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L^2-BOUNDEDNESS OF THE CAUCHY TRANSFORM ON SMOOTH NON-LIPSCHITZ CURVES

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1. Introduction and statements of results

Let Γ be a curve defined by $y = A(x)$ in $\mathbf{R}^2$. The Cauchy transform $\mathscr{C}_A$ on the curve Γ is a singular integral operator defined by the singular integral kernel

$$K(x, y) = \frac{1 + iA'(y)}{(x - y) + i(A(x) - A(y))}. \tag{1.1}$$

If A is a Lipschitz function, i.e., $\| A' \|_\infty < \infty$, then $\mathscr{C}_A$ makes a very significant example of non-convolution type singular integral operators. The problem of L^2-boundedness of the Cauchy transform was raised and solved when $\| A' \|_\infty$ is small by A. P. Calderón in relation to the Dirichlet problem on Lipschitz domains [Cal1, Cal2]. Since then, it has been a central problem in the theory of singular integral operators and several significant techniques has been developed to deal with this problem. Among them are the $T(1)$-Theorem of David and Journé, the technique of Coifman, McIntosh, and Meyer, and the technique of Coifman, Jones, and Semmes [D.J, C.M.M, C.J.S]. We refer to [Chi, Mur] for a history of development in the last decades on the theory of the Cauchy transform.

If $\| A' \|_\infty = \infty$, then the Cauchy kernel $K(x, y)$ given in (1.1) is not a standard kernel. An integral kernel on the line is called a standard kernel if it satisfies $| K(x, y) | \le C | x - y |^{-1}$ and $| \nabla_{x,y} K(x, y) | \le C | x - y |^{-2}$. If $\| A' \|_\infty = \infty$, then the Cauchy kernel does not satisfy both estimates. So, the theory of the singular integral operators may not be applied directly. Nevertheless, the question of L^2-boundedness of $\mathscr{C}_A$ is still an interesting one. In this paper, we deal with L^2-boundedness of $\mathscr{C}_A$ when A is smooth and $\| A' \|_\infty = \infty$.

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We first find two examples of curves on which the Cauchy transforms are not L^2-bounded. Those are curves defined by $A'(x) = x \sin x$ and $A(x) = \exp(x^2)$. In the first example, A' has too many zeros while the derivative of the second A grows too fast relatively to A. The fact that both $\log|x \sin x|$ and $\log \exp x^2$ are *not* BMO functions is relevant. We then consider the case when A is a polynomial. If A is a polynomial, then A' has only finitely many zeros and $|A'(x)/A(x)|$ behaves like $|1/x|$ as $x \to \infty$. In fact, if A is a polynomial, then $\log|A(x)|$ is a BMO function. In this paper, we prove the following theorem.

MAIN THEOREM. *Let $A(x)$ be a polynomial of the form*

$$
\begin{cases} A(x) \text{ is any polynomial if } d \text{ is an odd integer,} \\ A(x) = \sum_{i=1}^{n} a_i x^{2i} \text{ if } d = 2n \text{ is an even integer.} \end{cases} \tag{1.2}
$$

Then, the Cauchy transform $\mathscr{C}_A$ is bounded on L^p for any $1 < p < \infty$.

Among the polynomial which are not covered in (1.2) is $A(x) = x^4 - x^3$. This polynomial does not satisfy the estimate $|A(x) - A(y)| \approx |x - y||x + y|(|x|^2 + |y|^2)$ when $|x| + |y|$ is large which is a crucial estimate for our proofs. However, polynomials in (1.2) include a significantly large class of polynomials.

In order to explain ideas of proofs in this paper, let us consider an example. If $A(x) = x^2$, the kernel given in (1.1) does not satisfy the standard estimates. But, the kernel can be decomposed as

$$
K(x,y) = \frac{1 + 2iy}{(x-y) + i(x^2 - y^2)} = \frac{1}{x-y} + \frac{-i}{1 + (x+y)}. \tag{1.3}
$$

The first kernel of the right hand side is the Hilbert kernel while the second one is a kernel of Poisson type. So, if $A(x) = x^2$, then $\mathscr{C}_A$ is bounded on L^2. It turns out this decomposition can be performed for general kernels by using a proper cut-off function. Then, each one of the decomposed kernels is a standard kernel and we can apply the $T(1)$-Theorem to it.

This paper is organized as follows; In section 2, we give a sufficient condition for a function to belong to the BMO. In section 3, we collect some estimates on polynomials which will be used in later sections. In section 4, we decompose the kernel $K(x, y)$ into two standard kernels and show that both of them satisfy all the conditions of $T(1)$-Theorem. In section 5, we show that if $A'(x) = x \sin x$ or $A(x) = \exp(x^2)$, then $\mathscr{C}_A$ is *not* bounded on L^2.

Throughout this paper constants C may differ in each occurence and $A \approx B$

means that there are positive constants c and C such that $c \le A/B \le C$.

2. Preliminary lemma on BMO

Showing that a function is in BMO is a fairly hard task. One of the reasons is that being a BMO function is not just a size condition. For example, even if $|f| \in$ BMO, f may not be a BMO function. It can also be shown easily that even if $0 \le f \le g$ and $g \in$ BMO, f may not be a BMO function. In particular, that $f(x) = O(\log|x|)$ as $x \to \infty$ does not imply $f \in$ BMO. In this section we obtain a sufficient condition for a function to belong to BMO which will be used repeatedly in section 4. We show that if $f'(x) = O(|x|^{-1})$ as $x \to \infty$, then $f \in$ BMO.

LEMMA 2.1. *Suppose that there exists a positive number m such that f is bounded on $[-m, m]$ and f is continuously differentiable if $|x| \ge m$. If $|f'(x)| = O(|x|^{-1})$ as $x \to \infty$, then $f \in$ BMO.*

Proof. By the assumption, there are large constants L and C such that $|f'(x)| \le C|x|^{-1}$ if $|x| > L$. We write

$$f = f\chi_{(-\infty,-L)} + f\chi_{[-L,L]} + f\chi_{(L,\infty)} = f_1 + f_2 + f_3.$$

It is enough to show that $f_3 \in$ BMO since f_2 is bounded and that $f_1 \in$ BMO can be proved in the same way. For notational simplicity, we put $g = f_3$. We need to show that if $0 < a < b$, then

$$\frac{1}{b-a}\int_a^b |g(x) - g(b)|\,dx \le C.$$

We may assume $L < a < b$. If $b \le 2a$, then

$$\frac{1}{b-a}\int_a^b |g(x) - g(b)|\,dx \le C\int_a^b |g'(x)|\,dx \le C.$$

Suppose that $b \ge 2a$. Choose an integer N so that $2^{-N}b \le a \le 2^{-N+1}b$. Then,

$$\frac{1}{b-a}\int_a^b |g(x) - g(b)|\,dx \le \frac{1}{b-a}\sum_{j=1}^{N}\int_{2^{-j}b}^{2^{-j+1}b} |g(x) - g(b)|\,dx.$$

And we have, for each j,

$$\int_{2^{-j}b}^{2^{-j+1}b} |g(x) - g(b)|\,dx$$

$$\leq C\int_{2^{-j}b}^{2^{-j+1}b} | g(x) - g(2^{-j}b) | \, dx + 2^{-j}b \, | g(b) - g(2^{-j}b) |$$

$$\leq C2^{-j}b\int_{2^{-j}b}^{2^{-j+1}b} | g'(x) | \, dx + 2^{-j}b \sum_{j=1}^{j} | g(2^{-i+1}b) - g(2^{-j}b) |$$

$$\leq C2^{-j}b(\log 2 + j).$$

Therefore we obtain

$$\frac{1}{b-a}\int_a^b | g(x) - g(b) | \, dx \leq C\frac{1}{b-a}\sum_{j=1}^{N} 2^{-j}\, b(\log 2 + j)$$
$$\leq C\frac{b}{b-a} \leq C.$$

This completes the proof.

