L. Paris
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THE DELIGNE COMPLEX OF A REAL ARRANGEMENT OF HYPERPLANES

LUIS PARIS

1. Introduction

Let V be a real vector space. An *arrangement of hyperplanes* in V is a finite family $\mathscr{A}$ of hyperplanes of V through the origin. We say that $\mathscr{A}$ is *essential* if $\cap_{H\in\mathscr{A}} H = \{0\}$.

Let $V_{\mathbf{C}} = \mathbf{C} \otimes V$ be the *complexification* of V. Every element z of $V_{\mathbf{C}}$ can be written in a unique way $z = x + iy$, where $x, y \in 1 \otimes V = V$. We say that x is the *real part* of z and that y is its *imaginary part*. For two subsets $X, Y \subseteq V$, we write

$$X + iY = \{(x + iy) \in V_{\mathbf{C}} \mid x \in X \text{ and } y \in Y\}.$$

Let H be a hyperplane of V. The *complexification* $H_{\mathbf{C}}$ of H is the hyperplane of $V_{\mathbf{C}}$ spanned by H ; $H_{\mathbf{C}} = H + iH$.

Let $\mathscr{A}$ be an arrangement of hyperplanes in a real vector space V. We set

$$M(\mathscr{A}) = V_{\mathbf{C}} - \left(\bigcup_{H\in\mathscr{A}} H_{\mathbf{C}}\right).$$

This space is an open and connected submanifold of $V_{\mathbf{C}}$. We say that $\mathscr{A}$ is a $K(\pi, 1)$ *arrangement* if $M(\mathscr{A})$ is a $K(\pi, 1)$ space.

The *lattice* of a real arrangement $\mathscr{A}$ of hyperplanes is the poset

$$\mathscr{L}(\mathscr{A}) = \left\{\bigcap_{H\in\mathscr{B}} H \mid \mathscr{B} \subseteq \mathscr{A}\right\}$$

ordered by the reverse inclusion. $V = \cap_{H\in\emptyset} H$ is the smallest element of $\mathscr{L}(\mathscr{A})$, and $\cap_{H\in\mathscr{A}} H$ is the greatest one. For $X \in \mathscr{L}(\mathscr{A})$, we set

$$\mathscr{A}_X = \{H \in \mathscr{A} \mid H \supseteq X\}.$$

Let $\mathscr{A}$ be a real and essential arrangement of hyperplanes. A *chamber* of $\mathscr{A}$ is a connected component of $V - \cup_{H\in\mathscr{A}} H$. We say that $\mathscr{A}$ is *simplicial* if every

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chamber of $\mathscr{A}$ is an open simplicial cone. In [De], for a simplicial arrangement $\mathscr{A}$ of hyperplanes, Deligne constructs a cover $q : \hat{M}(\mathscr{A}) \rightarrow M(\mathscr{A})$, defines a simplicial complex $\mathrm{Del}(\mathscr{A})$ from $\mathscr{A}$, and proves that $\mathrm{Del}(\mathscr{A})$ has the same homotopy type as $\hat{M}(\mathscr{A})$, and that $\mathrm{Del}(\mathscr{A})$ is contractible. In particular, $q : \hat{M}(\mathscr{A}) \rightarrow M(\mathscr{A})$ is the universal cover of $M(\mathscr{A})$, and $\mathscr{A}$ is a $K(\pi, 1)$ arrangement.

In [Pa1], the author generalizes Deligne's construction of the universal cover $q : \hat{M}(\mathscr{A}) \rightarrow M(\mathscr{A})$ of $M(\mathscr{A})$ to any real arrangement $\mathscr{A}$ of hyperplanes using a new combinatorial tool: the *oriented systems.*

Our goal in this paper is to generalize the defintion of the Deligne complex $\mathrm{Del}(\mathscr{A})$ to any real and essential arrangement $\mathscr{A}$ of hyperplanes (in the general case, $\mathrm{Del}(\mathscr{A})$ is a regular and normal CW-complex), and to prove the following result.

MAIN THEOREM. *Let $\mathscr{A}$ be a real and essential arrangement of hyperplanes. The Deligne complex* $\mathrm{Del}(\mathscr{A})$ *of $\mathscr{A}$ has the same homotopy type as the universal cover $\hat{M}(\mathscr{A})$ of $M(\mathscr{A})$ if and only if $\mathscr{A}_X$ is a $K(\pi, 1)$ arrangement for every $X \in \mathscr{L}(\mathscr{A})$ different from* $\{0\}$.

In particular, if $\mathscr{A}$ is an essential arrangement of hyperplanes in a real vector space of dimension ≤ 3, then $\mathrm{Del}(\mathscr{A})$ has the same homotopy type as the universal cover $\hat{M}(\mathscr{A})$ of $M(\mathscr{A})$ (it is well known that any arrangement of hyperplanes in a real vector space of dimension ≤ 2 is a $K(\pi, 1)$ arrangement).

Note that the study of the topology of $M(\mathscr{A})$, where $\mathscr{A}$ is an arbitrary real arrangement of hyperplanes, can be easily reduced to the case of an essential arrangement. Thus the hypothesis "$\mathscr{A}$ is essential" is not a restriction.

At the end of this section we will prove that: "if $\mathscr{A}$ is a $K(\pi, 1)$ arrangement, then $\mathscr{A}_X$ is also a $K(\pi, 1)$ arrangement for every $X \in \mathscr{L}(\mathscr{A})$" (Lemma 1.1). It follows that, if $\mathscr{A}$ is a $K(\pi, 1)$ arrangement, then $\mathrm{Del}(\mathscr{A})$ has the same homotopy type as the universal cover $\hat{M}(\mathscr{A})$ of $M(\mathscr{A})$, and, consequently, $\mathrm{Del}(\mathscr{A})$ is contractible. In view of these facts, our complex $\mathrm{Del}(\mathscr{A})$ can certainly be used to prove that a given real arrangement of hyperplanes is a $K(\pi, 1)$ arrangement.

We refer to [FR] for a good exposition on $K(\pi, 1)$ arrangements, and to [Or] and [OT] for good expositions on the theory of arrangements of hyperplanes.

Our work is organized as follows.

Section 2 is a summary of [Pa1]. Its aim is to introduce our main combinatorial tool, the *oriented systems,* and to give the construction of the universal cover

$q : \hat{M}(\mathscr{A}) \to M(\mathscr{A})$ of $M(\mathscr{A})$. Although this section is almost identical to Section 2 of [Pa2], for convenience we reproduce it here rather than referring the reader to the original paper.

In Section 3, we define the complex $\mathrm{Del}(\mathscr{A})$ and prove the Main Theorem.

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LEMMA 1.1. *Let $\mathscr{A}$ be a real arrangement of hyperplanes, and let $X \in \mathscr{L}(\mathscr{A})$.* If *$\mathscr{A}$ is a $K(\pi, 1)$ arrangement, then $\mathscr{A}_X$ is also a $K(\pi, 1)$ arrangement.*

Proof. Let $\iota^1 : M(\mathscr{A}) \to M(\mathscr{A}_X)$ be the inclusion map of $M(\mathscr{A})$ into $M(\mathscr{A}_X)$. We are going to prove that ι^1 admits a right homotopy inverse. This shows that $(\iota^1)_* : \pi_n(M(\mathscr{A})) \to \pi_n(M(\mathscr{A}_X))$ is a surjective morphism of groups for every $n \geq 0$, and thus that $M(\mathscr{A}_X)$ is a $K(\pi, 1)$ space if $M(\mathscr{A})$ is a $K(\pi, 1)$ space.

Pick a point $z \in \cap_{H \in \mathscr{A}_X} H_{\mathbf{C}}$ such that $z \notin H_{\mathbf{C}}$ for any $H \in \mathscr{A} - \mathscr{A}_X$. Choose a small disk $\mathbf{B}$ in $V_{\mathbf{C}}$ centered in z and which does not intersect any hyperplane $H_{\mathbf{C}}$ with $H \in \mathscr{A} - \mathscr{A}_X$. Set

$$W = \mathbf{B} - \Big(\bigcup_{H \in \mathscr{A}_X} H_{\mathbf{C}}\Big) = \mathbf{B} - \Big(\bigcup_{H \in \mathscr{A}} H_{\mathbf{C}}\Big),$$

and let $\iota^0 : W \to M(\mathscr{A})$ denote the inclusion map of W into $M(\mathscr{A})$. Then $\iota = \iota^1 \circ \iota^0 : W \to M(\mathscr{A}_X)$ is obviously a homotopy equivalence, thus ι^1 admits a right homotopy inverse. $\square$

Note that Lemma 1.1 can be easily generalized to complex arrangements of hyperplanes.

2. The universal cover of $M(\mathscr{A})$

This section is divided into three subsections. In the first one we introduce our main combinatorial tool: the *oriented systems.* In the second subsection we define the oriented system $(\Gamma(\mathscr{A}), \sim)$ associated with a real arrangement $\mathscr{A}$ of hyperplanes. In the third subsection, using the universal cover $\rho : (\hat{\Gamma}(\mathscr{A}), \sim) \to (\Gamma(\mathscr{A}), \sim)$ of the oriented system $(\Gamma(\mathscr{A}), \sim)$, we give the construction of the universal cover $q : \hat{M}(\mathscr{A}) \to M(\mathscr{A})$ of $M(\mathscr{A})$.

All results stated in this section are derived from [Pa1], so we will not give any proofs.

2. A. Oriented systems

An *oriented graph* Γ is the following data:

1) a set $V(\Gamma)$ of *vertices,*

2) a subset $A(\Gamma) \subseteq (V(\Gamma) \times V(\Gamma)) - \{(v, v) \mid v \in V(\Gamma)\}$ of *arrows.*

The *origin* of an arrow $a = (v, w)$ is v and its *end* is w. An oriented graph Γ is *locally finite* if every vertex $v \in V(\Gamma)$ is the origin or the end of only a finite number of arrows.

A *path* of an oriented graph Γ is an expression

$$f = a_1^{\varepsilon_1} a_2^{\varepsilon_2} \cdots a_n^{\varepsilon_n},$$

where $a_i \in A(\Gamma)$ and $\varepsilon_i \in \{\pm 1\}$ (for $i = 1, \dots, n$), such that there exists a sequence $v_0, v_1, \dots, v_n$ of vertices of Γ with:

$a_i = (v_{i-1}, v_i)$ if $\varepsilon_i = 1$ and

$a_i = (v_i, v_{i-1})$ if $\varepsilon_i = -1$.

We say that v_0 is the *origin* of f and that v_n is its *end*. The integer n is its *length* and $\sum_{i=1}^{n} \varepsilon_i$ is its *weight*. Every vertex of Γ is assumed to be a path of length 0 and of weight 0. For a path $f = a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n}$, we write $f^{-1} = a_n^{-\varepsilon_n} \cdots a_1^{-\varepsilon_1}$. For two paths $f = a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n}$ and $g = b_1^{\mu_1} \cdots b_m^{\mu_m}$ with $\mathrm{end}(f) = \mathrm{origin}(g)$, we write $fg = a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n} b_1^{\mu_1} \cdots b_m^{\mu_m}$.

An oriented graph Γ is *connected* if, for every pair (v, w) of vertices of Γ, there exists a path of Γ which begins at v and ends in w.

We always assume the oriented graphs to be locally finite and connected.

Let Γ be an oriented graph. An *identification* of Γ is an equivalence relation $\sim$ in the set of paths of Γ with the following properties:

1) $f \sim g \Rightarrow \mathrm{origin}(f) = \mathrm{origin}(g)$, $\mathrm{end}(f) = \mathrm{end}(g)$ and $\mathrm{weight}(f) = \mathrm{weight}(g)$,

2) $ff^{-1} \sim \mathrm{origin}(f)$, for every path f,

3) $f \sim g \Rightarrow f^{-1} \sim g^{-1}$,

4) $f \sim g \Rightarrow h_1 f h_2 \sim h_1 g h_2$, for suitable paths h_1 and h_2.

An *oriented system* is a pair $(\Gamma, \sim)$, where Γ is an oriented graph and $\sim$ is an identification of Γ.

Let $\rho : \Theta \to \Gamma$ be a morphism of oriented graphs. We say that ρ is a *cover* of Γ if, for every vertex v of Θ and every path f of Γ beginning at $\rho(v)$, there exists a unique path $\hat{f}$ of Θ such that $\text{origin}(\hat{f}) = v$ and $\rho(\hat{f}) = f$.

Let $\rho : (\Theta, \sim) \to (\Gamma, \sim)$ be a morphism of oriented systems (i.e. $\hat{f} \sim \hat{g} \Rightarrow \rho(\hat{f}) \sim \rho(\hat{g})$). We say that ρ is a *cover* of $(\Gamma, \sim)$ if it has the following two properties.

