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THE UNIQUENESS OF POSITIVE SOLUTIONS OF PARABOLIC EQUATIONS OF DIVERGENCE FORM ON AN UNBOUNDED DOMAIN

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§1. Introduction

Let $R^{n+1} = R^n \times R$ be the $(n+1)$-dimensional Euclidean space $(n \geq 1)$. For $X \in R^{n+1}$, we write $X = (x, t)$ with $x \in R^n$ and $t \in R$. We consider parabolic operators of the following form:

$$L = \frac{\partial}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x, t) \frac{\partial}{\partial x_j}, \tag{1}$$

where the coefficients a_{ij} are measurable functions with $a_{ij} = a_{ji}$ and satisfy

$$M^{-1} |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x, t)\, \xi_i \xi_j,\ a_{ij}(x, t) \leq M \tag{2}$$

with some positive constant M, for every $\xi = (\xi_1, \ldots, \xi_n) \in R^n$ and almost all $(x, t) \in R^{n+1}$

For an unbounded domain Ω in R^{n+1}, we put

$$H_0(\Omega, L) = \{u \geq 0\,; Lu = 0 \text{ on } \Omega,\ u = 0 \text{ on } \partial_p \Omega\},$$

where $\partial_p \Omega$ denotes the parabolic boundary of Ω.

In this paper, we assume that for every $\tau \in R$, $D_\tau = \{x \in R^n\,; (x, \tau) \in \Omega\}$ is a bounded Lipschitz domain. Then $H_0(\Omega)$ coincides with $H_0(\Omega \cap R^n \times (-\infty, a))$ for every $a \in R$. For a bounded Lipschitz domain D in R^n and a continuous function $\varphi > 0$ on $(-\infty, a)$, we put

$$\Omega(D, \varphi) = \{(x, t) \in R^{n+1}\,; t < a,\ \varphi(t)^{-1} x \in D\}.$$

By using a special form of the boundary Harnack principle for $\Omega(D, \varphi)$, we shall show the following

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THEOREM 1. *Let* D *be a bounded Lipschitz domain in* R^n *and* $\varphi > 0$ *a* $1/2$*-Hölder continuous function on* $(-\infty, a)$ *for some* $a \in (-\infty, \infty]$. *If* $\liminf_{\tau \to -\infty} |\tau|^{-1/2}\varphi(\tau) < \infty$, *then there exists* $u \neq 0$ *such that*

$$H_0(\Omega(D, \varphi), L) = \{cu \,;\, c \geq 0\}.$$

§2. Some estimates of L-parabolic measures

For a domain Ω in R^{n+1} and a point (x, t) in Ω, we denote by $\omega_\Omega^{(x,t)}$ the L-parabolic measure at (x, t) with respect to Ω.

First we recall the Aronson estimate of the fundamental solution of L. For an $M > 0$, we denote by $\mathscr{L}(M)$ the class of the parabolic operators of the form (1) satisfying (2).

LEMMA 1 (see [1]). *Let* $\Gamma(x, t\,;\, y, s)$ *be the fundamental solution of* $L \in \mathscr{L}(M)$. *Then there exist positive constants* $C_1, C_2, \gamma_1, \gamma_2$ *depending only on* M, n *such that for all* $(x, t), (y, s) \in R^{n+1}$,

$$C_1\, g_{\gamma_1}(x, t\,;\, y, s) \leq \Gamma(x, t\,;\, y, s) \leq C_2\, g_{\gamma_2}(x, t\,;\, y, s),$$

where g_γ *is the fundamental solution of* $\partial/\partial t - \gamma\Delta$.

We shall use parabolic dilations. For $\alpha > 0$, we denote by τ_α the parabolic dilation defined by $\tau_\alpha(x, t) = (\alpha x, \alpha^2 t)$. We note that $\mathscr{L}(M)$ is invariant for every parabolic dilation, that is, for any $L \in \mathscr{L}(M)$ and $\alpha > 0$, $L_\alpha \in \mathscr{L}(M)$, where $L_\alpha(u \circ \tau_\alpha) = Lu$.