3. Estimates on polynomials

In this section we collect estimates on polynomials which will be used in later sections. Let $A(x)$ be d-th degree polynomial of the form:

$$\begin{cases} A(x) \text{ is any polynomial if } d \text{ is an odd integer} \\ A(x) = \sum_{i=1}^{n} a_i x^{2i} \quad \text{if } d = 2n \text{ is an even integer.} \end{cases} \tag{3.1}$$

For these polynomials we have the following elementary but significant estimates.

LEMMA. 3.1. *Let $A(x)$ be a polynomial of degree d as in (3.1). Then,*

(1) *If d is odd, then*

$$| A(x) - A(y) | \approx | x - y | (| x |^{d-1} + | y |^{d-1}).$$

Moreover, there exists a positive number M such that

(2) *If $| x | \geq M$, then $| A(x) | \approx | x |^d$, $| A'(x) | \approx | x |^{d-1}$, and $| A''(x) | \approx | x |^{d-2}$.*

(3) *If d is even and if either $| x | \geq M$ or $| y | \geq M$, then*

$$| A(x) - A(y) | \approx | x - y | \, | x + y | (| x |^{d-2} + | y |^{d-2}).$$

(4) *If $| x | < M$ and $| y | > 2M$, then*

$$| A(x) - A(y) | \approx | A(y) | \approx | y |^d.$$

Remark 3.2 We will fix M to be the number as in Lemma 3.1 throughout this paper.

Proof. That $|A(x)-A(y)| \leq C|x-y|(|x|^{d-1}+|y|^{d-1})$ is trivial for any polynomial. For (1), note that

$$\begin{aligned}
\frac{x^d-y^d}{x-y} &= \sum_{j=0}^{d-1} x^{d-j-1}y^j \\
&= \frac{1}{2}x^{d-1}+\frac{1}{2}y^{d-1}+\frac{1}{2}(x^2+2xy+y^2)\sum_{j=1}^{(d-1)/2} x^{d-(2j+1)}y^{2(j-1)} \\
&\geq \frac{1}{2}(x^{d-1}+y^{d-1})
\end{aligned}$$

since d is odd. Therefore we have

$$|A(x)-A(y)| \geq \frac{1}{2}|x-y|(|x|^{d-1}+|y|^{d-1}).$$

Let $A(x)=\sum_{j=1}^{d} a_j x^j$. Assume that $a_d=1$ without loss of generality. Then, we have

$$\begin{aligned}
|A(x)-A(y)| &\geq |x^d-y^d| - \sum_{j=1}^{d-1}|a_j||x^j-y^j| \\
&\geq \frac{1}{2}|x-y|(|x|^{d-1}+|y|^{d-1}) - C|x-y|(|x|^{d-2}+|y|^{d-2}) \\
&\geq C|x-y|(|x|^{d-1}+|y|^{d-1}).
\end{aligned}$$

For (3), we let $A(x)=\sum_{n=1}^{r} a_n x^{2n}$ with $2r=d$. Observe that

$$\begin{aligned}
|x^{2n}-y^{2n}| &= |x-y||x+y||x^{2n-2}+x^{2n-4}y^2+\cdots+x^2y^{2n-4}+y^{2n-2}| \\
&\approx |x-y||x+y|(|x|^{2n-2}+|y|^{2n-2})
\end{aligned}$$

by the same reason as before. Therefore, there exist constants C_1 and C_2 such that

$$\begin{aligned}
|A(x)-A(y)| &= \left|\sum_{n=1}^{r} a_n(x^{2n}-y^{2n})\right| \\
&\geq |x-y||x+y|[C_1(|x|^{2n-2}+|y|^{2n-2}) - C_2(|x|^{2n-4}+|y|^{2n-4})] \\
&\geq C|x-y||x+y|(|x|^{2n-2}+|y|^{2n-2})
\end{aligned}$$

as long as $|x|+|y|$ is large. It is easy to show that

$$|A(x)-A(y)| \leq C|x-y||x+y|(|x|^{2n-2}+|y|^{2n-2})$$

and hence we obtain (3). (2) and (4) are trivial. This completes the proof.

Remark 3.2. The polynomial $A(x) = x^4 - x^3$ is not included in (3.1). The estimate (3) in Lemma 3.1 is actually false for $A(x) = x^4 - x^3$. It would be interesting to see whether the Cauchy transform on the curve $y = x^4 - x^3$ is L^2-bounded or not.

4. L^2 boundedness

Throughout this paper $\mathscr{C}_A$ denotes the Cauchy transform on the curve $y = A(x)$. This section is devoted to the proof of Main Theorem:

MAIN THEOREM. *Let $A(x)$ be a polynomial of the form*

$$\begin{cases} A(x) \text{ is any polynomial if } p \text{ is an odd integer} \\ A(x) = \sum_{j=1}^{n} a_i x^{2i} \text{ if } p = 2n \text{ is an even integer.} \end{cases} \tag{4.1.1}$$

Then, the Cauchy transform $\mathscr{C}_A$ is bounded on L^p for any $1 < p < \infty$.

By the classical theory of singular integral operators, it suffices to prove when $p = 2$. Recall that the integral kernel $K(x, y)$ of $\mathscr{C}_A$ is given by

$$K(x,y) = \frac{1 + iA'(y)}{(x - y) + i(A(x) - A(y))}. \tag{4.1.2}$$

The kernel $K(x, y)$ does not satisfy the standard estimates. If $|y|$ is large and if x and y are close, then the estimate $K(x, y) \leq C|x - y|^{-1}$ does not hold. We overcome this obstacle by decomposing the kernel into two standard kernels by introducing an appropriate cut-off function.

Let $\tilde{\phi}$ be a C^∞ smooth function such that

$$\begin{cases} \tilde{\phi}(x) = 1 & \text{if } |x| < \frac{1}{2} \\ \tilde{\phi}(x) = 0 & \text{if } |x| \geq \frac{4}{5} \\ \|\tilde{\phi}'\|_{L^\infty} \leq 10 \end{cases} \tag{4.1.3}$$

and we let

$$\phi(x, y) = \tilde{\phi}\left(\frac{x - y}{1 + |x|}\right). \tag{4.1.4}$$

Let $K(x,y)$ be the Cauchy kernel as given in (4.1.1). We define K_1 and K_2 by $K_1(x, y) = K(x, y)\phi(x, y)$ and $K_2(x, y) = K(x, y)(1 - \phi(x, y))$. We denote by $\mathscr{C}_1$ and $\mathscr{C}_2$ the integral operators defined by K_1 and K_2 respectively. We show that both $\mathscr{C}_1$ and $\mathscr{C}_2$ are bounded on L^2 in the following subsections by using $T(1)$-theorem. Each subsection corresponds to a condition of $T(1)$-Theorem. Since $\mathscr{C}_A = \mathscr{C}_1 + \mathscr{C}_2$, Main Theorem follows. We now recall the Weak boundedness property and $T(1)$-Theorem of David and Journé:

$T(1)$-THEOREM (David and Journé). *Let T be the integral operator defined by*

$$\langle Tf, g\rangle = \int_{\mathbf{R}}\int_{\mathbf{R}} K(x, y) f(x) g(y)\, dxdy$$

for any bounded functions with compact supports f and g such that $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \emptyset$. *Suppose that an integral kernel $K(x, y)$ satisfies*

(1) *Standard Estimates:*

$$|K(x, y)| \leq C\frac{1}{|x-y|}$$

$$|\nabla_x K(x, y)| + |\nabla_y K(x, y)| \leq C\frac{1}{|x-y|^2}$$

for all $x \neq y \in \mathbf{R}$.

(2) *Weak Boundedness Property: there exist constants N and C such that for any pair of functions φ and ψ in $C_0^\infty(\mathbf{R})$ satisfying $\varphi(x) = \psi(x) = 0$ if $|x| > 1$ and $\|\varphi\|_{C^N} \leq 1$ and $\|\psi\|_{C^N} \leq 1$, for any $x \in \mathbf{R}$ and $t > 0$,*

$$|\langle T\varphi^{x,t}, \psi^{x,t}\rangle| \leq Ct$$

where $\varphi^{x,t}(y) = \varphi\left(\frac{x+y}{t}\right)$.

