G. Herbort
Nagoya Math. J.
Vol. 130(1993), 25–54

ON THE INVARIANT DIFFERENTIAL METRICS NEAR PSEUDOCONVEX BOUNDARY POINTS WHERE THE LEVI FORM HAS CORANK ONE

GREGOR HERBORT

0. Introduction

Let D be a bounded domain in $\mathbf{C}^n$; in the space $L^2(D)$ of functions on D which are square-integrable with respect to the Lebesgue measure $d^{2n}z$ the holomorphic functions form a closed subspace $H^2(D)$. Therefore there exists a well-defined orthogonal projection $P_D : L^2(D) \to H^2(D)$ with an integral kernel $K_D : D \times D \to \mathbf{C}$, the Bergman kernel function of D. An explicit computation of this function directly from the definition is possible only in very few cases, as for instance the unit ball, the complex "ellipsoids" $E_m = \{(z, w) \in \mathbf{C}^2 : |z|^2 + |w|^{2m} < 1\}$, or the annulus in the plane. Also, there is no hope of getting information about the function K_D in the interior of a general domain. Therefore the question for an asymptotic formula for the Bergman kernel near the boundary of D arises. Bergman [Be] was the first to study the behavior of the function $K_D(z) := K_D(z, z)$ near the boundary for certain classes of domains in $\mathbf{C}^2$. After the L^2-theory for the $\bar{\partial}$-operator, [Hör], and the $\bar{\partial}$-Neumann problem, [K 1], was developed a first precise description of the singularity of $K_D(z)$ and its derivatives became possible in case that D is a strongly pseudoconvex domain with smooth boundary, [Hör], [Di 1], [Di 2]. Since the work of Fefferman, [F], and Boutet de Monvel-Sjöstrand, [B-S], the asympotic behavior of K_D at the boundary of strongly pseudoconvex domains is completely understood.

The methods which worked well on strongly pseudoconvex domains cannot be extended to the weakly pseudoconvex case. A formula for the complete description of the singular behavior of $K_D(z)$ for general weakly pseudoconvex domains is unknown. Only partial results in this direction have been obtained, see for instance [Oh], [He 1], [He 2], [D-H-O]. In [C 1], however, Catlin gave a complete description of the singularity of $K_D(z)$ when D is a smooth bounded pseudoconvex domain of finite type in $\mathbf{C}^2$. His work contains also precise estimates from above

---

Received May 7, 1992.

and below for the invariant differential metrics of Caratheodory, Bergman and Kobayashi. It is by no means clear how to generalize these estimates to domains of finite type in the sense of d'Angelo, [A], in higher dimension. Here, similar as in the case of the Bergman kernel, a precise estimate for these metrics is known only in the strongly pseudoconvex case, [H], [Gr], [Di 1], [Di 2].

In the present article we investigate the behavior of $K_\Omega(z)$ and the invariant metrics of Caratheodory, Bergman, and Kobayashi on a smooth bounded pseudoconvex domain $\Omega \subset\subset \mathbf{C}^n$ near a point $q \in \partial\Omega$ of finite type where the Levi form of $\partial\Omega$ has at least $n-2$ positive eigenvalues. This extends the circle of ideas of [C 1] and, in a sense, also will complete it. Our main tool is a precise bumping theorem for Ω near the point q which is obtained from the bumping theorem of [F-S]. It allows us to simplyfy the techniques of [C 1] and to dispense with the estimates for the $\bar\partial$-Neumann operator when discussing the growth of the Caratheodory metric of Ω near q. The plan of the paper is as follows. In section 1 we set up the necessary notations and state the results. In sections 2 and 3 we will analyze the geometric properties of the boundary $\partial\Omega$ near a point q of finite type and introduce the appropriate local holomorphic coordinates. Contrary to the case $n = 2$ one has to deal with those terms in the Taylor series expansion of a defining function for Ω at q which reflect coupling effects between the variables in the "strongly pseudoconvex" directions and the "weakly pseudoconvex direction", see Theorems 3 and 4. Section 4 contains the analytic part of the proof of Theorems 1 and 2. In Main Lemma 4.2 the necessary holomorphic auxiliary functions are constructed by solving the $\bar\partial$-equation with weights, see Theorem 5. The desired precise estimates for the Bergman kernel on the diagonal and the invariant metrics are given in the normalized coordinates constructed in section 3. Finally, in section 5 we describe how to express the estimates obtained in section 4 in terms of the initial coordinates.