For a closed ball B in R^n, we put

$$T(B) = \{(x, t)\,;\, t < 0,\ (-t)^{-1/2}x \in B\},$$

and for $r > 0$ and a starlike open neighborhood V of 0 in R^n, we put

$$V_r = \{(x, t)\,;\, r^{-1}x \in V,\ |t| < r^2\}.$$

LEMMA 2. *Let* V *be a starlike open neighborhood of* 0 *in* R^n *and* B *a closed ball contained in* V. *For* $0 < s < 1$, *there exists* $\nu > 0$ *such that for any* $L \in \mathscr{L}(M)$ *and* $X \in V_s$,

$$\omega_{V_1}^X\,(\partial V_1 \cap T(B) > \nu.$$

Proof. Take a closed ball B_1 contained in the interior of B. Put

$$v(x, t) = \int_{R^n \setminus B_1} \Gamma(x, t; y, -1) dy$$

and

$$w(x, t) = \omega_{V_1}^{(x,t)} (\partial V_1 \cap T(B)).$$

By Lemma 1,

$$v(x, t) \geq C_1 \int_{R^n \setminus B_1} g_{\gamma_1}(x, t; y, -1) dy,$$

so that by the maximum principle there exists a constant $K > 0$ such that

$$1 - w \leq Kv \text{ on } V_1.$$

By Lemma 1, we can choose $(\xi, \tau) \in V_1$ with $-1 < \tau < -s^2$ such that

$$v(\xi, \tau) < \frac{1}{2K}.$$

By the Harnack inequality (see [4], p. 102), for any $(x, t) \in V_s$,

$$w(x, t) \geq Cw(\xi, \tau) > \frac{C}{2},$$

which shows Lemma 2.

Remark 1. By using parabolic dilations, Lemma 2 implies that for $r > 0$ and for $0 < s < 1$,

$$\omega_{V_r}^{X}(\partial V_r \cap T(B)) > \nu \quad \text{for} \quad X \in V_{sr},$$

where ν is the constant in Lemma 2.

The above lemma gives the following

LEMMA 3. *Let V be a starlike open neighborhood of* 0 *in R^n and B a closed ball contained in V. For any $\varepsilon > 0$, there exists $s > 0$ such that for any $L \in \mathcal{L}(M)$ and $X \in V_{sr} \setminus T(B)$,*

$$\omega_{V_r \setminus T(B)}^{X}(\partial V_r \setminus T(B)) < \varepsilon.$$

This shows that 0 is a regular point in $V_r \setminus T(B)$ with respect to the Dirichlet problem.

Proof. By using parabolic dilations, we may assume that $r = 1$. For $L \in \mathscr{L}(M)$, we put

$$u_L(x, t) = \omega^X_{V_1 \setminus T(B)}(\partial V_1 \setminus T(B)).$$

For $0 < s < 1$ and $(x, t) \in V_s$, we have

$$u_L(x, t) \leq \omega^{(x,t)}_{V_1}(\partial V_1 \setminus T(B)) \leq 1 - \nu,$$

where ν is the constant in Lemma 2. Since $u_L \circ \tau_s(x, t) = u_L(sx, s^2 t)$ is a solution of $L_s u = 0$, by the maximum principle,

$$u_L \circ \tau_s \leq (1 - \nu) u_{L_s} \quad \text{on} \quad V_1 \setminus T(B),$$

and inductively we have for every integer $k > 0$,

$$u_L \circ \tau_{s^k} \leq (1 - \nu)^k u_{L_{s^k}} \quad \text{on} \quad V_1 \setminus T(B),$$

which implies

$$u_L \leq (1 - \nu)^k \quad \text{on} \quad V_{s^k} \setminus T(B).$$

This shows Lemma 3.