(3) $T1 \in BMO$.

(4) $T^*1 \in BMO$.

Then T can be extended as an operator bounded on $L^2(\mathbf{R})$.

For notational convenience we put, throughout this paper,

$$Q(x, y) = \frac{A(x) - A(y)}{x - y}. \tag{4.1.5}$$

4.1. Standard estimates

In this subsection we show that the decomposed kernels K_1 and K_2 satisfy the standard estimates. We remark that if A is a polynomial of odd degree, there is no need of decomposing the kernel, namely, the Cauchy kernel $K(x, y)$ itself satisfy the standard estimates. However, we decompose the kernel even in this case since we want to deal with all the polynomials one time.

PROPOSITION 4.1.1. $K_1(x, y)$ satisfies the standard estimates: there exists a positive constant C such that

$$|K_1(x, y)| \leq C \frac{1}{|x-y|} \tag{4.1.6}$$

$$|\nabla_x K_1(x, y)| + |\nabla_y K_1(x, y)| \leq C \frac{1}{|x-y|^2}. \tag{4.1.7}$$

Proof. Note that $K_1(x, y) = 0$ if $|x-y| > \frac{4}{5}(1+|x|)$. Note also that $\nabla_{x,y}\phi(x, y) \neq 0$ only if $\frac{1}{2}(1+|x|) \leq |x-y| \leq \frac{4}{5}(1+|x|)$. Hence

$$|\nabla_{x,y}\phi(x, y)| \leq C \frac{1}{|x-y|}. \tag{4.1.8}$$

Let M be the number in Lemma 3.1. If $|x| > 5M$ and if $|x-y| \leq \frac{4}{5}(1+|x|)$, then $|y| > M$, x and y have the same signs, and $|x| \approx |y|$. It then follows from Lemma 3.1 that

$$|Q(x, y)| \approx |x|^{d-1} \approx |A'(y)|$$

and

$$|\nabla_{x,y}Q(x, y)| \leq \frac{(|A'(x)| + |A'(y)|)|x-y| + |A(x) - A(y)|}{|x-y|^2} \leq C \frac{|y|^{d-1}}{|x-y|}.$$

Therefore,

$$\left|\nabla_{x,y}\left(\frac{1+iA'(y)}{1+iQ(x, y)}\right)\right| \leq \frac{|A''(y)|(1+|Q(x, y)|^2)^{1/2}}{1+|Q(x, y)|^2} + \frac{(1+|A'(y)|^2)^{1/2}|\nabla_{x,y}(Q(x, y))|}{1+|Q(x, y)|^2}$$

$$\leq C\frac{1}{|x-y|}.$$

Combining all these estimates we have

$$K_1(x, y)| = \left|\frac{1}{|x-y|}\frac{1+iA'(y)}{1+iQ(x, y)}\phi(x,y)\right| \leq C\frac{1}{|x-y|}$$

and

$$|\nabla_{x,y}K_1(x, y)| = \left|\nabla_{x,y}\left(\frac{1}{|x-y|}\frac{1+iA'(y)}{1+iQ(x, y)}\phi(x,y)\right)\right| \leq C\frac{1}{|x-y|^2}$$

provided that $|x| > 5M$.

On the other hand, if $|x| \leq 5M$ and $|x-y| \leq \frac{4}{5}(1+|x|)$, then $|y| \leq 10M$ and hence $|A'(y)|$ is bounded. So, it is easy to derive the estimates (4.1.6) and (4.1.7) by using (4.1.8). This completes the proof.

Proposition 4.1.2. $K_2(x, y)$ satisfies the following estimates: there exists a constant C such that

$$|K_2(x, y)| \leq C\frac{1}{|x+y|} \tag{4.1.9}$$

$$|\nabla_x K_2(x, y)| + |\nabla_y K_2(x, y)| \leq C\frac{1}{|x+y|^2}. \tag{4.1.10}$$

Moreover, if either $|x| \leq M$ *or* $|y| \leq M$ *where* M *is the number in Lemma* 3.1, *then* $|K_2(x, y)| + |\nabla_{x,y}K_2(x, y)|$ *is bounded.*

Remark 3.5. We note that (4.1.9) and (4.1.10) are not standard estimates. But $K_2(x, -y)$ satisfies the standard estimates and we can apply $T(1)$-Theorem to $K_2(x, -y)$.

Proof. Since $K_2(x, y) = 0$ if $|x-y| \leq 1/2(|x|+1)$, we assume that $|x-y| > 1/2(|x|+1)$. Then, by the triangular inequality, we have

$$1+|x|+|y| \leq 4|x-y|. \tag{4.1.11}$$

Let M be the number in Lemma 3.1. We first deal with the easier case. Let A be a polynomial of odd degree. If either $|x| > M$ or $|y| > M$, then by Lemma 3.1 (1) and (4.1.8),

$$|K_2(x,y)| \le \left|\frac{1+iA'(y)}{A(x)-A(y)}\right| \tag{4.1.12}$$
$$\le C\frac{|y|^{d-1}+1}{|x-y|(|x|^{d-1}+|y|^{d-1})}$$
$$\le C\frac{1}{|x-y|} \le C\frac{1}{|x+y|+1}.$$

If $|x| \le M$ and $|y| \le M$, then it is easy to get $|K_2(x,y)| \le C$. (4.1.10) can be proved in the same way.

We now suppose that $A(x) = \sum_{k=1}^{n} a_k x^{2k}$. Recall that $|x-y| > 1/2(1+|x|)$. If $|y| > M$, then by Lemma 3.1 (2) and (3) and (4.1.11),

$$|K_2(x,y)| \le \left|\frac{1+iA'(y)}{A(x)-A(y)}\right| \tag{4.1.13}$$
$$\le C\frac{|y|^{d-1}+1}{|x-y|\,|x+y|\,(|x|^{d-2}+|y|^{d-2})} \le C\frac{1}{|x+y|}.$$

If $|y| < M$, then since $|x-y| > 1/2(1+|x|)$, we have

$$|K_2(x,y)| \le \left|\frac{1+iA'(y)}{(x-y)+i(A(x)-A(y))}\right| \tag{4.1.14}$$
$$\le C\frac{M^{d-1}}{|x-y|} \le C\frac{1}{|x+y|+1}.$$

In order to derive (4.1.10), we first observe as in the proof of Proposition 4.1.1 that $\nabla_{x,y}\phi(x,y) \neq 0$ only if $1/2(1+|x|) \le |x-y| \le 4/5(1+|x|)$ and hence

$$|\nabla_{x,y}\phi(x,y)| \le C\frac{1}{1+|x|} \le C\frac{1}{|x+y|+1}.$$

If $|y| > M$, then by Lemma 3.1 and (4.1.11)

$$\left|\nabla_{x,y}\left(\frac{1+iA'(y)}{(x-y)+i(A(x)-A(y))}\right)\right| \tag{4.1.15}$$
$$\le C\frac{(1+|A(x)|)|y|^{d-1}+|y|^{2d-2}}{|x-y|^2|x+y|^2(|x|^{d-2}+|y|^{d-2})^2} \le C\frac{1}{|x+y|^2}.$$

Hence

$$|\nabla_{x,y}K_2(x,y)| = \left|\nabla_{x,y}\left(\frac{1+iA'(y)}{(x-y)+i(A(x)-A(y))}(1-\phi(x,y))\right)\right| \tag{4.1.16}$$

$$\leq C \frac{1}{|x+y|^2}.$$

If $|y| \leq M$, $|A'(y)|$ is bounded and it is easy to derive

$$|\nabla_{x,y} K_2(x, y)| \leq C \frac{1}{|x-y|^2+1} \leq C \frac{1}{|x+y|^2}. \tag{4.1.17}$$

Combining estimates (4.1.12)-(4.1.16), we can derive (4.1.9) and (4.1.10). One also can see that if either $|x| \leq M$ or $|y| \leq M$, then $|K_2(x, y)| + |\nabla_{x,y} K_2(x, y)|$ is bounded. This completes the proof.