Note added in proof. The methods of this paper are also successful on a certain class of domains with Levi form of higher corank, (cf [He 3]).

1. Statement of the results

Let $\Omega \subset\subset \mathbf{C}^n$ be a smooth bounded pseudoconvex domain with a defining function r. Suppose $0 \in \partial\Omega$, and for a small ball B with centre 0 we have $\left| \frac{\partial r}{\partial z_1}(q) \right| > 1/2$, for all $q \in B$. On B we define the vector fields

$$L_a = \frac{\partial}{\partial z_a} - \frac{r_a}{r_1}\frac{\partial}{\partial z_1}, \tag{1.1}$$

for $2 \leq a \leq n$, and by $\bar{L}_b$ its conjugate, $2 \leq b \leq n$, where we abbreviate $r_a = \frac{\partial r}{\partial z_a}$, for all $a = 1, \dots, n$. The L_a, $a = 2, \dots, n$ form a basis for the holomorphic tangent bundle $T^{10}\partial\Omega$ restricted to B. By L_1 we denote the normal field

$$L_1 = \frac{1}{|\nabla r|^2}\sum_{b=1}^{n}\frac{\partial r}{\partial \bar{z}_b}\frac{\partial}{\partial z_b}. \tag{1.2}$$

Let us further write

$$\mathscr{L}_{a\bar{b}} := \partial r([L_a, \bar{L}_b]), \tag{1.3}$$

for $2 \leq a, b \leq n$, and denote by $\lambda_{\partial\Omega}$ the Levi function

$$\lambda_{\partial\Omega} = \det(\mathscr{L}_{a\bar{b}})_{a,b=2}^{n}. \tag{1.4}$$

Analogously to the definition in [C 1] we introduce the functions

$$A_l(z) := \max\{|L_n^{\alpha-1}\bar{L}_n^{\beta-1}\lambda_{\partial\Omega}(z)| \mid \alpha, \beta \geq 1, \alpha + \beta = l\}. \tag{1.5}$$

For a vector $X \in \mathbf{C}^n$ there are uniquely determined functions $s_1(X), \dots, s_n(X)$ satisfying $X = \sum_{j=1}^{n} s_j(X) L_j$.

With these notations we can state our result in the following

THEOREM 1. *Assume that the submatrix* $(\mathscr{L}_{a\bar{b}})_{a,b=2}^{n-1}$ *is strictly positive definite on* B, *and* 0 *is a point of finite type* $2k$ *in the sense of Kohn,* [K 2], (*this means* $A_{2k} > 0$ *on* B, *after shrinking* B, *if necessary*). *If we write*

$$\mathscr{C}_{2k}(z) := \sum_{l=2}^{2k}\left(\frac{A_l}{|r|}(z)\right)^{\frac{1}{l}}, \tag{1.6}$$

then the Bergman kernel function K_Ω *of* Ω *can be estimated on* $\Omega \cap B$ *by*

$$\frac{1}{C} \leq \frac{K_\Omega(z, \bar{z})}{|r(z)|^{-n}\mathscr{C}_{2k}^2(z)} \leq C, \tag{1.7}$$

where C *is a universal constant.*

We can also estimate the invariant pseudodifferential metrics of Caratheodory and Kobayashi, as well as the Bergman metric. In order to state the precise estimates for these metrics we define the pseudodifferential metric

$$(1.8)\qquad M_\Omega(z, X) = \frac{|s_1(X)|^2}{|r(z)|^2} + \sum_{a,b=2}^{n-1} \frac{\mathcal{L}_{a\bar b}(z) s_a(X) \overline{s_b(X)}}{|r(z)|} + \mathscr{C}_{2k}^2(z)\,|s_n(X)|^2.$$

With this notation we have

THEOREM 2. *Let the hypotheses be the same as in Theorem* 1. *If then H_Ω denotes one of the differential metrics of Caratheodory, Bergman or Kobayashi, we have on a small ball B_1 around $0 \in \partial\Omega$:*

$$(1.9)\qquad \frac{1}{C} M_\Omega(z, X)^{\frac{1}{2}} \le H_\Omega(z, X) \le C\, M_\Omega(z, X)^{\frac{1}{2}},$$

where again C is a universal positive constant.