1) $\rho : \Theta \to \Gamma$ is a cover of Γ.

2) Let $v \in V(\Theta)$, let f and g be two paths of Γ which both begin at $\rho(v)$, and let $\hat{f}$ and $\hat{g}$ be the lifts of f and g respectively into Θ beginning at v. If $f \sim g (\Rightarrow \text{end}(f) = \text{end}(g))$, then $\hat{f} \sim \hat{g} (\Rightarrow \text{end}(\hat{f}) = \text{end}(\hat{g}))$.

PROPOSITION 2.1. *Let $(\Gamma, \sim)$ be an oriented system. There exists a unique cover $\pi : (\hat{\Gamma}, \sim) \to (\Gamma, \sim)$ of $(\Gamma, \sim)$ (up to isomorphism) which has the following universal property.*

If $\rho : (\Theta, \sim) \to (\Gamma, \sim)$ is a cover of $(\Gamma, \sim)$, then there exists a unique cover $\pi' : (\Gamma, \dot{\sim}) \to (\Theta, \sim)$ of $(\Theta, \sim)$ (up to isomorphism) such that $\pi = \rho \circ \pi'$.

We call $\pi : (\hat{\Gamma}, \sim) \to (\Gamma, \sim)$ the *universal cover* of $(\Gamma, \sim)$.

PROPOSITION 2.2. *Let $\pi : (\hat{\Gamma}, \sim) \to (\Gamma, \sim)$ be the universal cover of an oriented system $(\Gamma, \sim)$. Two paths $\hat{f}$ and $\hat{g}$ of $\hat{\Gamma}$ are identified by $\sim$ if and only if* $\text{origin}(\hat{f}) = \text{origin}(\hat{g})$ and $\text{end}(\hat{f}) = \text{end}(\hat{g})$.

2. B. Definition of $(\Gamma(\mathscr{A}), \sim)$

Let $\mathscr{A}$ be an arrangement of hyperplanes in a real vector space V. The hyperplanes of $\mathscr{A}$ subdivide V into *facets*. We denote by $\mathscr{F}(\mathscr{A})$ the set of all the facets. The *support* $|F|$ of a facet F is the vector space $|F| \in \mathscr{L}(\mathscr{A})$ spanned by F. Every facet is open in its support. We denote by $\bar{F}$ the closure of F in V. There is a partial order in $\mathscr{F}(\mathscr{A})$ defined by $F \leq G$ if $F \subseteq \bar{G}$.

A *chamber* of $\mathscr{A}$ is a facet of codimension 0. A *face* is a facet of codimension 1. Two chambers C and D are *adjacent* if they have a common face (i.e. a common facet of codimension 1).

Now, let us define the oriented system $(\Gamma(\mathscr{A}), \sim)$ associated with $\mathscr{A}$.

The vertices of $\Gamma(\mathscr{A})$ are the chambers of $\mathscr{A}$. An arrow of $\Gamma(\mathscr{A})$ is a pair (C, D), where C and D are adjacent chambers. Note that, in this oriented graph, if (C, D) is an arrow, then (D, C) is also an arrow.

A *positive path* of an oriented graph Δ is a path $f = a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n}$ with $\varepsilon_1 = \ldots = \varepsilon_n = 1$. This positive path is *minimal* if there is no positive path in Δ having the same origin as f, the same end as f, and a length smaller than the one of f.

The relation $\sim$ is the smallest identification of $\Gamma(\mathscr{A})$ such that:

if f and g are both positive minimal paths with the same origin and the same end , then $f \sim g$.

2. C. Universal cover of $M(\mathscr{A})$

Let $\mathscr{A}$ be an arrangement of hyperplanes in a real vector space V. We set

$$M(\mathscr{A}) = V_{\mathbf{C}} - \Big(\bigcup_{H \in \mathscr{A}} H_{\mathbf{C}}\Big).$$

Our goal in this subsection is to explain the construction of the universal cover $q : \hat{M}(\mathscr{A}) \to M(\mathscr{A})$ of $M(\mathscr{A})$.

Let C be a chamber of $\mathscr{A}$. For a facet $F \in \mathscr{F}(\mathscr{A})$, we denote by C_F the unique chamber of $\mathscr{A}_{|F|}$ containing C. We write

$$M(C) = \bigcup_{F \in \mathscr{F}(\mathscr{A})} (F + iC_F) \subseteq V + iV) = V_{\mathbf{C}}.$$

Note that this union is disjoint.

LEMMA 2.3. *The set* $\{M(C) \mid C \in V(\Gamma(\mathscr{A}))\}$ *is a covering of* $M(\mathscr{A})$ *by open subsets.*

Now, consider the universal cover $\rho : (\hat{\Gamma}(\mathscr{A}), \sim) \to (\Gamma(\mathscr{A}), \sim)$ of $(\Gamma(\mathscr{A}), \sim)$. For every vertex v of $\hat{\Gamma}(\mathscr{A})$, write

$$M(v) = M(\rho(v)).$$

Set

$$M'(\mathscr{A}) = \coprod_{v \in V(\hat{\Gamma}(\mathscr{A}))} M(v),$$

and let

$$q' : M'(\mathscr{A}) \to M(\mathscr{A})$$

be the natural projection.

It is easy to see that, if two chambers C and D are adjacent, then there is only one hyperplane $H \in \mathcal{A}$ which separates C and D; it is the support of their common face. For a chamber C of $\mathcal{A}$ and a hyperplane $H \in \mathcal{A}$, we denote by H_C^+ the open half-space of V bordered by H and containing C.

Let $\mathcal{R}$ be the smallest equivalence relation on $M'(\mathcal{A})$ such that:

if $a = (v, w) \in \mathcal{A}(\hat{\Gamma}(\mathcal{A}))$, $z \in M(v)$, $z' \in M(w)$, and

$$q'(z) = q'(z') \in M(v) \cap M(w) \cap (H_{\rho(w)}^+ + iV),$$

where H is the unique hyperplane of $\mathcal{A}$ which separates $\rho(v)$ and $\rho(w)$, then

$$z\mathcal{R}z'.$$

The space $\hat{M}(\mathcal{A})$ is the quotient

$$\hat{M}(\mathcal{A}) = M'(\mathcal{A}) / \mathcal{R},$$

and

$$q : \hat{M}(\mathcal{A}) \to M(\mathcal{A})$$

is the map induced by q'.

THEOREM 2.4. *The map $q : \hat{M}(\mathcal{A}) \to M(\mathcal{A})$ is the universal cover of $M(\mathcal{A})$.*

The following Lemmas 2.5, 2.6 and 2.7 are in [Pa1] preliminary results to the proof of Theorem 2.4; nevertheless, we state them since they will be used later in this paper.

Fix a vertex $v \in V(\hat{\Gamma}(\mathcal{A}))$. Write $C = \rho(v)$. For every chamber D of $\mathcal{A}$, we choose a positive minimal path f_D of $\Gamma(\mathcal{A})$ beginning at C and ending in D. We denote by $\hat{f}_D$ the lift of f_D into $\hat{\Gamma}(\mathcal{A})$ beginning at v. Note that the end of $\hat{f}_D$ does not depend on the choice of f_D (see the definition of the identification $\sim$ of $\Gamma(\mathcal{A})$). We set

$$\sum(v) = \{\mathrm{end}(\hat{f}_D) \mid D \in V(\Gamma(\mathcal{A}))\}.$$

The restriction of ρ to $\sum(v)$ is clearly a bijection $\sum(v) \to V(\Gamma(\mathcal{A}))$.

Let v and w be two vertices of $\hat{\Gamma}(\mathcal{A})$. We write

$$\bar{Z}(v, w) = \bigcup_u \bar{\rho}(u),$$

where the union is over all vertices $u \in \sum(v) \cap \sum(w)$ and, for $u \in \sum(v) \cap \sum(w)$, the set $\bar{\rho}(u)$ is the closure of $\rho(u)$ in V. We denote by $Z(v, w)$ the interior of $\bar{Z}(v, w)$. Note that $Z(v, w)$ is a union of facets of $\mathscr{A}$.

Consider the natural projection

$$p : M'(\mathscr{A}) = \coprod_{v \in V(\hat{\Gamma}(\mathscr{A}))} M(v) \to \hat{M}(\mathscr{A}).$$

For every $v \in V(\hat{\Gamma}(\mathscr{A}))$, we write $\hat{M}(v) = p(M(v))$. Since $q' : M'(\mathscr{A}) \to M(\mathscr{A})$ sends $M(v)$ homeomorphically onto $M(v)$, and $q' : q \circ p$, the map $q : \hat{M}(\mathscr{A}) \to M(\mathscr{A})$ sends $\hat{M}(v)$ homeomorphically onto $M(v)$. Moreover, since q is a cover, $\hat{M}(v)$ is an open subset of $\hat{M}(\mathscr{A})$.

LEMMA 2.5. *Let v and w be two vertices of $\hat{\Gamma}(\mathscr{A})$. The border of $Z(v, w)$ is contained in the union of the hyperplanes $H \in \mathscr{A}$ which separate $\rho(v)$ and $\rho(w)$.*

LEMMA 2.6. *Let v and w be two vertices of $\hat{\Gamma}(\mathscr{A})$. Then*

$$q(\hat{M}(v) \cap \hat{M}(w)) = M(v) \cap M(w) \cap (Z(v, w) + iV).$$

COROLLARY. *Let v, w be two vertices of $\hat{\Gamma}(\mathscr{A})$. If $\Sigma(v) \cap \Sigma(w) = \emptyset$, then $\hat{M}(v) \cap M(w) = \emptyset$.*

LEMMA 2.7. *For every chamber C of $\mathscr{A}$, we have*

$$q^{-1}(M(C)) = \bigcup_{v \in \rho^{-1}(C)} \hat{M}(v),$$

and this union is disjoint.

3. The Deligne complex of $\mathscr{A}$

Throughout this section, $\mathscr{A}$ is an essential arrangement of hyperplanes in a real vector space V of dimension l, the map $q : \hat{M}(\mathscr{A}) \to M(\mathscr{A})$ is the universal cover of $M(\mathscr{A})$, the pair $(\Gamma(\mathscr{A}), \sim)$ is the oriented system associated with $\mathscr{A}$, and $\rho : (\hat{\Gamma}(\mathscr{A}), \sim) \to (\Gamma(\mathscr{A}), \sim)$ is the universal cover of $(\Gamma(\mathscr{A}), \sim)$.

We provide V with an arbitrary scalar product. Let $\mathbf{S}^{l-1} = \{x \in V \mid \|x\| = 1\}$ be the unit sphere. The arrangement $\mathscr{A}$ determines a cellular decomposition of $\mathbf{S}^{l-1}$. With a facet F of $\mathscr{A}$ of dimension d corresponds the (closed) cell $\Delta_{d-1}(F) = \bar{F} \cap \mathbf{S}^{l-1}$ of dimension $(d-1)$, and every cell of this decomposition has that form.

For every vertex v of $\hat{\Gamma}(\mathscr{A})$, we write

$$\Delta'_{l-1}(v) = \Delta_{l-1}(\rho(v))$$

(recall that $\rho(v)$ is a chamber of $\mathscr{A}$, so is a facet of dimension l). We set

$$\mathrm{Del}'(\mathscr{A}) = \coprod_{v} \Delta'_{l-1}(v),$$

where the union is over all the vertices v of $\hat{\Gamma}(\mathscr{A})$, and let

$$\pi' : \mathrm{Del}'(\mathscr{A}) \to \mathbf{S}^{l-1}$$

be the natural projection, The space $\mathrm{Del}'(\mathscr{A})$ is a disjoint union of $(l-1)$-cells, and each cell $\Delta'_{l-1}(v)$ has a natural cellular decomposition given by the embedding $\Delta'_{l-1}(v) \hookrightarrow \mathbf{S}^{l-1}$. Thus $\mathrm{Del}'(\mathscr{A})$ can be viewed as a cellular complex, and π' as a cellular map.