4.2. Weak boundedness

We now show that the operators $\mathscr{C}_1$ and $\mathscr{C}_2$ satisfy the weak boundedness property. We first show that $\mathscr{C}_A$ itself is weakly bounded and then show how the weak boundedness of $\mathscr{C}_1$ and $\mathscr{C}_2$ follows.

PROPOSITION 4.2.1 (Weak Boundedness). *Let $A(x)$ be a polynomial of the form (4.1.1). There exists a constant $C > 0$ such that*

$$|\langle \mathscr{C}_A \phi_1^{u,t}, \phi_2^{u,t} \rangle| \leq Ct$$

for any $u \in \mathbf{R}$, for any $t > 0$, and for any $\phi_i \in C_0^\infty$ supported in $\{x \in \mathbf{R} : |x| \leq 1\}$ and $\|\phi_i\|_{C^1} \leq 1$. Here $\phi^{u,t}(x) = \phi\left(\frac{x+u}{t}\right)$.

Proof. Let $u = tv$ and let $\phi^v(x) = \phi(x+v)$. By a change of variables, we have

$$\begin{aligned}\langle \mathscr{C}_A \phi_1^{u,t}, \phi_2^{u,t} \rangle &= \lim_{\varepsilon \to 0} \int\int_{|x-y|>\varepsilon} K(x,y)\, \phi_1^{u,t}(x) \phi_2^{u,t}(y)\, dxdy \\ &= t \lim_{\varepsilon \to 0} \int\int_{|x-y|>\varepsilon} \frac{1}{x-y} \left(\frac{1+iA'(ty)}{1+iQ(tx, ty)}\right) \phi_1^v(x) \phi_2^v(y)\, dxdy.\end{aligned}$$

We then write

$$\begin{aligned}\langle \mathscr{C}_A \phi_1^{u,t}, \phi_2^{u,t} \rangle &= t \lim_{\varepsilon \to 0} \int\int_{|x-y|>\varepsilon} \frac{1}{x-y} \phi_1^v(x) \phi_2^v(y)\, dxdy. \\ &\quad - it \lim_{\varepsilon \to 0} \int\int_{|x-y|>\varepsilon} \frac{tP(tx, ty)}{1+iQ(tx, ty)} \phi_1^v(x) \phi_2^v(y)\, dxdy \\ &:= t(\mathrm{I}_1(v) - i\mathrm{I}_2(v, t))\end{aligned} \tag{4.2.1}$$

where we put

$$P(x, y) = \frac{A'(y) - Q(x, y)}{y - x}. \tag{4.2.2}$$

Hence it suffices to show that I_1 and I_2 are bounded uniformly in v and t. Note that $\mathrm{I}_1(v) = \langle H\phi_1^v, \phi_2^v \rangle$ where H is the Hilbert transform and hence I_1 is bounded. So, it remains to show that I_2 is bounded. By using the polar coordinates, we have

$$\mathrm{I}_2(v, t) = \int_0^\infty \int_{-\pi}^{\pi} \frac{tr\, P(tr, \theta)}{1 + iQ(tr, \theta)} \phi(v, r, \theta)\, d\theta dr \tag{4.2.3}$$

where

$$Q(r, \theta) = Q(r\cos\theta, r\sin\theta)$$

$$P(r, \theta) = P(r\cos\theta, r\sin\theta)$$

$$\phi(v, r, \theta) = \phi_1^v(r\cos\theta)\phi_2^v(r\sin\theta).$$

Note that $\phi_1^v(x)\phi_2^v(y) \neq 0$ only if $|x + v| \leq 1$ and $|y + v| \leq 1$. Therefore, the set $\{r \in [0, \infty) : \phi(v, r, \theta) \neq 0\}$ is included in the interval $[|v| - 2, |v| + 2]$. In particular, $|\{r \in [0, \infty) : \phi(v, r, \theta) \neq 0\}| \leq 4$. Therefore, in order to prove that I_2 is bounded, it suffices to show that

$$F(v, s, r) := \int_{-\pi}^{\pi} \frac{sP(s, \theta)}{1 + iQ(s, \theta)} \phi(v, r, \theta)\, d\theta \tag{4.2.4}$$

is bounded.

Let $A(x) = \sum_{j=1}^d a_j x^j$ be a polynomial of the form (4.1.1). If we put

$$Q_j(x, y) = \frac{x^j - y^j}{x - y} = \sum_{k=1}^{j} x^{j-k} y^{k-1} \quad \text{for } j \geq 1,$$

then

$$Q(x, y) = \frac{A(x) - A(y)}{x - y} = \sum_{j=1}^{d} a_j Q_j(x, y).$$

Since

$$P(x, y) = \frac{A'(y) - Q(x, y)}{y - x} = \sum_{j=2}^{d} a_j \left(\sum_{k=1}^{j=1} y^{k-1} Q_{j-k}(x, y) \right),$$

one can see that

$$|P(s,\theta)| \leq Cs^{d-2} \quad \text{and} \quad \left|\frac{\partial P(s,\theta)}{\partial\theta}\right| \leq Cs^{d-2}. \tag{4.2.5}$$

Let $Q_j(\theta) = Q_j(\cos\theta, \sin\theta)$. Then, by the homogeneity, we have

$$F(v,s,r) = \int_{-\pi}^{\pi} \frac{sP(s,\theta)}{1 + i\sum_{j=1}^{d} s^{j-1} a_j Q_j(\theta)} \phi(v,r,\theta)\, d\theta. \tag{4.2.6}$$

Now suppose that d is an odd integer. Then,

$$Q_d(x,y) = 2^{-1}[x^{d-1} + x^{d-3}(x+y)^2 + \cdots + y^{d-3}(x+y)^2 + y^{d-1}]$$

and hence we have

$$Q_d(\theta) \geq 2^{-1}[(\cos\theta)^{d-1} + (\sin\theta)^{d-1}] \geq 2^{-d}. \tag{4.2.7}$$

Since $|s^{d-1} Q_d(\theta)| \geq 1/2\,|\sum_{j=1}^{d-1} s^{j-1} a_j Q_j(\theta)|$ for all large s by (4.2.7), it follows from (4.2.5) that

$$\left|\frac{sP(s,\theta)}{1 + i\sum_{j=1}^{d} s^{j-1} a_j Q_j(\theta)}\right| \leq C.$$

It also follows that $F(v,s,r)$ is bounded.

We now deal with the case when d is even. Let $2n = d$ and $A(x) = \sum_{j=1}^{n} a_j x^{2j}$. Then, we have

$$F(v,s,r) = \int_{-\pi}^{\pi} \frac{sP(s,\theta)}{1 + i\sum_{j=1}^{n} s^{2j-1} a_j Q_{2j}(\theta)} \phi(v,r,\theta)\, d\theta. \tag{4.2.8}$$

Using the identity $\sin\theta + \cos\theta = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$, we can write Q_{2j} as

$$Q_{2j}(\theta) = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)\sum_{l=1}^{j} (\cos\theta)^{2(j-l)}(\sin\theta)^{2(l-1)}. \tag{4.2.9}$$

Let us put

$$q_j(\theta) = \sum_{l=1}^{j} (\cos\theta)^{2(j-l)}(\sin\theta)^{2(l-1)} \quad \text{for } j = 1, 2, \cdots, n. \tag{4.2.10}$$

Observe that

$$q_j\left(-\theta - \frac{\pi}{4}\right) = q_j\left(\theta - \frac{\pi}{4}\right) \quad \text{and} \quad q_j(\theta) \geq 2^{-j}. \tag{4.2.11}$$