2. Normalization of the defining function

Assume $q \in \partial\Omega \cap \hat{B}$, where $\hat{B}$ is a ball around 0 which lies relatively compact in B. By the transformation

$$w_1^{(1)} = 2 \sum_{a=1}^{n} \frac{\partial r}{\partial z_a}(q)\,(z_a - q_a)$$
$$w_1^{(1)} = z_l - q_l,\ 2 \le l \le n$$

we absorb the linear term in the Taylor expansion of r around q. In the $w^{(1)}$-coordinates the equation for $\partial\Omega$ will be of the form

$$(2.1)\qquad \operatorname{Re} w_1^{(1)} + R^{(1)}(w_1^{(1)}, (w^{(1)})'; q) = 0,$$

where $R^{(1)}(\cdot\,; q)$ is a smooth function which is defined on a ball $\tilde{B} \subset\subset B$, with centre 0 and a radius independent of q, (and $w' = (w_2, \ldots, w_n)$ for all $w \in \mathbf{C}^n$). It vanishes of second order at 0, and, after multiplication with a positive affine linear function of the form $h = 1 + 2 \operatorname{Re} \sum_{j=1}^{n} \alpha_j w_j^{(1)}$ we can even achieve that $R_{1\bar l}^{(1)}(0\,; q) = 0$ for $1 = 1, \ldots, n$. Here $\Phi_{a\bar b} = \dfrac{\partial^2 \Phi}{\partial z_a \partial \bar z_b}$ for any differentiable function Φ, and $1 \le a, b \le n$. Obviously we can solve equation (2.1) for $\operatorname{Re} w_1^{(1)}$, and obtain

$$(2.2)\qquad \operatorname{Re} w_1^{(1)} + \tilde{R}^{(1)}(\operatorname{Im} w_1^{(1)}, (w^{(1)})'; q) = 0,$$

where again $\tilde{R}^{(1)}(\cdot\,; q)$ has the same properties as $R^{(1)}(\cdot\,; q)$. The Levi form of $R(\cdot\,; q)$ is described by a certain matrix $A = (a_{j\bar l}(q))_{j,l=2}^{n}$, the entries of which depend continuously on q, and for which the submatrix $(a_{j\bar l}(q))_{j,l=2}^{n-1}$ is positive definite. Thus we can choose a matrix $B \in GL(n-2, \mathbf{C})$, and continuous functions $c_2(q), \dots, c_{n-1}(q)$, $\tilde{a}_{n\bar{n}}(q)$ on $\hat{B} \cap \partial\Omega$ such that

$$\begin{pmatrix} B(q) & 0 \\ 0 & 1 \end{pmatrix}^T A \begin{pmatrix} \overline{B(q)} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} E_{n-2} & \bar{c}(q)^T \\ c(q) & \tilde{a}_{n\bar{n}}(q) \end{pmatrix},$$

where $c(q) = (c_2(q), \dots, c_{n-1}(q))$. If we therefore set

$$w_2^{(2)} = w_1^{(1)},$$

$$\begin{pmatrix} w_2^{(2)} \\ \vdots \\ w_n^{(2)} \end{pmatrix} = \begin{pmatrix} B(q)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_2^{(1)} \\ \vdots \\ w_n^{(1)} \end{pmatrix},$$

then $\partial\Omega$ will in the $w^{(2)}$-coordinates be described by the equation

$$\operatorname{Re} w_1^{(2)} + R^{(2)}(\operatorname{Im} w_1^{(2)}, (w^{(2)})'; q) = 0. \tag{2.3}$$