Let $\mathscr{R}$ be the smallest equivalence relation on $\mathrm{Del}'(\mathscr{A})$ such that:

if $a = (v, w) \in A(\Gamma(\mathscr{A}))$, $\alpha \in \Delta'_{l-1}(v)$, $\beta \in \Delta'_{l-1}(w)$, and $\pi'(\alpha) = \pi'(\beta)$, then

$$\alpha \mathscr{R} \beta.$$

We denote by $\mathrm{Del}^o(\mathscr{A})$ the quotient

$$\mathrm{Del}^o(\mathscr{A}) = \mathrm{Del}(\mathscr{A}) / \mathscr{R},$$

by

$$\tau : \mathrm{Del}'(\mathscr{A}) \to \mathrm{Del}^o(\mathscr{A})$$

the natural projection, and by

$$\pi^o : \mathrm{Del}^o(\mathscr{A}) \to \mathbf{S}^{l-1}$$

the map induced by π'. In other words, The space $\mathrm{Del}^o(\mathscr{A})$ is obtained from $\mathrm{Del}'(\mathscr{A})$ as follows: for every arrow $a = (v, w)$ of $\hat{\Gamma}(\mathscr{A})$, we identify the $(l-2)$-cell $\Delta_{l-1}(F) \subset \Delta'_{l-1}(v)$ with the $(l-2)$-cell $\Delta_{l-2}(F) \subseteq \Delta'_{l-1}(w)$, where F is the face of $\mathscr{A}$ common to $\rho(v)$ and $\rho(w)$. Thus $\mathrm{Del}^o(\mathscr{A})$ has a natural cellular decomposition where the maps τ and π^o are cellular maps.

For every vertex v of $\hat{\Gamma}(\mathscr{A})$, we write $\Delta^o_{l-1}(v) = \tau(\Delta'_{l-1}(v))$.

For every vertex v of $\hat{\Gamma}(\mathscr{A})$, we write

$$\mathbf{S}^{l-1}(v) = \bigcup_{u \in \Sigma(v)} \Delta^o_{l-1}(u) \subseteq \mathrm{Del}^o(\mathscr{A})$$

(the definition of $\Sigma(v)$ is given in Subsection 3.C). The restriction of π^o to $\mathbf{S}^{l-1}(v)$ is obviously an isomorphism $\mathbf{S}^{l-1}(v) \to \mathbf{S}^{l-1}$ of cellular complexes.

The *Deligne complex* of $\mathscr{A}$ is the cellular complex $\mathrm{Del}(\mathscr{A})$ obtained from $\mathrm{Del}^o(\mathscr{A})$ by attaching a l-cell $\mathbf{B}^l(v)$ to $\mathrm{Del}^o(\mathscr{A})$ having $\mathbf{S}^{l-1}(v)$ as border, for every vertex v of $\hat{\Gamma}(\mathscr{A})$.

The complexes $\mathbf{S}^{l-1}$, $\mathrm{Del}^o(\mathscr{A})$ and $\mathrm{Del}(\mathscr{A})$ are clearly regular and normal CW-complexes.

MAIN THEOREM. *Let $\mathscr{A}$ be a real and essential arrangement of hyperplanes. The Deligne complex $\mathrm{Del}(\mathscr{A})$ of $\mathscr{A}$ has the same homotopy type as the universal cover $\hat{M}(\mathscr{A})$ of $M(\mathscr{A})$ if and only if $\mathscr{A}_X$ is a $K(\pi, 1)$ arrangement for every $X \in \mathscr{L}(\mathscr{A})$ different from $\{0\}$.*

COROLLARY 1. *Let $\mathscr{A}$ be an essential arrangement of hyperplanes in a real vector space V of dimension ≤ 3. Then $\mathrm{Del}(\mathscr{A})$ has the same homotopy type as the universal cover $\hat{M}(\mathscr{A})$ of $M(\mathscr{A})$.*

COROLLARY 2. *Let $\mathscr{A}$ be a real, essential, and $K(\pi, 1)$ arrangement of hyperplanes. Then $\mathrm{Del}(\mathscr{A})$ has the same homotopy type as the universal cover $\hat{M}(\mathscr{A})$ of $M(\mathscr{A})$. In particular, $\mathrm{Del}(\mathscr{A})$ is contractible.*

Let N be a regular and normal CW-complex. The cellular decomposition of N determines a simplicial decomposition of N called the *barycentric subdivision* of N (see [LW, Ch. III, Theorem 1.7]). For every cell Δ_d of N we fix a point $w(\Delta_d) \in (\Delta_d - \partial\Delta_d)$, where $\partial\Delta_d$ is the border of Δ_d (we assume $\partial\Delta_d = \emptyset$ if $\dim(\Delta_d) = 0$). A chain $\Delta_{d_0} \subset \Delta_{d_1} \subset \ldots \subset \Delta_{d_r}$ of cells of N determines a simplex $\Phi = \omega(\Delta_{d_0}) \vee \omega(\Delta_{d_1}) \vee \ldots \vee \omega(\Delta_{d_r})$ having $\omega(\Delta_{d_0}), \omega(\Delta_{d_1}), \ldots, \omega(\Delta_{d_r})$ as vertices and included in $(\Delta_{d_r} - \partial\Delta_{d_r})$, and every simplex of this simplicial decomposition has that form. All the simplexes are assumed to be open.

From now on, we assume $\mathbf{S}^{l-1}$, $\mathrm{Del}^o(\mathscr{A})$ and $\mathrm{Del}(\mathscr{A})$ to be provided with their respective barycentric subdivisions; moreover, we assume all the simplexes of $\mathbf{S}^{l-1}$ to be convex subsets of $\mathbf{S}^{l-1}$, the complex $\mathrm{Del}^o(\mathscr{A})$ to be a simplicial subcomplex of $\mathrm{Del}(\mathscr{A})$, and $\pi^o : \mathrm{Del}^o(\mathscr{A}) \to \mathbf{S}^{l-1}$ to be a simplicial map.

NOTATIONS. Let ϕ be a simplex of $\mathbf{S}^{l-1}$. Then, by the construction of the barycentric subdivision of $\mathbf{S}^{l-1}$, the simplex ϕ is contained in a unique facet of $\mathscr{A}$ which we denote by $F(\phi)$. We write $X(\phi) = |F(\phi)|$. Note that $X(\phi) \neq \{0\}$.

For a simplex Φ^o of $\mathrm{Del}^o(\mathscr{A})$, we write $F(\Phi^o) = F(\pi^o(\Phi^o))$ and $X(\Phi^o) = X(\pi^o(\Phi^o))$.

The proof of the Main Theorem is divided in 5 parts.

In Part 1, we give some preliminary results on the oriented system associated with $\mathscr{A}$.

In Part 2, to every simplex Φ of $\mathrm{Del}(\mathscr{A})$ we associate a nonempty open subset $U(\Phi)$ of $\hat{M}(\mathscr{A})$.

In Part 3, we prove the following assertions.

1) *Let* $\omega_0, \omega_1, \ldots, \omega_r$ *be* $(r+1)$ *vertices of* $\mathrm{Del}(\mathscr{A})$. *If* $\cap_{i=0}^{r} U(\omega_i) \neq \emptyset$, *then* $\omega_0, \omega_1, \ldots, \omega_r$ *are the vertices of a simplex* Φ *of* $\mathrm{Del}(\mathscr{A})$.

2) *Let* $\omega_0, \omega_1, \ldots, \omega_r$ *be the vertices of a simplex* Φ *of* $\mathrm{Del}(\mathscr{A})$. *Then* $\cap_{i=0}^{r} U(\omega_i) = U(\Phi)$.

3) *The set* $\mathscr{U} = \{U(\omega) \mid \omega$ *a vertex of* $\mathrm{Del}(\mathscr{A})\}$ *is a covering of* $\hat{M}(\mathscr{A})$.

Assertions 1), 2) and 3) show that $\mathscr{U} = \{U(\omega) \mid \omega$ a vertex of $\mathrm{Del}(\mathscr{A})\}$ is a covering of $\hat{M}(\mathscr{A})$ having $\mathrm{Del}(\mathscr{A})$ as nerve.

In Part 4, we prove the following assertions.

1) *Let* v *be a vertex of* $\hat{\Gamma}(\mathscr{A})$. *Then* $U(\omega(\mathbf{B}^l(v)))$ *is contractible.*

2) *Let* v *be a vertex of* $\hat{\Gamma}(\mathscr{A})$, *and let* Φ^o *be a simplex of* $\mathrm{Del}^o(\mathscr{A})$ *contained in* $\mathbf{S}^{l-1}(v)$. *Write* $\Phi = \Phi^o \vee \omega(\mathbf{B}^l(v))$. *Then* $U(\Phi)$ *is contractible.*

3) *Let* Φ^o *be a simplex of* $\mathrm{Del}^o(\mathscr{A})$. *Then* $U(\Phi^o)$ *has the same homotopy type as the universal cover* $\hat{M}(\mathscr{A}_{X(\Phi^o)})$ *of* $M(\mathscr{A}_{X(\Phi^o)})$.

In particular, if $\mathscr{A}_X$ is a $K(\pi, 1)$ arrangement for every $X \in \mathscr{L}(\mathscr{A})$ different from $\{0\}$, then $U(\Phi^o)$ is contractible for every simplex Φ^o of $\mathrm{Del}^o(\mathscr{A})$ (since $U(\Phi^o)$ has the same homotopy type as $\hat{M}(\mathscr{A}_{X(\Phi^o)})$ and $X(\Phi^o) \neq \{0\}$). This fact, Assertion 2) of Part 3, and Assertions 1) and 2) of Part 4 show that every nonempty intersection of elements of $\mathscr{U}$ is contractible, thus, by [We], $\mathrm{Del}(\mathscr{A})$ has the same homotopy type as $\hat{M}(\mathscr{A})$ (since $\mathscr{U}$ is a covering of $\hat{M}(\mathscr{A})$ having $\mathrm{Del}(\mathscr{A})$ as nerve).

In Part 5, we assume that there exists an $X \in \mathscr{L}(\mathscr{A})$ different from $\{0\}$ such that $\mathscr{A}_X$ is not a $K(\pi, 1)$ arrangement. Then we construct a new space $\hat{M}_\infty$ by attaching cells to $\hat{M}(\mathscr{A})$ such that:

a) $\mathrm{Del}(\mathscr{A})$ has the same homotopy type as $\hat{M}_\infty$,

b) there exists an integer $n_0 > 0$ such that $\pi_{n_0}(\hat{M}(\mathscr{A})) \neq \pi_{n_0}(\hat{M}_\infty)$.

Part 1.

Let Γ be an oriented graph, and let W be a subset of $V(\Gamma)$. The *oriented subgraph* of Γ *generated* by W is the oriented graph Θ having W as set of vertices and $\{(v, w) \in A(\Gamma) \mid v, w \in W\}$ as set of arrows.

For a facet F of $\mathscr{A}$, we denote by Γ_F the oriented subgraph of $\Gamma(\mathscr{A})$

generated by $\{C \in V(\Gamma(\mathscr{A})) \mid C$ has F as facet$\}$. For a simplex Φ^o of $\mathrm{Del}^o(\mathscr{A})$, we denote by $\hat{\Gamma}_{\Phi^o}$ the oriented subgraph of $\hat{\Gamma}(\mathscr{A})$ generated by $\{v \in V(\hat{\Gamma}(\mathscr{A})) \mid \Delta^o_{l-1}(v) \supseteq \Phi^o\}$.

A *gallery* of $\mathscr{A}$ is a sequence $(C_0, C_1, \dots, C_n)$ of chambers of $\mathscr{A}$ such that C_{i-1} and C_i are adjacent for $i = 1, \dots, n$ (here we assume $C_{i-1} \neq C_i$). Any positive path $f = a_1 \dots a_n$ of $\Gamma(\mathscr{A})$ can be viewed as the gallery $G = (C_0, C_1, \dots, C_n)$, where $C_i = \mathrm{end}(a_1, \dots, a_i)$ for $i = 0, 1, \dots, n$. In particular, if $f = a_1 \dots a_n$ is a positive minimal path of $\Gamma(\mathscr{A})$ then $G = (C_0, C_1, \dots, C_n)$ is a minimal gallery (i.e. a gallery of minimal length among the galleries of $\mathscr{A}$ from C_0 to C_n). From this perspective, the following lemma is a well known result.

LEMMA 3.1. *Let F be a facet of $\mathscr{A}$, let C and D be two chambers having F as facet, and let f be a positive minimal path of $\Gamma(\mathscr{A})$ beginning at C and ending in D. Then f is a path of Γ_F.*

LEMMA 3.2. *Let Φ^o be a simplex of* $\mathrm{Del}^o(\mathscr{A})$. *Then $\hat{\Gamma}_{\Phi^o}$ is a connected component of $\rho^{-1}(\Gamma_{F(\Phi^o)})$.*

Proof. Fix a vertex v_0 of $\hat{\Gamma}_{\Phi^o}$. Let Θ denote the connected component of $\rho^{-1}(\Gamma_{F(\Phi^o)})$ with $v_0 \in V(\Theta)$. Let us prove that $V(\Theta) = V(\hat{\Gamma}_{\Phi^o})$.