We now begin to estimate F in (4.2.8). We split the integral in (4.2.8) as

$$F(v, s, r) = \int_{-\pi}^{\pi} \frac{sP(s, \theta)}{1 + i\left(\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) \sum_{j=1}^{n} s^{2j-1} a_j q_j(\theta)\right)} \phi(v, r, \theta)\, d\theta$$

$$= \int_{-\pi}^{-\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{0} + \int_{0}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi} \cdots d\theta = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4.$$

Since $|\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) q_n(\theta)| \geq 2^{-p}$ if $-\pi \leq \theta \leq -\pi/2$ or $0 \leq \theta \leq \pi/2$, I_1 and I_3 are bounded by the same reason as above. We now estimate I_2. By translating by $\frac{\pi}{4}$, we have

$$\mathrm{I}_2 = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{sP\left(s, \theta - \frac{\pi}{4}\right)}{1 + i\sqrt{2}\sin\theta \sum_{j=1}^{n} s^{2j-1} a_j q_j\left(\theta - \frac{\pi}{4}\right)} \phi\left(v, r, \theta - \frac{\pi}{4}\right) d\theta. \tag{4.2.12}$$

Let I and II be the real part and the imaginary part of I_2 respectively. Since $q_j\left(-\theta - \frac{\pi}{4}\right) = q_j\left(\theta - \frac{\pi}{4}\right)$ for any j, we have

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sqrt{2}\sin\theta \sum_{j=1}^{n} s^{2j-1} a_j q_j\left(\theta - \frac{\pi}{4}\right)}{1 + \left(\sqrt{2}\sin\theta \sum_{j=1}^{n} s^{2j-1} a_j q_j\left(\theta - \frac{\pi}{4}\right)\right)^2} d\theta = 0.$$

Therefore, by (4.2.5) and (4.2.9), we have

$$|\mathrm{II}| = \Bigg| \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sqrt{2}\sin\theta \sum_{j=1}^{d} s^{2j-1} a_j q_j\left(\theta - \frac{\pi}{4}\right)}{1 + \left(\sqrt{2}\sin\theta \sum_{j=1}^{n} s^{2j-1} a_j q_j\left(\theta - \frac{\pi}{4}\right)\right)^2}$$

$$\times \left(P\left(s, \theta - \frac{\pi}{4}\right)\phi\left(v, r, \theta - \frac{\pi}{4}\right) - P\left(-\frac{\pi}{4}\right)\phi\left(v, r, -\frac{\pi}{4}\right)\right) d\theta \Bigg|$$

$$\leq C \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\left|\sqrt{2}s\sin\theta \sum_{j=1}^{n} s^{2j-1} a_j q_j\left(\theta - \frac{\pi}{4}\right)\right|}{1 + \left(\sqrt{2}\sin\theta \sum_{j=1}^{d} s^{2j-1} a_j q_j\left(\theta - \frac{\pi}{4}\right)\right)^2} s^{d-2} |\theta|\, d\theta.$$

Since $q_n(\theta) \geq 2^{-n}$, we have

$$|\mathrm{II}| \leq C \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{s^{2d-2} |\theta\sin\theta|}{1 + [s^{d-1}\sin\theta]^2} d\theta \leq C.$$

Finally, for $\mathrm{I} = \Re\mathrm{I}_2$, we use the fact $d(\theta) > 2^{-p}$ to have

$$|\mathrm{I}| = \left| \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1 + \left(\sqrt{2} \sin\theta \sum_{j=1}^{n} s^{2j-1} a_j q_j\left(\theta - \frac{\pi}{4}\right)\right)^2} \times P\left(s, \theta - \frac{\pi}{4}\right) \phi\left(v, r, \theta - \frac{\pi}{4}\right) d\theta \right|$$

$$\leq C \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{s^{2d-1}}{1 + [s^{d-1} \sin(\theta)]^2} d\theta \leq C.$$

This proves that I_2 is bounded. I_2 can be proved to be bounded in a similar way.

PROPOSITION 4.2.2. *$\mathscr{C}_1$ and $\mathscr{C}_2$ satisfy the weak boundedness property.*

Proof. Proofs are similar to the proof of Proposition 4.2.1. As in (4.2.1), we have

$$\begin{aligned} \langle \mathscr{C}_1 \psi_1^{u,t}, \psi_2^{u,t} \rangle &= t \lim_{\varepsilon \to 0} \int \int_{|x-y|>\varepsilon} \frac{1}{x-y} \phi(tx, ty)\, \psi_1^v(x)\, \psi_2^v(y)\, dxdy \\ &\quad - it \lim_{\varepsilon \to 0} \int \int_{|x-y|>\varepsilon} \frac{tP(tx, ty)}{1 + iQ(tx, ty)} \phi(tx, ty)\, \psi_1^v(x)\, \psi_2^v(y)\, dxdy \\ &:= t(\mathrm{I}_1(t, v) - i\mathrm{I}_2(t, v)). \end{aligned}$$

Here ϕ is the cut-off function defined in (4.1.4). For I_2, (4.2.2) can be changed as

$$\mathrm{I}_2 = \int_0^\infty \int_{-\pi}^{\pi} \frac{trP(tr, \theta)}{1 + iQ(t_r, \theta)} \phi(t, v, r, \theta)\, d\theta dr$$

where

$$\psi(t, r, \theta, v) = \phi(tr\cos\theta, tr\sin\theta) \psi_1^v(r\cos\theta, r\sin\theta) \psi_1^v(r\cos\theta, r\sin\theta).$$

Again since the set $\{r \in [0, \infty) : \psi(t, v, r, \theta) \neq 0\}$ is included in the interval $[|v| - 2, |v| + 2]$, it suffices to show that

$$F(t, r, s) := \int_{-\pi}^{\pi} \frac{trP(tr, \theta)}{1 + iQ(tr, \theta)} \psi(t, v, r, \theta) d\theta$$

is bounded. Now we can repeat the same argument as in Proposition 4.2.1 to show that I_2 is bounded.

I_1 looks almost like a truncated Hilbert transform. However, unlike the usual truncation, the size of truncation in I_1 varies depending on x. So, we include the proof of the boundedness of I_1 even if it follows a standard argument. Note that, since $\phi(tx, ty) = 1$ if $|x - y| \leq 1/2(t^{-1} + |x|)$ and $\psi_1^v(x)\psi_2^v(y) = 0$ if $|x - y|$

≥ 2, $\phi(tx, ty)\psi_1^v(x)\psi_2^v(y) = \psi_1^v(x)\psi_2^v(y)$ if either $|x| > 4$ or $t \leq \frac{1}{4}$. Note also that, if $|v| > 8$, then $\psi_1^v(x) \neq 0$ only if $|x| \geq 7$. Hence if either $|v| > 8$ or $t \leq 1/4$, then $\phi(tx, ty)\psi_1^v(x)\psi_2^v(y) = \psi_1^v(x)\psi_2^v(y)$ and hence $\mathrm{I}_1(t, v) = \langle H\psi_1^v, \psi_2^v\rangle$ where H is Hilbert transform. Therefore, I_1 is bounded.