Here the function $R^{(2)}$ is smooth on a certain ball around 0, which we denote again by $\tilde{B}$. The following couple of steps is inspired by the method of section 1 in [F-S], where a precise bumping lemma of two-dimensional domains of finite type was established. At first we write (with $v'' := (v_2, \dots, v_{n-1})$ for $v \in \mathbf{C}^n$):

$$\begin{aligned} R^{(2)}(\operatorname{Im} w_1^{(2)}, (w^{(2)})'; q) &= \operatorname{Re} f_q^{(2)}(w_1, \dots, w_n) \\ &+ \operatorname{Im} w_1^{(2)} \Big(\sum_{j=2}^{k} Q_j^{(2)}(w_n; q) + \sigma_{k+1}(w_n^{(2)}) \Big) \\ &+ \operatorname{Im} w_1^{(2)} \, (\sigma_2((w^{(2)})') + \sigma_1(\operatorname{Im} w_1^{(2)})) \\ &+ \sum_{a=2}^{n-1} | w_a^{(2)} |^2 + 2 \operatorname{Re} \sum_{a=2}^{n-1} w_a^{(2)} \, g_a^{(2)}(w_n^{(2)}; q) \\ &+ \sigma_3((w^{(2)})'') + \sigma_1((w^{(2)})'') \, \sigma_{k+1}(w_n^{(2)}) + \sigma_2((w^{(2)})'')\sigma_1(w_n^{(2)}) \\ &+ \sum_{j=2}^{2k} P_j^{(2)}(w_n^{(2)}; q) + \sigma_{2k+1}(w_n^{(2)}). \end{aligned} \tag{2.4}$$

Here, $f_q^{(2)}$ is a holomorphic polynomial which vanishes at 0, the $P_j^{(2)}$, $Q_1^{(2)}$ are real-valued homogeneous polynomials of degree j, the $g_a^{(2)}$ are complex polynomials of degree at most k, which do not contain holomorphic terms. The symbol σ_i stands for smooth functions which vanish at 0 of order i. By means of another $2k$-1 steps we eliminate by transformations of the form

$$w_1 \rightarrow w_1 + \alpha w_n^j, \; w_a \rightarrow w_a, \quad a = 2, \dots, n$$

all the harmonic terms from the $P_j(\cdot)$'s. During this procedure the functions $P_j^{(\cdot)}$, $Q_j^{(\cdot)}$, $g_a^{(\cdot)}$, and $f_q^{(\cdot)}$ will change at each step. After that we let in another k steps all the harmonic terms in the $Q_j^{(\cdot)}$'s be absorbed by $\mathrm{Re}\, w_1$. The function $R^{(2)}$ will be changed at each step, but we can arrange that it retains the form (2.4). We will in the $(3k+2)^{\text{th}}$ step obtain a coordinate system $(w^{(3k+2)})$, with respect to which $\partial\Omega$ is given by the equation

$$\mathrm{Re}\, w_1^{(3k+2)} + R^{(3k+2)}(\mathrm{Im}\, w_1^{(3k+2)}\,;\,(R^{(3k+2)})'\,;\,q) = 0,$$

where the function $R^{(3k+2)}(\cdot\,;q)$ has the form (2.4) with $f_q^{(2)}$, $P_j^{(2)}$, $Q_j^{(2)}$, and $g_a^{(2)}$ replaced by $f_q^{(3k+2)}$, $P_j^{(3k+2)}$, $Q_j^{(3k+2)}$, and $g_a^{(3k+2)}$, respectively. We now will normalize the functions $g_a^{(3k+2)}$. For this we write

$$g_a^{(3k+2)}(w_n\,;q) = \overline{\tilde{h}_a^{(3k+2)}(w_n^{(3k+2)}\,;q)} + \tilde{g}_a^{(3k+2)}(w_n^{(3k+2)};\,q)$$

where $\tilde{h}_a^{(3k+2)}(\cdot\,;q)$ is a holomorphic polynomial and the polynomial $\tilde{g}_a^{(3k+2)}(\cdot\,;q)$ has no longer harmonic terms. Now we can write

$$\begin{aligned}\sum_{a=2}^{n-1} |\,w_a^{(3k+2)}\,|^2 + 2\,\mathrm{Re}\sum_{a=2}^{n-1} w_a^{(3k+2)}\,\overline{\tilde{h}_a^{(3k+2)}(w_n^{(3k+2)}\,;q)} &\\ = \sum_{a=2}^{n-1} |\,w_a^{(3k+2)} + \tilde{h}_a^{(3k+2)}(w_n^{(3k+2)}\,;q)\,|^2 &\\ - \sum_{a=2}^{n-1} |\,\tilde{h}_a^{(3k+2)}(w_n^{(3k+2)}\,;q)\,|^2, &\end{aligned}$$