Let $w \in V(\hat{\Gamma}_{\Phi^o})$. Choose a point $\alpha^o \in \Phi^o$, and write $\alpha = \pi^o(\alpha^o)$. Since $\alpha^o \in \Delta^o_{l-1}(v_0) \cap \Delta^o_{l-1}(w)$, by definition of $\mathrm{Del}^o(\mathscr{A})$, there exists a path $f = a_1^{\varepsilon_1} \dots a_n^{\varepsilon_n}$ of $\hat{\Gamma}(\mathscr{A})$ beginning at v_0, ending in w, and such that $\alpha \in \Delta_{l-1}(\rho(v_i))$ for every $i = 0, 1, \dots, n$, where $v_i = \mathrm{end}(a_1^{\varepsilon_1} \dots a_i^{\varepsilon_i})$ for $i = 0, 1, \dots, n$. We have $\alpha \in \pi^o(\Phi^o) \cap \Delta_{l-1}(\rho(v_i)) \subseteq F(\Phi^o) \cap \bar{\rho}(v_i)$, where $\bar{\rho}(v_i)$ is the closure of $\rho(v_i)$ in V, thus $F(\Phi^o) \cap \bar{\rho}(v_i) \neq \emptyset$, and therefore $F(\Phi^o)$ is a facet of $\rho(v_i)$ for every $i = 0, 1, \dots, n$. This implies that $\rho(v_i) \in V(\Gamma_{F(\Phi^o)})$, thus $\rho(f)$ is a path of $\Gamma_{F(\Phi^o)}$, and therefore f is a path of Θ (since $\mathrm{origin}(f) = v_0 \in V(\Theta)$). It follows that $\mathrm{end}(f) = w \in V(\Theta)$.

Now, let $w \in V(\Theta)$. Choose a path $f = a_1^{\varepsilon_1} \dots a_n^{\varepsilon_n}$ of Θ beginning at v_0 and ending in w. Write $v_i = \mathrm{end}(a_1^{\varepsilon_1} \dots a_i^{\varepsilon_i})$ for $i = 0, 1, \dots, n$. We have $\pi^o(\Phi^o) \subseteq \Delta_{l-1}(\rho(v_i)) \cap \Delta_{l-1}(\rho(v_{i+1}))$ for $i = 0, 1, \dots, n-1$ (since $\rho(f)$ is a path of $\Gamma_{F(\Phi^o)}$), thus, by the definition of $\mathrm{Del}^o(\mathscr{A})$, we successively have $\Phi^o \subseteq \Delta^o_{l-1}(v_i)$ for $i = 0, 1, \dots, n$. In particular, $\Phi^o \subseteq \Delta^o_{l-1}(w)$, namely, $w \in V(\hat{\Gamma}_{\Phi^o})$. □

Part 2.

For a simplex ϕ of $\mathbf{S}^{l-1}$, we denote by $K(\phi)$ the cone over ϕ;

$$K(\phi) = \{\lambda x \mid \lambda > 0 \text{ and } x \in \phi\}.$$

Note that $K(\phi) \subseteq F(\phi)$ for every simplex ϕ of $\mathbf{S}^{l-1}$, and $\{K(\phi) \mid \phi$ a simplex of $\mathbf{S}^{l-1}\}$ is a partition of $V - \{0\}$.

Let S be a simplicial complex, and let ψ and ϕ be two simplexes of S. We set $\psi \geq \phi$ if $\bar{\psi} \supset \phi$, where $\bar{\psi}$ is the closure of ψ in S. The relation "$\geq$" is a partial order in the set of simplexes of S.

Recall that, for a chamber C of $\mathscr{A}$ and for a facet F, we denote by C_F the unique chamber of $\mathscr{A}_{|F|}$ containing C.

For a simplex ϕ of $\mathbf{S}^{l-1}$ and for a chamber C of $\mathscr{A}$, we write

$$R(\phi, C) = \bigcup_{\psi \geq \phi} (K(\psi) + iC_{F(\psi)}).$$

We have $R(\phi, C) \subseteq M(C)$.

LEMMA 3.3. *Let ϕ be a simplex of $\mathbf{S}^{l-1}$, and let C be a chamber of $\mathscr{A}$. Then $R(\phi, C)$ is an open subset of $M(\mathscr{A})$.*

Proof. Pick $z = (x + iy) \in R(\phi, C)$. Let ψ be the simplex of $\mathbf{S}^{l-1}$ such that $x \in K(\psi)$. Then we have $y \in C_{F(\psi)}$. If $\psi' \geq \psi$, then $F(\psi') \geq F(\psi)$, thus $C_{F(\psi')} \supseteq C_{F(\psi)}$. Furthermore, the subset $\cup_{\psi' \geq \psi} K(\psi')$ is an open cone. It follows that

$$T(z) = \Big(\bigcup_{\psi' \geq \psi} K(\psi')\Big) + iC_{F(\psi)}$$

is an open neighbourhood of z, and $T(z) \subseteq R(\phi, C)$. □

Recall that, for every chamber C of $\mathscr{A}$,

$$q^{-1}(M(C)) = \bigcup_{v \in \rho^{-1}(C)} \hat{M}(v),$$

this union is disjoint, and q sends $\hat{M}(v)$ homeomorphically onto $M(v) = M(C)$ for every $v \in \rho^{-1}(C)$ (see Lemma 2.7). For a simplex ϕ of $\mathbf{S}^{l-1}$ and for a vertex v of $\hat{\Gamma}(\mathscr{A})$, we denote by $\hat{R}(\phi, v)$ the lift of $R(\phi, \rho(v))$ into $M(v)$. By Lemma 3.3, $\hat{R}(\phi, v)$ is an open subset of $\hat{\Gamma}(\mathscr{A})$.

Now, let us define $U(\Phi)$, where Φ is a simplex of $\mathrm{Del}(\mathscr{A})$.

If Φ is a simplex of $\mathrm{Del}^o(\mathscr{A})$, then

$$U(\Phi) = \bigcup_{v} \hat{R}(\pi^{o}(\Phi), v),$$

where the union is over all the vertices of $\hat{\Gamma}_{\Phi}$.

Assume that $\Phi = \omega(\mathbf{B}^{l}(v))$, where v is a vertex of $\hat{\Gamma}(\mathscr{A})$. Write $C = \rho(v)$. The set $U(\Phi) = U(\omega(\mathbf{B}^{l}(v)))$ is the lift of $(V + iC) \subseteq M(C)$ into $\hat{M}(v)$.

Assume that Φ has the form $\Phi = \Phi^{o} \vee \omega(\mathbf{B}^{l}(v))$, where v is a vertex of $\hat{\Gamma}(\mathscr{A})$ and Φ^{o} is a simplex of $\mathrm{Del}^{o}(\mathscr{A})$ contained in $\mathbf{S}^{l-1}(v)$. Write $\phi = \pi^{o}(\Phi^{o})$ and $C = \rho(v)$. Then $U(\Phi)$ is the lift of

$$\Big(\bigcup_{\psi \geq \phi} K(\psi)\Big) + iC \subseteq M(C)$$

into $\hat{M}(v)$.

Part 3.

LEMMA 3.4. i) *Let* $\omega_0, \omega_1, \ldots, \omega_r$ *be* $(r+1)$ *vertices of* $\mathrm{Del}^{o}(\mathscr{A})$. *If* $\bigcap_{i=0}^{r} U(\omega_i) \neq \emptyset$, *then* $\omega_0, \omega_1, \ldots, \omega_r$ *are the vertices of a simplex* Φ^{o} *of* $\mathrm{Del}^{o}(\mathscr{A})$.

ii) *Let* $\omega_0, \omega_1, \ldots, \omega_r$ *be the vertices of a simplex* Φ^{o} *of* $\mathrm{Del}^{o}(\mathscr{A})$. *Then* $\bigcap_{i=0}^{r} U(\omega_i) = U(\Phi^{o})$.

Proof. i) Let $\omega_0, \omega_1, \ldots, \omega_r$ be $(r+1)$ vertices of $\mathrm{Del}^{o}(\mathscr{A})$ such that $\bigcap_{i=0}^{r} U(\omega_i) \neq \emptyset$. Write $x_i = \pi^{o}(\omega_i)$ for $i = 0, 1, \ldots, r$. Pick $e \in \bigcap_{i=0}^{r} U(\omega_i)$. Write $z = (x + iy) = q(e)$. For every $i = 0, 1, \ldots, r$, we choose a vertex v_i of $\hat{\Gamma}_{\omega_i}$ such that $e \in \hat{R}(x_i, v_i)$, and we write $A_i = \rho(v_i)$.

Let ψ be the simplex of $\mathbf{S}^{l-1}$ such that $x \in K(\psi)$. By the definition of $R(x_i, A_i)$, we have $\psi \geq x_i$ for $i = 0, 1, \ldots, r$, thus $x_0, x_1, \ldots, x_r$ are vertices of ψ.

By the definition of $R(x_i, A_i)$, we have $y \in (A_i)_{F(\psi)}$ for every $i = 0, 1, \ldots, r$, thus $\bigcap_{i=0}^{r} (A_i)_{F(\psi)} \neq \emptyset$, therefore $(A_0)_{F(\psi)} = (A_1)_{F(\psi)} = \ldots = (A_r)_{F(\psi)}$. Let C be the chamber of $\mathscr{A}$ having $F(\psi)$ as facet and such that $C_{F(\psi)} = (A_0)_{F(\psi)} = \ldots = (A_r)_{F(\psi)}$.

Let $i \in \{0, 1, \ldots, r\}$. The facet $F(x_i)$ of $\mathscr{A}$ is common to A_i and C (since $F(\psi) \geq F(x_i)$). We fix a positive minimal path f_i of $\Gamma(\mathscr{A})$ beginning at A_i and ending in C. By Lemma 3.1, f_i is a path of $\Gamma_{F(x_i)}$. We denote by $\hat{f}_i$ the lift of f_i into $\hat{\Gamma}(\mathscr{A})$ beginning at v_i. By Lemma 3.2, $\hat{f}_i$ is a path of $\hat{\Gamma}_{\omega_i}$.

Write $w = \mathrm{end}(\hat{f}_0)$. First, let us prove that $w = \mathrm{end}(\hat{f}_i)$ for every $i = 1, \ldots, r$. By Lemma 2.6, we have $z \in R(x_0, v_0) \cap R(x_i, v_i) \cap (Z(v_0, v_i) + iV)$, therefore $x \in Z(v_0, v_i)$. Furthermore, $x \in F(\psi)$ and $Z(v_0, v_i)$ is a union of facets of $\mathscr{A}$, thus $F(\psi) \subseteq Z(v_0, v_i)$. Finally $F(\psi) \subseteq \bar{C}$ and $Z(v_0, v_1)$ is an open subset of V, therefore $C \subseteq Z(v_0, v_i)$. Thus, by the construction of $Z(v_0, v_i)$, there exists a vertex $u_i \in \sum(v_0) \cap \sum(v_i)$ such that $\rho(u_i) = C$. This can happen only if $u_i = \text{end}(\hat{f}_0) = \text{end}(\hat{f}_i)$.

Now, consider the simplex Ψ^o of $\text{Del}^o(\mathscr{A})$ such that $\Psi^o \subseteq \Delta^o_{l-1}(w)$ and $\pi^o(\Psi^o) = \psi$. Let us show that ω_i is a vertex of Ψ^o for every $i = 0, 1, \ldots, r$. Recall that $\hat{f}_i$ is a path of $\hat{\Gamma}_{\omega_i}$, thus $\text{end}(\hat{f}_i) = \omega \in V(\hat{\Gamma}_{\omega_i})$, therefore $\omega_i \in \Delta^o_{l=1}(w)$. It follows that ω_i is the unique vertex of $\Psi^o \subseteq \Delta^o_{l-1}(w)$ such that $\pi^o(\omega_i) = x_i$.

ii) Let $\omega_0, \omega_1, \ldots, \omega_r$ be the vertices of a simplex Φ^o of $\text{Del}^o(\mathscr{A})$. Write $x_i = \pi^o(\omega_i)$ for $i = 0, 1, \ldots, r$, and $\phi = \pi^o(\Phi^o)$.

Let $e \in \cup_{i=0}^{r} U(\omega_i)$. Write $z = (x + iy) = q(e)$. For every $i = 0, 1, \ldots, r$, we choose a vertex v_i of $\hat{\Gamma}_{\omega_i}$ such that $e \in \hat{R}(x_i, v_i)$, and we write $A_i = \rho(v_i)$. Let w be the vertex of $\hat{\Gamma}(\mathscr{A})$ defined in the proof of i). Let us prove that $w \in V(\hat{\Gamma}_{\Phi^o})$ and $e \in \hat{R}(\phi, w)$. This shows that $e \in U(\Phi^o)$.