Suppose now that $|v| \leq 8$ and $t \geq 1/4$. Then

$$\begin{aligned}\mathrm{I}_1(t, v) &= \frac{1}{2}\int\int \frac{1}{x-y}\left(\phi(tx, ty)\psi_1^v(x)\psi_2^v(y) - \phi(ty, tx)\psi_1^v(y)\psi_2^v(x)\right) dxdy \\ &= \frac{1}{2}\int\int \frac{1}{x-y}\left(\phi(tx, ty) - \phi(ty, tx)\right)\psi_1^v(x)\psi_2^v(y)\, dxdy \\ &+ \frac{1}{2}\int\int \frac{1}{x-y}\phi(tx, ty)\left(\psi_1^v(x)\psi_2^v(y) - \psi_1^v(y)\psi_2^v(x)\right) dxdy.\end{aligned}$$

But from (4.1.3) and (4.1.4), we obtain

$$|\phi(tx, ty) - \phi(ty, tx)| \leq 10|x-y|(h(t, x, y) + h(t, y, x))$$

where

$$h(t, x, y) = \begin{cases} \dfrac{1}{\frac{1}{t} + |x|} & \text{for } \frac{1}{2}\left(\frac{1}{t} + |x|\right) \leq |x-y| \leq \frac{4}{5}\left(\frac{1}{t} + |x|\right), \\ 0 & \text{otherwise.}\end{cases}$$

Since $\psi_1^v(x)\psi_2^v(y) = 0 = \psi_1^v(y)\psi_2^v(x)$ for $|x| + |y| > 20$,

$$\begin{aligned}|\mathrm{I}_1(t, v)| &\leq C\int\int_{|x|+|y|\leq 20}(h(t, x, y) + h(t, y, x))dxdy \\ &\quad + \int\int_{|x|+|y|\leq 20}\phi(ty, tx)(\|\psi_1^v\|_{C^1} + \|\psi_2^v\|_{C^1})dxdy \\ &\leq C.\end{aligned}$$

This completes the proof.

4.3. Estimates for $\mathscr{C}_1 1$

We now show that $\mathscr{C}_1 1, \mathscr{C}_1^* 1 \in \mathrm{BMO}$.

PROPOSITION 4.3.1. *$\mathscr{C}_1 1 \in BMO$.*

Proof. Since $\phi(x, y) = 0$ if $|x-y| > \frac{4}{5}(|x| + 1)$ and 1 if $|x-y|$

$< \frac{1}{2}(|x|+1)$, we can divide $\mathscr{C}_1 1(x)$ as follows;

$$\begin{aligned}\mathscr{C}_1 1(x) &= \text{p.v.} \int_{-\infty}^{\infty} K_1(x, y)dy \\ &= \text{p.v.} \int_{|x-y|\leq \frac{1}{2}} \frac{1}{x-y}\left(\frac{1+iA'(y)}{1+iQ(x, y)} - 1\right) \\ &\quad + \int_{\frac{1}{2}\leq |x-y| \leq \frac{1}{2}(|x|+1)} \frac{1}{x-y} \frac{1+iA'(y)}{1+iQ(x, y)} dy \\ &\quad + \int_{\frac{1}{2}(|x|+1) \leq (|x-y|) \leq \frac{4}{5}(|x|+1)} \frac{1}{x-y} \frac{1+iA'(y)}{1+iQ(x, y)} \phi(x, y)dy \\ &:= \mathrm{I}_1(x) + \mathrm{I}_2(x) + \mathrm{I}_3(x).\end{aligned}$$

Let

$$\Psi(x, y) = \frac{1+iA'(y)}{1+iQ(x, y)} - 1 = \frac{1+iA'(y)}{1+iQ(x, y)} - \frac{1+iA'(y)}{1+iA'(y)}.$$

Then, by the mean value theorem and Lemma 3.1 (1) and (3), we have

$$|\Psi(x, y)| \leq |x-y| \sup_{|x-y|<1} \left|\frac{\partial \Psi}{\partial y}(x, y)\right| \leq C|x-y|.$$

Therefore, $\mathrm{I}_1(x)$ is bounded. Let M be the large number given in Lemma 3.1. If $|x| \leq M$ and if $|x-y| \leq \frac{4}{5}(|x|+1)$, then, $|y| \leq 2M$. Hence, I_2 and I_3 are bounded if $|x| \leq M$.

We now handle the case when $|x| > M$. Assume that $x > M$ without loss of generality. If $\frac{1}{2}(x+1) \leq |x-y| \leq \frac{4}{5}(1+x)$ and if $|x| \geq M$, then by Lemma 3.1

$$\left|\frac{1}{x-y}\left(\frac{1+iA'(y)}{1+iQ(x, y)}\right)\right| \leq C|x|^{-1}$$

and therefore I_3 is bounded for $x > M$. Hence we only need to show $\mathrm{I}_2 \in \mathrm{BMO}$. We decompose I_2 as

$$\begin{aligned}\text{(4.3.1)} \qquad \mathrm{I}_2(x) &= \int_{1\leq |x-y| \leq \frac{1}{2}(x+1)} \frac{1}{x-y}\left(\frac{1+iA'(y)}{1+iQ(x, y)} - \frac{A'(y)}{Q(x, y)}\right) dy \\ &\quad + \int_{1\leq |x-y| \leq \frac{1}{2}(x+1)} \frac{A'(y)}{A(x)-A(y)} dy\end{aligned}$$

$$:= F(x) + G(x).$$

If $d = 1$, clearly $\mathrm{I}_2 \in$ BMO. Assume that $d > 1$. Then, by Lemma 3.1, we have

$$\left| \frac{1 + iA'(y)}{1 + iQ(x, y)} - \frac{A'(y)}{Q(x, y)} \right| \leq C \frac{| Q(x,y) - A'(y) |}{| Q(x, y) |^2} \leq Cx^{-d+1}$$

if $x > M$ and $1 \leq | x - y | \leq \frac{1}{2}(x + 1)$. Hence $F(x)$ is bounded for $x \geq M$. So, in order to prove that $\mathrm{I}_2 \in$ BMO, it suffices to show

(4.3.2) $$| G'(x) | < Cx^{-1} \quad \text{for sufficiently large } x$$

by Lemma 2.1. To prove (4.3.2), note that

$$\begin{aligned} G'(x) &= \int_{1 \leq |x-y| \leq \frac{1}{2}(x+1)} \left(\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right)\left(\frac{A'(y)}{A(x) - A(y)}\right) dy + J(x) \\ &= \int_{1 \leq |x-y| \leq \frac{1}{2}(x+1)} \frac{A''(y)[A(x) - A(y)] - A'(y)[A'(x) - A'(y)]}{[A(x) - A(y)]^2} dy + J(x) \end{aligned}$$

where

$$J(x) = \frac{1}{2} \frac{A'\left(\frac{1}{2}x - \frac{1}{2}\right)}{A(x) - A\left(\frac{1}{2}x - \frac{1}{2}\right)} + \frac{1}{2} \frac{A'\left(\frac{3}{2}x + \frac{1}{2}\right)}{A(x) - A\left(\frac{3}{2}x + \frac{1}{2}\right)}.$$

Clearly $J(x)$ is bounded for $x > M$. If $1 \leq | x - y | \leq \frac{1}{2}(x + 1)$ and $x \geq M$, then

$$\begin{aligned} | A(x) - A(y) | &\geq C | x |^{d-1} | y - x | \\ A''(y)[A(x) - A(y)] &= A''(y)[A'(y)(x - y) + (x - y)^2 O(| x |^{d-2})] \\ A'(y)[A'(x) - A'(y)] &= A'(y)[A''(y)(x - y) + (x - y)^2 O(| x |^{d-2})] \end{aligned}$$

and therefore

$$| A''(y)[A(x) - A(y)] - A'(y)[A'(x) - A'(y)]| \leq C | x - y |^2 | x |^{d-2}.$$

Hence for $x > M$,

$$| G'(x) | \leq C \int_{1 \leq |x-y| \leq \frac{1}{2}x} \frac{x^{2d-4}}{x^{2d-2}} dy \leq Cx^{-1}.$$

This completes the proof.

PROPOSITION 4.3.2. $\mathscr{C}_1^* 1 \in BMO$.