§3. The existence of positive solutions

A domain Ω in R^{n+1} is said to be spatially bounded if for every $\tau \in R$, $D_\tau = \{x \in R^n\,;\,(x, \tau) \in \Omega\}$ is bounded. A domain Ω in R^{n+1} is called a $(1, 1/2)$-Lipschitz domain with the Lipschitz constant m if for every boundary point $(y, s) \in \partial\Omega$, there exist a coordinate system $(x_1, \ldots, x_n)$ of R^n, a function f on $R^{n-1} \times R$ and a neighborhood U of (y, s) such that for every $x^*, \xi^* \in R^{n-1}$ and every $t, \tau \in R$,

$$|f(x^*, t) - f(\xi^*, \tau)| \leq m(|x^* - \xi^*| + |t - \tau|^{1/2})$$

and

$$\Omega \cap U = \{(x^*, x_n, t) \in U\,;\, x_n > f(x^*, t)\}. \tag{3}$$

Let D be a bounded Lipschitz domain in R^n, $\tau \in R$ and $m > 0$. A point $X \in R^{n+1}$ is called a proper inner point with respect to (D, τ, m) if $X \in \Omega$ for every $(1, 1/2)$-Lipschitz domain Ω with the Lipschitz constant m satisfying $\{x \in R^n\,;\,(x, \tau) \in \Omega\} = D$.

Hereafter we shall give a special form of the boundary Harnack principle, which is used to show the existence of a non-zero solution in $H_0(\Omega, L)$.

LEMMA 4. *Let Ω be a spatially bounded $(1, 1/2)$-Lipschitz domain in R^{n+1} with the Lipschitz constant m. For $\tau \in R$, we put $D = D_\tau$. For $x_0 \in R^n$ and $\tau_0 > 0$, we assume that $(x_0, \tau + \tau_0)$ is a proper inner point with respect to (D, τ, m). Then there exists a constant $C > 0$ such that for any solution $u \geq 0$ of $Lu = 0$ on $\Omega^{(\tau)} = \Omega \cap R^n \times (\tau, \infty)$ which vanishes continuously on $\partial\Omega \cap R^n \times [\tau, \infty)$,*

$$u(x, t) \leq C\, u(x_0, \tau + t_0) \text{ for } (x, t) \in \Omega^{(\tau+\tau_0)},$$

where C depends only on n, M, m, D, x_0 and τ_0.

Proof. Put $V = \{(x_1, \ldots, x_n) ; |x_j| < 3m, j = 1, \ldots, n\}$. For $r > 0$ and $Y_0 \in R^{n+1}$, we set $V_r(Y_0) = \{Y_0\} + V_r$ (for the notation V_r, see the paragraph 2). If a solution $u \geq 0$ of $Lu = 0$ on $\Omega^{(\tau)}$ vanishes continuously on $\partial\Omega \cap R^n \times [\tau, \infty)$, then for any $(x, t) \in \Omega^{(\tau)}$

$$u(x, t) = \int_{D \times \{\tau\}} u(y, \tau)\, d\omega_{\Omega^{(\tau)}}^{(x,t)}(y),$$

and the parabolic measure $\omega_{\Omega^{(\tau)}}^{(x,t)}$ is absolutely continuous with respect to $\omega_{\Omega^{(\tau)}}^{(x_0,\tau+\tau_0)}$ on $D \times \{\tau\}$. Hence it suffices to show that

$$\omega_{\Omega^{(\tau)}}^{(x,t)}(V_r(y_0, \tau)) \leq C\, \omega_{\Omega^{(\tau)}}^{(x_0,\tau+\tau_0)}(V_r(y_0, \tau)) \tag{4}$$

for $(x, t) \in \Omega^{(\tau+\tau_0)}$ and sufficiently small $r > 0$. As Ω is $(1, 1/2)$-Lipschitz, there exist a finite family (U_k) of open sets in R^{n+1} with $\cup\, U_k \supset \partial D \times \{\tau\}$ such that U_k associates with a coordinate system and a function satisfying (3). If $(y_0, \tau) \notin D \times \{\tau\} \setminus \cup\, U_k$, we put $A_r(y_0, \tau) = (y_0, \tau + 2r^2)$. Otherwise we choose another open set U in R^{n+1}, an associated coordinate system in R^{n+1} and a function f satisfying (3). Put $A_r(y_0, \tau) = (y_0^*, y_{0n} + 3mr, \tau + 2r^2)$, where $y_0 = (y_0^*, y_{0n}) \in R^{n-1} \times R$, and