Consider the simplex Ψ^o defined in the proof of i), and write $\psi = \pi^o(\Psi^o)$. The simplex ψ is the (unique) simplex of $\mathbf{S}^{l-1}$ such that $x \in K(\psi)$. Since $\omega_0, \omega_1, \ldots, \omega_r$ are vertices of Ψ^o, we have $\Psi^o \geq \Phi^o$, thus $V(\hat{\Gamma}_{\Psi^o}) \subseteq V(\hat{\Gamma}_{\Phi^o})$, therefore $w \in V(\hat{\Gamma}_{\Phi^o})$ (since $w \in V(\hat{\Gamma}_{\Psi^o})$).

In order to prove that $e \in \hat{R}(\phi, w)$, by Lemma 2.6, it suffices to show that

$$z \in R(x_0, A_0) \cap R(\phi, C) \cap (Z(v_0, w) + iV),$$

where $A_0 = \rho(v_0)$ and $C = \rho(w)$. By the starting hypothesis, we have $z \in R(x_0, A_0)$. The inequality $\psi \geq \phi$ and the inclusions $x \in K(\psi)$ and $y \in C_{F(\psi)} = (A_0)_{F(\psi)}$ imply $z \in R(\phi, C)$. Now, $C \subseteq Z(v_0, w)$ (since $w \in \sum(v_0) \cap \sum(w)$) and $F(\psi) \subseteq \bar{C}$, thus $F(\psi) \subseteq \bar{Z}(v_0, w)$. Since $(A_0)_{F(\psi)} = C_{F(\psi)}$, no hyperplane of $\mathscr{A}$ which separates A_0 and C contains $F(\psi)$, thus, by Lemma 2.5, $x \in F(\psi) \subseteq Z(v_0, w)$. It follows that $z = (x + iy) \in (Z(v_0, w) + iV)$.

Now, let $e \in U(\Phi^o)$. We choose a vertex v of $\hat{\Gamma}_{\Phi^o}$ such that $e \in \hat{R}(\phi, v)$. Then we have $v \in V(\hat{\Gamma}_{\omega_i})$ and $\hat{R}(\phi, v) \subseteq \hat{R}(x_i, v)$ for every $i = 0, 1, \ldots, r$, thus $e \in \cap_{i=0}^{r} U(\omega_i)$. □

LEMMA 3.5. i) *Let v and w be two vertices of $\hat{\Gamma}(\mathscr{A})$. If $v \neq w$, then $U(\omega(\mathbf{B}^l(v))) \cap U(\omega(\mathbf{B}^l(w))) = \emptyset$.*

ii) *Let Φ^o be a simplex of* $\text{Del}^o(\mathscr{A})$, *and let v be a vertex of $\hat{\Gamma}(\mathscr{A})$. If $U(\Phi^o) \cap U(\omega(\mathbf{B}^l(v))) \neq \emptyset$, then $\Phi^o \subseteq \mathbf{S}^{l-1}(v)$.*

iii) *Let v be a vertex of $\hat{\Gamma}(\mathscr{A})$, and let Φ^o be a simplex of* $\text{Del}^o(\mathscr{A})$ *such that $\Phi^o \subseteq \mathbf{S}^{l-1}(v)$. Write $\Phi = \Phi^o \vee \omega(\mathbf{B}^l(v))$. Then $U(\Phi^o) \cap U(\omega(\mathbf{B}^l(v))) = U(\Phi)$.*

Proof. i) Let v and w be two vertices of $\hat{\Gamma}(\mathscr{A})$. Assume $U(\omega(\mathbf{B}^l(v))) \cap U(\omega(\mathbf{B}^l(w))) \neq \emptyset$, and let us prove that $v = w$.

We have

$$\begin{aligned} & q(U(\omega(\mathbf{B}^l(v)))) \cap q(U(\omega(\mathbf{B}^l(w)))) = (V + i\rho(v)) \cap (V + i\rho(w)) \neq \emptyset \\ \Rightarrow \quad & \rho(v) \cap \rho(w) \neq \emptyset \\ \Rightarrow \quad & \rho(v) = \rho(w). \end{aligned}$$

Write $C = \rho(v) = \rho(w)$. We know that

$$q^{-1}(M(C)) = \bigcup_{u \in \rho^{-1}(C)} \hat{M}(u),$$

this union is disjoint, $U(\omega(\mathbf{B}^l(v))) \subseteq \hat{M}(v)$, and $U(\omega(\mathbf{B}^l(w))) \subseteq \hat{M}(w)$. Thus $v = w$.

ii) Let v be a vertex of $\hat{\Gamma}(\mathscr{A})$, and let Φ^o be a simplex of $\mathrm{Del}^o(\mathscr{A})$. Assume $U(\Phi^o) \cap U(\omega(\mathbf{B}^l(v))) \neq \emptyset$. Write $\phi = \pi^o(\Phi^o)$. Pick an $e \in U(\Phi^o) \cap U(\omega(\mathbf{B}^l(v)))$, and write $z = (x + iy) = q(e)$. We choose a vertex w of $\hat{\Gamma}_{\Phi^o}$ such that $e \in \hat{R}(\phi, w)$. We write $A = \rho(v)$ and $B = \rho(w)$. Let ψ be the simplex of $\mathbf{S}^{l-1}$ such that $x \in K(\psi)$.

We have $y \in A$ (since $z \in (V + iA)$) and $y \in B_{F(\psi)}$ (since $z \in R(\phi, B)$), thus $A_{F(\psi)} \cap B_{F(\psi)} \neq \emptyset$, therefore $A_{F(\psi)} = B_{F(\psi)}$. Let C be the chamber of $\mathscr{A}$ having $F(\psi)$ as facet and such that $C_{F(\psi)} = A_{F(\psi)} = B_{F(\psi)}$. Let f be a positive minimal path of $\Gamma(\mathscr{A})$ beginning at A and ending in C, and let g be a positive minimal path of $\Gamma(\mathscr{A})$ beginning at B and ending in C. By the definition of $R(\phi, B)$, we have $\psi \geq \phi$ (since $(x + iy) \in R(\phi, B)$ and $x \in K(\psi)$), thus $F(\psi) \geq F(\phi)$, therefore $F(\phi)$ is a facet of C. On the other hand, we have $\Phi^o \subseteq \Delta^o_{l-1}(w)$, thus $F(\Phi^o) = F(\phi)$ is a facet of $\rho(w) = B$. It follows that B and C are vertices of $\Gamma_{F(\phi)}$ and, consequently, by Lemma 3.1, g is a path of $\Gamma_{F(\phi)}$.

We denote by $\hat{f}$ the lift of f into $\hat{\Gamma}(\mathscr{A})$ beginning at v, and by $\hat{g}$ the lift of g into $\hat{\Gamma}(\mathscr{A})$ beginning at w. First, let us prove that end $(\hat{f}) = \mathrm{end}(\hat{g})$. By Lemma 2.6, we have

$$z = (x + iy) \in (V + iA) \cap R(\phi, B) \cap (Z(v, w) + iV),$$

thus $x \in Z(v, w)$. Furthermore, $x \in F(\psi)$ and $Z(v, w)$ is a union of facets of $\mathscr{A}$, thus $F(\psi) \subseteq Z(v, w)$. Finally, $F(\psi) \subseteq \bar{C}$ and $Z(v, w)$ is an open subset of V, therefore $C \subseteq Z(v, w)$. This implies, by the definition of $Z(v, w)$, that there exists a vertex $u \in \Sigma(v) \cap \Sigma(w)$ such that $\rho(u) = C$. This can happen only if $\mathrm{end}(\hat{f}) = \mathrm{end}(\hat{g}) = u$.

Now, let us prove that $\Phi^o \subseteq \Delta^o_{l-1}(u) \subseteq \mathbf{S}^{l-1}(v)$. The path g is a path of

$\Gamma_{F(\Phi^o)} = \Gamma_{F(\phi)}$, the vertex w is a vertex of $\hat{\Gamma}_{\Phi^o}$, and $\hat{\Gamma}_{\Phi^o}$ is a connected component of $\rho^{-1}(\Gamma_{F(\Phi^o)})$ (Lemma 3.2), thus $\hat{g}$ is a path of $\hat{\Gamma}_{\Phi^o}$, and, consequently, $u = \mathrm{end}(\hat{g}) \in V(\hat{\Gamma}_{\Phi^o})$. It follows, by the definition of $\hat{\Gamma}_{\Phi^o}$, that $\Phi^o \subseteq \Delta^o_{l-1}(u)$. On the other hand, $u \in \sum(v)$, therefore, by the definition of $\mathbf{S}^{l-1}(v)$, we have $\Delta^o_{l-1}(u) \subseteq \mathbf{S}^{l-1}(v)$.

iii) Let v be a vertex of $\hat{\Gamma}(\mathscr{A})$, and let Φ^o be a simplex of $\mathrm{Del}^o(\mathscr{A})$ such that $\Phi^o \subseteq \mathbf{S}^{l-1}(v)$. We write $\Phi = \Phi^o \vee \omega(\mathbf{B}^l(v))$ and $\phi = \pi^o(\Phi^o)$.

Let $e \in U(\Phi^o) \cap U(\omega(\mathbf{B}^l(v)))$. Pick a vertex w of $\hat{\Gamma}_{\Phi^o}$ such that $e \in \hat{R}(\phi, w)$. Write $A = \rho(v)$ and $B = \rho(w)$. We have

$$
\begin{aligned}
& e \in U(w(\mathbf{B}^l(v))) \cap \hat{R}(\phi, w) \\
\Rightarrow\ & q(e) \in (V + iA) \cap R(\phi, B) \cap (Z(v, w) + iV) \quad \text{(Lemma 2.6)} \\
\Rightarrow\ & q(e) \in ((\cap_{\psi \geq \phi} K(\psi)) + iA) \cap R(\phi, B) \cap (Z(v, w) + iV) \\
& \quad \text{(indeed, if } (x + iy) \in R(\phi, B)\text{, then } x \in \cap_{\psi \geq \phi} K(\psi)) \\
\Rightarrow\ & e \in U(\Phi) \cap \hat{R}(\phi, B) \quad \text{(Lemma 2.6)} \\
\Rightarrow\ & e \in U(\Phi).
\end{aligned}
$$

Now, let $e \in U(\Phi)$. Write $z = (x + iy) = q(e)$ and $A = \rho(v)$. Let ψ be the simplex of $\mathbf{S}^{l-1}$ such that $x \in K(\psi)$, and let B be the chamber of $\mathscr{A}$ having $F(\psi)$ as facet and such that $A_{F(\psi)} = B_{F(\psi)}$. Pick a positive minimal path f of $\Gamma(\mathscr{A})$ beginning at A and ending in B, and denote by $\hat{f}$ the lift of f into $\hat{\Gamma}(\mathscr{A})$ beginning at v. Set $w = \mathrm{end}(\hat{f})$. Let us prove that $w \in V(\hat{\Gamma}_{\Phi^o})$ and $e \in \hat{R}(\phi, w)$, This shows that $e \in U(\Phi^o)$, and, consequently, $e \in U(\Phi^o) \cap U(\omega(\mathbf{B}^l(v)))$ (we obviously have $e \in U(\Phi) \subseteq U(\omega(\mathbf{B}^l(v)))$.

Since $\psi \geq \phi$ and $\psi \subseteq \Delta_{l-1}(B)$, we have $\phi \subseteq \Delta_{l-1}(B)$. Thus there exists a simplex $\phi'^o \subseteq \Delta^o_{l-1}(w)$ such that $\pi^o(\phi'^o) = \phi$. Moreover, $\Delta^o_{l-1}(w) \subseteq \mathbf{S}^{l-1}(v)$ (since $w \in \sum(v)$) and the restriction of π^o to $\mathbf{S}^{l-1}(v)$ is an isomorphism $\mathbf{S}^{l-1}(v) \rightarrow \mathbf{S}^{l-1}$, therefore $\phi'^o = \Phi^o$. It follows that $w \in V(\hat{\Gamma}_{\Phi^o})$.

In order to prove that $e \in \hat{R}(\phi, w)$, by Lemma 3.6, it suffices to show that

$$z \in (V + iA) \cap R(\phi, B) \cap (Z(v, w) + iV).$$

By the starting hypothesis, we have $z \in (V + iA)$ and $z = (x + iy) \in (K(\psi) + iB_{F(\psi)}) \subseteq R(\phi, B)$. Now, $w \in \sum(v) \cap \sum(w)$, thus $C \in Z(v, w)$. Moreover, $F(\psi) \subseteq \bar{C}$, therefore $F(\psi) \subseteq \bar{Z}(v, w)$. Finally, since $A_{F(\psi)} = B_{F(\psi)}$, no hyperplane of $\mathscr{A}$ containing $F(\psi)$ separates A and B, thus, by Lemma 2.5, $x \in F(\psi) \subseteq Z(v, w)$, therefore $z \in (Z(v, w) + iV)$. □

LEMMA 3.6. *The set* $\mathscr{U} = \{U(\omega) \mid \omega$ *a vertex of* $\mathrm{Del}(\mathscr{A})\}$ *is a covering of* $\hat{M}(\mathscr{A})$.

