\mathrm{odd}(l\in I) \\ \rho_l:\mathrm{even}(l\in K)}} = \sum_{\substack{(\rho_l)_{l\in\Pi} \\ 0\le\rho_l\le\nu_l}} \sum\nolimits'_{\Pi=I+J+K}.$$

Here, $\sum_{\Pi=I+J+K}$ is the sum extended over all partitions $\Pi = I + J + K$ and $\sum'_{\Pi=I+J+K}$ is the sum extended over all partitions $\Pi = I + J + K$ such that $I \subseteq \Pi(\rho)_{\mathrm{odd}}$ and $K \subseteq \Pi(\rho)_{\mathrm{even}}$.

From these results, we obtain the following.

$$(2.7) \quad \begin{Bmatrix} \mathrm{tr}(\tilde{T}(n^2)\,;\, S^{\emptyset,\kappa}) & \text{if } k \ge 2 \\ \mathrm{tr}(\tilde{T}(n^2)\,;\, V^{\emptyset,\kappa}) & \text{if } k = 1 \end{Bmatrix}$$

$$= 2^{-\#\Pi} \prod_{l\in\Pi} \chi_l(n) \sum_{\beta=0}^{\omega} (\omega - \beta + 1) \sum_{\substack{(w_p)_{p|M_1} \\ w_p=0,1}} \prod_{p|M_1} (2 - w_p)$$

$$\times \sum_{\substack{(\rho_l)_{l\in\Pi} \\ 0\le\rho_l\le\nu_l}} \sum\nolimits'_{\Pi=I+J+K} \kappa_I \left(\frac{-1}{l_I}\right)^k \prod_{p|2M_1} \chi_p(-l_I) \prod_{l\in J} \chi_l(-l_I n)$$

$$\times \mathrm{tr}\left([W_{I+K}]_{2k} T(n)\,;\, S^0\Big(2k,\, 2^\beta \prod_{p|M_1} p^{w_p} \prod_{l\in\Pi} l^{\rho_l}\Big)\right).$$

Here, n is a positive integer such that $(n, N) = 1$ and κ is any element of $\{\pm 1\}^{\Pi}$. $\sum'_{\Pi=I+J+K}$ is the same meaning as in the above.

(III) In the following, we shall introduce subspaces $S^*(2k, N')$ $(0 < N' \mid N)$ and continue the calculation of traces of $\tilde{T}(n^2)$ on $S^{\emptyset,\kappa}$ and $V^{\emptyset,\kappa}$ (see the appendix for the definition of $S^*(2k, N')$).

In this part (III), we fix the following letters in the formula (2.7): β, $(w_p)_{p|M_1}$, $(\rho_l)_{l\in\Pi}$, I, and K. Then we decompose any subset P of Π with respect to $(\rho_l)_{l\in\Pi}$ as follows:

$P = P_{3+} + P_2 + P_1 + P_0$, where for any non-negative integer i,

$$P(\rho)_i = P_i := \{l \in P \mid \rho_l = i\} \text{ and } P(\rho)_{i+} = P_{i+} := \{l \in P \mid \rho_l \ge i\}.$$

Moreover for a partition $\Pi_2 = A + B + C$, we denote

$$\tilde{N} = \tilde{N}(\rho) := 2^\beta \prod_{p|M_1} p^{w_p} \prod_{l\in\Pi} l^{\rho_l} \text{ and}$$

$$\tilde{N}(B, C) = \tilde{N}(\rho\,;B, C) := 2^\beta \prod_{p|M_1} p^{w_p} \prod_{l\in\Pi-\Pi_2} l^{\rho_l} \prod_{l\in A} l^2 \prod_{l\in B} l.$$

From Proposition (A.8), the following orthogonal direct sum decomposition holds.

$$(2.8) \qquad S^0(2k, \tilde{N}) = \bigoplus_{\Pi_2=A+B+C} S^*(2k, \tilde{N}(B, C)) \mid R_{B+C}.$$

Here, $\bigoplus_{\Pi_2=A+B+C}$ is the orthogonal direct sum extended over all partitions $\Pi_2 = A + B + C$. From Propositions (A.2), (A.3), and (A.6), it follows that each subspace $S^*(2k, \tilde{N}(B, C)) \mid R_{B+C}$ in the formula (2.8) is stable under the action of $T(n)$ ($n \in \mathbf{Z}_+$ prime to N) and W_l ($l \in \Pi$).

We shall decompose $S^*(2k, \tilde{N}(B, C))$ into smaller subspaces by the operators W_l and R_l.

We discuss in a little general. Let θ_l ($l \in \Pi$) be a non-negative integer. We put $\tilde{N}(\theta) := 2^{\beta} \prod_{p|M_1} p^{w_p} \prod_{l\in\Pi} l^{\theta_l}$ and $\Pi(\theta)_{2+} := \{l \in \Pi \mid \theta_l \geq 2\}$. Then from Propositions (A.2), (A.6), and (A.7), we know that $S^*(2k, \tilde{N}(\theta))$ is stable under the operators $W_l = W(l^{\theta_l})$ ($l \in \Pi$), $R_l = R_{\left(\frac{}{l}\right)}$ ($l \in \Pi(\theta)_{2+}$), and $T(n)$ ($n \in \mathbf{Z}_+$ prime to $\tilde{N}(\theta)$) and also that $S^*(2k, \tilde{N}(\theta))$ has a basis consisting of primitive forms of conductor $\tilde{N}(\theta)$. Take a primitive form f in such a basis.

By Proposition (A.2), $T(n)$ commutes with W_l ($l \in \Pi$) and $R_l W_l R_l$ ($l \in \Pi(\theta)_{2+}$) as operators on $S^*(2k, \tilde{N}(\theta))$ for all $n \in \mathbf{Z}_+$ prime to $\tilde{N}(\theta)$. Hence $f \mid W_l$ ($l \in \Pi$) and $f \mid R_l W_l R_l$ ($l \in \Pi(\theta)_{2+}$) are constant multiples of f by [M, Lemma 4.6.12]. Since W_l ($l \in \Pi$) and R_l ($l \in \Pi(\theta)_{2+}$) induce $\mathbf{C}$-linear automorphisms of $S^*(2k, \tilde{N}(\theta))$ of order 2 (Propositions (A.2) and (A.7)), it is easily seen that $f \mid W_l = \pm f$ for all $l \in \Pi$ and $f \mid R_l W_l R_l = \pm f$ for all $l \in \Pi(\theta)_{2+}$. Hence the following subspace is well-defined: For each $\tau \in \mathrm{Map}(\Pi, \{\pm 1\})$, $\sigma \in \mathrm{Map}(\Pi(\theta)_{2+}, \{\pm 1\})$, we define

$$S^{*(\tau,\sigma)}(2k, \tilde{N}(\theta)) := \left\{ \begin{array}{l} f \in S^*(2k, \tilde{N}(\theta)) \,; \\ f \mid W_l = \tau(l) f \text{ for all } l \in \Pi, \\ f \mid R_l W_l = \sigma(l) f \mid R_l \text{ for all } l \in \Pi(\theta)_{2+} \end{array} \right\}.$$

We remark that if $\Pi(\theta)_{2+} = \emptyset$, the last condition on $\Pi(\theta)_{2+}$ has no meaning and so this subspace depends only on τ.

Then from the above argument, we get the following orthogonal direct sum decomposition:

$$S^*(2k, \tilde{N}(\theta)) = \bigoplus_{\substack{\tau \in \mathrm{Map}(\Pi, \{\pm 1\}) \\ \sigma \in \mathrm{Map}(\Pi(\theta)_{2+}, \{\pm 1\})}} S^{*(\tau,\sigma)}(2k, \tilde{N}(\theta)). \tag{2.9}$$

We also get that the subspace $S^{*(\tau,\sigma)}(2k, \tilde{N}(\theta))$ has a basis consisting of primitive forms of conductor $\tilde{N}(\theta)$ for any $\tau \in \mathrm{Map}(\Pi, \{\pm 1\})$ and $\sigma \in \mathrm{Map}(\Pi(\theta)_{2+}, \{\pm 1\})$.

Next we study the behavior of the subspaces $S^{*(\tau,\sigma)}(2k, \tilde{N}(\theta))$ under the twisting operator R_q ($q \in \Pi(\theta)_{2+}$).

Take any $q \in \Pi(\theta)_{2+}$ and $f \in S^{*(\tau,\sigma)}(2k, \tilde{N}(\theta))$ for $\tau \in \mathrm{Map}(\Pi, \{\pm 1\})$ and $\sigma \in \mathrm{Map}(\Pi(\theta)_{2+}, \{\pm 1\})$. We put $g = f \mid R_q$. Then $g \in S^*(2k, \tilde{N}(\theta))$ by

Proposition (A.7). From Proposition (A.2), it follows that for any $l \in \Pi$,

$$g \mid W_l = f \mid R_q W_l = \begin{cases} \tau(l) \left(\frac{l}{q}\right)^{\theta_l} g, & \text{if } l \neq q, \\ \sigma(q) g, & \text{if } l = q. \end{cases}$$

Similarly, by using Propositions (A.2) and (A.7), we have for any $l \in \Pi(\theta)_{2+}$,

$$g \mid R_l W_l = f \mid R_q R_l W_l = \begin{cases} \left(\frac{l}{q}\right)^{\theta_l} \sigma(l) g, & \text{if } l \neq q, \\ f \mid W_q = \tau(q) f = \tau(q) g \mid R_q, & \text{if } l = q. \end{cases}$$

Now we put

$$\tau_1(l) := \begin{cases} \left(\frac{l}{q}\right)^{\theta_l} \tau(l), & \text{if } l \in \Pi \text{ and } l \neq q, \\ \sigma(l), & \text{if } l = q. \end{cases}$$

$$\sigma_1(l) := \begin{cases} \left(\frac{l}{q}\right)^{\theta_l} \sigma(l), & \text{if } l \in \Pi(\theta)_{2+} \text{ and } l \neq q, \\ \tau(l), & \text{if } l = q. \end{cases}$$

Then the twisting operator R_q induces a map $S^{*(\tau,\sigma)}(2k, \tilde{N}(\theta)) \ni f \mapsto f \mid R_q \in S^{*(\tau_1,\sigma_1)}(2k, \tilde{N}(\theta))$. Since R_q is a $\mathbf{C}$-linear automorphism of $S^*(2k, \tilde{N}(\theta))$ of order 2 (by Proposition (A.7)), this map gives an isomorphism from $S^{*(\tau,\sigma)}(2k, \tilde{N}(\theta))$ to $S^{*(\tau_1,\sigma_1)}(2k, \tilde{N}(\theta))$.

We return to the previous situation. From Proposition (A.5) and (2.9), we have for a partition $\Pi_2 = A + B + C$,

(2.10)

$$S^*(2k, \tilde{N}(B, C)) \mid R_{B+C} = \bigoplus_{\substack{\tau \in \mathrm{Map}(\Pi, \{\pm 1\}) \\ \sigma \in \mathrm{Map}(\Pi_{3+} + A, \{\pm 1\})}} (S^{*(\tau,\sigma)}(2k, \tilde{N}(B, C)) \mid R_{B+C}).$$

Moreover, it follows that each subspace $S^{*(\tau,\sigma)}(2k, \tilde{N}(B, C)) \mid R_{B+C}$ in the formula (2.10) is fixed by all the operators $T(n)$ ($n \in \mathbf{Z}_+$ prime to N) and W_l ($l \in \Pi$) from Propositions (A.2), (A.3), and the remark after the formula (2.9).

Now, by using the above formulas, we shall modify the right hand side of the formula (2.7). Then we have the following: For all $n \in \mathbf{Z}_+$ prime to N,

$$\prod_{l \in \Pi} \chi_l(n) \prod_{l \in J} \chi_l(n) \operatorname{tr}(W_{I+K} T(n) \,;\, S^0(2k, \tilde{N}))$$

$$= \prod_{l \in I+K} \chi_l(n) \sum_{\Pi_2 = A+B+C} \mathrm{tr}(W_{I+K}T(n) \,;\, S^*(2k, \tilde{N}(B, C)) \mid R_{B+C}) \quad \text{(by (2.8))}$$

$$= \sum_{\Pi_2 = A+B+C} \sum_{\substack{\tau \in \mathrm{Map}(\Pi, \{\pm 1\}) \\ \sigma \in \mathrm{Map}(\Pi_{3+}+A, \{\pm 1\})}} \prod_{l \in I+K} \chi_l(n)$$

$$\times \mathrm{tr}(W_{I+K}T(n) \,;\, S^{*(\tau,\sigma)}(2k, \tilde{N}(B, C)) \mid R_{B+C}) \quad \text{(by (2.10))}$$

$$= \sum_{\Pi_2 = A+B+C} \sum_{\tau,\sigma} \prod_{l \in I} \left\{ \tau(l) \prod_{q \in B+C} \left(\frac{l}{q}\right) \right\} \prod_{l \in K-(B+C)} \tau(l) \prod_{l \in K \cap (B+C)} \left(\frac{-1}{l}\right)$$

$$\times \prod_{l \in I+K} \chi_l(n)\, \mathrm{tr}(T(n) \,;\, S^{*(\tau,\sigma)}(2k, \tilde{N}(B, C)) \mid R_{B+C}) \quad \text{(by Propositions (A.2–3))}$$

$$= \sum_{\Pi_2 = A+B+C} \sum_{\tau,\sigma} \prod_{l \in I} \left\{ \tau(l) \prod_{q \in B+C} \left(\frac{l}{q}\right) \right\} \prod_{l \in K-(B+C)} \tau(l) \prod_{l \in K \cap (B+C)} \left(\frac{-1}{l}\right)$$

$$\times \prod_{l \in I+K} \chi_l(n) \prod_{l \in B+C} \left(\frac{n}{l}\right) \mathrm{tr}(T(n) \,;\, S^{*(\tau,\sigma)}(2k, \tilde{N}(B, C))) \quad \text{(by Propositions (A.2, A.5))}.$$

Here, we use the conditions: $I \subseteq \Pi(\rho)_{\mathrm{odd}}$ and $K \subseteq \Pi(\rho)_{\mathrm{even}}$ at the third equality.

We introduce the following notation: Put $\Pi^1 = \{l \in \Pi \mid \chi_l \neq 1\}$ and we decompose any supset P of Π into two pieces as follows.

$$P = P^1 + P^0, \quad P^1 = P \cap \Pi^1, \quad P^0 = P \cap (\Pi - \Pi^1). \tag{2.11}$$

By using this notation, we can express for a partition $\Pi_2 = A + B + C$,

$$\prod_{l \in I+K} \chi_l(n) \prod_{l \in B+C} \left(\frac{n}{l}\right) = \prod_{l \in I^1+K^1} \left(\frac{n}{l}\right) \prod_{l \in B+C} \left(\frac{n}{l}\right) = \prod_{l \in X+Y} \left(\frac{n}{l}\right).$$

Here, $X := I^1_{3+} + K^1_{3+} + (K^1_2 \cap A)$ and $Y := I^1_1 + K^1_0 + (\Pi_2 - (A \cup K^1_2))$.

Since $(\Pi_{3+} + A) \supseteq X$, we have for any (τ, σ),

$$S^{*(\tau,\sigma)}(2k, \tilde{N}(B, C)) \mid R_X = S^{*(\tau',\sigma')}(2k, \tilde{N}(B, C)),$$

$$\tau'(l) = \begin{cases} \tau(l) \prod_{q \in X} \left(\frac{l}{q}\right)^{\phi_l}, & \text{if } l \in \Pi - X, \\ \sigma(l) \prod_{l \neq q \in X} \left(\frac{l}{q}\right)^{\phi_l}, & \text{if } l \in X, \end{cases} \tag{2.12}$$

$$\sigma'(l) = \begin{cases} \sigma(l) \prod_{q \in X} \left(\frac{l}{q}\right)^{\phi_l}, & \text{if } l \in (\Pi_{3+} + A) - X, \\ \tau(l) \prod_{l \neq q \in X} \left(\frac{l}{q}\right)^{\phi_l}, & \text{if } l \in X, \end{cases}$$

where $\phi_l = \rho_l$, 1, or 0 according as $l \in \Pi - (B + C)$, B, or C.

Therefore

(2.13)

$$\prod_{l\in I+K}\chi_l(n)\,\mathrm{tr}(W_{I+K}T(n)\,;\,S^0(2k,\tilde{N}))$$

$$=\sum_{\Pi_2=A+B+C}\ \sum_{\substack{\tau\in\mathrm{Map}(\Pi,\{\pm1\})\\ \sigma\in\mathrm{Map}(\Pi_{3+}+A,\{\pm1\})}} C_3\prod_{l\in X+Y}\left(\frac{n}{l}\right)\mathrm{tr}(T(n)\,;\,S^{*(\tau,\sigma)}(2k,\tilde{N}(B,C)))$$

$$=\sum_{\Pi_2=A+B+C}\ \sum_{\tau,\sigma} C_3\,\mathrm{tr}(T(n)\,;\,S^{*(\tau,\sigma)}(2k,\tilde{N}(B,C))\mid R_{X+Y})\quad\text{(by Proposition (A.5))}$$

$$=\sum_{\Pi_2=A+B+C}\ \sum_{\tau,\sigma} C_3\,\mathrm{tr}(T(n)\,;\,S^{*(\tau',\sigma')}(2k,\tilde{N}(B,C))\mid R_{Y}),$$

where $C_3=\Pi_{l\in I}\left\{\tau(l)\,\Pi_{q\in B+C}\left(\frac{l}{q}\right)\right\}\Pi_{l\in K-(B+C)}\,\tau(l)\,\Pi_{l\in K\cap(B+C)}\left(\frac{-1}{l}\right)$ and (τ',σ') is determined by the formula (2.12) for any (τ,σ).

It is easily shown that if (τ,σ) runs over the set $\mathrm{Map}(\Pi,\{\pm1\})\times\mathrm{Map}(\Pi_{3+}+A,\{\pm1\})$, then also (τ',σ') runs over the same set. Hence we can express (τ,σ) by (τ',σ'). In fact, we easily get the following.

$$\tau(l)=\begin{cases}\tau'(l)\,\Pi_{q\in X}\left(\frac{l}{q}\right), & \text{if } l\in I_{3+}^0+I_1,\\ \sigma'(l)\,\Pi_{l\neq q\in X}\left(\frac{l}{q}\right), & \text{if } l\in I_{3+}^1,\\ \tau'(l), & \text{if } l\in K_{3+}^0+(K_2^0\cap A)+K_0,\\ \sigma'(l), & \text{if } l\in K_{3+}^1+(K_2^1\cap A).\end{cases}$$

By applying this results to the formula (2.13) and also replacing the notation (τ',σ') by (τ,σ), we obtain the following formula.