$$v(x, t) = \omega_{\Omega^{(\tau)}}^{(x,t)}(V_r(y_0, \tau)).$$

We shall show that there exists $C_0 > 0$ such that

$$v(x, t) = C_0\, \omega_{\Omega^{(\tau)}}^{(x,t)}(A_{2^k r}(y_0, \tau)), \quad (x, t) \in \Omega^{(\tau)} \setminus V_{2^k r}(y_0, \tau) \tag{5}$$

for every integer $k \geq 0$ with $2^{2k+1} r^2 \geq t_0/2$. By Remark 1 and the Harnack inequality, we have for some $C_1 > 0$,

$$v(x, t) \leq 1 \leq \frac{1}{\nu} v(A_{r/2}(y_0, \tau)) \leq \frac{C_1}{\nu} v(A_r(y_0, \tau)).$$

Similarly

$$v(A_r(y_0, \tau)) \leq C_1\, v(A_{2r}(y_0, \tau)),$$

so that

$$v(x, t) \leq \frac{C_1^2}{\nu}\, v(A_{2r}(y_0, \tau)), \quad (x, t) \in \Omega^{(\tau)} \setminus V_r(y_0, \tau).$$

By using Lemma 3 for $\varepsilon = 1/C_1$ and for $B = \{(x^*, x_n) \in R^n ; |x^*|^2 + (x_n + 2m)^2 \leq m^2/(1 + m^2)\}$, there exists $0 < s < 1$ such that

$$\omega^X_{V_r(Y)\setminus(\{Y\}+T(B))}(\{Y\} + T(B)) < \frac{1}{C_1}, \quad X \in V_{sr}(Y) \setminus (\{Y\} + T(B))$$

for every $Y \in R^{n+1}$. Hence for every $Y \in \partial\Omega^{(\tau)} \setminus V_{2r}(y_0, \tau)$ and $(x, t) \in V_{sr}(Y)$,

$$\begin{aligned} v(x, t) &\leq \frac{C_1^2}{\nu}\, v(A_{2r}(y_0, \tau))\, \omega^{(x,t)}_{V_r(Y)\setminus(\{Y\}+T(B))}(\{Y\} + T(B)) \\ &\leq \frac{C_1}{\nu}\, v(A_{2r}(Y_0, \tau)). \end{aligned}$$

On the other hand, for every $(x, t) \in \partial V_{2r}(y_0, \tau)$ which is not included in any $V_{sr}(Y)$ with $Y \in \partial\Omega^{(\tau)} \setminus V_{2r}(y_0, \tau)$, the Harnack inequality gives

$$v(x, t) \leq C_2\, v(A_{2r}(y_0, \tau))$$

with some constant $C_2 > 0$. Therefore by the maximum principle, we have

$$v(x, t) \leq C_0\, v(A_{2r}(y_0, \tau)), \quad (x, t) \in \Omega^{(\tau)} \setminus V_{2r}(y_0, \tau)$$

for $C_0 = \max(C_1/\nu, C_2)$, which shows (5) for $k = 1$. Thus inductively we have (5) for every integer $k \geq 0$.

Furthermore we have

$$v(A_{t_0^{1/2}/2}(y_0, \tau)) \leq C_3\, v(x_0, \tau + t_0) \tag{6}$$

by the Harnack inequality, where $C_3 > 0$ is a constant depending only on $\boldsymbol{n}$, $\boldsymbol{M}$, $\boldsymbol{m}$, $\boldsymbol{D}$, x_0 and t_0. Combining (5) and (6), we obtain (4), which shows Lemma 4.

This gives the following

LEMMA 5. *In the same situation as in Lemma 4, we have*

$$u(x, t) \leq C\, u(x_0, \tau + \tau_0)\, \omega^{(x,t)}_{\Omega^{(\tau+\tau_0)}}(D_{\tau+\tau_0} \times \{\tau + \tau_0\})$$

for every $(x, t) \in \Omega^{(\tau+\tau_0)}$, *where* $C > 0$ *is the constant in Lemma* 4.

Using the above two lemmas, we obtain the Harnack inequality of the following form.