Proof. Let $e \in \hat{M}(\mathscr{A})$. Write $z = (x + iy) = q(e)$.

Case a : $x = 0$.

Then there exists a chamber C of $\mathscr{A}$ such that $y \in C$. We have $z = (x + iy) \in (V + iC) \subseteq M(C)$. By Lemma 2.7,

$$q^{-1}(M(C)) = \bigcup_{v \in \rho^{-1}(C)} \hat{M}(v),$$

and this union is disjoint, so there exists a unique vertex $v \in \rho^{-1}(C)$ such that $e \in q^{-1}(V + iC) \cap \hat{M}(v) = U(\omega(\mathbf{B}^l(v)))$.

Case b : $x \neq 0$.

Let ϕ be the simplex of $\mathbf{S}^{l-1}$ such that $x \in K(\phi)$. Let C be the chamber of $\mathscr{A}$ having $F(\phi)$ as facet and such that $y \in C_{F(\phi)}$ (recall that $K(\phi) \subseteq F(\phi)$). We have $z = (x + iy) \in (K(\phi) + iC_{F(\phi)}) \subseteq R(\phi, C) \subseteq M(C)$. By Lemma 2.7,

$$q^{-1}(M(C)) = \bigcup_{v \in \rho^{-1}(C)} \hat{M}(v),$$

and this union is disjoint, so there exists a vertex $v \in \rho^{-1}(C)$ such that $e \in q^{-1}(R(\phi, C)) \cap \hat{M}(v) = \hat{R}(\phi, v)$. We have $\phi \subseteq \Delta_{l-1}(C)$, thus there exists a simplex $\Phi^o \subseteq \Delta^o_{l-1}(v)$ such that $\pi^o(\Phi^o) = \phi$. We have $e \in \hat{R}(\phi, v)$ and $v \in (\hat{\Gamma}_{\Phi^o})$, therefore $e \in U(\Phi^o)$. By Lemma 3.4, $e \in U(\omega)$, where ω is any vertex of Φ^o. □

Part 4.

LEMMA 3.7. i) *Let v be a vertex of $\hat{\Gamma}(\mathscr{A})$. Then $U(\omega(\mathbf{B}^l(v)))$ is contractible.*

ii) *Let v be a vertex of $\hat{\Gamma}(\mathscr{A})$, and let Φ^o be a simplex of* $\mathrm{Del}^o(\mathscr{A})$ *contained in $\mathbf{S}^{l-1}(v)$. Write $\Phi = \Phi^o \vee \omega(\mathbf{B}^l(v))$. Then $U(\Phi)$ is contractible.*

Proof. i) Write $A = \rho(v)$. Then

$$q(U(\omega(\mathbf{B}^l(v))) = (V + iA)$$

is clearly contractible, thus the lift $U(\omega(\mathbf{B}^l(v)))$ of $q(U(\omega(\mathbf{B}^l(v))))$ into $\hat{M}(v)$ is also contractible.

ii) Write $A = \rho(v)$ and $\phi = \pi^o(\Phi^o)$. Then

$$q(U(\Phi)) = \Big(\bigcup_{\psi \geq \phi} K(\psi)\Big) + iA$$

is clearly contractible, thus the lift $U(\Phi)$ of $q(U(\Phi))$ into $\hat{M}(v)$ is also contractible. □

LEMMA 3.8. *Let* Φ^o *be a simplex of* $\mathrm{Del}^o(\mathscr{A})$. *Then* $U(\Phi^o)$ *is homotopically equivalent to* $\hat{M}(\mathscr{A}_{X(\Phi^o)})$.

Following Lemmas 3.9 and 3.10 are preliminary results to the proof of Lemma 3.8.

For a simplex ϕ of $\mathbf{S}^{l-1}$, we write

$$W(\phi) = \bigcup_C R(\phi, C),$$

where the union is over all the chambers C of $\mathscr{A}$ having $F(\phi)$ as facet (i.e. over all the vertices of $V(\Gamma_{F(\phi)})$). The set $W(\phi)$ is an open subset of $M(\mathscr{A})$. We denote by $\iota_\phi^0 : W(\phi) \to M(\mathscr{A})$ the inclusion map of $W(\phi)$ into $M(\mathscr{A})$, by $\iota_\phi^1 : M(\mathscr{A}) \to M(\mathscr{A}_{X(\phi)})$ the inclusion map of $M(\mathscr{A})$ into $M(\mathscr{A}_{X(\phi)})$, and by $\iota_\phi = \iota_\phi^1 \circ \iota_\phi^0 : W(\phi) \to M(\mathscr{A}_{X(\phi)})$ the inclusion map of $W(\phi)$ into $M(\mathscr{A}_{X(\phi)})$.

LEMMA 3.9. *Let* ϕ *be a simplex of* $\mathbf{S}^{l-1}$. *Then* $\iota_\phi : W(\phi) \to M(\mathscr{A}_{X(\phi)})$ *is a homotopy equivalence.*

Proof. We have to define a continuous family $(h_t)_{0\le t\le 1} : M(\mathscr{A}_{X(\phi)}) \to M(\mathscr{A}_{X(\phi)})$ of maps such that:

a) $h_0(z) = z$ for all $z \in M(\mathscr{A}_{X(\phi)})$,
b) $h_1(z) \in W(\phi)$ for all $z \in M(\mathscr{A}_{X(\phi)})$,
c) $h_t(z) \in W(\phi)$ for all $z \in W(\phi)$ and all $t \in [0, 1]$.

We set

$$K = \bigcup_{\phi \ge \phi} K(\phi),$$

and we fix a point $x_0 \in \phi$. Since K is an open cone of V and $x_0 \in K$, there exists a continuous map $\lambda : V \to [0, +\infty[$ such that $(x + \lambda(x)x_0) \in K$ for all $x \in V$.

For every $z = (x + iy) \in M(\mathscr{A}_{X(\phi)})$ and for every $t \in [0, 1]$, we set

$$h_t(z) = (x + t\lambda(x)x_0) + iy.$$

The family $(h_t)_{0\le t\le 1} : M(\mathscr{A}_{X(\phi)}) \to V_{\mathbf{C}}$ is a continuous family of maps, and $h_0(z) = z$ for all $z \in M(\mathscr{A}_{X(\phi)})$. It remains to prove:

1) $h_t(z) \in M(\mathscr{A}_{X(\phi)})$ for all $z \in M(\mathscr{A}_{X(\phi)})$ and all $t \in [0, 1]$,
2) $h_1(z) \in W(\phi)$ for all $z \in M(\mathscr{A}_{X(\phi)})$,
3) $h_t(z) \in W(\phi)$ for all $z \in W(\phi)$ and all $t \in [0, 1]$.

1) Let $z = (x + iy) \in M(\mathscr{A}_{X(\phi)})$. Suppose that there exists a $t \in [0, 1]$ such that $h_t(z) \notin M(\mathscr{A}_{X(\phi)})$. Then there exists a hyperplane $H \in \mathscr{A}_{X(\phi)}$ such that $h_t(z) \in H_{\mathbf{C}}$ (i.e. $(x + t\lambda(x)x_0) \in H$ and $y \in H$). Since $x_0 \in \phi \subseteq H$ and H is a linear space, we have $x \in H$ and $y \in H$, thus $z \in H_{\mathbf{C}}$. This contradicts the fact $z \in M(\mathscr{A}_{X(\phi)})$.

2) Let $z = (x + iy) \in M(\mathscr{A}_{X(\phi)})$. We have $(x + \lambda(x)x_0) \in K$, so there exists a simplex ψ of $\mathbf{S}^{l-1}$ such that $\psi \geq \phi$ and $(x + \lambda(x)x_0) \in K(\psi)$.

Let G be the facet of $\mathscr{A}_{X(\phi)}$ with $\psi \subseteq G$. Let us prove that $|G| = |F(\psi)|$ (recall that $F(\psi)$ is a facet of $\mathscr{A}$ but not necessarily of $\mathscr{A}_{X(\phi)}$). If a hyperplane $H \in \mathscr{A}$ contains $F(\psi)$, then $H \supseteq X(\phi)$ (since $\psi \geq \phi$), thus H is a hyperplane of $\mathscr{A}_{X(\phi)}$ containing ψ, therefore $H \supseteq G$. This shows that $|G| \subseteq |F(\psi)|$. If a hyperplane $H \in \mathscr{A}_{X(\phi)}$ contains G, then $H \in \mathscr{A}$ and $H \supseteq F(\psi)$. This shows that $|F(\psi)| \subseteq |G|$.

Now, since $(x + \lambda(x)x_0) + iy \in M(\mathscr{A}_{X(\phi)})$ and $(x + \lambda(x)x_0) \in G$, there exists a chamber D of $\mathscr{A}_{|G|} = \mathscr{A}_{|F(\psi)|}$ such that $y \in D$. Let C be the chamber of $\mathscr{A}$ having $F(\psi)$ as facet and such that $D = C_{F(\psi)}$. The inequality $\psi \geq \phi$ implies $F(\psi) \geq F(\phi)$, thus C has also $F(\phi)$ as facet. It follows that $h_1(z) \in (K(\psi) + iC_{F(\psi)}) \subseteq R(\phi, C) \subseteq W(\phi)$.

3) Let $z = (x + iy) \in W(\phi)$. There are a chamber $C \in V(\Gamma_{F(\phi)})$ and a simplex $\psi \geq \phi$ of $\mathbf{S}^{l-1}$ such that $z \in (K(\psi) + iC_{F(\psi)})$. Since $x_0 \in \phi \subseteq \bar{K}(\psi)$ (where $\bar{K}(\psi)$ is the closure of $K(\psi)$ in V) and $K(\psi)$ is a convex cone, we have $(x + t\lambda(x)x_0) \in K(\psi)$, thus $h_t(z) = ((x + t\lambda(x)x_0) + iy) \in (K(\psi) + iC_{F(\psi)}) \subseteq W(\phi)$ for every $t \in [0, 1]$. □

Let Φ^o be a simplex of $\mathrm{Del}^o(\mathscr{A})$. We denote by $q_{\Phi^o}: U(\Phi^o) \to M(\mathscr{A})$ the restriction of q to $U(\Phi^o)$. Note that q_{Φ^o} can be viewed as a map $q_{\Phi^o}: U(\Phi^o) \to W(\pi^o(\Phi^o))$ onto $W(\pi^o(\Phi^o))$.

LEMMA 3.10. *Let* Φ^o *be a simplex of* $\mathrm{Del}^o(\mathscr{A})$. *Then* $q_{\Phi^o}: U(\Phi^o) \to W(\pi^o(\Phi^o))$ *is a cover.*

Proof. Write $\phi = \pi^o(\Phi^o)$. In order to prove Lemma 3.10, it suffices to show, for every chamber A of $\mathscr{A}$ having $F(\phi)$ as facet, that

$$q_{\Phi^o}^{-1}(R(\phi, A)) = \bigcup_v \hat{R}(\phi, v),$$

where the union is over all the vertices v of $\rho_{\Phi^o}^{-1}(A)$; indeed, this union is disjoint (Lemma 2.7), the sets $\hat{R}(\phi, v)$ are copies of $R(\phi, A)$, the map q_{ϕ^o} is surjective, and $\{R(\phi, A) \mid A \in V(\Gamma_{F(\phi)})\}$ is a covering of $W(\phi)$ by open subsets.

Fix $A \in V(\Gamma_{F(\phi)})$, and pick $e \in q_{\Phi^o}^{-1}(R(\phi, A))$. By the definition of $U(\Phi^o)$, there exists a vertex w of $\hat{\Gamma}_{\Phi^o}$ such that $e \in R(\phi, w)$. On the other hand, by Lemma 2.7,

$$q_{\Phi^o}^{-1}(R(\phi, A)) \subseteq q^{-1}(R(\phi, A)) = \bigcup_{v \in \rho^{-1}(A)} \hat{R}(\phi, v),$$

thus there exists a vertex $v \in \rho^{-1}(A)$ such that $e \in \hat{R}(\phi, v)$. Write $z = (x + iy) = q(e)$ and $B = \rho(w)$. Let ψ be the simplex of $\mathbf{S}^{l-1}$ such that $x \in K(\psi)$. Since $z \in R(\phi, A) \cap R(\phi, B)$, we have $y \in A_{F(\psi)} \cap B_{F(\psi)}$, thus $A_{F(\psi)} = B_{F(\psi)}$. Let C be the chamber of $\mathscr{A}$ having $F(\psi)$ as facet and such that $C_{F(\psi)} = A_{F(\psi)} = B_{F(\psi)}$.