(2.14) For all $n\in\mathbf{Z}_+$ prime to N,

$$\prod_{l\in I+K}\chi_l(n)\,\mathrm{tr}(W_{I+K}T(n)\,;\,S^0(2k,\tilde{N}))$$
$$=\sum_{\Pi_2=A+B+C}\ \sum_{\substack{\tau\in\mathrm{Map}(\Pi,\{\pm1\})\\ \sigma\in\mathrm{Map}(\Pi_{3+}+A,\{\pm1\})}}\left(\prod_{l\in I+K}\zeta_l\right)\mathrm{tr}(T(n)\,;\,S^{*(\tau,\sigma)}(2k,\tilde{N}(B,C))\mid R_Y),$$

where $X=I_{3+}^1+K_{3+}^1+(K_2^1\cap A)$, $Y=I_1^1+K_0^1+(\Pi_2-(A\cup K_2^1))$, and

$$\zeta_l=\begin{cases}\tau(l)\,\Pi_{q\in X+B+C}\left(\frac{l}{q}\right), & \text{if } l\in I_{3+}^0+I_1,\\ \sigma(l)\,\Pi_{l\neq q\in X+B+C}\left(\frac{l}{q}\right), & \text{if } l\in I_{3+}^1,\\ \tau(l), & \text{if } l\in K_{3+}^0+(K_2^0\cap A)+K_0,\\ \sigma(l), & \text{if } l\in K_{3+}^1+(K_2^1\cap A),\\ \left(\frac{-1}{l}\right), & \text{if } l\in K\cap(B+C).\end{cases}$$

(IV) In this part, we fix the letters β and $(w_p)_{p|M_1}$ as the part (III) and we consider that the letters $(\rho_l)_{l\in\Pi}$, I, and K vary satisfying the conditions in the formula (2.7).

Let $(\alpha_l)_{l\in\Pi}$ be a system of integers satisfying the conditions $0 \leq \alpha_l \leq \nu_l$ for all $l \in \Pi$. We put $\tilde{N}(\alpha) = 2^\beta \Pi_{p|M_1} p^{w_p} \Pi_{l\in\Pi} l^{\alpha_l}$ and for a non-negative integer i, $\Pi(\alpha)_i := \{l \in \Pi \mid \alpha_l = i\}$ and $\Pi(\alpha)_{i+} := \{l \in \Pi \mid \alpha_l \geq i\}$. Let Ψ be a subset of $\Pi(\alpha)_1 + \Pi(\alpha)_0$ and $(\hat{\tau}, \hat{\sigma}) \in \mathrm{Map}(\Pi, \{\pm 1\}) \times \mathrm{Map}(\Pi(\alpha)_{2+}, \{\pm 1\})$.

Remark. The letter α_l was already used at Part I in a different meaning. We hope that there will be no confusion.

It is easy to see that all the subspaces in the right hand side of the formula (2.14) are of the form $S^{*(\hat{\tau},\hat{\sigma})}(2k, \tilde{N}(\alpha)) \mid R_\Psi$ for suitable (α_l), Ψ, and $(\hat{\tau}, \hat{\sigma})$.

We take any $(\alpha_l)_{l\in\Pi}$, Ψ, and $(\hat{\tau}, \hat{\sigma})$ and fix them for a while. Then we shall find all $(\rho_l)_{l\in\Pi}$, I, K, a partition $\Pi_2 = A + B + C$, and $(\tau, \sigma) \in \mathrm{Map}(\Pi, \{\pm 1\}) \times \mathrm{Map}(\Pi_{3+} + A, \{\pm 1\})$ which satisfy the condition:

$$S^{*(\tau,\sigma)}(2k, \tilde{N}(B, C)) \mid R_Y = S^{*(\hat{\tau},\hat{\sigma})}(2k, \tilde{N}(\alpha)) \mid R_\Psi.$$

Here, $Y := I_1^1 + K_0^1 + (\Pi_2 - (A \cup K_2^1))$.

Obviously, this condition is equivalent to the following three conditions.

$$\text{(i)} \qquad \alpha_l = \begin{cases} \rho_l, & \text{if } l \in (\Pi - \Pi_2) + A, \\ 1, & \text{if } l \in B, \\ 0, & \text{if } l \in C. \end{cases}$$

$$\text{(ii)} \qquad \Psi = \mathrm{Y}.$$

$$\text{(iii)} \qquad (\tau, \sigma) = (\hat{\tau}, \hat{\sigma}).$$

We shall simplify these conditions. First we have

$$\text{(2.15)} \qquad \text{the condition (i)} \Leftrightarrow \begin{cases} \Pi(\alpha)_{3+} = \Pi_{3+}, & \Pi(\alpha)_2 = A, \\ \Pi(\alpha)_1 = \Pi_1 + B, & \Pi(\alpha)_0 = \Pi_0 + C. \end{cases}$$

Hence Π_{3+} and A are determined by (α_l), and B (resp. C) is a subset of $\Pi(\alpha)_1$ (resp. $\Pi(\alpha)_0$). Moreover

$$\rho_l = \begin{cases} \alpha_l, & \text{if } l \in \Pi(\alpha)_{3+}, \\ 2, & \text{if } l \in \Pi(\alpha)_2 + B + C, \\ 1, & \text{if } l \in \Pi(\alpha)_1 - B, \\ 0, & \text{if } l \in \Pi(\alpha)_0 - C. \end{cases}$$

Next, under the condition (i), we shall discuss conditions for B, C, I, and K coming from the condition (ii).

Since $\Psi \subseteq \Pi(\alpha)_1 + \Pi(\alpha)_0$ and (2.15),

$$\Psi = \Psi_1^0 + (\Psi \cap B)^0 + \Psi_0^0 + (\Psi \cap C)^0 + \Psi_1^1 + (\Psi \cap B)^1 + \Psi_0^1 + (\Psi \cap C)^1.$$

On the other hand, we have

$$Y = I_1^1 + K_0^1 + (B^1 - (K_2 \cap B)^1) + (C^1 - (K_2 \cap C)^1) + B^0 + C^0.$$

From these facts, we have

(2.16)

the condition (ii)

$$\Leftrightarrow \begin{cases} I_1^1 = \Psi_1^1,\ K_0^1 = \Psi_0^1,\ B^0 + C^0 = \Psi^0, \\ (K_2 \cap B)^1 = B^1 - (\Psi \cap B)^1,\ (K_2 \cap C)^1 = C^1 - (\Psi \cap C)^1. \end{cases}$$

From this, we have $B^0 = \Psi^0 \cap \Pi(\alpha)_1$ and $C^0 = \Psi^0 \cap \Pi(\alpha)_0$ and hence B^0 and C^0 are determined by (α_l) and Ψ.

Moreover, we know from (2.15), $\Psi_1^1 = (\Psi \cap (\Pi(\alpha)_1 - B))^1 = \Psi^1 \cap (\Pi(\alpha)_1^1 - B^1)$ and $\Psi_0^1 = \Psi^1 \cap (\Pi(\alpha)_0^1 - C^1)$. Moreover, since $K_2 = K \cap \Pi_2 = K \cap (\Pi(\alpha)_2 + B + C)$, $K_2 \cap B = K \cap B$ and $K_2 \cap C = K \cap C$. Hence if B^1 and C^1 are given, then I_1^1, K_0^1, $(K \cap B)^1$, and $(K \cap C)^1$ are determined by (α_l) and Ψ.

Finally, we remark that under the condition (i), the definition domain of $\hat{\sigma}$ is equal to those of σ.

Now combining the above results, all $(\rho_l)_{l\in\Pi}$, I, K, a partition $\Pi_2 = A + B + C$, and (τ, σ) satisfying the condition:

$$S^{*(\tau,\sigma)}(2k, \tilde{N}(B, C)) \mid R_Y = S^{*(\hat{\tau},\hat{\sigma})}(2k, \tilde{N}(\alpha)) \mid R_\Psi,$$

are described in the following form.

(2.17)

$$\rho_l := \begin{cases} \alpha_l, & \text{if } l \in \Pi(\alpha)_{3+}, \\ 2, & \text{if } l \in \Pi(\alpha)_2 + (Q_1^1 + Q_1^0) + (Q_0^1 + Q_0^0), \\ 1, & \text{if } l \in \Pi(\alpha)_1 - (Q_1^1 + Q_1^0), \\ 0, & \text{if } l \in \Pi(\alpha)_0 - (Q_0^1 + Q_0^0). \end{cases}$$

$$I := D_{3+} + D_1^0 + D_1^1.$$

$$K := E_{3+} + E_2^0 + E_2^1 + E_0^0 + E_0^1 + P_1^0 + P_1^1 + P_0^0 + P_0^1.$$

$$A := \Pi(\alpha)_2,\ B := Q_1^1 + Q_1^0,\ C := Q_0^1 + Q_0^0.$$

$\tau := \hat{\tau}, \sigma := \hat{\sigma}.$

Here, the meaning of each letter is as follows.

(2.18) $Q_1^0 := \Psi^0 \cap \Pi(\alpha)_1^0, Q_0^0 := \Psi^0 \cap \Pi(\alpha)_0^0.$

Q_1^1 (resp. Q_0^1) runs over all subsets of $\Pi(\alpha)_1^1$ (resp. $\Pi(\alpha)_0^1$).

P_1^0 (resp. P_0^0) runs over all subsets of Q_1^0 (resp. Q_0^0).

$P_1^1 := Q_1^1 - (\Psi^1 \cap Q_1^1), P_0^1 := Q_0^1 - (\Psi^1 \cap Q_0^1).$

$D_1^1 := \Psi^1 \cap (\Pi(\alpha)_1^1 - Q_1^1), E_0^1 := \Psi^1 \cap (\Pi(\alpha)_0^1 - Q_0^1).$

$D_{3+}, D_1^0, E_{3+}, E_2^0, E_2^1$, and E_0^0 run over all subsets of $\Pi(\alpha)_{3+,\mathrm{odd}}, \Pi(\alpha)_1^0 - Q_1^0, \Pi(\alpha)_{3+,\mathrm{even}}, \Pi(\alpha)_2^0, \Pi(\alpha)_2^1, \Pi(\alpha)_0^0 - Q_0^0$ respectively, where we denote $\Pi(\alpha)_{3+,\mathrm{odd}} = \{l \in \Pi \mid \alpha_l \geq 3, \alpha_l \text{ is odd}\}$ and $\Pi(\alpha)_{3+,\mathrm{even}} = \{l \in \Pi \mid \alpha_l \geq 3, \alpha_l \text{ is even}\}$.

Now we shall calculate the following.

(2.19)
$$2^{-\#\Pi} \prod_{l\in\Pi} \chi_l(n) \sum_{\substack{(\rho_l)_{l\in\Pi} \\ 0\leq\rho_l\leq\nu_l}} {\sum}'_{\Pi=I+J+K} \kappa_I \left(\frac{-1}{l_I}\right)^k \times \prod_{p|2M_1} \chi_p(-l_I) \prod_{l\in J} \chi_l(-l_I n)\, \mathrm{tr}(W_{I+K}T(n)\,;\, S^0(2k, \tilde{N}))$$
$$= 2^{-\#\Pi} \sum_{(\rho_l)} {\sum}'_{\Pi=I+J+K} \sum_{\Pi_2=A+B+C} \sum_{\tau,\sigma} \kappa_I \left(\frac{-1}{l_I}\right)^k \prod_{p|2M_1} \chi_p(-l_I) \times \prod_{l\in J} \chi_l(-l_I) \left(\prod_{l\in I+K} \zeta_l\right) \mathrm{tr}(T(n)\,;\, S^{*(\tau,\sigma)}(2k, \tilde{N}(B, C)) \mid R_Y).$$

Here, n, κ, ${\sum}'_{\Pi=I+J+K}$ are the same as in (2.7), ζ_l and Y are the same as in (2.14), and (τ, σ) runs over $\mathrm{Map}(\Pi, \{\pm 1\}) \times \mathrm{Map}(\Pi_{3+} + A, \{\pm 1\})$.

From the above results, we can express the fromula (2.19) in terms of (α_l), Ψ, and (τ, σ) as follows.

(2.20)
$$\sum_{\substack{(\alpha_l)_{l\in\Pi} \\ 0\leq\alpha_l\leq\nu_l}} \sum_{\Psi\subseteq\Pi(\alpha)_1+\Pi(\alpha)_0} \sum_{\substack{\tau\in\mathrm{Map}(\Pi,\{\pm1\}) \\ \sigma\in\mathrm{Map}(\Pi(\alpha)_{2+},\{\pm1\})}} \Xi((\alpha_l), \Psi, (\tau, \sigma)) \times \mathrm{tr}(T(n)\,;\, S^{*(\tau,\sigma)}(2k, \tilde{N}(\alpha)) \mid R_\Psi).$$

Here, $\Xi = \Xi((\alpha_l), \Psi, (\tau, \sigma))$ is defined by the following:

$$\Xi = 2^{-\#\Pi} \sum_{Q_1^1} \sum_{Q_0^1} \sum_{P_1^0} \sum_{P_0^0} \sum_{D_{3+}} \sum_{D_1^0} \sum_{E_{3+}} \sum_{E_2^0} \sum_{E_2^1} \sum_{E_0^0} \times \kappa_I \left(\frac{-1}{l_I}\right)^k \prod_{p|2M_1} \chi_p(-l_I) \prod_{l\in J} \chi_l(-l_I) \prod_{l\in I+K} \zeta_l,$$

and Q_1^1, Q_0^1, P_1^0, P_0^0, D_{3+}, D_1^0, E_{3+}, E_2^0, E_2^1, and E_0^0 run over the same subsets as in (2.18) respectively. Moreover I, K, A, B, and C are defined by (2.17), $J := \Pi - (I + K)$, and for such I, K, A, B, and C, ζ_l $(l \in I + K)$ is defined by (2.14).

Next, we must calculate the value of Ξ. First, we have

$$\prod_{p|2M_1} \chi_p(-1) \prod_{l\in J} \chi_l(-1) = \chi(-1) \prod_{l\in I+K} \chi_l(-1) = \prod_{l\in I^1+K^1} \left(\frac{-1}{l}\right),$$

because χ is an even character modulo N. Also $\Pi_{l\in J}\chi_l(l_I) = \Pi_{l\in J^1}\left(\frac{l_I}{l}\right)$ and $J^1 = (\Pi(\alpha)^1_{3+,\mathrm{odd}} - D^1_{3+}) + (\Pi(\alpha)^1_{3+,\mathrm{even}} - E^1_{3+}) + (\Pi(\alpha)^1_2 - E^1_2) + ((\Pi(\alpha)^1_1 - Q^1_1) - D^1_1) + ((\Pi(\alpha)^1_0 - Q^1_0) - E^1_0) + (\Psi^1 \cap Q^1_1) + (\Psi^1 \cap Q^1_0)$.

For simplicity, we put $\chi_{2M_1} = \Pi_{p|2M_1}\chi_p$ and divide the variables D_{3+} and E_{3+} as follows: $D_{3+} = D^0_{3+} + D^1_{3+}$, $E_{3+} = E^0_{3+} + E^1_{3+}$.

Then we have

$$(2.21)\quad \Xi = 2^{-\#\Pi} \sum_{Q_1^1}\sum_{Q_0^1}\sum_{D_{3+}}\sum_{D_1^0} \kappa_I \left(\frac{-1}{l_I}\right)^k \prod_{p|2M_1} \chi_p(l_I) \prod_{q\in\Lambda^{(1)}} \left(\frac{l_I}{q}\right)$$
$$\times \prod_{l\in D^0_{3+}+D^0_1+D^1_1} \left(\prod_{q\in D^1_{3+}} \left(\frac{l}{q}\right)\right) \prod_{l\in D^1_{3+}} \left(\prod_{l\neq q\in D^1_{3+}} \left(\frac{l}{q}\right)\right) \prod_{l\in D^1_{3+}+D^1_1+E^1_0} \left(\frac{-1}{l}\right)$$
$$\times \prod_{l\in D^0_{3+}+D^0_1+D^1_1+E^1_0} \tau(l) \prod_{l\in D^1_{3+}} \sigma(l) \times C_4,$$

where

$$\Lambda^{(1)} := (\Pi(\alpha)^1_{3+,\mathrm{odd}} - D^1_{3+}) + ((\Pi(\alpha)^1_1 - Q^1_1) - D^1_1) + ((\Pi(\alpha)^1_0 - Q^1_0) - E^1_0)$$
$$+ (Q^1_1 - (\Psi^1 \cap Q^1_1)) + (Q^1_0 - (\Psi^1 \cap Q^1_0)) + Q^0_1 + Q^0_0.$$

$$C_4 := \sum_{E^1_{3+}} \prod_{q\in\Pi(\alpha)^1_{3+,\mathrm{even}}-E^1_{3+}} \left(\frac{l_I}{q}\right) \prod_{q\in E^1_{3+}} \left\{\left(\frac{-1}{q}\right)\sigma(q)\left(\frac{l_I}{q}\right)\right\}$$
$$\times \sum_{E^1_2} \prod_{q\in\Pi(\alpha)^1_2-E^1_2} \left(\frac{l_I}{q}\right) \prod_{q\in E^1_2} \left\{\left(\frac{-1}{q}\right)\sigma(q)\left(\frac{l_I}{q}\right)\right\}$$
$$\times \sum_{E^0_{3+}} \prod_{l\in E^0_{3+}} \tau(l) \sum_{E^0_2} \prod_{l\in E^0_2} \tau(l) \sum_{E^0_0} \prod_{l\in E^0_0} \tau(l) \sum_{P^0_1} \prod_{l\in P^0_1} \left(\frac{-1}{l}\right) \sum_{P^0_0} \prod_{l\in P^0_0} \left(\frac{-1}{l}\right).$$

We can easily calculate the value of C_4 as follows:

$$C_4 = C_5 \times \prod_{q\in\Pi(\alpha)^1_{3+,\mathrm{even}}+\Pi(\alpha)^1_2} \left(\frac{l_I}{q}\right).$$

$$C_5 := \prod_{l\in\Pi(\alpha)^0_{3+,\mathrm{even}}+\Pi(\alpha)^0_2+(\Pi(\alpha)^0_0-Q^0_0)} (1+\tau(l)) \prod_{l\in Q^0_1+Q^0_0} \left(1+\left(\frac{-1}{l}\right)\right)$$
$$\times \prod_{l\in\Pi(\alpha)^1_{3+,\mathrm{even}}+\Pi(\alpha)^1_2} \left(1+\left(\frac{-1}{l}\right)\sigma(l)\right).$$

Splitting the summand of (2.21) into the factors depending on prime numbers in the variables Q_1^1, Q_0^1, D_{3+}^1, D_{3+}^0, and D_1^0, we have

$$\begin{aligned}\Xi := 2^{-\#\Pi} C_5 \sum_{Q_1^1} \sum_{Q_0^1} \prod_{l\in D_1^1} &\left\{\kappa(l)\left(\frac{-1}{l}\right)^{k-1} \chi_{2M_1}(l)\tau(l) \prod_{q\in\Phi}\left(\frac{l}{q}\right)\right\} \prod_{l\in E_0^1}\left\{\left(\frac{-1}{l}\right)\tau(l)\right\} \\ \times \sum_{D_{3+}^1} \prod_{l\in D_{3+}^1} &\left\{\kappa(l)\left(\frac{-1}{l}\right)^{k-1} \chi_{2M_1}(l)\sigma(l) \prod_{l\neq q\in\Phi}\left(\frac{l}{q}\right)\right\} \\ \times \sum_{D_{3+}^0} \prod_{l\in D_{3+}^0} &\left\{\kappa(l)\left(\frac{-1}{l}\right)^{k} \chi_{2M_1}(l)\tau(l) \prod_{q\in\Phi}\left(\frac{l}{q}\right)\right\} \\ \times \sum_{D_1^0} \prod_{l\in D_1^0} &\left\{\kappa(l)\left(\frac{-1}{l}\right)^{k} \chi_{2M_1}(l)\tau(l) \prod_{q\in\Phi}\left(\frac{l}{q}\right)\right\},\end{aligned}$$

where $\Phi := (\Pi^1 - \Psi^1) + \Psi^0$.