PROPOSITION 1. *Let Ω be a spatially bounded $(1, 1/2)$-Lipschitz domain in R^{n+1}, $\tau \in R$ and K a compact subset of $\Omega^{(\tau)}$. Then there exists a constant $C > 0$ such that for every $L \in \mathcal{L}(M)$ and every solution $u \geq 0$ of $Lu = 0$ on $\Omega^{(\tau)}$ which vanishes continuously on $\partial\Omega \cap R^n \times [\tau, \infty)$,*

$$\max_K u \leq C \min_K u.$$

In [2], E.B. Fabes, N. Garofalo and S. Salsa show a similar Harnack inequality in the case Ω is a Lipschitz cylinder.

We shall prove the existence of non-zero $u \in H_0(\Omega, L)$ by using Lemma 5 and Proposition 1.

PROPOSITION 2. *Let Ω be a spatially bounded $(1, 1/2)$-Lipschitz domain in R^{n+1}. Then there exists a non-zero positive solution u of $Lu = 0$ on Ω such that u vanishes continuously on $\partial\Omega$.*

Proof. Let $Y_0 = (y_0, s_0) \in \Omega$ be fixed. For $\tau < s_0$, we put

$$u_\tau(x, t) = \frac{\omega_{\Omega^{(\tau)}}^{(x,t)}(D_\tau \times \{\tau\})}{\omega_{\Omega^{(\tau)}}^{Y_0}(D_\tau \times \{\tau\})}.$$

Then $u_\tau(Y_0) = 1$. Therefore by Proposition 1, for every $t_0 < s_0$, the sequence $\{u_\tau\}_{\tau < t_0}$ is uniformly bounded and hence equicontinuous on every compact set in $\Omega^{(t_0)}$. Then there exist a decreasing sequence $\{\tau_k\}_{k=1}^\infty$ tending to $-\infty$ and a solution u of $Lu = 0$ on Ω such that

$$\lim_{k \to \infty} u_{\tau_k} = u \quad \text{(compact uniformly)}.$$

Using Lemma 5 for u_{τ_k} and letting k tend to the infinity, we see that u vanishes continuously on $\partial\Omega$, so that $u \in H_0(\Omega, L)$. This completes the proof.

§4. The uniqueness of positive solutions

Let D be a bounded Lipschitz domain in R^n and φ a strictly positive $1/2$-Hölder continuous function on R.

Remark 2. $\Omega(D, \varphi)$ is a $(1, 1/2)$-Lipschitz domain with Lipschitz constant $\max(c, m(1+c)d(0, \partial D))$, where c is the Lipschitz constant of D, m is the $1/2$-Hölder constant and $d(0, \partial D)$ is the distance from 0 to ∂D.

The following lemma is a kind of boundary Harnack principle.

LEMMA 6. *For a bounded Lipschitz domain D in R^n and a $1/2$-Hölder continuous function $\varphi > 0$ on R, we put $\Omega = \Omega(D, \varphi)$. Let $\tau_0 > 0$, $\tau \in R$ and Δ be a non-empty subdomain of D with $\bar{\Delta} \subset D$. Then there exists a constant $C > 0$ independent of τ such that*

$$\sup_{\varphi(\tau)D \times \{\tau\}} u \leq C \inf_{\varphi(\tau)\Delta \times \{\tau\}} u$$

for every solution $u \geq 0$ of $Lu = 0$ on $\Omega^{(\tau - \tau_0\varphi(\tau)^2)}$ which vanishes continuously on $\partial\Omega \cap R^n \times [\tau - \tau_0\varphi(\tau)^2, \infty)$.

Proof. Let $x_0 \in \Delta$ be fixed. Put $t_0 = (\tau_0^{-1/2} + m)^{-2}$, where m is the $1/2$-Hölder constant of φ. Then there exists $0 < T \leq \tau_0\varphi(\tau)^2$ such that

$$\frac{T}{\varphi(\tau - T)^2} = t_0.$$

Applying Lemma 4 to $\tau_0 = t_0/2$ and using the parabolic dilation $\tau_{\varphi(\tau - T)}$, we have for any solution $u \geq 0$ of $Lu = 0$ on $\Omega^{(\tau - \tau_0\varphi(\tau)^2)}$ which vanishes continuously on $\partial\Omega \cap R^n \times [\tau - \tau_0\varphi(\tau)^2, \infty)$,

$$\sup_{x \in \varphi(\tau)D} u(x, \tau) \leq C_1 u\left(\varphi\left(\tau - \frac{T}{2}\right)x_0, \tau - \frac{T}{2}\right) \leq C_1C_2 \inf_{x \in \varphi(\tau)\Delta} u(x, \tau),$$

which shows Lemma 6.

