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A CHARACTERIZATION OF COMPLETE RIEMANNIAN MANIFOLDS MINIMALLY IMMERSED IN THE UNIT SPHERE*

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§1. Introduction

Let M^n be an n-dimensional Riemannian manifold minimally immersed in the unit sphere $S^{n+p}(1)$ of dimension $n+p$. When M^n is compact, Chern, do Carmo and Kobayashi [1] proved that if the square $\|h\|^2$ of length of the second fundamental form h in M^n is not more than $\frac{n}{2-1/p}$, then either M^n is totally geodesic, or M^n is the Veronese surface in $S^4(1)$ or M^n is the Clifford torus $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$ in $S^{n+1}(1)\,(0 < k < n)$.

In this paper, we generalize the results due to Chern, do Carmo and Kobayashi [1] to complete Riemannian manifolds.

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§2. Preliminaries

Let M^n be an n-dimensional Riemannian manifold which is minimally immersed in the unit sphere $S^{n+p}(1)$ of dimension $n+p$. Then the second fundamental form h of the immersion is given by $h(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$ and it satisfies $h(X, Y) = h(Y, X)$, where $\tilde{\nabla}$ and ∇ denote the covariant differentiations on $S^{n+p}(1)$ and M^n respectively, X and Y are vector fields on M^n. We choose a local field of orthonormal frames $e_1, \ldots, e_{n+p}$ in $S^{n+p}(1)$ such that, restricted to M^n, the vectors $e_1, \ldots, e_n$ are tangent to M^n. We use the following convention on the range of indices unless otherwised stated: $A, B, C, \cdots = 1, 2, \ldots, n+p$; $i, j, k, \cdots = 1, 2, 3, \ldots, n$; $\alpha, \beta, \cdots = n+1, \ldots, n+p$. We agree the

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repeated indices under a summation sign without indication are summed over the respective range. With respect to the frame field of $S^{n+p}(1)$ chosen above, let $\omega_1, \ldots, \omega_{n+p}$ be the dual frame. Then the structure equations of $S^{n+p}(1)$ are given by

$$d\omega_A = \sum \omega_{AB} \wedge \omega_B, \ \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = \sum \omega_{AC} \wedge \omega_{CB} - \omega_A \wedge \omega_B. \tag{2.2}$$

Restricting these forms to M^n, we have the structure equations of the immersion:

$$\omega_\alpha = 0, \tag{2.3}$$

$$\omega_{i\alpha} = \sum h^\alpha_{ij} \omega_j, \quad h_{ij} = h_{ji}, \tag{2.4}$$

$$d\omega_i = \sum \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.5}$$

$$d\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj} - \Omega_{ij}, \quad \Omega_{ij} = \frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l, \tag{2.6}$$

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}), \tag{2.7}$$

$$d\omega_{\alpha\beta} = \sum \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \Omega_{\alpha\beta}, \quad \Omega_{\alpha\beta} = \frac{1}{2} \sum R_{\alpha\beta ij} \omega_i \wedge \omega_j, \tag{2.8}$$

$$R_{\alpha\beta ij} = \sum (h^\alpha_{ik} h^\beta_{kj} - h^\alpha_{jk} h^\beta_{ki}). \tag{2.9}$$

Then, the second fundamental form h can be written as

$$h(e_i, e_j) = \sum h^\alpha_{ij} e_\alpha. \tag{2.10}$$

We denote the square of the length of h by $\| h \|^2$. Then $\| h \|^2$ is intrinsic and given by $\| h \|^2 = n(n-1) - R$, where R is the scalar curvature. If we define h^α_{ijk} by

$$\sum h^\alpha_{ijk} \omega_k = dh^\alpha_{ij} + \sum h^\alpha_{jk} \omega_{ki} + \sum h^\alpha_{ik} \omega_{kj} + \sum h^\beta_{ij} \omega_{\beta\alpha}, \tag{2.11}$$

then, from (2.2), (2.3) and (2.4), we have $h^\alpha_{ijk} = h^\alpha_{ikj}$.

In this paper, we denote the image of the immersion by M^n for simplicity.

LEMMA 1 (cf. [2]). *Let M^n be a Riemannian manifold minimally immersed in $S^{n+p}(1)$. Then for any unit vector v on M^n,*

$$\mathrm{Ric}(v, v) \geq \frac{n-1}{n} (n - \| h \|^2), \tag{2.12}$$

where $\mathrm{Ric}(v, v)$ denotes the Ricci curvature in the v direction.