We can easily calculate the each term of the above formula.

Thus we obtain the following.

$$\Xi = \Xi((\alpha_l), \Psi, (\tau, \sigma)) = \prod_{p\in\Pi} \Xi_p((\alpha_l), \Psi, (\tau, \sigma)), \tag{2.22}$$

where

$$2 \times \Xi_p((\alpha_l), \Psi, (\tau, \sigma))$$

$$:= \begin{cases} 1+\tau(p), & \text{if } p \in \Pi(\alpha)^0_{3+,\text{even}} + \Pi(\alpha)^0_2 + (\Pi(\alpha)^0_0 - Q^0_0), \\ 1+\left(\frac{-1}{p}\right)\sigma(p), & \text{if } p \in \Pi(\alpha)^1_{3+,\text{even}} + \Pi(\alpha)^1_2, \\ 1+\kappa(p)\left(\frac{-1}{p}\right)^k \chi_{2M_1}(p)\tau(p)\prod_{q\in\Phi}\left(\frac{p}{q}\right), & \text{if } p \in \Pi(\alpha)^0_{3+,\text{odd}} + (\Pi(\alpha)^0_1 - Q^0_1), \\ 1+\kappa(p)\left(\frac{-1}{p}\right)^{k-1} \chi_{2M_1}(p)\sigma(p)\prod_{p\neq q\in\Phi}\left(\frac{p}{q}\right), & \text{if } p \in \Pi(\alpha)^1_{3+,\text{odd}}, \\ 1+\kappa(p)\left(\frac{-1}{p}\right)^{k-1} \chi_{2M_1}(p)\tau(p)\prod_{q\in\Phi}\left(\frac{p}{q}\right), & \text{if } p \in \Pi(\alpha)^1_1 \cap \Psi^1, \\ 1+\left(\frac{-1}{p}\right)\tau(p), & \text{if } p \in \Pi(\alpha)^1_0 \cap \Psi^1, \\ 2, & \text{if } p \in (\Pi(\alpha)^1_1 + \Pi(\alpha)^1_0) - \Psi^1, \\ 1+\left(\frac{-1}{p}\right), & \text{if } p \in Q^0_1 + Q^0_0, \end{cases}$$

and $\Phi = (\Pi^1 - \Psi^1) + \Psi^0$, $Q_1^0 = \Psi^0 \cap \Pi(\alpha)_1^0$, $Q_0^0 = \Psi^0 \cap \Pi(\alpha)_0^0$.

From (2.7), (2.19), (2,20), and (2.22), we obtain the following final result.

PROPOSITION (2.23). *We assume that* $2 \leq \mu = \mathrm{ord}_2(N) \leq 4$ *and also* $\mathfrak{f}(\chi_2) = 8$ *if* $\mu = 4$. *Let* n *be a positive integer prime to* N *and* κ *an element of* $\{\pm 1\}^{\Pi}$. *Then we have*

$$(1)\quad \begin{Bmatrix} \mathrm{tr}(\tilde{T}(n^2)\,;\,S^{\emptyset,\kappa}) & \text{if } k \geq 2 \\ \mathrm{tr}(\tilde{T}(n^2)\,;\,V^{\emptyset,\kappa}) & \text{if } k = 1 \end{Bmatrix}$$

$$= \sum_{\beta=0}^{\omega} (\omega - \beta + 1) \sum_{\substack{(w_p)_{p|M_1} \\ w_p = 0,1}} \prod_{p|M_1} (2 - w_p) \sum_{\substack{(\alpha_l)_{l \in \Pi} \\ 0 \leq \alpha_l \leq \nu_l}} \sum_{\Psi \subseteq \Pi(\alpha)_1 + \Pi(\alpha)_0} \sum_{\substack{\tau \in \mathrm{Map}(\Pi, \{\pm 1\}) \\ \sigma \in \mathrm{Map}(\Pi(\alpha)_{2+}, \{\pm 1\})}}$$

$$\times\, \Xi((\alpha_l),\, \Psi,\, (\tau,\, \sigma))\, \mathrm{tr}\Big(T(n)\,;\,S^{*(\tau,\sigma)}\Big(2k,\, 2^{\beta} \prod_{p|M_1} p^{w_p} \prod_{l \in \Pi} l^{\alpha_l}\Big) \Big|\, R_{\Psi}\Big),$$

where ω *is the same as in* (2.7), $\Pi(\alpha)_i = \{l \in \Pi \mid \alpha_l = i\}\,(i = 0,1)$, $\Pi(\alpha)_{2+} = \{l \in \Pi \mid \alpha_l \geq 2\}$. $\Xi((\alpha_l),\, \Psi,\, (\tau,\, \sigma))$ *are the constants explicitly determined by the formula* (2.22).

(2) $\Xi((\alpha_l),\, \Psi,\, (\tau,\, \sigma))$ *does not depend on* β *and* (w_p).

(3) *The value of* $\Xi((\alpha_l),\, \Psi,\, (\tau,\, \sigma))$ *is either* 0 *or* 1. *By using the formula* (2.22), *we can explicitly find* (α_l), Ψ, *and* $(\tau,\, \sigma)$ *such that* $\Xi((\alpha_l),\, \Psi,\, (\tau,\, \sigma)) = 1$. □

We have some comments on this proposition. The subspaces in the right-hand side of (2.23) (1), $S^{*(\tau,\sigma)}(2k,\, 2^{\beta}\Pi_{p|M_1} p^{w_p}\, \Pi_{l \in \Pi}\, l^{\alpha_l}) \mid R_{\Psi}$'s, are orthogonal with each other (cf. Proposition (A.8)) and these subspaces are spanned by primitive forms. Moreover, the multiplicities for these subspaces in the right-hand side of the formula of (2.23) are at most 1 if we consider only Kohnen spaces $S^{\emptyset,\kappa}(k + 1/2,\, N,\, \chi)_K$ and $V^{\emptyset,\kappa}(N,\, \chi)_K$ and if $M_1 = 1$.

Hence in such a case, from the above proposition and the theory of newforms of integral weight, we can deduce that $S^{\emptyset,\kappa}(k + 1/2,\, N,\, \chi)_K$ and $V^{\emptyset,\kappa}(N,\, \chi)_K$ have an orthogonal $\mathbf{C}$-basis consisting of common eigenforms for all operators $\tilde{T}(n^2)$ $((n,\, N) = 1)$ which are uniquely determined up to multiplication with non-zero scalars.

Actually, we also have similar and more exact results for Kohnen spaces in general cases. See Theorems (3.10−11) for precise statements of results. In next §3, we shall consider only Kohnen spaces and investigate in detail.

§3. The space of newforms and the strong multiplicity one theorem

We keep to the notations in §1 and §2. In this and next sections, we shall consider only Kohnen spaces $S^{\emptyset,\chi}(k+1/2, N, \chi)_K$ and $V^{\emptyset,\chi}(N, \chi)_K$.

In this case, we can write $N = 4M$ and $\chi = \left(\frac{M_0}{\ }\right)$ with an odd positive integer M and a squarefree positive divisor M_0 of M (cf. §0(d)). Then we decompose

(3.1) $M_0 = m_1 m_{2+}$ and $\chi = \chi^1 \chi^{2+}$, $0 < m_1 \mid M_1$, $0 < m_{2+} \mid M_{2+}$, $\chi^1 = \left(\frac{m_1}{\ }\right)$, $\chi^{2+} = \left(\frac{m_{2+}}{\ }\right)$.

The characters χ^1, χ^{2+} are defined with modulo $4M_1$, $4M_{2+}$ respectively.

Under the above notation, for a positive divisor d of M_1 and any $\kappa \in \{\pm 1\}^{\Pi}$, we put

(3.2)

$$S^{\emptyset,\chi}(k+1/2, 4dM_{2+}, \chi)_K := S^{\emptyset,\chi'}(k+1/2, 4dM_{2+}, \chi^{2+})_K \mid U(m_1), \text{ if } k \geq 2,$$

$$V^{\emptyset,\chi}(4dM_{2+}, \chi)_K := V^{\emptyset,\chi'}(4dM_{2+}, \chi^{2+})_K \mid U(m_1), \text{ if } k = 1,$$

where $\kappa' = \kappa \cdot \left(\frac{m_1}{\ }\right)$. We can easily check the well-definedness of this definition by using Proposition (1.28).

For any $\kappa \in \{\pm 1\}^{\Pi}$, we define the space of "oldforms" by:

$$\hat{\mathfrak{S}}^{\emptyset,\chi}(k+1/2, N, \chi)_K := \begin{cases} \sum_{p \mid M_1} \{S^{\emptyset,\chi}(k+1/2, N/p, \chi)_K + S^{\emptyset,\chi}(k+1/2, N/p, \chi)_K \mid U(p^2)\}, & \text{if } k \geq 2, \\ \sum_{p \mid M_1} \{V^{\emptyset,\chi}(N/p, \chi)_K + V^{\emptyset,\chi}(N/p, \chi)_K \mid U(p^2)\}, & \text{if } k = 1, \end{cases}$$

where $\sum_{p \mid M_1}$ is the sum extended over all prime divisors p of M_1.

Then we denote by $\mathfrak{S}^{\emptyset,\chi}(k+1/2, N, \chi)_K$ the orthogonal complement of $\hat{\mathfrak{S}}^{\emptyset,\chi}(k+1/2, N, \chi)_K$ in the space $S^{\emptyset,\chi}(k+1/2, N, \chi)_K$ (if $k \geq 2$) resp. $V^{\emptyset,\chi}(N, \chi)_K$ (if $k = 1$).

We have the following:

(3.3) For any $\kappa \in \{\pm 1\}^{\Pi}$, the mapping $f \mapsto f \mid U(m_1)$ gives an isomorphism from $\mathfrak{S}^{\emptyset,\chi'}(k+1/2, N, \chi^{2+})_K$ onto $\mathfrak{S}^{\emptyset,\chi}(k+1/2, N, \chi)_K$, where $\kappa' = \kappa \cdot \left(\frac{m_1}{\ }\right)$.

Proof. From the definitions, we can see

$$\hat{\mathfrak{S}}^{\emptyset,\chi}(k+1/2, N, \chi)_K = \hat{\mathfrak{S}}^{\emptyset,\chi'}(k+1/2, N, \chi^{2+})_K \mid U(m_1).$$

By using this fact and Proposition (1.28), it is sufficient to prove the inclusion:

$\mathfrak{S}^{\emptyset,\chi'}(k+1/2, N, \chi^{2+})_K \mid U(m_1) \subseteq \mathfrak{S}^{\emptyset,\chi}(k+1/2, N, \chi)_K$.

Take any $f \in \mathfrak{S}^{\emptyset,\chi'}(k+1/2, N, \chi^{2+})_K$ and $g \in \hat{\mathfrak{S}}^{\emptyset,\chi'}(k+1/2, N, \chi^{2+})_K$. Then by using Lemma (3.4) (see below), $0 = \langle f, g\rangle = (m_1)^{-k+1/2}\langle f \mid U(m_1), g \mid U(m_1)\rangle$. The proof is completed. □

LEMMA (3.4). *Let p be a prime divisor of M_1 such that $\chi_p = 1$. For any $F_1, F_2 \in S(k+1/2, N, \chi)$,*

$$\langle F_1 \mid U(p), F_2 \mid U(p)\rangle = p^{k-1/2} \langle F_1, F_2\rangle.$$

Proof. By using Proposition (1.18), [U2, Lemma (1.9)], and Proposition (1.27) in order,

$$\begin{aligned}\langle F_1 \mid U(p), F_2 \mid U(p)\rangle &= \langle F_1 \mid U(p)\tilde{W}(p)^2, F_2 \mid U(p)\tilde{W}(p)^2\rangle \\ &= \langle F_1 \mid U(p)\tilde{W}(p), F_2 \mid U(p)\tilde{W}(p)\rangle = p^{k-3/2}\langle F_1 \mid Y_p, F_2 \mid Y_p\rangle \\ &= p^{k-3/2}\langle F_1, F_2 \mid Y_p Y_p\rangle = p^{k-1/2}\langle F_1, F_2\rangle.\end{aligned}$$

□

From (3.3), we can extend the definition as follows: For a positive divisor d of M_1 and any $\kappa \in \{\pm 1\}^{\Pi}$, we put

(3.5) $$\mathfrak{S}^{\emptyset,\chi}(k+1/2, 4dM_{2+}, \chi)_K := \mathfrak{S}^{\emptyset,\chi'}(k+1/2, 4dM_{2+}, \chi^{2+})_K \mid U(m_1),$$

where $\kappa' = \kappa \cdot \left(\frac{m_1}{\cdot}\right)$.

Now we shall define a hermitian involution $\mathbf{w}_p$ on $S(k+1/2, N, \chi)$.

Let p be an odd prime divisor of N with $\mathrm{ord}_p(N) = 1 (\Leftrightarrow p \mid M_1)$. From Proposition (1.29), $X_p := p^{-1/2}\left(\frac{-1}{p}\right)^{k+1/2}\left(\frac{m_{2+}}{p}\right) Y_p$ is a hermitian involution on $S(k+1/2, N, \chi^{2+})$.

Then we define an operator on $S(k+1/2, N, \chi) = S(k+1/2, N, \chi^{2+}) \mid U(m_1)$ by:

(3.6) $$\mathbf{w}_p := U(m_1)^{-1} X_p U(m_1) = p^{-1/2}\left(\frac{-1}{p}\right)^{k+1/2}\left(\frac{m_{2+}}{p}\right) U(m_1)^{-1} Y_p U(m_1).$$

From Lemma (3.4), it is easily shown that this operator $\mathbf{w}_p$ is a hermitian involution on $S(k+1/2, N, \chi)$.

Moreover if $(p, m_1) = 1$, we can simplify $\mathbf{w}_p$ as follows:

(3.7) $$f \mid \mathbf{w}_p = p^{-1/2}\left(\frac{-1}{p}\right)^{k+1/2}\left(\frac{m_1 m_{2+}}{p}\right) f \mid Y_p, \quad f \in S(k+1/2, N, \chi),$$

by using Proposition (1.20) (1).

We shall also use the operator $\tilde{T}_{k+1/2,N,\chi}(4)$ on $S(k+1/2, N, \chi)_K$ (cf. §0(d) and [K]). This operator has the following properties:

PROPOSITION (3.8). *Let $N = 4M$ with an odd positive integer M and χ an even character modulo N such that $\chi^2 = 1$. Then $\tilde{T}_{k+1/2,N,\chi}(4)$ is a hermitian operator on $S(k+1/2, N, \chi)_K$. Moreover for any $f = \sum_{n\geq 1} a(n)\mathbf{e}(nz) \in S(k+1/2, N, \chi)_K$, we have the following.*

(1) $f \mid \tilde{T}_{k+1/2,N,\chi}(4)$

$$= \sum_{\substack{\varepsilon(-1)^k n \equiv 0,1(4) \\ n\geq 1}} \left\{a(4n) + \chi_M(2)\left(\frac{\varepsilon(-1)^k n}{2}\right) 2^{k-1} a(n) + 2^{2k-1} a(n/4)\right\}\mathbf{e}(nz),$$

where $\varepsilon = \chi_2(-1)$ and χ_M is the M-primary component of χ.
(2) *If n is a positive integer prime to N,*

$$f \mid \tilde{T}_{k+1/2,N,\chi}(n^2)\tilde{T}_{k+1/2,N,\chi}(4) = f \mid \tilde{T}_{k+1/2,N,\chi}(4)\tilde{T}_{k+1/2,N,\chi}(n^2).$$

(3) *If p is an odd prime divisor of N,*

$$f \mid U(p)\tilde{T}_{k+1/2,N,\chi\left(\frac{p}{}\right)}(4) = f \mid \tilde{T}_{k+1/2,N,\chi}(4)\, U(p).$$

(4) *Suppose $l \in \Pi$. Then*

$$f \mid R_l \tilde{T}_{k+1/2,N,\chi}(4) = f \mid \tilde{T}_{k+1/2,N,\chi}(4) R_l.$$

(5) *Let m be an odd positive integer.*

$$f \mid \tilde{T}_{k+1/2,N,\chi}(4) = f \mid \tilde{T}_{k+1/2,Nm,\chi}(4).$$

(6) *For any $\kappa \in \{\pm 1\}^{\Pi}$, $\tilde{T}_{k+1/2,N,\chi}(4)$ fixes the subspaces $U(N;\chi)$, $V(N;\chi)_K$, $S^{\emptyset,\kappa}(k+1/2, N, \chi)_K$, and $V^{\emptyset,\kappa}(N, \chi)_K$.*

Proof. See [K, §3] for proofs of the hermitian property and the assertion (1). The assertion (2) follows from straightforward computation in the abstract Hecke algebra. The assertions (3), (4), and (5) are easily verified by checking the coincidence of the Fourier coefficients of the both sides (cf. (1)). From (1), we have for any $(\rho, t) \in \Omega^1(N, \chi)$, $h^1(\rho; tz) \mid \tilde{T}_{k+1/2,N,\chi}(4) = 3\rho(2)h^1(\rho; tz)$.

Hence $\tilde{T}_{k+1/2,N,\chi}(4)$ fixes $U(N;\chi)$. Since $\tilde{T}_{k+1/2,N,\chi}(4)$ is hermitian, $V(N;\chi)_K$

is also fixed. The rest of the assertion (6) follows from the assertion (4). □

Now we can state the main results of this paper.