Let f be a positive minimal path of $\Gamma(\mathscr{A})$ beginning at A and ending in C, and let g be a positive minimal path of $\Gamma(\mathscr{A})$ beginning at B and ending in C. The facet $F(\phi)$ is common to A (since $A \in V(\Gamma_{F(\phi)})$, to B (since $w \in V(\hat{\Gamma}_{\Phi^o})$), and to C (since $F(\psi) \geq F(\phi)$), so, by Lemma 3.1, the paths f and g are paths of $\Gamma_{F(\phi)}$.

Let $\hat{f}$ denote the lift of f into $\hat{\Gamma}(\mathscr{A})$ beginning at v, and let $\hat{g}$ denote the lift of g into $\hat{\Gamma}(\mathscr{A})$ beginning at w. Let us prove that $\mathrm{end}(\hat{f}) = \mathrm{end}(\hat{g})$. This shows that $v \in \rho_{\Phi^o}^{-1}(A)$, thus ends the proof of Lemma 3.10; indeed, gf^{-1} is a path of $\Gamma_{F(\phi)}$, the oriented graph $\hat{\Gamma}_{\Phi^o}$ is a connected component of $\rho^{-1}(\Gamma_{F(\phi)})$ (Lemma 3.2), and $w \in V(\hat{\Gamma}_{\Phi^o})$, thus $\hat{g}\hat{f}^{-1}$ is a path of $\hat{\Gamma}_{\Phi^o}$, and, consequently, $v = \mathrm{end}(\hat{g}\hat{f}^{-1}) \in V(\hat{\Gamma}_{\Phi^o})$.

By Lemma 2.6,

$$z \in R(\phi, A) \cap R(\phi, B) \cap (Z(v, w) + iV),$$

thus $x \in Z(v, w)$. Moreover, $Z(v, w)$ is a union of facets of $\mathscr{A}$ and $x \in F(\psi)$, therefore $F(\psi) \subseteq Z(v, w)$. Finally $Z(v, w)$, is an open subset of V and $F(\psi) \subseteq \bar{C}$, thus $C \subseteq Z(v, w)$. By the definition of $Z(v, w)$, there exists a vertex $u \in \Sigma(v) \cap \Sigma(w)$ such that $\rho(u) = C$. This can happen only if $u = \mathrm{end}(\hat{f}) = \mathrm{end}(\hat{g})$. □

Proof of Lemma 3.8. Let Φ^o be a simplex of $\mathrm{Del}^o(\mathscr{A})$. Write $\phi = \pi^o(\Phi^o)$ and $X = X(\Phi^o)$. We denote by $q_X : \hat{M}(\mathscr{A}_X) \to M(\mathscr{A}_X)$ the universal cover of $M(\mathscr{A}_X)$. Since q is the universal cover of $M(\mathscr{A})$ and q_X is a cover, there exists a map

$\hat{\iota}^{1}_{\Phi^o} : \hat{M}(\mathscr{A}) \to \hat{M}(\mathscr{A}_X)$ such that the following diagram commutes.

$$\begin{array}{ccc} \hat{M}(\mathscr{A}) & \xrightarrow{\hat{\iota}^{1}_{\Phi^o}} & \hat{M}(\mathscr{A}_X) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q_X} \\ M(\mathscr{A}) & \xrightarrow{\iota^{1}_{\Phi}} & M(\mathscr{A}_X) \end{array}$$

We denote by $\hat{\iota}^{0}_{\Phi^o} : U(\Phi^o) \to \hat{M}(\mathscr{A})$ the inclusion map of $U(\Phi^o)$ into $\hat{M}(\mathscr{A})$. Then the following diagram commutes.

$$\begin{array}{ccc} U(\Phi^o) & \xrightarrow{\hat{\iota}^{0}_{\Phi^o}} & \hat{M}(\mathscr{A}) \\ {\scriptstyle q_{\Phi^o}}\downarrow & & \downarrow{\scriptstyle q} \\ W(\phi) & \xrightarrow{\iota^{0}_{\Phi}} & M(\mathscr{A}) \end{array}$$

We write $\hat{\iota}_{\Phi^o} = \hat{\iota}^{1}_{\Phi^o} \circ \hat{\iota}^{0}_{\Phi^o}$. By the above considerations, the following diagram commutes.

$$\begin{array}{ccc} U(\Phi^o) & \xrightarrow{\hat{\iota}_{\Phi^o}} & \hat{M}(\mathscr{A}_X) \\ {\scriptstyle q_{\Phi^o}}\downarrow & & \downarrow{\scriptstyle q_X} \\ W(\phi) & \xrightarrow{\iota_{\phi}} & M(\mathscr{A}_X) \end{array}$$

The map ι_ϕ is a homotopy equivalence (Lemma 3.9), q_{Φ^o} is a cover (Lemma 3.10), and q_X is the universal cover of $M(\mathscr{A}_X)$, thus q_{Φ^o} is the universal cover of $W(\phi)$ and $\hat{\iota}_{\Phi^o}$ is a homotopy equivalence. □

PROPOSITION 3.11. *Let $\mathscr{A}$ be a real and essential arrangement of hyperplanes. Assume $\mathscr{A}_X$ to be a $K(\pi, 1)$ arrangement for every $X \in \mathscr{L}(\mathscr{A})$ different from $\{0\}$.*

Then $\mathrm{Del}(\mathscr{A})$ *has the same homotopy type as the universal cover* $\hat{M}(\mathscr{A})$ *of* $M(\mathscr{A})$.

Proof. Lemmas 3.4, 3.5 and 3.6 show that $\mathscr{U} = \{U(\omega) \mid \omega$ a vertex of $\mathrm{Del}(\mathscr{A})\}$ is a covering of $\hat{M}(\mathscr{A})$ having $\mathrm{Del}(\mathscr{A})$ as nerve. Lemmas 3.7 and 3.8 and the hypothesis "$\mathscr{A}_X$ is a $K(\pi, 1)$ arrangement for every $X \in \mathscr{L}(\mathscr{A})$ different from $\{0\}$" show that every nonempty intersection of elements of $\mathscr{U}$ is contractible. It follows, by [We], that $\mathrm{Del}(\mathscr{A})$ is homotopically equivalent to $\hat{M}(\mathscr{A})$. □

Part 5.

PROPOSITION 3.12. *Let* $\mathscr{A}$ *be a real and essential arrangement of hyperplanes. Assume that there exists an* $X \in \mathscr{L}(\mathscr{A})$ *different from* $\{0\}$ *such that* $\mathscr{A}_X$ *is not a* $K(\pi, 1)$ *arrangement. Then* $\mathrm{Del}(\mathscr{A})$ *is not homotopically equivalent to the universal cover* $\hat{M}(\mathscr{A})$ *of* $M(\mathscr{A})$.

Proof. We are going to construct a space $\hat{M}_\infty$ by attaching cells to $\hat{M}(\mathscr{A})$, and a covering $\mathscr{U}_\infty = \{U_\infty(\omega) \mid \omega$ a vertex of $\mathrm{Del}(\mathscr{A})\}$ of $\hat{M}_\infty$ by open subsets, having $\mathrm{Del}(\mathscr{A})$ as nerve, and such that every nonempty intersection of elements of $\mathscr{U}_\infty$ is contractible. By [We], the space $\hat{M}_\infty$ will be homotopically equivalent to $\mathrm{Del}(\mathscr{A})$. Afterwards, we will prove that there exists an integer $n_0 > 0$ such that the inclusion map $\hat{M}(\mathscr{A}) \to \hat{M}_\infty$ determines a surjective morphism $\pi_{n_0}(\hat{M}(\mathscr{A})) \to \pi_{n_0}(\hat{M}_\infty)$ which is not injective. This shows that $\pi_{n_0}(\mathrm{Del}(\mathscr{A})) = \pi_{n_0}(\hat{M}_\infty) \neq \pi_{n_0}(\hat{M}(\mathscr{A}))$.

Choose an $X \in \mathscr{L}(\mathscr{A})$ different from $\{0\}$ such that $\mathscr{A}_X$ is not a $K(\pi, 1)$ arrangement. Pick a simplex Φ^o of $\mathrm{Del}^o(\mathscr{A})$ such that $X(\Phi^o) = X$. By Lemma 3.8, $U(\Phi^o)$ has the same homotopy type as $\hat{M}(\mathscr{A}_X)$, so is not contractible.

It follows that there exists an integer $n_0 > 0$ such that:

i) $\pi_n(U(\Phi^o)) = \{0\}$ for every simplex Φ^o of $\mathrm{Del}^o(\mathscr{A})$ and every $n \in \{0, 1, \ldots, n_0 - 1\}$,

ii) there exists a simplex Φ^o of $\mathrm{Del}^o(\mathscr{A})$ such that $\pi_{n_0}(U(\Phi^o)) \neq \{0\}$.

Recall that, if Φ is a simplex of $\mathrm{Del}(\mathscr{A})$ not contained in $\mathrm{Del}^o(\mathscr{A})$, then $U(\Phi)$ is contractible (Lemma 3.7).

We set $\hat{M}_{n_0-1} = \hat{M}(\mathscr{A})$, and $U_{n_0-1}(\Phi) = U(\Phi)$ for every simplex Φ of $\mathrm{Del}(\mathscr{A})$.

First, we are going to define, by induction on $k \geq n_0$,

a) a space $\hat{M}_k$,

b) an open subspace $U_k(\Phi)$ of $\hat{M}_k$ for every simplex Φ of $\mathrm{Del}(\mathscr{A})$,

such that:

1) $\hat{M}_{k-1} \subseteq \hat{M}_k$,

2) $U_{k-1}(\Phi) = U_k(\Phi) \cap \hat{M}_{k-1}$ for every simplex Φ of $\mathrm{Del}(\mathscr{A})$,

3) the inclusion map $\hat{M}_{k-1} \to \hat{M}_k$ induces an isomorphism of groups $\pi_n(\hat{M}_{k-1}) \to \pi_n(\hat{M}_k)$ for every $n \in \{0, 1, \ldots, k-1\}$, and induces a surjective morphism $\pi_k(\hat{M}_{k-1}) \to \pi_k(\hat{M}_k)$,

4) $\pi_n(U_k(\Phi)) = \{0\}$ for every simplex Φ of $\mathrm{Del}(\mathscr{A})$ and every $n \in \{0, 1, \ldots, k\}$,

5) let $\omega_0, \omega_1, \ldots, \omega_r$ be $(r+1)$ vertices of $\mathrm{Del}(\mathscr{A})$, if $\cap_{j=0}^r U_k(\omega_j) \neq \emptyset$, then $\omega_0, \omega_1, \ldots, \omega_r$ are the vertices of a simplex Φ of $\mathrm{Del}(\mathscr{A})$,

6) let $\omega_0, \omega_1, \ldots, \omega_r$ be the vertices of a simplex Φ of $\mathrm{Del}(\mathscr{A})$ then $\cap_{j=0}^r U_k(\omega_j) = U_k(\Phi)$,

7) $\{U_k(\Phi) \mid \omega \text{ a vertex of } \mathrm{Del}(\mathscr{A})\}$ is a covering of $\hat{M}_k$.

Assume $\hat{M}_{k-1}$ to be defined. Let Φ be a simplex of $\mathrm{Del}(\mathscr{A})$ such that $\pi_k(U_{k-1}(\Phi)) \neq \{0\}$. We fix a base point $e_\Phi \in U_{k-1}(\Phi)$. We choose a generator system $\{\gamma_i\}_{i \in I_\Phi}$ of $\pi_r(U_{k-1}(\Phi), e_\Phi)$, and, for every $i \in I_\Phi$, we fix a representative map $f_i : \mathbf{S}^k \to U_{k-1}(\Phi)$ for γ_i. We write $I_\Phi = \emptyset$ if $\pi_k(U_{k-1}(\Phi)) = \{0\}$. We set

$$I = \bigcup_{\Phi} I_\Phi,$$

where the union is over all the simplexes Φ of $\mathrm{Del}(\mathscr{A})$. The space $\hat{M}_k$ is obtained by attaching a $(k+1)$-cell E_i to $\hat{M}_{k-1}$ by means of the map $f_i : \mathbf{S}^k \to \hat{M}_{k-1}$ defined on the boundary of E_i for every $i \in I$. In other words, for every $i \in I$, we fix a copy $\mathbf{B}_i^{k+1} = \{x \in \mathbf{R}^{k+1} \mid \|x\| \leq 1\}$ of $\mathbf{B}^{k+1}$. Then

$$\hat{M}_k = \left\{\hat{M}_{k-1} \amalg \left(\coprod_{i \in I} \mathbf{B}_i^{k+1}\right)\right\} \Big/ \sim,$$

where $\sim$ is the equivalence relation on $\hat{M}_{k-1} \amalg (\amalg_{i \in I} \mathbf{B}_i^{k+1})$ defined by $x \sim f_i(x)$ for every $i \in I$ and for every $x \in \partial\mathbf{B}_i^{k+1} = \mathbf{S}^k$. We denote by $g_t : \mathbf{B}_i^{k+1} \to \hat{M}_k$ the natural map, and by E_i the image of g_i (where $i \in I$). We have $g_i|_{\partial\mathbf{B}_i^{k+1}} = f_i$.