LEMMA 2 (cf. [3]). *Let M^n be a complete Riemannian manifold with Ricci curvature bounded from below. Let f be a C^2-function bounded from above on M^n, then for all $\varepsilon > 0$, there exists a point x in M^n such that at x,*

$$f(x) > \sup f - \varepsilon, \tag{2.13}$$

$$\|\nabla f\| < \varepsilon, \tag{2.14}$$

$$\Delta f < \varepsilon. \tag{2.15}$$

§3. Main results

THEOREM 1. *Let M^n be an n-dimensional complete Riemannian manifold minimally immersed in the unit sphere $S^{n+p}(1)$ of dimension $n+p$. Then either M^n is totally geodesic and M^n is globally isometric to $S^n(1)$, or* $\inf R \leq n(n-1) - \dfrac{n}{2-1/p}$.

Proof. Following the computation in [1], we have

$$\frac{1}{2}\Delta\|h\|^2 = \sum (h^\alpha_{ijk})^2 - K_N - L_N + n\|h\|^2. \tag{3.1}$$

Because

$$\sum_{ij}\Big(\sum_k (h^\alpha_{ik}h^\beta_{kj} - h^\alpha_{jk}h^\beta_{ki})\Big)^2 \leq 2\sum_{ij}(h^\alpha_{ij})^2 \sum_{ij}(h^\beta_{ij})^2, \tag{3.2}$$

we get

$$\begin{aligned} K_N &= \sum \Big(\sum_k (h^\alpha_{ik}h^\beta_{kj} - h^\alpha_{jk}h^\beta_{ki})\Big)^2 \\ &\leq 2\sum_{\alpha\neq\beta}\sum_{ij}(h^\alpha_{ij})^2\sum_{ij}(h^\beta_{ij})^2 = 2\|h\|^4 - 2\sum\Big(\sum_{ij}(h^\alpha_{ij})^2\Big)^2. \end{aligned} \tag{3.3}$$

(3.1) and (3.3) imply

$$\frac{1}{2}\Delta\|h\|^2 \geq \|h\|^2\left[n - \left(2 - \frac{1}{p}\right)\|h\|^2\right]. \tag{3.4}$$

1) If $\inf R \leq n(n-1) - \dfrac{n}{2-1/p}$, then Theorem 1 is true.

2) If $\inf R > n(n-1) - \dfrac{n}{2-1/p}$, then $R > n(n-1) - \dfrac{n}{2-1/p}$. We have

$$\|h\|^2 = n(n-1) - R < \frac{n}{2-1/p}. \tag{3.5}$$

Hence, $\|h\|^2$ is bounded. According to Lemma 1, we know that the Ricci curvature of M^n is bounded from below. In fact, from (2.12) and (3.5), we have, for any unit vector v,

$$\mathrm{Ric}(v, v) \geq \frac{n-1}{n}(n - \|h\|^2) \geq (n-1)\left[1 - \frac{1}{2-1/p}\right].$$

We define $f = \|h\|^2$, $F = (f+a)^{1/2}$ (where $a > 0$ is any positive constant number). F is bounded because $\|h\|^2$ is bounded.

$$dF = \frac{1}{2}(f+a)^{-1/2}df,$$

$$\begin{aligned} \Delta F &= \frac{1}{2}\left[-\frac{1}{2}(f+a)^{-3/2}\|df\|^2 + (f+a)^{-1/2}\Delta f\right] \\ &= \frac{1}{2}[-2\|dF\|^2 + \Delta f](f+a)^{-1/2} = \frac{1}{2F}[-2\|dF\|^2 + \Delta f]. \end{aligned}$$

Hence, $F\Delta F = -\|dF\|^2 + \frac{1}{2}\Delta f$, namely,

$$\frac{1}{2}\Delta f = F\Delta F + \|dF\|^2. \tag{3.6}$$

Applying the Lemma 2 to F, we have for all $\varepsilon > 0$, there exists a point x in M^n such that at x,

$$\|dF(x)\| < \varepsilon, \tag{3.7}$$

$$\Delta F(x) < \varepsilon, \tag{3.8}$$

$$F(x) > \sup F - \varepsilon. \tag{3.9}$$

(3.6), (3.7) and (3.8) imply

$$\frac{1}{2}\Delta f < \varepsilon^2 + F\varepsilon = \varepsilon(\varepsilon + F) \quad (\text{by } F > 0). \tag{3.10}$$

We take a sequence $\{\varepsilon_m\}$ such that $\varepsilon_m \to 0$ $(m \to \infty)$ and for all m, there exists a point x_m in M^n such that (3.7), (3.8) and (3.9) hold good. Hence, $\varepsilon_m(\varepsilon_m + F(x_m)) \to 0$ $(m \to \infty)$ because F is bounded.