THEOREM (3.9). *Let p be a prime divisor p of N with* $\mathrm{ord}_p(N) = 1$ *and* $\kappa \in \{\pm 1\}^{\Pi}$. *The operators* $U(p^2)$, $\mathbf{w}_p$, *and* $\tilde{T}_{k+1/2,N,\chi}(4)$ *fix* $\mathfrak{S}^{\emptyset,\chi}(k+1/2, N, \chi)_K$. *Moreover we have* $U(p^2) = -p^{k-1}\mathbf{w}_p$ *on* $\mathfrak{S}^{\emptyset,\chi}(k+1/2, N, \chi)_K$. □

THEOREM (3.10). *Let the notations be the same as above and let* $\kappa \in \{\pm 1\}^{\Pi}$. *Then, in particular, we suppose that* $\mathrm{ord}_2(N) = 2$.

(1) *We have the following direct sum decomposition*:

$$\begin{Bmatrix} S^{\emptyset,\chi}(k+1/2, N, \chi)_K, & \text{if } k \geq 2 \\ V^{\emptyset,\chi}(N, \chi)_K, & \text{if } k = 1 \end{Bmatrix} = \bigoplus_{\substack{0<e,d \\ ed|M_1}} \mathfrak{S}^{\emptyset,\chi}(k+1/2, 4dM_{2+}, \chi)_K \mid U(e^2),$$

where $\bigoplus_{\substack{0<e,d \\ ed|M_1}}$ *is the sum extended over all pairs* (e, d) *of positive divisors of* M_1 *such that* $ed \mid M_1$.

(2) *Let n be a posisive integer prime to N. Then* $\tilde{T}_{k+1/2,N,\chi}(n^2)$ *fixes the space* $\mathfrak{S}^{\emptyset,\chi}(k+1/2, N, \chi)_K$ *and the trace is given by the following*:

$$\mathrm{tr}(\tilde{T}_{k+1/2,N,\chi}(n^2)\,;\,\mathfrak{S}^{\emptyset,\chi}(k+1/2, N, \chi)_K)$$

$$= \sum_{((\alpha_l),\Psi,(\tau,\sigma))\in P} \Xi((\alpha_l), \Psi, (\tau, \sigma))\,\mathrm{tr}\Big(T(n)\,;\,S^{*(\tau,\sigma)}\Big(2k, M_1 \prod_{l\in\Pi} l^{\alpha_l}\Big) \mid R_\Psi\Big),$$

where the notations are as follows: $\sum_{((\alpha_l),\Psi,(\tau,\sigma))\in P}$ *is the sum extended over all elements of the following set*:

$$P := \left\{\begin{matrix} ((\alpha_l), \Psi, (\tau, \sigma))\,;\,(\alpha_l) = (\alpha_l)_{l\in\Pi} \text{ is a system of integers} \\ \text{such that } 0 \leq \alpha_l \leq \nu_l := \mathrm{ord}_l(N) \text{ for any } l \in \Pi, \\ \Psi \subseteq \Pi(\alpha)_0 + \Pi(\alpha)_1,\ \tau \in \mathrm{Map}(\Pi, \{\pm 1\}),\ \sigma \in \mathrm{Map}(\Pi(\alpha)_{2+}, \{\pm 1\}) \end{matrix}\right\}.$$

$\Pi(\alpha)_i = \{l \in \Pi \mid \alpha_l = i\}\ (i = 0,1)$, $\Pi(\alpha)_{2+} = \{l \in \Pi \mid \alpha_l \geq 2\}$, $\Xi((\alpha_l), \Psi, (\tau, \sigma))$ *are the constants explicitly determined by the formula* (2.22) *which are either* 0 *or* 1 (*cf. Proposition* (2.23)). □

THEOREM (3.11) *Let the notations be the same as in Theorem* (3.10).

(1) *The space* $\mathfrak{S}^{\emptyset,\chi}(k+1/2, N, \chi)_K$ *has an orthogonal* $\mathbf{C}$-*basis consisting of common eigenforms for all operators* $\tilde{T}_{k+1/2,N,\chi}(p^2)$ (p : *prime*, $p \nmid M$) *and* $U(p^2)$ (p : *prime*, $p \mid M$), *which are uniquely determined up to multiplication with non-zero complex*

numbers. If f is such an eigenform and λ_p the eigenvalue of f with respect to $\tilde{T}_{k+1/2,N,\chi}(p^2)$ $(p \nmid M)$ resp. $U(p^2)$ $(p \mid M)$, then there exist a positive divisor M' of M_{2+} and a primitive form (of conductor M_1M') $F \in S^0(2k, M_1M')$, which is uniquely determined and satisfies the following: For a prime p,

$$F \mid T(p) = \lambda_p F \text{ if } (p, M) = 1, \text{ and } F \mid U(p) = \lambda_p F \text{ if } p \mid M_1.$$

Moreover $\lambda_p = \pm p^{k-1}$ for any prime divisor p of M_1.

(2) *(The strong multiplicity one theorem.) Let f, g be non-zero elements of $\mathfrak{S}^{\emptyset,\chi}(k + 1/2, N, \chi)_K$. If f and g are common eigenforms of $\tilde{T}_{k+1/2,N,\chi}(p^2)$ with the same eigenvalue for all prime numbers p prime to some integer A, then $\mathbf{C}f = \mathbf{C}g$.* □

Remark. We have $\lambda_p = 0$ for all primes p dividing M_{2+}. On the other hand, there exists a case that $F \mid U(p) \neq 0$ for a certain prime $p \mid M_{2+}$. Therefore we cannot claim that $F \mid U(p) = \lambda_p F$ for any prime $p \mid M_{2+}$ in general cases. We shall discuss this topic in the next section.

Proof of Theorem (3.9). Observing (3.3), the definition (3.6), and Proposition (3.8) (3), we see that it is enough to check these statements for the case of $m_1 = 1$.

From (3.8) ((3), (5), and (6)), $\tilde{T}(4) = \tilde{T}_{k+1/2,N,\chi}(4)$ fixes the space of old forms $\hat{\mathfrak{S}}^{\emptyset,\chi} = \hat{\mathfrak{S}}^{\emptyset,\chi}(k + 1/2, N, \chi)_K$. Since $\tilde{T}(4)$ is hermitian, $\mathfrak{S}^{\emptyset,\chi} = \mathfrak{S}^{\emptyset,\chi}(k + 1/2, N, \chi)_K$ is also fixed by $\tilde{T}(4)$.

Now we shall check the statement for $U(p)$ and $\mathbf{w}_p$ only for the case of $k \geq 2$, since the proof for the case of $k = 1$ is completely similar to that.

For $\mathbf{w}_p$, it is also enough to show that $\mathbf{w}_p$ fixes $\hat{\mathfrak{S}}^{\emptyset,\chi}$. Hence it is sufficient to prove that for any prime divisor q of M_1, Y_p maps $S^{\emptyset,\chi}(k + 1/2, N/q, \chi)_K$ and $S^{\emptyset,\chi}(k + 1/2, N/q, \chi)_K \mid U(q^2)$ to $\hat{\mathfrak{S}}^{\emptyset,\chi}$.

If $q \neq p$, Y_p is an automorphism of $S^{\emptyset,\chi}(k + 1/2, N/q, \chi)_K$ (from (1.28) (3)). Moreover from (1.20) (1), $f \mid U(q^2) Y_p = f \mid Y_p U(q^2)$ for any $f \in S(k + 1/2, N, \chi)$. Hence we get the assertion for $q \neq p$.

Next suppose $q = p$. Take any $f = \sum_{n\geq 1} a(n)\mathbf{e}(nz) \in S^{\emptyset,\chi}(k + 1/2, N/p, \chi)_K$ $(\subseteq S(k + 1/2, N, \chi))$. From (1.25-26), we have

$$f \mid Y_p = \left(\frac{-1}{p}\right)^{-k-1/2} \chi(p) f \mid \tilde{W}(p)\tilde{\delta}_p + \left(\frac{-1}{p}\right)^{-1/2} p^{1/2} \sum_{n\geq 1} a(n) \left(\frac{n}{p}\right) \mathbf{e}(nz).$$

Here, we note that $\chi(p)$ is meaningful because $\chi_p = 1$.

Since $f \in S(k + 1/2, N/p, \chi)$, $f \mid \tilde{W}(p) = f \mid \tilde{\delta}_p$ and then

(3.12)

$$f \mid Y_p = \left(\frac{-1}{p}\right)^{-k-1/2} p^{1/2}\chi(p) \sum_{n\geq 1} \left\{\left(\frac{(-1)^k n}{p}\right)\chi(p)a(n) + p^k a(n/p^2)\right\} \mathbf{e}(nz)$$

$$= \left(\frac{-1}{p}\right)^{-k-1/2} p^{-k+3/2}\chi(p)\ \{f \mid \tilde{T}_{k+1/2,N/p,\chi}(p^2) - f \mid U(p^2)\}$$

(cf. [Sh 1, Theorem (1.7)]). This shows $f \mid Y_p \in \hat{\mathfrak{S}}^{\emptyset,\chi}$.

Next applying Y_p on both sides of (3.12) and observing $Y_p^2 = \left(\frac{-1}{p}\right)p$, we get

$$(3.13) \qquad f \mid U(p^2) Y_p = f \mid \tilde{T}_{k+1/2,N/p,\chi}(p^2) Y_p - \left(\frac{-1}{p}\right)^{k+3/2} p^{k-1/2}\chi(p) f.$$

This shows $f \mid U(p^2) Y_p \in \hat{\mathfrak{S}}^{\emptyset,\chi}$.

We must show the relation between $U(p^2)$ and $\mathbf{w}_p$. We use the trace operator $Tr^N_{N/p} : S(k+1/2, N, \chi) \to S(k+1/2, N/p, \chi)$ (adjoint to the inclusion map). See Appendix 2 for its definition and properties.

$Tr^N_{N/p}$ maps $S^{\emptyset,\chi}(k+1/2, N, \chi)_K$ to $S^{\emptyset,\chi}(k+1/2, N/p, \chi)_K$ (cf. Proposition (A.10)). Let $f \in \mathfrak{S}^{\emptyset,\chi}$. Since f is orthogonal to $S^{\emptyset,\chi}(k+1/2, N/p, \chi)_K$, $f \mid Tr^N_{N/p} = 0$.

A system of representatives for $\Delta_0(N, \chi) \backslash \Delta_0(N/p, \chi)$ is formed by the elements $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^*$ and $\gamma_p^* \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}^* (i \in \mathbf{Z}/p\mathbf{Z})$, where we choose an element γ_p such that

$$SL_2(\mathbf{Z}) \ni \gamma_p \equiv \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & (\mathrm{mod}\, N/p), \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & (\mathrm{mod}\, p). \end{cases}$$

Hence, $0 = f \mid Tr^N_{N/p} = (p+1)^{-1}\left(f + \sum_{i\in\mathbf{Z}/p\mathbf{Z}} f \mid \gamma_p^* \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}^*\right)$.

On the other hand, we have from (1.19),

$$f \mid \tilde{W}(p)\, U(p) = p^{k/2-3/4} \sum_{i\in\mathbf{Z}/p\mathbf{Z}} f \mid \gamma_p^* \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}^* = -p^{k/2-3/4} f.$$

We already proved that Y_p is an automorphism of $\mathfrak{S}^{\emptyset,\chi}$. By replacing f with $f \mid Y_p$, $-p^{k/2-3/4} f \mid Y_p = p^{-k/2+3/4} f \mid U(p)\tilde{W}(p)^2 U(p)$. The last assertion is easily shown from this and Proposition (1.18). □

Proof of Theorem (3.10). From (1.20) (3) and (1.8), we can see that $\hat{\mathfrak{S}}^{\emptyset,\kappa} = \hat{\mathfrak{S}}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$ is fixed by the operators $\tilde{T}(n^2) = \tilde{T}_{k+1/2,N,\chi}(n^2)((n, N) = 1)$. Since these operators are hermitian, we have $\mathfrak{S}^{\emptyset,\kappa} = \mathfrak{S}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$ is also fixed by the operators $\tilde{T}_{k+1/2,N,\chi}(n^2)$ for all $n \in \mathbf{Z}_+$ with $(n, N) = 1$.

Now we assume $m_1 = 1$ for a while and we shall give a proof for this case, first. The general cases are dealt with after that.

We shall prove the statements (1) and (2) by using induction with respect to the number of prime divisors of M_1 $(=: a)$.

Suppose $a = 0$. Since then $\hat{\mathfrak{S}}^{\emptyset,\kappa} = \{0\}$, (1) is trivial and (2) is just the Proposition (2.23).

Next we suppose that $a > 0$ and that the assertions (1) and (2) hold good if the number of prime divisors of $M_1 \leq a - 1$ (the assumption of induction).

For any positive divisor d of M_1 with $d < M_1$, $\mathfrak{S}^{\emptyset,\kappa}(k+1/2, 4dM_{2+}, \chi)_K$ has a $\mathbf{C}$-basis $\mathscr{B}_d$ consisting of common eigenforms for all operators $\tilde{T}_{k+1/2,4dM_{2+},\chi}(n^2)$ ($n \in \mathbf{Z}_+$ prime to $4dM_{2+}$). From the assumption of induction, the system of eigenvalues for each element in $\mathscr{B}_d$ corresponds to a primitive form of weight $2k$, trivial character, and conductor of the form dM' $(0 < M' \mid M_{2+})$. By using the strong multiplicity one theorem of weight $2k$ (cf. [M, Theorem 4.6.19]), we have $\mathscr{B}_{d_1} \perp \mathscr{B}_{d_2}$ for any distinct divisors d_1 and d_2.

Moreover for any $d(0 < d \mid M_1, d \neq M_1)$ and $f \in \mathscr{B}_d$, any element of the space $\mathscr{A}(f) := \sum_{0<e\mid(M_1/d)} \mathbf{C}f \mid U(e^2)$ has the same eigenvalues as those of f for all operators $\tilde{T}_{k+1/2,N,\chi}(n^2)$ $((n, N) = 1)$ (cf. (1.20) (3)). Hence the spaces $\mathscr{A}(f)$ $(f \in \mathscr{B}_d)$ are orthogonal to each other, from the strong multiplicity one theorem of weight $2k$ (cf. [M, Theorem 4.6.19]) and the later half of the assertion of induction.

From these results and the assumption of induction, we get

$$\begin{aligned}\hat{\mathfrak{S}}^{\emptyset,\kappa} &= \sum_{\substack{0<ed\mid M_1\\0<d<M_1}} \mathfrak{S}^{\emptyset,\kappa}(k+1/2, 4dM_{2+}, \chi)_K \mid U(e^2)\\ &= \bigoplus_{\substack{0<d\mid M_1\\d\neq M_1}} \left\{\sum_{0<e\mid(M_1/d)} \mathfrak{S}^{\emptyset,\kappa}(k+1/2, 4dM_{2+}, \chi)_K \mid U(e^2)\right\}\\ &= \bigoplus_{\substack{0<d\mid M_1\\d\neq M_1}} \left\{\bigoplus_{f\in\mathscr{B}_d} \sum_{0<e\mid(M_1/d)} \mathbf{C}f \mid U(e^2)\right\}.\end{aligned}$$

Therefore it is sufficient to prove that the elements $f \mid U(e^2)$ $(0 < e \mid (M_1/d))$ are linearly independent for each $f \in \mathscr{B}_d$.

Let d be a positive divisor of M_1 with $d \neq M_1$ and f an element of $\mathscr{B}_d$. Then we can take a prime p such that $d \mid (M_1/p)$. Applying the assumption of induction to (M_1/p), we can see that the elements $f \mid U(e'^2)$, $0 < e' \mid (M_1/pd)$ are linearly independent.

We need to decompose the space $\langle f \mid U(e')^2 ; 0 < e' \mid M_1/(pd)\rangle_{\mathbf{C}}$ with respect to prime divisors of $M_1/(pd)$. If $M_1/(pd) = 1$, the decomposition is trivial.

Suppose $M_1/(pd) \neq 1$. Denote the set of all prime divisors of $M_1/(pd)$ by $\mathscr{P} = \{q_1, \ldots, q_t\}$. For any s $(1 \leq s \leq t)$, put

$$\mathscr{A}'_s := \langle f \mid U(e')^2 ; 0 < e' \mid (q_1 \cdots q_s)\rangle_{\mathbf{C}}.$$

From (3.12-13), we have the following:

(3.14) Let $q \in \mathscr{P}$ and h any element of $S^{\emptyset,\chi}(k+1/2, N/q, \chi)_K$ (resp. $V^{\emptyset,\chi}(N/q, \chi)_K$) if $k \geq 2$ (resp. $k = 1$). Then

$$h \mid Y_q = c_q \{h \mid \tilde{T}_{k+1/2,N/q,\chi}(q^2) - h \mid U(q^2)\},$$

$$h \mid U(q^2)\, Y_q = h \mid \tilde{T}_{k+1/2,N/q,\chi}(q^2)\, Y_q - c_q q^{2k-2} h,$$

where $c_q = \left(\frac{-1}{q}\right)^{-k-1/2} q^{-k+3/2} \chi(q)$.

Since f is an eigenform of $\tilde{T}_{k+1/2,N/q,\chi}(q^2)$ for any $q \in \mathscr{P}$, Y_q preserves the space $\langle f, f \mid U(q^2)\rangle_{\mathbf{C}}$ from the above. By using (1.20) (1), we also see that all operators $Y_{q_i} (1 \leq i \leq s)$ fix the space $\mathscr{A}'_s$.

Since these Y_q $(q \in \mathscr{P})$ commute with each other, we have the decomposition into common eigen subspaces.

(3.15) Let s be an integer such that $1 \leq s \leq t$. We have

$$\mathscr{A}'_s = \bigoplus_{\rho_s \in \mathrm{Map}(\{q_1,\ldots,q_s\},\{\pm 1\})} \mathscr{A}'_s(\rho_s),$$

$$\mathscr{A}'_s(\rho_s) = \left\{x \in \mathscr{A}'_s \,;\, x \mid Y_q = \rho_s(q) \left\{\left(\frac{-1}{q}\right) q\right\}^{1/2} x \text{ for all } q \in \{q_1, \ldots, q_s\}\right\}.$$

Then we claim: $\dim_{\mathbf{C}} \mathscr{A}'_s(\rho_s) = 1$ for all $s = 1, \ldots, t$ and $\rho_s \in \mathrm{Map}(\{q_1, \ldots, q_s\}, \{\pm 1\})$.

We inductively prove this claim.

Let s be an integer with $1 \leq s \leq t$. Assume that the above claims for all s_0 $(s_0 \leq s-1)$ hold good. Then $\mathscr{A}'_s = \bigoplus_{\rho_{s-1}} \{\mathscr{A}'_{s-1}(\rho_{s-1}) \oplus \mathscr{A}'_{s-1}(\rho_{s-1}) \mid U(q_s)^2\}$. Here, ρ_{s-1} runs over $\mathrm{Map}(\{q_1, \ldots, q_{s-1}\}, \{\pm 1\})$. We can take an element g such that $\mathscr{A}'_{s-1}(\rho_{s-1}) = \mathbf{C}g$ by the assumption of induction. In particular, if $s = 1$, we take $g = f$.