Let L^* be the adjoint operator of $L \in \mathscr{L}(M)$. Then for any solution u of $L^*u = 0$, $v(x, t) = u(x, -t)$ is a solution of $\tilde{L}v = 0$ for some $\tilde{L} \in \mathscr{L}(M)$, so that the analogous assertions to Lemma 6 hold. This yields Lemma 7, which plays an important role to show the uniqueness.

LEMMA 7. *For a bounded Lipschitz domain D in R^n and a $1/2$-Hölder continuous function $\varphi > 0$ on R, we put $\Omega = \Omega(D, \varphi)$. Let $\tau_0 > 0$, $\tau \in R$ and Δ be a non-empty subdomain of D with $\bar{\Delta} \subset D$. Then there exists a constant $C > 0$ independent of τ such that*

$$\omega_{\Omega^{(\tau)}}^{(x,t)}(\varphi(\tau)\,D\times\{\tau\})\leq C\,\omega_{\Omega^{(\tau)}}^{(x,t)}(\varphi(\tau)\,\Delta\times\{\tau\})$$

for every $(x,\ t)\in\Omega^{(\tau+\tau_0\varphi(\tau)^2)}$.

Proof. Let $G(x,\ t\,;\,y,\ s)$ be the Green function of L with respect to $\Omega(D,\ \varphi)$. Then for $(x,t)\in\Omega(D,\ \varphi)$,

$$\omega_{\Omega^{(\tau)}}^{(x,t)}=G(x,\ t\,;\,y,\ \tau)\,dy\quad\text{on}\quad\varphi(\tau)D\times\{\tau\},$$

where dy denotes the n-dimensional Lebesgue measure. For $(x,\ t)\in\Omega^{(\tau+\tau_0\varphi(\tau)^2)}$, $G(x,\ t\,;\,\cdot\ ,\cdot\ )$ is a solution of the adjoint operator L^* of L on $\Omega\cap R^n\times(-\infty,\ t)$. Applying Lemma 6 to L^*, we obtain

$$\sup_{y\in\varphi(\tau)D}G(x,\ t\,;\,y,\ \tau)\leq C\inf_{y\in\varphi(\tau)\Delta}G(x,\ t\,;\,y,\ \tau),$$

which shows our lemma.

We shall show our main theorem, which implies the preceding assertion in the paragraph 1.

THEOREM 2. *Let D be a bounded Lipschitz domain in R^n and $\varphi>0$ a locally $1/2$-Hölder continuous function on $(-\infty,\ a)$ with $a\in(-\infty,\ \infty]$. Suppose that there exist $m>0$, $\tau_0>0$ and a sequence $\{t_k\}_{k=1}^{\infty}$ tending to $-\infty$ as $k\to\infty$ such that*

$$\liminf_{k\to\infty}|\,t_k\,|^{-1/2}\varphi(t_k)<\infty \tag{7}$$

and that for every $k=1,2,\dots$,

$$|\,\varphi(t)-\varphi(s)\,|<m\,|\,t-s\,|^{1/2} \tag{8}$$

for $t,\ s\in[t_k,\ t_k+\tau_0\varphi(t_k)^2]$. Then there exists $u\neq0$ such that

$$H_0(\Omega(D,\ \varphi),\ L)=\{cu\,;\,c\geq0\}.$$

Proof. By Proposition 1 and Remark 2, $H_0(\Omega(D,\ \varphi),\ L)\neq\{0\}$. Hence it suffices to show that there exist $C>0$ and $h\in H_0(\Omega(D,\ \varphi),\ L)$ with $h(Y_0)=1$ for fixed $Y_0\in\Omega(D,\ \varphi)$ such that $u\geq C\,h$ for every $u\in H_0(\Omega(D,\ \varphi),\ L)$ with $u(Y_0)$ (see [3], p.253).