On the other hand, from (3.9),

$$F(x_m) > \sup F - \varepsilon_m.$$

Since F is bounded, $\{F(x_m)\}$ is a bounded sequence, and we get

$$F(x_m) \to F_0,$$

if necessary, we can choose subsequence. Hence,

$$F_0 \geq \sup F.$$

According to the definition of supremum, we have

$$F_0 = \sup F. \tag{3.11}$$

From the definition of F, we get

$$f(x_m) \to f_0 = \sup f \quad (\text{by } F_0 = \sup F). \tag{3.12}$$

From (3.4) and (3.10), we obtain

$$f[n - (2 - 1/p)f] \leq \frac{1}{2}\Delta f < \varepsilon^2 + \varepsilon F,$$

$$f(x_m)[n - (2 - 1/p)f(x_m)] < \varepsilon_m^2 + \varepsilon_m F(x_m) \leq \varepsilon_m^2 + \varepsilon_m F_0.$$

Let $m \to \infty$, we have $\varepsilon_m \to 0$, $f(x_m) \to f_0$. Hence,

$$f_0[n - (2 - 1/p)f_0] \leq 0.$$

1) If $f_0 = 0$, we have $f = \| h \|^2 = 0$. Hence M^n is totally geodesic, and we know that M^n is globally isometric to $S^n(1)$.

2) If $f_0 > 0$, we have

$$n - (2 - 1/p)f_0 \leq 0, \quad f_0 \geq \frac{n}{2 - 1/p},$$

that is, $\sup \| h \|^2 \geq \frac{n}{2 - 1/p}$. From (2.15),

$$\inf R \leq n(n-1) - \frac{n}{2 - 1/p}.$$

This completes the proof of Theorem 1.

THEOREM 2. *Let M^n be an n-dimensional complete Riemannian manifold minimally immersed in the unit sphere $S^{n+p}(1)$ of dimension $n + p$. If $n > 1$, $p > 1$, then either M^n is totally geodesic and M^n is globally isometric to $S^n(1)$, or M^n is the*

Veronese surface in $S^4(1)$ *or* $\inf R < n(n-1) - \frac{n}{2-1/p}$.

Proof. According to the proof of Theorem 1, we know

$$\| h \|^2 = 0 \quad \text{or} \quad \sup \| h \|^2 \geq \frac{n}{2-1/p}.$$

1) If $\| h \|^2 = 0$, then M^n is totally geodesic and M^n is globally isometric to $S^n(1)$ from Theorem 1.

2) If $\sup \| h \|^2 \geq \frac{n}{2-1/p}$, then we have

$$\inf R = n(n-1) - \sup \| h \|^2 \leq n(n-1) - \frac{n}{2-1/p}.$$

When $\inf R < n(n-1) - \frac{n}{2-1/p}$, we know that Theorem 2 holds.
When $\inf R = n(n-1) - \frac{n}{2-1/p}$, we have

$$\sup \| h \|^2 = \frac{n}{2-1/p}.$$

Hence,

$$\| h \|^2 \leq \frac{n}{2-1/p}.$$

According to Lemma 1, we get, for any unit vector v in M^n,

$$\begin{aligned} \mathrm{Ric}(v, v) &\geq \frac{n-1}{n}\left[n - \frac{n}{2-1/p}\right] \\ &\geq (n-1)\left[1 - \frac{1}{2-1/p}\right] > 0 \quad (\text{by } p > 1,\ n > 1). \end{aligned}$$

From Myers' Theorem, we know that M^n is compact. Main theorem, Corollary and theorem 3 in [1] yield $p = n = 2$ and M^n is the Veronese surface in $S^4(1)$. This completes the proof of Theorem 2.

THEOREM 3. *Let* M^n *be an* n*-dimensional connected complete Riemannian manifold immersed in the unit sphere* $S^{n+1}(1)$ *of dimension* $n+1$. *If there is a point* p *in* M^n *and a unit vector* v *such that* $\mathrm{Ric}(v, v)(p) = 0$, *then either* M^n *is totally geodesic and* M^n *is globally isometric to* $S^n(1)$, *or* M^n *is locally the Clifford torus* $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$ *in* $S^{n+1}(1)\,(0 < k < n)$, *or* $\inf R < n(n-2)$.

Proof. According to Theorem 1, we know that either M^n is totally geodesic and M^n is globally isometric to $S^n(1)$, or $\inf R \leq n(n-1) - n = n(n-2)$ (from $p = 1$).

1) If M^n is totally geodesic or $\inf R < n(n-2)$, then Theorem 3 is true.

2) If $\inf R = n(n-2)$, then $\sup \| h \|^2 = n$. Hence, $\| h \|^2 \leq n$. When $\| h \|^2$ get its maximum in M^n, that is, there is a point p in M^n such that $\| h(p) \|^2 = \sup \| h \|^2$, we have $\| h \|^2 = n$ from E. Hopf's Theorem. Theorem 2 of [1] implies that Theorem 3 is true. When $\| h \|^2 < n$, we will show that it is impossible. In fact, if $\| h \|^2 < n$, we have

$$\mathrm{Ric}(v, v) \geq (n-1)\left(1 - \frac{\| h \|^2}{n}\right) > 0.$$

This is a contradiction.

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