Since $U(q_s)^2$ commutes with all Y_q's$(q \in \{q_1, \ldots, q_{s-1}\})$ (cf. Proposition(1.20) (1)), $h \mid Y_q = \rho_{s-1}(q) \left\{\left(\frac{-1}{q}\right) q\right\}^{1/2} h$ for any $q \in \{q_1, \ldots, q_{s-1}\}$ and any $h \in \mathscr{A}'_{s-1}(\rho_{s-1}) \oplus \mathscr{A}'_{s-1}(\rho_{s-1}) \mid U(q_s)^2$. Moreover we can see that f and g have the same eigenvalue λ on $\tilde{T}_{k+1/2,N/q_s,\chi}(q_s^2)$ from Proposition (1.8) and Proposition (1.20) (3).

Now applying (3.14) to $q_s \in \mathscr{P}$ and g,

$$(g \mid Y_{q_s}, g \mid U(q_s)^2 Y_{q_s}) = (g, g \mid U(q_s)^2) \begin{pmatrix} c_{q_s}\lambda & c_{q_s}(\lambda^2 - q_s^{2k-2}) \\ -c_{q_s} & -c_{q_s}\lambda \end{pmatrix}.$$

Since the characteristic polynomial of this matrix is $X^2 - \left(\frac{-1}{q_s}\right) q_s$, there are two distinct eigen subspaces of $\mathscr{A}'_{s-1}(\rho_{s-1}) \oplus \mathscr{A}'_{s-1}(\rho_{s-1}) \mid U(q_s)^2$ with respect to Y_{q_s} and both of them are one-dimensional. Thus, the proof of the claim is completed.□

In either case, we get from (1.24) (1) and the above,

$$\mathscr{A} := \langle f \mid U(e^2); 0 < e \mid (M_1/d) \rangle_{\mathbf{C}} = \bigoplus_{\rho} \{\mathscr{A}(\rho) + \mathscr{A}(\rho) \mid U(p^2)\},$$

where ρ (in the direct sum) runs over $\mathrm{Map}(\mathscr{P}, \{\pm 1\})$, $\mathscr{P} =$ the set of all prime divisors of $M_1/(pd)$ and for any $\rho \in \mathrm{Map}(\mathscr{P}, \{\pm 1\})$,

$$\mathscr{A}(\rho) = \left\{ x \in \mathscr{A}\, ; x \mid Y_q = \rho(q) \left\{ \left(\frac{-1}{q}\right) q \right\}^{1/2} x \quad \text{for any } q \in \mathscr{P} \right\}.$$

Since $\dim_{\mathbf{C}} \mathscr{A}(\rho) = 1$, we can take a basis $g_\rho = \sum_{n \geq 1} a_\rho(n) \mathbf{e}(nz)$ of $\mathscr{A}(\rho)$ for each $\rho \in \{\pm 1\}^{\mathscr{P}}$.

Suppose that g_ρ and $g_\rho \mid U(p^2)$ are linearly dependent. Then we may put that $g_\rho \mid U(p^2) = \alpha g_\rho$ with $0 \neq \alpha \in \mathbf{C}$. g_ρ and f have the same eigenvalue λ_p on $\tilde{T}_{k+1/2,N/p,\chi}(p^2)$. Then

(3.16)

$$\left(\lambda_p - \alpha - \chi(p) \left(\frac{(-1)^k n}{p}\right) p^{k-1}\right) a_\rho(n) = p^{2k-1} a_\rho(n/p^2) \text{ for all } n \in \mathbf{Z}_+$$

(cf. [Sh 1, (1.7)]).

We have the following lemma.

LEMMA (3.17). *Let d be a positive divisor of M_1 such that $\chi_p = 1$ for any prime p dividing d. Then for each non-zero element $f = \sum_{n \geq 1} a(n) \mathbf{e}(nz) \in S(k + 1/2, N, \chi)$, there exists $n \in \mathbf{Z}_+$ such that $(n, d) = 1$ and $a(n) \neq 0$.*

Proof. We use an induction on the number r of prime divisors of d. If $d = 1$, the assertion is trivial.

Suppose $r > 0$. We choose and fix a prime divisor p of d. We consider $f \mid R_{\left(\frac{\cdot}{p}\right)}^{\;2}$

$\in S(k+1/2, pN, \chi)$. Let us suppose $f \mid R_{\left(\frac{\cdot}{p}\right)}{}^2 = 0 \Leftrightarrow a(n) = 0$ for all $n \in \mathbf{Z}_+$ prime to p. Then since $p^2 \mid p \cdot \mathfrak{f}\left(\chi\left(\frac{p}{\cdot}\right)\right) \nmid N$, we have $f = 0$ (cf. Proposition (1.10)) which is a contradiction. Hence $f \mid R_{\left(\frac{\cdot}{p}\right)}{}^2 \neq 0$. Applying the assumption of the induction to d/p and $f \mid R_{\left(\frac{\cdot}{p}\right)}{}^2$, there exists $n \in \mathbf{Z}_+$ such that $(n, d/p) = 1$ and $a(n)\left(\frac{n}{p}\right)^2 \neq 0$. Such n is also prime to p and $a(n) \neq 0$. Then proof of Lemma is completed. □

From the above, we can take $n' \in \mathbf{Z}_+$ such that $(n', M_1) = 1$ and $a_\rho(n') \neq 0$. Since g_ρ is a common eigenform of Y_q $(q \in \mathscr{P})$, we get $\left(\frac{n'}{q}\right) = \rho(q)\left(\frac{-1}{q}\right)$ for all $q \in \mathscr{P}$ (cf. Proposition (1.29)). Then

$$\lambda_p = \alpha + \chi(p)\left(\frac{(-1)^k n'}{p}\right) p^{k-1}. \tag{3.18}$$

Substituting this into (3.16),

$$\chi(p)\left(\frac{-1}{p}\right)^k\left(\left(\frac{n'}{p}\right) - \left(\frac{n}{p}\right)\right) a_\rho(n) = p^k a_\rho(n/p^2) \text{ for all } n \in \mathbf{Z}_+. \tag{3.19}$$

Set $n = p^2 n'$. By using $\alpha a_\rho(n') = a_\rho(p^2 n')$ and $a_\rho(n') \neq 0$, we have

$$\alpha = \chi(p)\left(\frac{(-1)^k n'}{p}\right) p^k. \tag{3.20}$$

From (3.18) and this,

$$|\lambda_p| = p^k + p^{k-1} > 2p^{k-1/2}. \tag{3.21}$$

We note that λ_p is an eigenvalue of f and so λ_p is an eigenvalue of a primitive form of weight $2k$, trivial character, and conductor dM' $(0 < M' \mid M_{2+})$ from the assumption of induction. Hence (3.21) contradicts Deligne's theorem. Thus we see that g_ρ and $g_\rho \mid U(p^2)$ are linearly independent and the elements $f \mid U(e^2)$, $0 < e \mid (M_1/d)$, are also linearly independent. This prove the assertion (1).

Next we prove the assertion (2). All positive divisors d of M_1 are uniquely expressed as $d = \Pi_{p \mid M_1} p^{w_p}$, $w_p = 0,1$. Then from Proposition (1.20) (3), (1.28), and the assertion (1), we have for all $n \in \mathbf{Z}_+$ prime to N,

$$\begin{Bmatrix} \mathrm{tr}(\tilde{T}(n^2)\,;\, S^{\emptyset,\kappa}(k+1/2,\,N,\,\chi)_K), & \text{if } k \geq 2 \\ \mathrm{tr}(\tilde{T}(n^2)\,;\, V^{\emptyset,\kappa}(N,\,\chi)_K), & \text{if } k = 1 \end{Bmatrix}$$

$$= \sum_{\substack{(w_p)_{p|M_1} \\ w_p=0,1}} \prod_{p|M_1} (2-w_p)\mathrm{tr}\Big(\tilde{T}(n^2)\,;\, \mathfrak{S}^{\emptyset,\kappa}\Big(k+1/2,\, 4\,\Pi_{p|M_1} p^{w_p} M_{2+},\, \chi\Big)_K\Big).$$

From the assumption of induction, we can express the trace $\mathrm{tr}(\tilde{T}(n^2)\,;$ $\mathfrak{S}^{\emptyset,\kappa}(k+1/2,\, 4\,\Pi_{p|M_1} p^{w_p} M_{2+},\, \chi)_K)$ with the formula in the assertion (2) for any $\Pi_{p|M_1} p^{w_p} < M_1$. Then it is easily seen that the range P of the sums in that expression formula does not depend on $(w_p)_{p|M_1}$. Moreover we can check that the coefficients $\Xi((\alpha_l),\, \Psi,\, (\tau,\, \sigma))$ in that expression formula also do not depend on $(w_p)_{p|M_1}$. In fact, observing the definition (2.22) of the values of $\Xi((\alpha_l),\, \Psi,$ $(\tau,\, \sigma))$, it is possible that only the value of $\chi_{2d}(p)$ may depend on $(w_p)_{p|M_1}$, where $d = \Pi_{p|M_1} p^{w_p}$.

But its value does not depend on $(w_p)_{p|M_1}$, either. Because we have $\chi_{2d} = \chi_2$ $= \chi_{2M_1}$ from the assumption $m_1 = 1$.

Thus we can take $\mathrm{tr}(\tilde{T}(n^2)\,;\, \mathfrak{S}^{\emptyset,\kappa}(k+1/2,\, N,\, \chi)_K)$ out of the formula of Proposition (2.23) (1) by using these above facts. That is the assertion (2).

Now we shall deal with the general cases.

By using Proposition (1.28) (3) and the definition (3.5), the assertion (1) for the general cases is easily deduced from those for the case of $m_1 = 1$.

Next we consider the assertion (2). It follows from (1.20) (3) and (3.3) that for each $n \in \mathbf{Z}_+$ prime to N,

$$\mathrm{tr}(\tilde{T}(n^2)\,;\, \mathfrak{S}^{\emptyset,\kappa}(k+1/2,\, N,\, \chi)_K) = \mathrm{tr}(\tilde{T}(n^2)\,;\, \mathfrak{S}^{\emptyset,\kappa'}(k+1/2,\, N,\, \chi^{2+})_K),$$

where $\kappa' = \kappa \cdot \left(\frac{m_1}{\cdot}\right)$.

We already proved that the latter trace is expressed with the formula in the assertion (2). In that expression, the range P of the sums does not depend on κ' and χ^{2+}. Therefore, for the proof, it is enough to show that all coefficients $\Xi((\alpha_l),$ $\Psi,\, (\tau,\, \sigma))$, determined by the formula (2.22), for κ' and χ^{2+} coincide with those for κ and χ.

We easily check it in case by case, by observing the fact: For any $p \in \Pi$,

$$\chi_{2M_1}(p) = (\chi^1\chi^{2+})_{2M_1}(p) = \chi^1(p)(\chi^{2+})_{2M_1}(p) = \left(\frac{m_1}{p}\right)(\chi^{2+})_{2M_1}(p).$$

Thus we completed the proof of Theorem (3.10). □

Proof of Theorem (3.11). For any $p \in \Pi$ and any $f = \sum_{n\geq 1} a(n)\mathbf{e}(nz) \in \mathfrak{S}^{\emptyset,\chi}(k+1/2, N, \chi)_K$, we see that $f \mid R_p^2 = f$ and hence $a(n) = 0$ if $p \mid n$. Therefore $f \mid U(p^2) = 0$ for any prime p dividing M_{2+}. Thus the operators $\tilde{T}(p^2) = \tilde{T}_{k+1/2,N,\chi}(p^2)$ (p : prime, $p \nmid M$) and $U(p^2)$ (p : prime, $p \mid M$) fix the space $\mathfrak{S}^{\emptyset,\chi} = \mathfrak{S}^{\emptyset,\chi}(k+1/2, N, \chi)_K$ and commute with each other (cf. (3.8-10)). Moreover all these operators are hermitian on $\mathfrak{S}^{\emptyset,\chi}$. Form these facts, $\mathfrak{S}^{\emptyset,\chi}$ has an orthogonal $\mathbf{C}$-basis consisting of common eigenforms for all operators $\tilde{T}(p^2)\,(p \nmid M)$ and $U(p^2)\,(p \mid M)$.

Then from Theorem (3.10), the strong multiplicity one theorem of integral weight ([M, Theorem 4.6.19]), and the following fact: The subspace $S^{*(\tau,\sigma)}(2k, M_1 \Pi_{l\in\Pi} l^{\alpha_l}) \mid R_\Psi$ in the right-hand side of (3.10) (2) are orthogonal with each other (cf. Proposition (A.8)), the uniqueness property (stated in (1)) of such common eigenforms and also the assertion (2) follow.

Let f be any element of such a basis consisting of common eigenforms. Set its eigenvalues as follows: $f \mid \tilde{T}(p^2) = \lambda_p f$ (p: prime, $p \nmid M$) resp. $f \mid U(p^2) = \lambda_p f$ (p: prime, $p \mid M$). We know that $\lambda_p = 0$ if $p \mid M_{2+}$ and also $\lambda_p = \pm p^{k-1}$ if $p \mid M_1$ (cf. Theorem (3.9)). Moreover the system of eigenvalues $\{\lambda_p\,;\, p \nmid N\}$ corresponds to a certain primitive form F of weight $2k$, character trivial, and conductor $M_1 M'$ $(0 < M' \mid M_{2+})$. This follows from Theorem (3.10).

Now we shall prove that λ_2 and λ_p (any $p \mid M_1$) also become the eigenvalues of F with respect to Hecke operators.

First, we claim the following: There exists a fundamental discriminant D such that $\varepsilon(-1)^k D = |D| > 0$ and $a(|D|) \neq 0$. Here we put $\varepsilon = \chi_2(-1)$ for simplicity.

This claim follows from [Sh 1, Theorem 1.7], Proposition (3.8) (1), and the definition of Kohnen space.

We take such a fundamental discriminant D. Then a formal computation as like [Sh 1, p.452] shows that

$$(3.22)\quad \sum_{n\geq 1} a(|D|n^2)n^{-s}$$
$$= a(|D|)\prod_p \left(1 - \chi_M(p)\left(\frac{D}{p}\right)p^{k-1-s}\right)\prod_p \left(1 - \lambda_p p^{-s} + \left(\frac{p}{M}\right)^2 p^{2k-1-2s}\right)^{-1},$$

where $\chi_M \cdot \left(\frac{D}{\ }\right)$ is considered as a character modulo MD.

Put

$$t := \begin{cases} \varepsilon(-1)^k D, & \text{if } D \equiv 1 \pmod 4\ ; \\ \varepsilon(-1)^k D/4, & \text{if } D \equiv 0 \pmod 4. \end{cases}$$

Then t is a squarefree positive integer.

We also define a cusp form $S_t(f) := \sum_{n\geq 1} A_t(n)\mathbf{e}(nz)$ by:

$$\sum_{n\geq 1} A_t(n)n^{-s} = \sum_{n\geq 1} a(tn^2)n^{-s}\cdot L\left(s-k+1,\ \chi\cdot\left(\frac{(-1)^k t}{}\right)\right), \tag{3.23}$$

where $\chi\cdot\left(\frac{(-1)^k t}{}\right)$ is considered as a character modulo $4Mt$.

Then $S_t(f) \in S(2k, 2M)$ (cf. [C, Theorem 4.3] if $k \geq 2$ and [C, Corollary 4.10] if $k = 1$). Now we divide into two cases.

Case 1. We suppose that $D \equiv 1 \pmod 4$. Then

(3.24)

$$\sum_{n\geq 1} A_t(n)n^{-s} = a(t)\prod_p\left(1-\lambda_p p^{-s}+\left(\frac{p}{M}\right)^2 p^{2k-1-2s}\right)^{-1}(1-c_{k,\chi}(2)2^{k-1-s}),$$

where $c_{k,\chi}(2) = \chi_M(2)\left(\frac{D}{2}\right)$.

Hence $S_t(f) \in S(2k, 2M)$ is a common eigenform of all operators $T(p)$ (p: prime, $p \nmid 2M$) and $U(p)$ (p: prime, $p \mid M$) and its eigenvalue is λ_p for each odd prime p.

The primitive form $F \in S^0(2k, M_1M')$ has the same eigenvalues λ_p with respect to all operators $T(p)\,(p \nmid 2M)$. From this and the standard theory of newforms of integral weight, we have the following expression:

$$S_t(f) = \sum_e{}' \alpha_e F(ez),\ \alpha_e \in \mathbf{C}. \tag{3.25}$$

Here, $\sum_e'$ is the sum extended over all positive divisors e of $2M_{2+}/M'$.

For simplicity, we put $F(z) = \sum_{n\geq 1} c(n)\mathbf{e}(nz)$ and for each prime p, μ_p is the eigenvalue of F with respect to $T(p)\,(p \nmid M_1M')$ resp. $U(p)\,(p \mid M_1M')$.

From (3.25), we have $A_t(n) = \sum_e' \alpha_e c(n/e)$ for all $n \in \mathbf{Z}_+$. Hence from (3.24),

$$a(t)\prod_p\left(1-\lambda_p p^{-s}+\left(\frac{p}{M}\right)^2 p^{2k-1-2s}\right)^{-1}(1-c_{k,\chi}(2)2^{k-1-s}) \tag{3.26}$$
$$= \left(\sum_e{}' \alpha_e e^{-s}\right)\prod_p(1-\mu_p p^{-s}+1_{M_1M'}(p)p^{2k-1-2s})^{-1}.$$

Here $1_{M_1M'}$ is the trivial character modulo M_1M'.

We can easily deduce from this identity that $\lambda_2 = \mu_2$ and $\lambda_p = \mu_p$ for any prime divisor p of M_1.

Case 2. We suppose that $D \equiv 0 \pmod 4$. Then $A_t(n) = 0$ for each odd integer $n \in \mathbf{Z}_+$ because of the definition of Kohnen space. On the other hand, we get for each $n \in \mathbf{Z}_+$ that $A_t(2n) = \sum_{0<d|n} a(|D|n^2/d^2)\chi_M(d)\left(\frac{D}{d}\right)d^{k-1}$, because $\chi\cdot\left(\frac{(-1)^k t}{}\right) = \chi_M\cdot\left(\frac{D}{}\right)$ as characters modulo MD.

Therefore

$$a(|D|)\prod_p\left(1-\lambda_p p^{-s}+\left(\frac{p}{M}\right)^2 p^{2k-1-2s}\right)^{-1} = \sum_{n\geq 1} A_t(2n)n^{-s} = 2^s\sum_{n\geq 1} A_t(n)n^{-s}.$$

By the same method as in the case 1, we can deduce from this identity that λ_2 and λ_p (any $p\mid M_1$) also become the eigenvalues of F with respect to Hecke operators in this case.