Proof. Proposition 4.3.2 can be proved in the same way as Proposition 4.3.1. We include a sketch of the proof for reader's sake. As in the proof of Proposition 4.3.1, we divide $\mathscr{C}_1^* 1$ as follows;

$$\begin{aligned}
\mathscr{C}_1^* 1(x) &= \text{p.v.} \int_{-\infty}^{\infty} K_1(y, x)\, dy \\
&= \text{p.v.} \int_{|x-y|<\frac{1}{2}} \frac{1}{x-y} \left(\frac{1 + iA'(x)}{1 + iQ(x, y)} - 1 \right) \\
&\quad + \int_{\frac{1}{2} \le |x-y| \le \frac{1}{2}(|y|+1)} \frac{1}{x-y} \left(\frac{1 + iA'(x)}{1 + iQ(x, y)} \right) dy \\
&\quad + \int_{\frac{1}{2}(|y|+1) \le |x-y| \le \frac{4}{5}(|y|+1)} \frac{1}{x-y} \left(\frac{1 + iA'(x)}{1 + iQ(x, y)} \right) \phi(y,x)\, dy \\
&:= \mathrm{I}_1(x) + \mathrm{I}_2(x) + \mathrm{I}_3(x).
\end{aligned}$$

Then, by the same reasons as in the proof of Proposition 4.3.1, I_1 is bounded and I_2 and I_3 are bounded if $|x| \le M$.

Assume that $|x| \ge M$ and that $x > 0$. Then, by Lemma 3.1, we have

$$\begin{aligned}
|\mathrm{I}_3(x)| &\le \int_{\frac{1}{2}(|y|+1) \le |x-y| \le \frac{4}{5}(|y|+1)} \frac{1}{|x-y|} \left| \frac{1 + iA'(x)}{1 + iQ(x, y)} \right| dy \\
&\le Cx^{d-1} \int_{\frac{1}{2}x \le |y|} \frac{1}{|y|^d}\, dy \le C.
\end{aligned}$$

In order to prove that $\mathrm{I}_2 \in \mathrm{BMO}$, it suffices to show that

$$G_*(x) = \int_{\frac{1}{2} \le |x-y| \le \frac{1}{2}(x+1)} \left(\frac{A'(x)}{A(x) - A(y)} \right) dy$$

satisfies the estimates $|G_*'(x)| < Cx^{-1}$ for $x \ge M$ as in the proof of Proposition 4.3.1. We then note that

$$G_*'(x) = \int_{1 \le |x-y| \le \frac{1}{4}x} \left(\frac{\partial}{\partial y} + \frac{\partial}{\partial x} \right) \left(\frac{A'(x)}{A(x) - A(y)} \right) dy + J_*(x)$$

where

$$J_*(x) = \frac{2}{3} \frac{A'(x)}{A(x) - A\left(\frac{2}{3}x - \frac{1}{3}\right)} + \frac{A'(x)}{A(x) - A(2x+1)}.$$

One can see as in the proof of Proposition 4.3.1 that if $1 \leq |x-y| \leq \frac{1}{2}(|y|+1)$ and $x \geq M$, then

$$\left| \left(\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right) \left(\frac{A'(x)}{A(x) - A(y)}\right) \right| \leq Cx^{-2}$$

and therefore $G'_*(x) = O(x^{-1})$ for $x > M$. This completes the proof.

4.4 Estimates for $\mathscr{C}_2 1$

In this subsection, we finally show that $\mathscr{C}_2 1$, $\mathscr{C}_2^* 1 \in$ BMO.

PROPOSITION 4.4.1 *$\mathscr{C}_2 1 \in BMO$.*

Proof. For given x, we let $Q = [-2|x|, 2|x|]$. Then, $\mathscr{C}_2 \chi_Q(0)$ is finite. So, $\mathscr{C}_2 1(x)$ is understood to be

$$\mathscr{C}_2 1(x) = \mathscr{C}_2 \chi_Q(x) - \mathscr{C}_2 \chi_Q(0) + \int_{Q^c} [K_2(x, y) - K_2(0, y)]\, dy$$

where χ_Q is the characteristic function for Q. Then, by Proposition 4.1.2

$$\int_{Q^c} [K_2(x, y) - K_2(0, y)]\, dy \leq \int_{|y|>2|x|} \sup_{|\xi| \leq |x|} |\nabla_x K_2(\xi, y)|\, |x|\, dy$$

$$\leq C|x| \int_{|y|>2|x|} \frac{1}{|y|^2}\, dy \leq C.$$

So, it remains to show that $\mathscr{C}_2 \chi_Q(x)$ and $\mathscr{C}_2 \chi_Q(0)$ belong to BMO. If $|x| < 2M$, then both $\mathscr{C}_2 \chi_Q(x)$ and $\mathscr{C}_2 \chi_Q(0)$ are bounded. Suppose that $|x| \geq 2M$. Then,

$$\mathscr{C}_2 \chi_Q(0) = \int_{1/2<|y|<4/5} K_2(0, y)\, dy + \int_{4/5 \leq |y| \leq 2|x|} \frac{1 + iA'(y)}{y + iA(y)}\, dy.$$

The first integral in the right hand side is bounded and the second one is $\log(2|x| + iA(|x|)) + C$ which belongs to BMO by Lemma 2.1. Finally we show that $\mathscr{C}_2 \chi_Q(x) \in$ BMO.

$$\mathscr{C}_2 \chi_Q(x) = \int_{\substack{|y| \leq 2|x| \\ 1/2(1+|x|)<|x-y|<4/5(1+|x|)}} K_2(x, y)\, dy + \int_{\substack{|y| \leq 2|x| \\ 4/5(1+|x|)<|x-y|}} K_2(x, y)\, dy$$

$$= \mathrm{I}_1(x) + \mathrm{I}_2(x).$$

If $|y| \leq 2|x|$ and $1/2(1+|x|) < |x-y| < 4/5(1+|x|)$, then $|K_2(x, y)|$

$\leq C|x+y|^{-1} \leq C|x|^{-1}$ and hence

$$|\mathrm{I}_1(x)| \leq C \int_{1/2|x| \leq |y| \leq 9/5|x|} |K_2(x, y)|\, dy \leq C.$$

If $4/5(1+|x|) \leq |x-y|$, then $\phi(x, y) = 0$ and hence

$$\begin{aligned}
\mathrm{I}_2(x) &= \int_{-2|x|}^{1/5|x|-4/5} + \int_{9/5|x|+4/5}^{2|x|} \frac{1+iA'(y)}{(x-y)+i(A(x)-A(y))}\, dy \\
&= \log((x+2|x|) + i(A(x) - A(-2|x|))) \\
&\quad - \log((x - 1/5|x| + 4/5) + i(A(x) - A(1/5|x| - 4/5))) \\
&\quad - \log((x - 2|x|) + i(A(x) - A(2|x|))) \\
&\quad + \log((x - 9/5|x| - 4/5) + i(A(x) - A(9/5|x| - 4/5))) \in \mathrm{BMO}.
\end{aligned}$$

This completes the proof.

Proposition 4.4.2. *$\mathscr{C}_2^*1 \in BMO$.*

Proof. We may assume $d = \deg A \geq 2$. Recall that

$$\mathscr{C}_2^*1(x) = \int_{-\infty}^{\infty} \frac{1+iA'(y)}{(x-y)+i(A(x)-A(y))} (1 - \phi(y, x))\, dy.$$

If $|x| < 2M$, then

$$\begin{aligned}
|\mathscr{C}_2^*1(x)| &\leq C \int_{|x-y|>\frac{1}{2}} \frac{1}{|x-y| + |A(x) - A(y)|}\, dy \\
&\leq C \int_{-\infty}^{\infty} \frac{1}{1+|y|^d}\, dy < C.
\end{aligned}$$

We now suppose that $|x| \geq 2M$ and assume that $x > 0$ without loss of generality. We then split $\mathscr{C}_2^*1$ as

$$\begin{aligned}
\mathscr{C}_2^*1(x) &= \int_{|y|>2x} + \int_{-2x \leq y \leq -M} + \int_{-M<y\leq 2x} \overline{K_2(y, x)}\, dy \\
&= \mathrm{I}_1(x) + \mathrm{I}_2(x) + \mathrm{I}_3(x).
\end{aligned}$$