Let $u\in H_0(\Omega(D,\ \varphi),\ L)$ with $u(Y_0)=1$ and put $\Omega=\Omega(D,\ \varphi)$. Taking a subsequence of $\{t_k\}_{k=1}^{\infty}$ and replacing τ_0 by smaller one if necessary, we may assume that

$$t_k + \tau_0 \varphi(t_k)^2 < \frac{t_k}{2}$$

for every positive integer k. Put

$$T_k = t_k + \frac{\tau_0 \varphi(t_k)^2}{2}.$$

Let Δ be a non-empty subdomain of D and take $x_0 \in \Delta$. Then by Lemmas 6 and 7, we have for every positive integer k and every $(x, t) \in \Omega^{(t_k+\tau_0\varphi(t_k)^2)}$

$$\begin{aligned} u(x, t) &= \int_{\varphi(T_k)D\times\{T_k\}} u(y, T_k)\, d\omega^{(x,t)}_{\Omega^{(T_k)}}(y) \\ &\geq \int_{\varphi(T_k)\Delta\times\{T_k\}} u(y, T_k)\, d\omega^{(x,t)}_{\Omega^{(T_k)}}(y) \\ &\geq \left(\inf_{\varphi(T_k)\Delta\times\{T_k\}} u\right) \omega^{(x,t)}_{\Omega^{(T_k)}}(\varphi(T_k)\,\Delta \times \{T_k\}) \\ &\geq C_1^{-1} u(\varphi(T_k)\,x_0, T_k)\, \omega^{(x,t)}_{\Omega^{(T_k)}}(\varphi(T_k) D \times \{T_k\}), \end{aligned}$$

where $C_1 > 0$ is a constant independent of k, u and (x, t). On the other hand, by Lemma 5, there exists a constant $C_2 > 0$ such that

$$1 = u(Y_0) \leq C_2\, u(\varphi(T_k)x_0, T_k)\, \omega^{Y_0}_{\Omega^{(T_k)}}(\varphi(T_k)\, D \times \{T_k\}),$$

so that

$$u \geq C_1^{-1} C_2^{-1} h_k \quad \text{on} \quad \Omega^{(t_k+\tau_0\varphi(t_k)^2)},$$

where

$$h_k(x, t) = \frac{\omega^{(x,t)}_{\Omega^{(T_k)}}(\varphi(T_k)D \times \{T_k\})}{\omega^{Y_0}_{\Omega^{(T_k)}}(\varphi(T_k)D \times \{T_k\})}.$$

Similarly to Proposition 2, we can take a subsequence of $\{h_n\}_{n=1}^{\infty}$ which converges a certain $h \in H_0(\Omega, L)$ with $h(Y_0) = 1$, which shows

$$u \geq C_1^{-1} C_2^{-1} h \quad \text{on} \quad \Omega.$$

This completes the proof.

Remark 3. The assumptions (7), (8) in Theorem 2 can be replaced by

$$|\varphi(t) - \varphi(s)| < m\,|t - s|^{1/2}$$

for $t, s \in [t_k - \tau_0\varphi(t_k)^2, t_k + \tau_0\varphi(t_k)^2]$.

Applying Theorem 1 to $\varphi_\alpha(t) = (-t)^\alpha$ $(t < 0)$, we have

COROLLARY. *Let* $-\infty < \alpha \leq 1/2$. *For a bounded Lipschitz domain* D *in* R^n, *put*

$$\Omega_\alpha = \{(x, t)\,;\, t < 0,\ (-t)^{-\alpha}x \in D\}.$$

Then every non-zero elements in $H_0(\Omega_\alpha, L)$ *are mutually proportional.*

EXAMPLE. Let D be a bounded Lipschitz domain in R^n and put $\Omega = D \times R$. Then

$$H_0\left(\Omega, \frac{\partial}{\partial t} - \Delta\right) = \{ce^{-\lambda t}f(x)\,;\, c \geq 0\},$$

where λ is the first eigenvalue of $-\Delta$ (Laplacian) and f is the eigenfunction.

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