Thus we completed the proof of Theorem (3.11). □

§4. A more elaborate decomposition

We keep to the notations in §1-§3. We shall also consider only Kohnen spaces in this section.

As we proved in Theorems (3.10) and (3.11), $\mathfrak{S}^{\emptyset,\chi}(k+1/2, N, \chi)_K$ may have an eigenform corresponding to a primitive form $F \in S^0(2k, M_1M')$ ($M' \mid M_{2+}$ and $M' < M_{2+}$). Then, under a certain assumption (cf. (4.1)), we shall define a subspace of $\mathfrak{S}^{\emptyset,\chi}(k+1/2, N, \chi)_K$ which corresponds to only primitive forms of conductor $M = M_1M_{2+}$, and give a more elaborate decomposition of Kohnen spaces. The author believes that there exists a similar decomposition without any assumptions. But we cannot prove it yet. We shall discuss this topic at the end of this section.

Now we assume the following in this section.

ASSUMPTION (4.1). The conductor of χ divides $4M_1$, i.e., $\chi = \chi^1 = \left(\frac{m_1}{}\right)$ (cf. (3.1) for the notations).

We prepare some notations. For any $\kappa \in \{\pm 1\}^{\Pi}$ and any $l \in \Pi$, we define operators by $e_l^\kappa := \frac{1}{2}(R_l^2 + \kappa(l)R_l) \in \mathscr{R}_{\mathbf{C}}$ and $e_\Pi^\kappa := \Pi_{l\in\Pi} e_l^\kappa$.

Let $(\alpha_l)_{l\in\Pi}$ be a system of integers such that $0 \leq \alpha_l \leq \nu_l := \mathrm{ord}_l(N)$ for all $l \in \Pi$. For simplicity, put $M_{2+}^{(\alpha)} := \Pi_{l\in\Pi} l^{\alpha_l}$, $\tilde{N}(\alpha) = 4M_1M_{2+}^{(\alpha)}$, $D = \Pi(\alpha)_0$, $E = \Pi(\alpha)_1$, and $F = \Pi(\alpha)_{2+}$ (cf. §2(IV) for the definitions of $\Pi(\alpha)_i$ and $\Pi(\alpha)_{i+}$ $(0 \leq i \in \mathbf{Z})$). We note that every positive divisor of M_{2+} is of the form $M_{2+}^{(\alpha)}$ for some $(\alpha_l)_{l\in\Pi}$.

We choose and fix $\kappa \in \{\pm 1\}^\Pi$ until the end of this section. Then $\mathfrak{S}^{\emptyset,\kappa|F} = \mathfrak{S}^{\emptyset,\kappa|F}(k+1/2, \tilde{N}(\alpha), \chi)_K$ is well-defined. Here $\kappa|_F \in \{\pm 1\}^F$ is the restriction of κ to F. Put

$$B = B^{(\alpha)} := \bigoplus_{0<a|l_D} \mathfrak{S}^{\emptyset,\kappa|F}(k+1/2, \tilde{N}(\alpha), \chi)_K \mid U(a^2), \tag{4.2}$$

where $l_D := \Pi_{l\in D}\, l$ (cf. the beginning of §1).

From Theorem (3.10), we know

$$B \subseteq \begin{cases} S^{\emptyset,\kappa|F}(k+1/2, \tilde{N}(\alpha)\cdot l_D, \chi)_K & \text{if } k \geq 2, \\ V^{\emptyset,\kappa|F}(\tilde{N}(\alpha)\cdot l_D, \chi)_K & \text{if } k = 1, \end{cases} \tag{4.3}$$

and also have that $\mathfrak{S}^{\emptyset,\kappa|F}$ has a $\mathbf{C}$-basis $\mathscr{B}$ consisting of common eigenforms for all operators $\tilde{T}(n^2) = \tilde{T}_{k+1/2,\tilde{N}(\alpha),\chi}(n^2)$ $(n \in \mathbf{Z}_+, (n, \tilde{N}(\alpha)) = 1)$ and $U(p^2)$ $(p \in E)$ (cf. (3.11)). Hence we can write as follows:

$$B = \bigoplus_{f\in\mathscr{B}} B_f, \quad B_f := \bigoplus_{0<a|l_D} \mathbf{C}f \mid U(a^2). \tag{4.4}$$

We shall consider the hermitian involution $\mathbf{w}_p$ on $S(k+1/2, \tilde{N}(\alpha)\cdot l_D, \chi)$ for each $p \in D+E$ (cf. (3.6)). Then we can claim the following: For each $f \in \mathscr{B}$, the space B_f can be decomposed into common eigen subspaces with respect to $\mathbf{w}_p$ $(p \in D+E)$;

$$B_f = \bigoplus_\rho B_f(\rho),\ B_f(\rho) := \{g \in B_f\, ;\, g \mid \mathbf{w}_p = \rho(p)g \text{ for all } p \in D+E\}, \tag{4.5}$$

where ρ runs over $\{\pm 1\}^{D+E}$ in the direct sum.

Moreover we have $\dim_{\mathbf{C}} B_f(\rho) \leq 1$ for each $f \in \mathscr{B}$ and $\rho \in \{\pm 1\}^{D+E}$, and also have an equivalence relation:

$$\dim_{\mathbf{C}} B_f(\rho) = 1 \Leftrightarrow f|\mathbf{w}_p = \rho(p) f \text{ for all } p \in E. \tag{4.6}$$

Proof of the claim. We can prove the following fact in the same way as the proof of Theorem (3.10):

$$B_f = \bigoplus_{\rho' \in \{\pm 1\}^D} B'_f(\rho'),\ B'_f(\rho') := \{g \in B_f\,;\, g \mid \mathbf{w}_p = \rho'(p) g \text{ for all } p \in D\}$$

(cf. the claim after (3.15)).

Next we consider any prime p in E. Since f is a common eigenform of $U(p^2)$ $(p \in E)$, we get $f \mid \mathbf{w}_p = \pm f$ from Theorem (3.9).

Each operator $U(q^2)\,(q \in D)$ commutes with any Y_p $(p \in E)$ on the space $S(k + 1/2, \tilde{N}(\alpha) \cdot l_D, \chi)$ (cf. (1.24) (1)). Hence we have for any $p \in E$ and a $(0 < a \mid l_D)$,

$$f \mid U(a^2)\mathbf{w}_p = p^{-1/2}\left(\frac{-1}{p}\right)^{k+1/2}\left(\frac{m_1}{p}\right) f \mid U(a^2) Y_p = f \mid \mathbf{w}_p U(a^2)$$

(cf. (3.7)). Therefore any element of B_f is a common eigenform for $\mathbf{w}_p$ $(p \in E)$ with the same eigenvalues as f. The claim follows from these results. □

From (4.3), we know that B is contained in $S(k + 1/2, N, \chi)_K$ (if $k \geq 2$) resp. $V(N\,;\chi)_K$ (if $k = 1$). Applying the operator e_Π^κ to the both sides, we see that

$$B \mid e_\Pi^\kappa \subseteq \begin{cases} S(k + 1/2, N, \chi)_K \mid e_\Pi^\kappa = S^{\emptyset,\kappa}(k + 1/2, N, \chi)_K, & \text{if } k \geq 2, \\ V(N\,;\chi)_K \mid e_\Pi^\kappa = V^{\emptyset,\kappa}(N, \chi)_K, & \text{if } k = 1. \end{cases}$$

Each generator $f \mid U(a^2) e_\Pi^\kappa (f \in \mathcal{B},\ 0 < a \mid l_D)$ of $B \mid e_\Pi^\kappa$ is a common eigenform for $\tilde{T}(n^2)\,((n, N) = 1)$ with the same eigenvalues as f. The system of eigenvalues of f corresponds to a primitive form in $S^0(2k, M_1 M')\,(0 < M' \mid M_{2+}^{(\alpha)})$ (cf. Theorem (3.10)). Then it follows that $B \mid e_\Pi^\kappa \subseteq \mathfrak{S}^{\emptyset,\kappa}(k + 1/2, N, \chi)_K$ from Theorem (3.10) and the strong multiplicity one theorem of weight $2k$ (cf. [M, Theorem 4.6.19]).

We shall closely study $B \mid e_\Pi^\kappa$. We fix $f \in \mathcal{B}$ and $\rho \in \{\pm 1\}^{D+E}$ for a while. Take any $g = \sum_{n \geq 1} b(n)\mathbf{e}(nz) \in B_f(\rho)$. Then from Proposition (1.29), we can see that

$$\text{(4.7)} \qquad b(n) = 0 \text{ if } \left(\frac{n}{q}\right) = -\left(\frac{(-1)^k m_1}{q}\right)\rho(q) \text{ for some } q \in D + E.$$

Suppose that there exists $q \in D + E$ such that $\kappa(q) = -\left(\dfrac{(-1)^k m_1}{q}\right)\rho(q)$.

Then for such a prime q, we get that $b(n) = 0$ if $\left(\dfrac{n}{q}\right) = \kappa(q)$. Hence

$$g \mid \prod_{q\in D+E} e_q^{\varkappa} = 2^{-\#(D+E)} \sum_{n\geq 1} b(n) \prod_{q\in D+E} \left\{\left(\frac{n}{q}\right)^2 + \kappa(q)\left(\frac{n}{q}\right)\right\} \mathbf{e}(nz) = 0.$$

Since $U(l^2)\,(l \in D)$ commutes with $R_q\ (q \in F)$ (cf. (1.20)), each operator $e_q^{\varkappa}\ (q \in F)$ trivially acts on B and so $g \mid e_{\Pi}^{\varkappa} = 0$.

Thus we proved that

$$B_f(\rho) \mid e_{\Pi}^{\varkappa} = \{0\} \quad \text{unless } \kappa = \left(\frac{(-1)^k m_1}{}\right)\cdot \rho \text{ on } D+E.$$

Moreover $B_f(\rho) \mid e_{\Pi}^{\varkappa}$ is contained in a common eigen subspace for $\tilde{T}(n^2)\,((n, N) = 1)$ with the same eigenvalues as f.

Put $\tilde{\kappa} :=$ the restriction of $\kappa\cdot\left(\frac{(-1)^k m_1}{}\right)$ to $D+E$ ($\in \{\pm 1\}^{D+E}$) and $(*)$ is the following condition for any element $h \in \mathscr{B}$:

$$(*) \qquad h \mid \mathbf{w}_p = \tilde{\kappa}(p) h \quad \text{for all } p \in E,$$

where $\mathbf{w}_p\ (p \in E)$ is the hermitian involution on $S(k+1/2, \tilde{N}(\alpha), \chi)$.

Remark. For $p \in E$, $\mathbf{w}_p$ on $S(k+1/2, \tilde{N}(\alpha), \chi)$ becomes the restriction of those on $S(k+1/2, \tilde{N}(\alpha)\cdot l_D, \chi)$ (see the definition of $\mathbf{w}_p$).

Then from the above, we have a decomposition

$$B \mid e_{\Pi}^{\varkappa} = \bigoplus_{f\in\mathscr{B}, f:(*)} B_f(\tilde{\kappa}) \mid e_{\Pi}^{\varkappa}.$$

Here f in the direct sum runs over all elements of $\mathscr{B}$ satisfying the condition $(*)$.

The operator $e_{\Pi}^{\varkappa}$ is injective on $B_f(\tilde{\kappa})$. In fact, let us assume $g \mid e_{\Pi}^{\varkappa} = 0$ for $g = \sum_{n\geq 1} b(n)\mathbf{e}(nz) \in B_f(\tilde{\kappa})$. Then $b(n) = 0$ for all $n \in \mathbf{Z}_+$ prime to $\prod_{q\in D+E} q$ (cf. (4.7)). From this and Lemma (3.17), we have $g = 0$.

Now we define a subspace of $\mathfrak{S}^{\emptyset,\varkappa|F}$ by:

$$\mathfrak{S}^{\emptyset,\varkappa|F}_{\tilde{\varkappa}|E} = \mathfrak{S}^{\emptyset,\varkappa|F}_{\tilde{\varkappa}|E}(k+1/2, \tilde{N}(\alpha), \chi)_K$$
$$:= \{g \in \mathfrak{S}^{\emptyset,\varkappa|F}(k+1/2, \tilde{N}(\alpha), \chi)_K\, ;\, g \mid \mathbf{w}_p = \tilde{\kappa}(p) g \quad \text{for all } p \in E\}.$$

This space is generated by the set $\{f \in \mathscr{B}\, ;\, f \text{ satisfies the condition } (*)\}$.

Hence we have:

(4.8) For any $n \in \mathbf{Z}_+$ prime to N,

$$\mathrm{tr}(\tilde{T}(n^2)\, ;\, B \mid e_{\Pi}^{\varkappa}) = \sum_{f\in\mathscr{B}, f:(*)} \mathrm{tr}(\tilde{T}(n^2)\, ;\, B_f(\tilde{\kappa})) = \sum_{f\in\mathscr{B}, f:(*)} \mathrm{tr}(\tilde{T}(n^2)\, ;\, \mathbf{C}f)$$
$$= \mathrm{tr}(\tilde{T}(n^2)\, ;\, \mathfrak{S}^{\emptyset,\varkappa|F}_{\tilde{\varkappa}|E}(k+1/2, \tilde{N}(\alpha), \chi)_K).$$

We gave the expression of $\mathrm{tr}(\tilde{T}(n^2)\,;\mathfrak{S}^{\emptyset,\chi|F})((n,\tilde{N}(\alpha))=1)$ in terms of primitive forms of weight $2k$ (cf. Theorem (3.10)). Let us find primitive forms of weight $2k$ which correspond to the subspace $\mathfrak{S}^{\emptyset,\chi|F}_{\tilde{\chi}|E}$.

Take any $f\in\mathscr{B}$ satisfying (∗). From Theorem (3.9), $f\mid U(p^2)=-p^{k-1}\tilde{\kappa}(p)f$ for all $p\in E$. Let g be the primitive form of weight $2k$ which corresponds to f (cf. (3.10-11)). So $g\mid U(p)=-p^{k-1}\tilde{\kappa}(p)g$ $(p\in E)$. By using [M, Corollary 4.6.18], we have $g\mid W(p)=\tilde{\kappa}(p)g$ for all $p\in E$. Hence $\mathfrak{S}^{\emptyset,\chi|F}_{\tilde{\chi}|E}$ corresponds to all such g's.

From the expression of $\mathrm{tr}(\tilde{T}(n^2)\,;\mathfrak{S}^{\emptyset,\chi|F})$ and Proposition (A.2) (3), we can deduce the following expression:

(4.9) For any $n\in\mathbf{Z}_+$ prime to $\tilde{N}(\alpha)$,

$$\mathrm{tr}(\tilde{T}(n^2)\,;\mathfrak{S}^{\emptyset,\chi|F}_{\tilde{\chi}|E}(k+1/2,\tilde{N}(\alpha),\chi)_K)$$
$$=\sum_{((\beta_l),\Psi',(\tau',\sigma'))\in P'}\ \prod_{q\in F}\Xi_q((\beta_l),\Psi',(\tau',\sigma'))$$
$$\times\ \mathrm{tr}\Big(T(n)\,;S^{*(\hat{\tau}',\sigma')}(2k,M_1l_E\prod_{l\in F}l^{\beta_l})\mid R_{\Psi'}\Big),$$

where the notations are as follows: $\sum_{((\beta),\Psi',(\tau',\sigma'))\in P'}$ is the sum extended over all elements of the following set:

$$P':=\left\{\begin{array}{l}((\beta_l),\Psi',(\tau',\sigma'))\,;(\beta_l)=(\beta_l)_{l\in F}\text{ is a system of integers}\\ \text{such that }0\le\beta_l\le\alpha_l\text{ for any }l\in F,\\ \Psi'\subseteq F(\beta)_0+F(\beta)_1,\ \tau'\in\mathrm{Map}(F,\{\pm1\}),\ \sigma'\in\mathrm{Map}(F(\beta)_{2+},\{\pm1\})\end{array}\right\}.$$

$F(\beta)_i=\{l\in F\mid\beta_l=i\}$ for $i=0,1$ and $F(\beta)_{2+}=\{l\in F\mid\beta_l\ge2\}$. $l_E=\prod_{l\in E}l$. Each $\Xi_q((\beta_l),\Psi',(\tau',\sigma'))$ is the constant determined by (2.22). $\hat{\tau}'\in\{\pm1\}^{\Pi}$ is the following extension of τ': $\hat{\tau}'(l)=\tau'(l)$, $\tilde{\kappa}(l)\prod_{q\in\Psi'}\left(\frac{l}{q}\right)$, or 1 according to $l\in F$, E, or D. Finally,

$$S^{*(\hat{\tau}',\sigma')}\Big(2k,M_1l_E\prod_{l\in F}l^{\beta_l}\Big):=\left\{\begin{array}{l}f\in S^*\Big(2k,M_1l_E\prod_{l\in F}l^{\beta_l}\Big)\,;\\ f\mid W_l=\hat{\tau}'(l)f\text{ for all }l\in\Pi,\\ f\mid R_lW_l=\sigma'(l)f\mid R_l\text{ for all }l\in F(\beta)_{2+}\end{array}\right\}.$$

We extend each β_l $(l\in F)$ to Π as follows: $\tilde{\beta}_l=\beta_l$, 1, or 0 according to $l\in F$, E, or D. Then we have $\Pi(\tilde{\beta})_{2+}=F(\beta)_{2+}$. Finally replacing $\hat{\tau}'$, σ' with τ, σ, and combining (4.8) with (4.9), we can obtain an expression of $\mathrm{tr}(\tilde{T}(n^2)\,;B\mid e^{\chi}_{\Pi})((n,N)=1)$ in terms of primitive forms of weight $2k$.