Let Φ be a simplex of $\mathrm{Del}(\mathscr{A})$. The set $U_k(\Phi)$ is defined by:

a) $U_k(\Phi) \cap \hat{M}_{k-1} = U_{k-1}(\Phi)$,

b) let $i \in I$, if $\partial E_i \subseteq U_{k-1}(\Phi)$, then $E_i \subseteq U_k(\Phi)$,

c) let $i \in I$, if $\partial E_i \not\subseteq U_{k-1}(\Phi)$, then

$$U_k(\Phi) \cap E_i = g_i(\{\lambda x \mid 0 < \lambda \leq 1 \text{ and } x \in f_t^{-1}(U_{k-1}(\Phi))\}).$$

Let $i \in I$, and let Φ be a simplex of $\mathrm{Del}(\mathscr{A})$. Then $g_i(0) \in U_k(\Phi)$ if and only if $\partial E_i \subseteq U_{k-1}(\Phi)$, and $g_i(\lambda x) \in U_k(\Phi)$ if and only if $g_i(x) = f_i(x) \in U_{k-1}(\Phi)$, where $\lambda \in [0, 1]$ and $x \in \mathbf{S}^{l-1}$.

Now, let us prove Properties 1) to 7).

1) and 2) are obvious.

3) The space $\hat{M}_k$ is obtained by attaching $(k+1)$-cells to $\hat{M}_{k-1}$, so $\pi_n(\hat{M}_k, \hat{M}_{k-1}) = \{0\}$ for every $n \in \{0, 1, \ldots, k\}$, thus the inclusion map $\hat{M}_{k-1} \to \hat{M}_k$ induces a group isomorphism $\pi_n(\hat{M}_{k-1}) \to \pi_n(\hat{M}_k)$ for every $n \in \{0, 1, \ldots, k-1\}$, and induces a surjective morphism $\pi_k(\hat{M}_{k-1}) \to \pi_k(\hat{M}_k)$.

4) Let Φ be a simplex of $\mathrm{Del}(\mathscr{A})$. We denote by $U'_k(\Phi)$ the subset of $\hat{M}_k$ defined by:

a) $U'_k(\Phi) \cap \hat{M}_{k-1} = U_{k-1}(\Phi)$,

b) let $i \in I$, if $\partial E_i \subseteq U_{k-1}(\Phi)$, then $E_i \subseteq U_k(\Phi)$,

c) let $i \in I$, if $\partial E_i \not\subseteq U_{k-1}(\Phi)$, then $\dot{E}_i \cap U'_k(\Phi) = \emptyset$, where $\dot{E}_i$ is the interior of E_i.

The set $U'_k(\Phi)$ is a strong deformation retract of $U_k(\Phi)$ and is obtained by attaching $(k+1)$-cells to $U_{k-1}(\Phi)$. If follows that the inclusion map $U_{k-1}(\Phi) \to U_k(\Phi)$ induces a group isomorphism $\pi_n(U_{k-1}(\Phi)) \to \pi_n(U_k(\Phi))$ for every $n \in \{0, 1, \ldots, k-1\}$, and induces a surjective morphism $\xi_k^\Phi : \pi_k(U_{k-1}(\Phi)) \to \pi_k(U_k(\Phi))$. A first consequence is, by the inductive hypothesis, that $\pi_n(U_k(\Phi)) = \pi_n(U_{k-1}(\Phi)) = \{0\}$ for every $n \in \{0, 1, \ldots, k-1\}$. On the other hand, by the construction of $\hat{M}_k$, every generator γ_i of $\pi_k(U_{k-1}(\Phi), e_\Phi)$ is sent by ξ_k^Φ onto 0, thus the image of ξ_k^Φ is $\{0\} = \pi_k(U_k(\Phi))$.

5) Let $\omega_0, \omega_1, \ldots, \omega_r$ be $(r+1)$ vertices of $\mathrm{Del}(\mathscr{A})$ such that $\cap_{j=0}^r U_k(\omega_j) \neq \emptyset$. Pick an $e \in \cap_{j=0}^r U_k(\omega_j)$.

Case a: $e \in \hat{M}_{k-1}$. Then $e \in \cap_{j=0}^r U_{k-1}(\omega_j)$, thus, by the inductive hypothesis, $\omega_0, \omega_1, \ldots, \omega_r$ are the vertices of a simplex Φ of $\mathrm{Del}(\mathscr{A})$.

Case b: There exists an $i \in I$ such that $e \in E_i$ and $e = g_i^{-1}(0)$. Then, by the construction of $U_k(\omega_j)$, we have $\partial E_i \in U_{k-1}(\omega_j)$ for every $j = 0, 1, \ldots, r$, therefore $\cap_{j=0}^r U_{k-1}(\omega_j) \neq \emptyset$. It follows, by the inductive hypothesis, that $\omega_0, \omega_1, \ldots, \omega_r$ are the vertices of a simplex Φ of $\mathrm{Del}(\mathscr{A})$.

Case c: There exists an $i \in I$ such that $e \in E_i$ and $e \neq g_i^{-1}(0)$. There are an $x \in \mathbf{S}^k$ and a $\lambda \in]0, 1]$ such that $e = g_i(\lambda x)$. By the construction of $U_k(\omega_j)$, we have $g_i(x) = f_i(x) \in U_{k-1}(\omega_j)$ for every $j = 0, 1, \ldots, r$, therefore

$\cap_{j=0}^{r} U_{k-1}(\omega_j) \neq \emptyset$. It follows, by the inductive hypothesis, that $\omega_0, \omega_1, \ldots, \omega_r$ are the vertices of a simplex Φ of $\mathrm{Del}(\mathscr{A})$.

6) Let $\omega_0, \omega_1, \ldots, \omega_r$ be the vertices of a simplex Φ of $\mathrm{Del}(\mathscr{A})$.

a) $(\cap_{j=0}^{r} U_k(\omega_j)) \cap \hat{M}_{k-1} = \cap_{j=0}^{r} U_{k-1}(\omega_j) = U_{k-1}(\Phi) = U_k(\Phi) \cap \hat{M}_{k-1}$.

b) let $i \in I$ such that $\partial E_i \subseteq U_{k-1}(\omega_j)$ for every $j = 0, 1, \ldots, r$. Then $\partial E_i \subseteq \cap_{j=0}^{r} U_{k-1}(\omega_j) = U_{k-1}(\Phi)$, and, consequently,

$$(\cap_{j=0}^{r} U_k(\omega_j)) \cap E_i = E_i = U_k(\Phi) \cap E_i.$$

c) Let $i \in I$ such that there exists a $j \in \{0, 1, \ldots, r\}$ with $\partial E_i \not\subseteq U_{k-1}(\omega_j)$. then $\partial E_i \not\subseteq U_{k-1}(\Phi)$, and, consequently,

$$\begin{aligned}(\cap_{j=0}^{r} U_k(\omega_j)) \cap E_i &= g_i(\{\lambda x \mid 0 < \lambda \leq 1 \text{ and } x \in f_i^{-1}(\cap_{j=0}^{r} U_{k-1}(\omega_j))\}) \\ &= g_i(\{\lambda x \mid 0 < \lambda \leq 1 \text{ and } x \in f_i^{-1}(U_{k-1}(\Phi))\}) \\ &= U_k(\Phi) \cap E_i.\end{aligned}$$

a), b) and c) show that $\cap_{j=0}^{r} U_k(\omega_j) = U_k(\Phi)$.

7) Let $e \in \hat{M}_k$. If $e \in \hat{M}_{k-1}$, then, by the inductive hypothesis, there exists a vertex ω of $\mathrm{Del}(\mathscr{A})$ such that $e \in U_{k-1}(\omega) \subseteq U_k(\omega)$. Assume now that there exists an $i \in I$ such that $e \in E_i$. Let Φ denote the simplex of $\mathrm{Del}(\mathscr{A})$ such that $i \in I_\Phi$. By the construction of $\hat{M}_k$, we have $\partial E_\iota \subseteq U_{k-1}(\Phi)$, and, by the construction of $U_k(\Phi)$, we have $e \in E_i \subseteq U_k(\Phi)$. By Property 6), $e \in U_k(\omega)$, where ω is any vertex of Φ.

Now , we set:

a) $\hat{M}_\infty = \varinjlim \hat{M}_k$

b) $U_\infty(\Phi) = \varinjlim U_k(\Phi)$ for every simplex of $\mathrm{Del}(\mathscr{A})$.

We have the following properties.

1) $\pi_n(\hat{M}_\infty) = \pi_n(\hat{M}(\mathscr{A}))$ for every $n \in \{0, 1, \ldots, n_0 - 1\}$, and $\pi_n(\hat{M}_\infty) = \pi_n(\hat{M}_n)$ for every $n \geq n_0$.

2) $\pi_n(U_\infty(\Phi)) = \{0\}$ for every $n \geq 0$ and for every simplex Φ of $\mathrm{Del}(\mathscr{A})$.

3) Let $\omega_0, \omega_1, \ldots, \omega_r$ be $(r+1)$ vertices of $\mathrm{Del}(\mathscr{A})$. If $\cap_{j=0}^{r} U_\infty(\omega_j) \neq \emptyset$, then $\omega_0, \omega_1, \ldots, \omega_r$ are the vertices of a simplex Φ of $\mathrm{Del}(\mathscr{A})$.

4) Let $\omega_0, \omega_1, \ldots, \omega_r$ be the vertices of a simplex Φ of $\mathrm{Del}(\mathscr{A})$. Then $\cap_{j=0}^{r} U_\infty(\omega_j) = U_\infty(\Phi)$.

5) $\mathscr{U}_\infty = \{U_\infty(\omega) \mid \omega$ a vertex of $\mathrm{Del}(\mathscr{A})\}$ is a covering of $\hat{M}_\infty$ by open subsets.

Properties 3), 4) and 5) show that $\mathcal{U}_\infty$ is a covering of $\hat{M}_\infty$ having $\mathrm{Del}(\mathcal{A})$ as nerve. Properties 2) and 4) show that any nonempty intersection of elements of $\mathcal{U}_\infty$ is contractible. It follows, by [We], that $\mathrm{Del}(\mathcal{A})$ is homotopically equivalent to $\hat{M}_\infty$.

Since $\pi_{n_0}(\hat{M}_\infty) = \pi_{n_0}(\hat{M}_{n_0})$ and the inclusion map $\hat{M}(\mathcal{A}) \to \hat{M}_{n_0}$ induces a surjective morphism $\xi_{n_0} : \pi_{n_0}(\hat{M}(\mathcal{A})) \to \pi_{n_0}(\hat{M}_{n_0})$, in order to prove that $\mathrm{Del}(\mathcal{A})$ is not homotopically equivalent to $\hat{M}(\mathcal{A})$, it suffices to show that ξ_{n_0} is not injective.

Choose a simplex Φ^o of $\mathrm{Del}^o(\mathcal{A})$ such that $\pi_{n_0}(U(\Phi^o)) \neq \{0\}$. Let $\overset{0}{\iota}_{\Phi^o} : U(\Phi^o) \to \hat{M}(\mathcal{A})$ be the inclusion map of $U(\Phi^o)$ into $\hat{M}(\mathcal{A})$, and let $\overset{1}{\iota}_{\Phi^o} : \hat{M}(\mathcal{A}) \to \hat{M}(\mathcal{A}_{X(\Phi^o)})$ be the map defined in the proof of Lemma 3.8. Then $\hat{\iota}_{\Phi^o} = \overset{1}{\iota}_{\Phi^o} \circ \overset{0}{\iota}_{\Phi^o}$ is a homotopy equivalence (see the proof of Lemma 3.8), thus $(\overset{0}{\iota}_{\Phi^o})_* : \pi_{n_0}(U(\Phi^o)) \to \pi_{n_0}(\hat{M}(\mathcal{A}))$ is injective. Furthermore, by construction of $\hat{M}_{n_0}$, the morphism $\xi_{n_0} \circ (\overset{0}{\iota}_{\Phi^o})_* : \pi_{n_0}(U(\Phi^o)) \to \pi_{n_0}(\hat{M}_{n_0})$ sends $\pi_{n_0}(U(\Phi^o))$ onto $\{0\}$. This shows that ξ_{n_0} is not injective. □

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Département de Mathématiques
Universite de Nantes
2 rue de la Houssinière
44072 Nantes Cedex 03
France