Estimates of I_1 and I_3 are easier parts. In fact, by Lemma 3.1,

$$|\mathrm{I}_1(x)| \leq C|A'(x)| \int_{|y|>2x} \frac{1}{|y|^{d-1}}\, dy < C,$$

and

$$|\mathrm{I}_3(x)| \leq C \int_{\substack{-M \leq y \leq M \\ 1/2(1+|y|)<|x-y|}} \frac{|A'(x)|}{|A(x)|} dy$$

$$+ C \int_{\substack{M \leq y \leq 2x \\ 1/2(1+|y|)<|x-y|}} \frac{|A'(x)|}{|x-y|[1+|x+y|(x^{d-2}+|y|^{d-2})]} dy$$

$$\leq \frac{C}{x}^{-1} + Cx^{d-1} \int_{\frac{1}{10}x}^{2x} \frac{1}{y^d} dy \leq C.$$

For $\mathrm{I}_2(x)$, we let

$$f(x) = A'(x) \int_{-2x}^{-M} \frac{1}{(x-y)+i(A(x)-A(y))} (1-\phi(y, x))\, dy.$$

Note that if $-2x \leq y \leq -M$ and $x > 2M$, then $4/5(1+|y|) < |x-y|$ and hence $\phi(y, x) = 0$. If $A(x)$ is of odd degree, then it is easy to see that $f(x)$ is bounded. Therefore, we assume that A is of even degree. We use Lemma 2.1. Note that

$$f'(x) = A''(x) \int_{-2x}^{-M} \frac{1}{(x-y)+i(A(x)-A(y))} dy$$

$$+ A'(x) \int_{-2x}^{-M} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) \left(\frac{1}{(x-y)+i(A(x)-A(y))}\right) dy + E(x)$$

where

$$E(x) = \frac{A'(x)}{(x-M)-i(A(x)-A(M))} - \frac{A'(x)}{3x-i(A(x)-A(-2x))}.$$

It then follows that

$$f'(x) = \int_{-2x}^{-M} \frac{A''(x)(x-y)-2A'(x)}{[(x-y)+i(A(x)-A(y))]^2} dy$$

$$+ i \int_{-2x}^{-M} \frac{A''(x)(A(y)-A(x))+A'(x)(A'(y)+A'(x))}{[(x-y)+i(A(x)-A(y))]^2} dy + E(x)$$

$$= J_1(x) + iJ_2(x) + E(x).$$

Since $|A''(x)(x-y) - 2A'(x)| \leq C(x^{d-2}|x-y| + x^{d-1})$ by Lemma 3.1 (2) and $|x-y| \approx |x|$ if $-2x < y < -M$, we have

$$
\begin{aligned}
|J_1(x)| &\le C\int_{-2x}^{-M} \frac{x^{d-2}|x-y| + x^{d-1}}{|x-y|^2[1+|x+y|(x^{d-2}+|y|^{d-2})]^2}\,dy \\
&\le Cx^{d-3}\int_{-2x}^{-M} \frac{1}{1+|x+y|^2 x^{2(d-2)}}\,dy \le Cx^{-1}.
\end{aligned}
$$

We now estimate $J_2(x)$. Since A is even, we have

$$
\begin{aligned}
&A''(x)(A(y) - A(x)) + A'(x)(A'(y) + A'(x)) \\
&\quad = A''(-x)(A(y) - A(-x)) - A'(-x)(A'(y) - A'(-x)) \\
&\quad = (x+y)^2 \sum_{j=2}^{d} \frac{1}{j!}[A''(-x)A^{(j)}(-x) - A'(-x)A^{(j+1)}(-x)](x+y)^{j-2}
\end{aligned}
$$

and hence

$$
\begin{aligned}
&|A''(x)[A(y) - A(x)] + A'(x)[A'(y) + A'(x)]| \\
&\quad \le C|x+y|^2[|x+y|^{d-2} + x^{2(d-2)}].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|J_2(x)| &\le C\int_{-2x}^{-M} \frac{|x+y|^2(|x+y|^{d-2} + x^{2(d-2)})}{|x|^2[1+|x+y|(|x|^{d-2}+|y|^{d-2})]^2}\,dy \\
&\le \frac{C}{x^2}\int_{-x}^{-M+x} \frac{t^2(t^{d-2} + |x|^{2(d-2)})}{(1+|t||x|^{d-2})^2}\,dt \le x^{-1}.
\end{aligned}
$$

It is easy to see that $|E(x)| \le C|x|^{-1}$. In conclusion, we have $|f'(x)| \le C|x|^{-1}$ if $|x| \ge 2M$. By Lemma 2.1, $f \in \mathrm{BMO}$. It follows that $\mathrm{I}_2 \in \mathrm{BMO}$. This completes the proof.

5. Non-L^2-boundedness

In this section, we give two examples of A for which $\mathscr{C}_A$ are not L^2-bounded. The first example of A has two many zeros while the derivative of the second A grows too fast relatively to A itself.

THEOREM 5.1. *Let $A'(x) = x\sin x$. Then, $\mathscr{C}_A$ is not bounded on L^2.*

Proof. For each positive integer n, we let f_n be the characteristic function on $[2n\pi + \pi/4, 2n\pi + 3\pi/4]$. Then, $\|f_n\|_2 = \pi/2$ for each n. Note that

$$|A(x) - A(y)| = |-x\cos x + y\cos y + \sin x - \sin y| \le 2(|x-y| + |y| + 1).$$

If $n \ge 2, j \ge 2$, and if $2n\pi + \pi/4 \le y \le 2n\pi + 3\pi/4$ and $2(n+j)\pi \le x \le 2(n+j+1)\pi$, then

$$|A(x) - A(y)| \le 2(|x-y| + |y| + 1) \le \frac{1}{2} y \sin y\,(x-y).$$

It then follows that

$$\begin{aligned}
|\mathscr{C}_A f_n(x)| &\ge |\Im \mathscr{C}_A f_n(x)| \\
&= \left| \int_{2n\pi+\pi/4}^{2n\pi+3\pi/4} \frac{y\sin y(x-y) + (A(x)-A(y))}{(x-y)^2 + (A(x)-A(y))^2}\,dy \right| \\
&\ge \int_{2n\pi+\pi/4}^{2n\pi+3\pi/4} \frac{y\sin y\,|x-y| - |A(x)-A(y)|}{(x-y)^2 + (A(x)-A(y))^2}\,dy \\
&\ge \frac{1}{10}\int_{2n\pi+\pi/4}^{2n\pi+3\pi/4} \frac{y\sin y\,|x-y|}{(|x-y| + |y| + 1)^2}\,dy \ge C\frac{nj}{(n+j)^2}
\end{aligned}$$

for some constant C. Therefore,

$$\|\mathscr{C}_A f_n\|_2^2 \ge \sum_{j=2}^{\infty} \int_{2(n+j)\pi}^{2(n+j+1)\pi} |\mathscr{C}_A f_n(x)|^2 dx \ge C \sum_{j=2}^{\infty} \frac{n^2 j^2}{(n+j)^4}.$$

Since

$$\sum_{j=2}^{\infty} \frac{j^2}{(n+j)^4} \ge \frac{C}{n},$$

we have $\|\mathscr{C}_A f_n\|_2 \ge C\sqrt{n}\,\|f_n\|_2$ for each $n \ge 2$. This completes the proof.

THEOREM 5.2. *Let $A(x) = \exp(x^2)$. Then, $\mathscr{C}_A$ is not bounded on L^2.*

Proof. For each positive integer n, we let f_n be the characteristic function on $[n, n+1]$. Then, $\|f_n\|_2 = 1$ for each n. If $x \in [0, 1]$, we have

$$|\mathscr{C}_A f_n(x)| \ge C\int_n^{n+1} \frac{y\exp(y^2)\,|A(x)-A(y)|}{(x-y)^2 + (A(x)-A(y))^2}\,dy \ge C$$

for some constant C independent of n. So $\|\mathscr{C}_A f_n\|_2 \ge Cn\,\|f_n\|_2$ for each n. This completes the proof.

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