(4.10) For $n \in \mathbf{Z}_+$ prime to N,

$$\begin{aligned}&\operatorname{tr}(\tilde{T}(n^2)\,;\,B\,|e_{\Pi}^{\chi})\\&=\sum_{((\beta_l),\Psi',(\tau,\sigma))\in P''}\prod_{q\in D}\frac{1}{2}(1+\tau(q))\prod_{q\in E}\frac{1}{2}\left(1+\tau(q)\tilde{\kappa}(q)\prod_{l\in\Psi'}\left(\frac{q}{l}\right)\right)\\&\quad\times\prod_{q\in F}\Xi_q((\beta_l),\,\Psi',\,(\tau|_F,\,\sigma))\operatorname{tr}(T(n)\,;\,S^{*(\tau,\sigma)}(2k,\,M_1M_{2+}^{(\tilde{\beta})})\,|\,R_{\Psi'}),\end{aligned}$$

where $\sum_{((\beta_l),\Psi',(\tau,\sigma))\in P''}$ is the sum extended over all elements of the following set:

$$P'':=\left\{\begin{array}{l}((\beta_l),\,\Psi',\,(\tau,\,\sigma))\,;\,(\beta_l)=(\beta_l)_{l\in F}\text{ is a system of integers}\\ \text{such that }0\le\beta_l\le\alpha_l\text{ for any }l\in F,\ \Psi'\subseteq F(\beta)_0+F(\beta)_1,\\ \tau\in\operatorname{Map}(\Pi,\,\{\pm1\}),\ \sigma\in\operatorname{Map}(\Pi(\tilde{\beta})_{2+},\,\{\pm1\})\\ =\operatorname{Map}(F(\beta)_{2+},\,\{\pm1\})\end{array}\right\}.$$

$\tau|_F$ is the restriction of τ to F, $\Xi_q((\beta_l),\,\Psi',\,(\tau|_F,\,\sigma))$ $(q\in F)$ are the same as in (4.9) and $S^{*(\tau,\sigma)}(2k,\,M_1M_{2+}^{(\tilde{\beta})})$ are the same as in §2(III) (or Proposition (2.23)).

We claim that the coefficient of $\operatorname{tr}(T(n)\,;\,S^{*(\tau,\sigma)}(2k,\,M_1M_{2+}^{(\tilde{\beta})})\,|\,R_{\Psi'})$ in (4.10) equals to those in Theorem (3.10) (2) for any (β_l), Ψ', $(\tau,\,\sigma)$, i.e.,

$$\begin{aligned}(4.11)\quad&\prod_{q\in D}\frac{1}{2}(1+\tau(q))\prod_{q\in E}\frac{1}{2}\left(1+\tau(q)\tilde{\kappa}(q)\prod_{l\in\Psi'}\left(\frac{q}{l}\right)\right)\prod_{q\in F}\Xi_q((\beta_l),\,\Psi',\,(\tau|_F,\,\sigma))\\&=\Xi((\tilde{\beta}_l)_{l\in\Pi},\,\Psi',\,(\tau,\,\sigma)).\end{aligned}$$

Here, the right-hand side is the constant with respect to $\mathfrak{S}^{\emptyset,\chi}(k+1/2,\,N,\,\chi)_K$ determined by (2.22).

This claim is easily verified in case by case from the facts: $\chi=\chi_{2M_1}$, $\Pi^1=\emptyset$, and $\Pi^0=\Pi$.

Observing $\Pi(\tilde{\beta})_0=D+F(\beta)_0$ and $\Pi(\tilde{\beta})_1=E+F(\beta)_1$, we can see the following from (4.11).

(4.12) The expression (4.10) of $\operatorname{tr}(\tilde{T}(n^2)\,;\,B\,|\,e_{\Pi}^{\chi})$ $((n,\,N)=1)$ is a part of the expression (3.11) (2) of $\operatorname{tr}(\tilde{T}(n^2)\,;\,\mathfrak{S}^{\emptyset,\chi}(k+1/2,\,N,\,\chi)_K)$.

Now we can state a more elaborate decomposition.

Put

$$\hat{\mathfrak{N}}^{\emptyset,\chi}=\hat{\mathfrak{N}}^{\emptyset,\chi}(k+1/2,\,N,\,\chi)_K:=\sum_{(\alpha_l)\neq(\nu_l)}B^{(\alpha)}\,|\,e_{\Pi}^{\chi}.$$

Here, (α_l) in the sum runs over all systems of integers such that $0\le\alpha_l\le\nu_l\,(l\in\Pi)$ and $(\alpha_l)_{l\in\Pi}\neq(\nu_l)_{l\in\Pi}$. We denote by $\mathfrak{N}^{\emptyset,\chi}=\mathfrak{N}^{\emptyset,\chi}(k+1/2,\,N,\,\chi)_K$ the orthogonal complement of $\hat{\mathfrak{N}}^{\emptyset,\chi}(k+1/2,\,N,\,\chi)_K$ in $\mathfrak{S}^{\emptyset,\chi}(k+1/2,\,N,\,\chi)_K$.

Obviously, $\mathfrak{N}^{\emptyset,\chi}$ and $\hat{\mathfrak{N}}^{\emptyset,\chi}$ are stable under the action of $\tilde{T}(n^2)\,((n,\,N)=1)$.

THEOREM (4.13). *Let the notations be the same as above and let* $\kappa \in \{\pm 1\}^{\Pi}$. *Then, in particular, we suppose that* $\mathrm{ord}_2(N) = 2$. *Under the assumption* (4.1), *we have the following.*

(1) *For any* $n \in \mathbf{Z}_+$ *prime to* N,

$$\begin{aligned}&\mathrm{tr}(\tilde{T}_{k+1/2,N,\chi}(n^2)\,;\,\mathfrak{N}^{\emptyset,\kappa}(k+1/2,\,N,\,\chi)_K)\\ &\quad= \sum_{I+J+K=\Pi(\nu)_2}\ \sum_{\substack{\tau\in \mathrm{Map}(\Pi,\{\pm1\})\\ \sigma\in\mathrm{Map}(\Pi-(I+J),\{\pm1\})}} \Xi((\nu(I,J)_l),\,I+J,\,(\tau,\sigma))\\ &\qquad\times \mathrm{tr}\Big(T(n)\,;\,S^{*(\tau,\sigma)}\Big(2k,\,M_1\prod_{l\in J} l \prod_{l\in\Pi-(I+J)} l^{\nu_l}\Big)\,|\,R_{I+J}\Big),\end{aligned}$$

where $\Pi(\nu)_2 = \{l \in \Pi \mid \nu_l = 2\}$, $\sum_{I+J+K=\Pi(\nu)_2}$ *is the sum extended over all partitions* $\Pi(\nu)_2 = I + J + K$, $\nu(I,J)_l = 0,\,1$, *or* $\nu_l\ (:= \mathrm{ord}_l(N))$ *according to* $l \in I,\,J$, *or* $\Pi - (I+J)$, $\Xi((\nu(I,J)_l),\,I+J,\,(\tau,\sigma))$ *are the constants determined by* (2.22).

(2) *Let* $\mathscr{B}$ *be the orthogonal basis of* $\mathfrak{S}^{\emptyset,\kappa}(k+1/2,\,N,\,\chi)_K$ *which is stated in Theorem* (3.11) (1). *Let* $\mathscr{B}_0$ *(resp.* $\mathscr{B}_1$*) be the set of all* $f \in \mathscr{B}$ *which correspond to primitive forms* $\in S^0(2k, M)$ *(resp.* $\notin S^0(2k, M)$*) in the sense of Theorem* (3.11) (1). *Then* $\mathscr{B}_0$ *(resp.* $\mathscr{B}_1$*) generates the space* $\mathfrak{N}^{\emptyset,\kappa}(k+1/2,\,N,\,\chi)_K$ *(resp.* $\hat{\mathfrak{N}}^{\emptyset,\kappa}(k+1/2,\,N,\,\chi)_K$*).*

(3) *Let* f *be any element of* $\mathscr{B}_0$ *and* λ_p *the eigenvalue of* f *with respect to* $\tilde{T}_{k+1/2,N,\chi}(p^2)$ *(*p*: prime,* $p \nmid M$*) resp.* $U(p^2)$ *(*p*: prime,* $p \mid M$*). Then the primitive form* F, *which corresponds to* f *in the sense of Theorem* (3.11) (1), *satisfies* $F \mid T(p) = \lambda_p F$ *resp.* $F \mid U(p) = \lambda_p F$ *for all primes* p *with* $p \nmid M$ *resp.* $p \mid M$.

Proof. Let $\mathscr{B}$ be the same $\mathbf{C}$-basis of $\mathfrak{S}^{\emptyset,\kappa}(k+1/2,\,N,\,\chi)_K$ as in the above statement (2) and P the same set of parameters as in Theorem (3.10) (2). For any $(\alpha_l)_{l\in\Pi}$, $B^{(\alpha)} \mid e^{\kappa}_{\Pi}$ is stable under the action of all operators $\tilde{T}(n^2)\,((n, N) = 1)$. It follows from Theorem (3.11) (2) that $B^{(\alpha)} \mid e^{\kappa}_{\Pi}$ is genarated by the set $\mathscr{B} \cap (B^{(\alpha)} \mid e^{\kappa}_{\Pi})$. Similarly, we can see that $\mathfrak{N}^{\emptyset,\kappa}$, $\hat{\mathfrak{N}}^{\emptyset,\kappa}$ are generated by $\mathscr{B} \cap \mathfrak{N}^{\emptyset,\kappa}$, $\mathscr{B} \cap \hat{\mathfrak{N}}^{\emptyset,\kappa}$ respectively.

Let us find $\cup_{(\alpha_l)\neq(\nu_l)}\{\mathscr{B} \cap (B^{(\alpha)} \mid e^{\kappa}_{\Pi})\}$.

For any $((\rho_l),\,\Psi,\,(\tau,\sigma)) \in P$, we denote by $\mathscr{B}_{((\rho_l),\Psi,(\tau,\sigma))}$ the subset of $\mathscr{B}$ which corresponds to the space $S^{*(\tau,\sigma)}(2k,\,M_1\prod_{l\in\Pi} l^{\rho_l}) \mid R_{\Psi}$ in the sense of Theorem (3.11).

Then from (4.12), there exist only two possible cases: $\mathscr{B}_{((\rho_l),\Psi,(\tau,\sigma))} \subseteq B^{(\alpha)} \mid e^{\kappa}_{\Pi}$, or $\mathscr{B}_{((\rho_l),\Psi,(\tau,\sigma))} \cap (B^{(\alpha)} \mid e^{\kappa}_{\Pi}) = \emptyset$; and whether the former case occurs or not depends only on the parameter $((\rho_l),\,\Psi,\,(\tau,\sigma))$.

We define the subset of P^* of P by:

$$P^*:=\left\{\begin{array}{l}((\rho_l),\ \Psi,\ (\tau,\ \sigma))\in P\,;\ ((\rho_l),\ \Psi,\ (\tau,\ \sigma))\text{ does not satisfy}\\ \text{(at least) one of the following three conditions:}\\ \text{(i) }\rho_l=\nu_l\text{ on }\Pi(\rho)_{2+}\,;\ \text{(ii) }\ \Psi=\Pi(\rho)_0+\Pi(\rho)_1\,;\\ \text{(iii) }\nu_l=2\text{ on }\Pi(\rho)_0+\Pi(\rho)_1\end{array}\right\}.$$

Take a system $(\alpha_l)_{l\in\Pi}\neq(\nu_l)_{l\in\Pi}$ and a parameter $((\rho_l),\ \Psi,\ (\tau,\ \sigma))\in P$ such that $\mathscr{B}_{((\rho_l),\Psi,(\tau,\sigma))}\subseteq B^{(\alpha)}\mid e_\Pi^\chi$ for this (α_l).

Then $((\rho_l),\ \Psi,\ (\tau,\ \sigma))\in P^*$. In fact, suppose that $((\rho_l),\ \Psi,\ (\tau,\ \sigma))$ satisfies the above conditions (i) and (ii). We get from (4.10) that $\rho_l\leq\alpha_l$ for all $l\in\Pi$ and $\Psi\subseteq\Pi(\alpha)_{2+}$. Hence by using the condition (i), $(\rho_l=)\ \alpha_l=\nu_l$ for all $l\in\Pi(\rho)_{2+}$ and so $\Pi(\rho)_{2+}\subseteq\Pi(\alpha)_{2+}$. From the condition (ii), $\Pi(\alpha)_{2+}\supseteq\Psi=\Pi-\Pi(\rho)_{2+}\supseteq\Pi-\Pi(\alpha)_{2+}$. This means $\Pi=\Pi(\alpha)_{2+}$.

Since $(\alpha_l)\neq(\nu_l)$, there exists $l\in\Pi-\Pi(\rho)_{2+}$ $(\subseteq\Pi(\alpha)_{2+})$ such that $2\leq\alpha_l<\nu_l$. Hence the condition (iii) is not satisfied.

The contrary is also true. Take any $((\rho_l),\ \Psi,\ (\tau,\ \sigma))\in P^*$. Put $\alpha_l=\rho_l$, 2,1, or 0 according to $l\in\Pi(\rho)_{2+}$, Ψ, $\Pi(\rho)_1-\Psi$, or $\Pi(\rho)_0-\Psi$. Then $(\alpha_l)\neq(\nu_l)$. Put $\beta_l=\rho_l$, 1, or 0 according to $l\in\Pi(\rho)_{2+}$, $\Psi\cap\Pi(\rho)_1$, or $\Psi\cap\Pi(\rho)_0$.

Let P'' be the same set of parameters as in (4.10) and we define $(\tilde{\beta}_l)_{l\in\Pi}$ by the above (β_l) in the same manner as in (4.10). Then $((\beta_l)_{l\in\Pi(\alpha)_{2+}},\ \Psi,\ (\tau,\ \sigma))\in P''$ and $(\tilde{\beta}_l)=(\rho_l)$. Hence $\mathscr{B}_{((\rho_l),\Psi,(\tau,\sigma))}\subseteq B^{(\alpha)}\mid e_\Pi^\chi$.

Thus we get that $\hat{\mathfrak{N}}^{\emptyset,\chi}$ is generated by $\cup_{((\rho_l),\Psi,(\tau,\sigma))\in P^*}\mathscr{B}_{((\rho_l),\Psi,(\tau,\sigma))}$. Hence $\mathfrak{N}^{\emptyset,\chi}$ is generated by $\cup_{((\rho_l),\Psi,(\tau,\sigma))\in P-P^*}\mathscr{B}_{((\rho_l),\Psi,(\tau,\sigma))}$. The assertion (1) is easily deduced from this.

Next we have for any $((\rho_l),\ \Psi,\ (\tau,\ \sigma))\in P$,

$$S^{*(\tau,\sigma)}\Big(2k,\ M_1\prod_{l\in\Pi}l^{\rho_l}\Big)\mid R_\Psi\subseteq S^0\Big(2k,\ M_1\prod_{l\in\Pi-\Psi}l^{\rho_l}\prod_{l\in\Psi}l^2\Big).$$

Since

$$((\rho_l),\ \Psi,\ (\tau,\ \sigma))\in P^*\Leftrightarrow\prod_{l\in\Pi-\Psi}l^{\rho_l}\prod_{l\in\Psi}l^2\neq M_{2+},$$

we have the assertion (2).

The assertion (3) can be proved by the same method as in Theorem (3.11) (1).

□

We shall discuss on an extension of this theorem to general cases. The key point of this proof is the expression formulae (4.10-12) and they come from Proposition (1.29). We needed the assumption (4.1) in order to use this proposition.

Now we give an example for a speculation of general cases.

EXAMPLE (4.14). Let p be an odd prime and $\kappa \in \{\pm 1\}$. Then we have an isomorphism of restricted Hecke algebras (cf. Theorem (3.11) (1) and [U1 §3]):

$$\mathfrak{S}^{\emptyset,\chi}\left(k+1/2,\ 4p^2,\ \left(\frac{p}{\cdot}\right)\right)_\kappa \simeq A(2k,\ p^2) \oplus S^0(2k,\ p) \oplus S(2k,\ 1),$$

where $A(2k,\ p^2)$ is a certain Hecke-submodule of $S^0(2k,\ p^2)$. □

The following question for this example is very natural: Can the part of $S^0(2k,\ p) \oplus S(2k,\ 1)$ be constructed with $S\left(k+1/2,\ 4p,\ \left(\frac{p}{\cdot}\right)\right)_\kappa$ and twisting operators as the space like $B \mid e_\Pi^\chi$?

We can affirmatively answer this question by using the following proposition.

PROPOSITION (4.15). *Let p be an odd prime. If a non-zero element $f = \sum_{n\geq 1} a(n)\mathbf{e}(nz) \in S\left(k+1/2,\ 4p,\ \left(\frac{p}{\cdot}\right)\right)_\kappa$ is orthogonal to the space $\{g(pz)\ ;\ g \in S(k+1/2,\ 4)_\kappa\}$, there exists n_+ (resp. n_-) $\in \mathbf{Z}_+$ such that $\left(\frac{n_+}{p}\right) = 1$ and $a(n_+) \neq 0$ (resp. $\left(\frac{n_-}{p}\right) = -1$ and $a(n_-) \neq 0$).* □

We shall prove this proposition and more general results in the forthcoming paper [U5]. The method of the proofs is completely different from those of Proposition (1.29).

From the above example (4.14) and some numerical examples (cf. [U4]), it seems that, in every case, there exists an elaborate decomposition of $\mathfrak{S}^{\emptyset,\chi}(k+1/2,\ N,\ \chi)_\kappa$ like Theorem (4.13).

We hope to get such an elaborate decomposition by proving the generalization of Proposition (4.15).

Appendix 1

In this appendix, we collect several properties and notations on cusp forms of integral weight, which are used in the previous sections. We state almost results with no proof, because we can prove them by straightforward calculations and the standard theory of newforms of integral weight.

In this appendix, we keep to the notations of §0 (a) and (b) and we fix the following notations: $k,\ N \in \mathbf{Z}_+$. Moreover, we simply write $T(n) = T_{2k,N}(n)$ and $W(Q) = [W(Q)]_{2k}$, etc., if the subcripts are obvious and any confusion does not occur.

PROPOSITION (A.1). *Let A, B be finite sets consisting of prime numbers such that $A \cap B = \emptyset$ and also let a_p $(p \in A)$, b_q $(q \in B)$ be any non-negative integers. Then for a positive integer n prime to $\prod_{p \in A} p^{a_p} \prod_{q \in B} q^{b_q}$, we have the following identity.*

$$\operatorname{tr}\left([W_A]_{2k} T(n) \, ; \, S\left(2k, \prod_{p \in A} p^{a_p} \prod_{q \in B} q^{b_q}\right)\right)$$
$$= \sum_{\substack{(t_p)_{p \in A} \\ 0 \le t_p \le [a_p/2]}} \sum_{\substack{(u_q)_{q \in B} \\ 0 \le u_q \le b_q}} \prod_{q \in B} (b_q - u_q + 1)$$
$$\times \operatorname{tr}\left([W_A]_{2k} T(n) \, ; \, S^0\left(2k, \prod_{p \in A} p^{a_p - 2t_p} \prod_{q \in B} q^{u_q}\right)\right).$$

□

PROPOSITION (A.2) ([Sa], [Atkin-Li]). *Let χ be a primitive character modulo $\mathfrak{f}(\chi)$ such that $\chi^2 = 1$ and M a positive divisor of N such that $(M, N/M) = 1$. For any $f \in S(2k, N)$, we have the following.*

(1) *If n is a positive integer prime to $N\mathfrak{f}(\chi)$,*

$$f \mid T(n) R_\chi = \chi(n) f \mid R_\chi T(n).$$

(2) *If n is a positive integer prime to N,*

$$f \mid T(n) W(M) = f \mid W(M) T(n).$$

(3) *Suppose $(M, \mathfrak{f}(\chi)) = 1$. Then*

$$f \mid R_\chi W(M) = \chi(M) f \mid W(M) R_\chi.$$

(4) *Let M' be a positive divisor of N such that $(M', N/M') = 1$ and $(M, M') = 1$. Then*

$$f \mid W(M) W(M') = f \mid W(MM'), \quad f \mid W(M) W(M) = f.$$

□

PROPOSITION (A.3). *Let p be an odd prime and M a positive integer prime to p. For any $f \in S(2k, pM)$, we have*

$$f \mid R_{(\frac{\cdot}{p})} \in S(2k, p^2 M) \text{ and } f \mid R_{(\frac{\cdot}{p})} W(p^2) = \left(\frac{-1}{p}\right) f \mid R_{(\frac{\cdot}{p})}.$$

□

Proposition (A.4). *Let M be a positive divisor of N such that $(M, N/M) = 1$, n a positive integer prime to N, and ψ a primitive character modulo r such that r is odd, $r^2 \mid N$, and $\psi^2 = 1$. For any f, $g \in S(2k, N)$, we have the following.*

(1) $$\langle f \mid W(M), g\rangle = \langle f, g \mid W(M)\rangle.$$

(2) $$\langle f \mid T(n), g\rangle = \langle f, g \mid T(n)\rangle.$$

(3) $$\langle f \mid R_\psi, g\rangle = \langle f, g \mid R_\psi\rangle.$$

□

Proposition (A.5). *Let χ be a primitive character modulo r. If $f \mid R_\chi = 0$ for $f \in S^0(2k, N)$, then $f = 0$. In other words, a twisting operator R_χ induces a $\mathbf{C}$-linear isomorphism from $S^0(2k, N)$ onto $S^0(2k, N) \mid R_\chi$.*

Proof. By [M, Theorem 4.6.8]. □

Let Ω be a finite set consisting of odd prime numbers and $\tilde{\Omega} = \{2\} \cup \Omega$ and also let α_p $(p \in \tilde{\Omega})$ be non-negative integers. Then we shall define a subspace $S^*(2k, \Pi_{p\in\tilde{\Omega}} p^{\alpha_p})$ of $S^0(2k, \Pi_{p\in\tilde{\Omega}} p^{\alpha_p})$.

We put $\Omega_2 := \{p \in \Omega \mid \alpha_p = 2\}$. We simply write $R_A = \Pi_{p\in A} R_{\left(\frac{}{p}\right)}$ for any subset A of Ω_2. In particular, $R_\emptyset = 1$. For a partition $\Omega_2 = A + B + C$, we denote

$$\tilde{N} := \prod_{p\in\tilde{\Omega}} p^{\alpha_p} \text{ and } \tilde{N}(B, C) := \prod_{p\in\tilde{\Omega}-\Omega_2} p^{\alpha_p} \prod_{p\in A} p^2 \prod_{p\in B} p = \prod_{p\in\tilde{\Omega}-(B+C)} p^{\alpha_p} \prod_{p\in B} p.$$

Then we know for any partition $\Omega_2 = A + B + C$, $S^0(2k, \tilde{N}(B, C)) \mid R_{B+C} \subseteq S^0(2k, \tilde{N})$ (see [Atkin-Li, p.228, Theorem 4.1, and Corollary 4.1]). We take the sum of these subspaces $S^0(2k, \tilde{N}(B, C)) \mid R_{B+C}$ over all partitions $\Omega_2 = A + B + C$ such that $\Omega_2 \neq A$, and put

$$S^2(2k, \tilde{N}) := \sum_{\substack{\Omega_2=A+B+C\\ \Omega_2\neq A}} S^0(2k, \tilde{N}(B, C)) \mid R_{B+C}.$$

Then we define $S^*(2k, \tilde{N})$ by the orthogonal complement of the subspace $S^2(2k, \tilde{N})$ in $S^0(2k, \tilde{N})$ with respect to the Petersson inner product.

Proposition (A.6). *The notations being as above, the following assertions hold.*

(1) $$S^2(2k, \tilde{N}) = \sum_{\substack{\Omega_2=A+B+C\\ \Omega_2\neq A}} S^*(2k, \tilde{N}(B, C)) \mid R_{B+C}.$$

(2) $S^*(2k, \tilde{N})$ *is fixed by the operator* $T(n)$ *for all positive integer* n *such that* $(n, \tilde{N}) = 1$. *In particular,* $S^*(2k, \tilde{N})$ *is spanned by primitive forms of conductor* $\tilde{N}$.
(3) $S^*(2k, \tilde{N})$ *is also fixed by the operator* $W(p^{\alpha_p})$ *for all* $p \in \Omega$.

Proof (sketch). (1) We use an induction with respect to $\# \Omega_2$. (2), (3) Since $T(n)$ and $W(p^{\alpha_p})$ are self-adjoint operators, it is sufficient to show that $S^2(2k, \tilde{N})$ is fixed by those operators. We use Proposition (A.2) for $T(n)$ and $W(p^{\alpha_p})$ ($p \notin B + C$), and (A.3) for $W(p^{\alpha_p})$ ($p \in B + C$). □

PROPOSITION (A.7). *The notations are the same as above. Put* $\Omega_{2+} := \{p \in \Omega \mid \alpha_p \geq 2\}$ *and* $R_p := R_{\left(\frac{\cdot}{p}\right)}$ *for any* $p \in \Omega_{2+}$. *Then each* R_p ($p \in \Omega_{2+}$) *induces a* $\mathbf{C}$-*linear automorphism of* $S^*(2k, \tilde{N})$ *of order 2.*

Proof. It is sufficient to show the following two facts:
(1) $S^*(2k, \tilde{N}) \mid R_p \subseteq S^*(2k, \tilde{N})$ for all $p \in \Omega_{2+}$.
(2) $f \mid R_p R_p = f$ for all $f \in S^*(2k, \tilde{N})$ and all $p \in \Omega_{2+}$.
We prove these facts.

(1) We denote the subspace of old forms in $S(2k, \tilde{N})$ by $S^1(2k, \tilde{N})$. Then $S^*(2k, \tilde{N})$ is the orthogonal complement of $S^1(2k, \tilde{N}) \oplus S^2(2k, \tilde{N})$ in $S(2k, \tilde{N})$. Since R_p ($p \in \Omega_{2+}$) is a self-adjoint operator, it is sufficient for the proof to show that $S^1(2k, \tilde{N}) \oplus S^2(2k, \tilde{N})$ is stable under R_p.

(i) First we discuss the space of old forms $S^1(2k, \tilde{N})$. Take systems of integers $(\beta_q)_{q \in \tilde{\Omega}}$ and $(s_q)_{q \in \tilde{\Omega}}$ such that $0 \leq \beta_q \leq \alpha_q$ ($q \in \tilde{\Omega}$), $(\beta_q)_{q \in \tilde{\Omega}} \neq (\alpha_q)_{q \in \tilde{\Omega}}$, and $0 \leq s_q \leq \alpha_q - \beta_q$ ($q \in \tilde{\Omega}$).

For a positive integer m, put $\delta(m) = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$. It is easy to show that for any $f \in S^0(2k, \Pi_{q \in \tilde{\Omega}} q^{\beta_q})$,

$$f \mid \delta\Big(\prod_{q \in \tilde{\Omega}} q^{s_q}\Big) R_p = \begin{cases} \Pi_{q \in \tilde{\Omega}} \left(\frac{q}{p}\right)^{s_q} f \mid R_p \mid \delta(\Pi_{q \in \tilde{\Omega}} q^{s_q}), & \text{if } (p, \Pi_{q \in \tilde{\Omega}} q^{s_q}) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence it is sufficient to consider only the case of $(p, \Pi_{q \in \tilde{\Omega}} q^{s_q}) = 1$. In this case, the level of $f \mid R_p$ is at most $p^{\max(2, \beta_p)} \Pi_{p \neq q \in \tilde{\Omega}} q^{\beta_q}$. Therefore if this number is less than $\tilde{N}$, $f \mid R_p$ is an old form and so is $f \mid \delta(\Pi_{q \in \tilde{\Omega}} q^{s_q}) R_p$.

Next we see

$$\tilde{N} = p^{\max(2, \beta_p)} \prod_{p \neq q \in \tilde{\Omega}} q^{\beta_q} \Leftrightarrow \begin{cases} \alpha_q = \beta_q \text{ if } q \neq p, \\ \alpha_p = 2, \ \beta_p = 0, 1, \end{cases}$$

because $(\alpha_q)_{q \in \tilde{\Omega}} \neq (\beta_q)_{q \in \tilde{\Omega}}$ and $\alpha_p \geq 2$. Moreover in this case, we have $s_q = 0$ for all $q \in \tilde{\Omega}$ and then $f|\delta(\Pi_{q\in\tilde{\Omega}} q^{s_q})R_p = f|R_p \in S^2(2k, \tilde{N})$.

Combining these results, we have $S^1(2k, \tilde{N}) | R_p \subseteq S^1(2k, \tilde{N}) \oplus S^2(2k, \tilde{N})$.

(ii) We consider $S^2(2k, \tilde{N})$. We use an induction with respect to $\# \Omega_2$. If $\# \Omega_2 = 0$, $S^2(2k, \tilde{N}) = \{0\}$. In this case, the assertion is trivial. Next we discuss the case of $\# \Omega_2 = a > 0$. We put the following assumption of the induction: If $\# \Omega_2 \leq a-1$, $S^2(2k, \tilde{N}) | R_p \subseteq S^1(2k, \tilde{N}) \oplus S^2(2k, \tilde{N})$ for all $p \in \Omega_{2+}$.

Then combining this assumption and the previous part (i) of this proof, we get the following: If $\# \Omega_2 \leq a-1$, $S^*(2k, \tilde{N}) | R_p \subseteq S^*(2k, \tilde{N})$ for all $p \in \Omega_{2+}$.

By (1) of Proposition (A.6), it is sufficient to show that $S^*(2k, \tilde{N}(B, C)) | R_{B+C}R_p \subseteq S^1(2k, \tilde{N}) \oplus S^2(2k, \tilde{N})$ for any $p \in \Omega_{2+}$ and any partition of $\Omega_2 = A + B + C$ with $\Omega_2 \neq A$.

Suppose either $p \in A$ or $\alpha_p \geq 3$. In this case, we can use the assumption of the induction because of $\# A < \# \Omega_2 = a$. Hence $S^*(2k, \tilde{N}(B, C)) | R_{B+C}R_p = S^*(2k, \tilde{N}(B, C)) | R_p R_{B+C} \subseteq S^*(2k, \tilde{N}(B, C)) | R_{B+C} \subseteq S^2(2k, \tilde{N})$.

Next suppose $p \in B + C$. Then $\alpha_p = 2$ and $S^*(2k, \tilde{N}(B, C)) | R_{B+C} | R_p = S^*(2k, \tilde{N}(B, C)) | R_{(B+C)-\{p\}} | R_p R_p$. Take any $f = \sum_{n\geq 1} a(n)\mathbf{e}(nz) \in S^*(2k, \tilde{N}(B, C)) | R_{(B+C)-\{p\}} \subseteq S(2k, p^{\beta} \Pi_{p\neq q\in\tilde{\Omega}} q^{\alpha_q})$, where $\beta = 1$ or 0 according as $p \in B$ or C. Hence $f \in S^1(2k, \tilde{N})$. We put $g(z) = \sum_{n\geq 1} a(pn)\mathbf{e}(nz)$. Then $g(pz) = f(z) - f|R_pR_p(z) \in S(2k, \tilde{N})$. From [M, Theorem 4.6.4], $g \in S(2k, \tilde{N}/p)$ and $g(pz) \in S^1(2k, \tilde{N})$. Hence $f|R_pR_p(z) = f(z) - g(pz) \in S^1(2k, \tilde{N})$. Therefore $S^*(2k, \tilde{N}(B, C)) | R_{B+C} | R_p \subseteq S^1(2k, \tilde{N})$.

Combining these results, we have $S^2(2k, \tilde{N}) | R_p \subseteq S^1(2k, \tilde{N}) \oplus S^2(2k, \tilde{N})$ for any $p \in \Omega_{2+}$.

(2) Take any $p \in \Omega_{2+}$ and any $f = \sum_{n\geq 1} a(n)\mathbf{e}(nz) \in S^*(2k, \tilde{N})$. We put $g(z) := \sum_{n\geq 1} a(pn)\mathbf{e}(nz)$. Then $f|R_pR_p(z) = f(z) - g(pz)$. Since $g(pz) = f(z) - f|R_pR_p(z) \in S(2k, \tilde{N})$, we have $g \in S(2k, \tilde{N}/p)$ from [M, Theorem 4.6.4]. Hence $g(pz)$ belongs to the space of old forms of level $\tilde{N}$. On the other hand, $g(pz) = f(z) - f|R_pR_p(z) \in S^*(2k, \tilde{N}) \subseteq S^0(2k, \tilde{N})$ as we showed in the previous part of this proof. Therefore $f - f|R_pR_p = 0$. □

PROPOSITION (A.8). *The notations being as above, the following orthogonal direct sum decomposition holds.*

$$S^0(2k, \tilde{N}) = \bigoplus_{\Omega_2 = A+B+C} S^*(2k, \tilde{N}(B, C)) | R_{B+C}.$$

Here, $\bigoplus_{\Omega_2=A+B+C}$ *is the orthogonal direct sum extended over all partitions* $\Omega_2 = A + B + C$.

Proof. By the strong multiplicity one theorem [M, Theorem 4.6.19]. □

Appendix 2

In this appendix, we give the definition and properties of the trace operator.

Let $k, N, m \in \mathbf{Z}_+$ with $4 \mid N$ and χ an even character modulo N with $\chi^2 = 1$. We define the trace operator $Tr_N^{Nm} : S(k+1/2, Nm, \chi) \to S(k+1/2, N, \chi)$ by:

$$f \mid Tr_N^{Nm} := \mid \Gamma_0(N) : \Gamma_0(Nm) \mid^{-1} \sum_{\xi \in \Delta_0(Nm,\chi) \backslash \Delta_0(N,\chi)} f \mid \xi, \quad f \in S(k+1/2, Nm, \chi).$$

Then we have the following properties.

PROPOSITION (A.9). *The notations are the same as above,*

(1) *For any* $f \in S(k+1/2, Nm, \chi)$ *and* $g \in S(k+1/2, N, \chi)$, *we get*

$$\langle f, g \rangle = \langle f \mid Tr_N^{Nm}, g \rangle,$$

i. e., the adjoint operator of Tr_N^{Nm} *is the inclusion map*: $S(k+1/2, N, \chi) \to S(k+1/2, Nm, \chi)$.

(2) $$V(Nm\,;\chi) \mid Tr_N^{Nm} \subseteq V(N\,;\chi).$$

(3) *Suppose that* $\mathrm{ord}_2(N) = 2$ *and* m *is odd.*

$$S(k+1/2, Nm, \chi)_K \mid Tr_N^{Nm} \subseteq S(k+1/2, N, \chi)_K.$$

Proof. (1) follows from straightforward computation.

(2) We easily see $U(N\,;\chi) \subseteq U(Nm\,;\chi)$ from the definition. Hence from (1), $0 = \langle v, u \rangle = \langle v \mid Tr_N^{Nm}, u \rangle$ for any $v \in V(Nm\,;\chi)$ and $u \in U(N\,;\chi)$.

(3) Put $\xi = \left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \chi_2(-1)^{k+1/2}\mathbf{e}((2k+1)/8)\right)$, $\Delta = \Delta_0(N, \chi)$, and $\Delta' = \Delta_0(Nm, \chi)$. Then $S(k+1/2, Nm, \chi)_K$ (resp. $S(k+1/2, N, \chi)_K$) is the α-eigen subspace with respect to the hermitian operator $Q_{Nm} = [\Delta'\xi\Delta']$ (resp. $Q_N = [\Delta\xi\Delta]$), $\alpha = (-1)^{[(k+1)/2]}2\sqrt{2}\chi_2(-1)$ (cf. §0(d)). Moreover, we can choose the set $\{\xi\zeta_\nu^*\,;\nu \in \mathbf{Z}/4\mathbf{Z}\}$ as a common system of representatives for $\Delta \backslash \Delta\xi\Delta$ and $\Delta' \backslash \Delta'\xi\Delta'$, where $\zeta_\nu = \begin{pmatrix} 1 & 0 \\ Nm\nu & 1 \end{pmatrix}$.

For any $f \in S(k+1/2, Nm, \chi)_K$ and $g \in S(k+1/2, N, \chi)$,

$$\begin{aligned}\langle f \mid Tr_N^{Nm} Q_N, g \rangle &= \langle f \mid Tr_N^{Nm}, g \mid Q_N \rangle = \langle f, g \mid Q_N \rangle = \langle f, g \mid Q_{Nm} \rangle \\ &= \langle f \mid Q_{Nm}, g \rangle = \alpha\langle f, g \rangle = \alpha\langle f \mid Tr_N^{Nm}, g \rangle.\end{aligned}$$

Hence $(f \mid Tr_N^{Nm})\, Q_N = \alpha(f \mid Tr_N^{Nm})$ and so $f \mid Tr_N^{Nm} \in S(k+1/2, N, \chi)_K$. □

Let us prove the claim which are used in §3. For the letter N, we define the letters M_1, M_{2+}, and Π in the same manner as in §1.

PROPOSITION (A.10). *Notations are the same as above. Suppose that* $\mathrm{ord}_2(N) = 2$. *Let* $\kappa \in \{\pm 1\}^{\Pi}$ *and* p *a prime divisor of* M_1 *with* $\chi_p = 1$. *Then* χ *can be defined with modulo* N/p *and we have the following*:

$$S^{\emptyset,\kappa}(k+1/2, N, \chi)_K \mid Tr_{N/p}^{N} \subseteq S^{\emptyset,\kappa}(k+1/2, N/p, \chi)_K,$$
$$V^{\emptyset,\kappa}(N, \chi)_K \mid Tr_{N/p}^{N} \subseteq V^{\emptyset,\kappa}(N/p, \chi)_K.$$

Proof. For any $l \in \Pi, f \in S(k+1/2, N, \chi)$, and $g \in S(k+1/2, N/p, \chi)$,

$$\langle f \mid Tr_{N/p}^{N} R_l, g \rangle = \langle f \mid Tr_{N/p}^{N}, g \mid R_l \rangle = \langle f, g \mid R_l \rangle = \langle f \mid R_l, g \rangle = \langle f \mid R_l Tr_{N/p}^{N}, g \rangle,$$

(cf. [U2, Proposition (1.10)]).

Hence $f \mid Tr_{N/p}^{N} R_l = f \mid R_l Tr_{N/p}^{N}$. The assertions are deduced from this fact and Proposition (A.9). □

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