and all the $\tilde{h}_a^{(3k+2)}(\cdot\,;q)$ vanish at 0. Thus, if we set

$$\begin{aligned} w_1^{(3k+3)} &= w_1^{(3k+2)}, \\ w_a^{(3k+3)} &= w_a^{(3k+2)} + \tilde{h}_a^{(3k+2)}(w_n^{(3k+2)}\,;q),\ 2 \le a \le n-1 \\ w_n^{(3k+3)} &= w_n^{(3k+2)}, \end{aligned}$$

we obtain new holomorphic coordinates with respect to which $\partial\Omega$ is described by equation (2.5), and no harmonic terms appear in the P_j, Q_j, or g_a-polynomials. Finally we let all the Taylor terms of the form

$$\frac{\partial^\gamma f_q^{(3k+3)}(0;q)}{\partial(w^{(3k+3)})^\gamma}\cdot(w^{(3k+3)})^\gamma,$$

where $\gamma \in \mathbf{N}^n$, and $\gamma_1 = 0$, $\frac{1}{2}\sum_{a=2}^{n-1}\gamma_a + \frac{1}{2k}\gamma_n \le 1$, appearing in $f_q^{(3k+3)}$, be absorbed by $w_1^{(3k+3)}$. This will neither introduce new harmonic terms in the P_j, Q_j, or g_a's nor change the form of (2.4) or (2.5). The result of our transformations can now be summarized in

THEOREM 3. *There exists an open neighborhood U of the origin and a mapping $F : \mathbf{C}^n \times (\partial\Omega \cap U) \rightarrow \mathbf{C}^n$ with the following properties:*

(1) *For any $q \in \partial\Omega \cap U$ the mapping $F(\cdot\,; q) : \mathbf{C}^n \rightarrow \mathbf{C}^n$ is biholomorphic, and $F(q\,; q) = 0$.*

(2) *The Jacobi matrix of $F(\cdot\,; q)$ is of the form*

$$F'(z\,;q) = \begin{pmatrix} \frac{\partial F_1}{\partial z_1}(z\,;q) & \frac{\partial F_1}{\partial z_2}(z\,;q) & \cdots & \frac{\partial F_1}{\partial z_{n-1}}(z\,;q) & \frac{\partial F_1}{\partial z_n}(z\,;q) \\ 0 & \frac{\partial F_2}{\partial z_2}(z\,;q) & \cdots & \frac{\partial F_2}{\partial z_{n-1}}(z\,;q) & h_2(z_n - q_n\,;q) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \frac{\partial F_{n-1}}{\partial z_2}(z\,;q) & \cdots & \frac{\partial F_{n-1}}{\partial z_{n-1}}(z\,;q) & h_{n-1}(z_n - q_n\,;q) \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

with certain holomorphic polynomials $h_a(z_n - q_n\,; q)$.

(3) *For each $q \in \partial\Omega \cap U$ we have $\Omega_q = F(\Omega\,; q) = \{\hat{r}_q < 0\}$, where $\hat{r}_q = r \circ F(\cdot\,; q)^{-1}$ has the following form*

$$\begin{aligned} \hat{r}_q(w) = {} & \mathrm{Re}(w_1 + f(w\,;q)) + \mathrm{Im}\, w_1 \sum_{j=2}^{k} Q_j(w_n\,;q) \\ & + \mathrm{Im}\, w_1\, [\sigma_{k+1}(w_n) + \sigma_1(w'')\sigma_1(w_n)] + \sigma_2(\mathrm{Im}\, w_1) \\ & + \sum_{a=2}^{n-1} |\, w_a\,|^2 + 2\mathrm{Re} \sum_{a=2}^{n-1} w_a g_a(w_n\,;q) + \sigma_3(w'') \\ & + \sigma_1(w'')\sigma_{k+1}(w_n) + \sigma_2(w'')\sigma_1(w_n) \\ & + \sum_{j=2}^{2k} P_j(w_n\,;q) + \sigma_{2k+1}(w_n). \end{aligned}$$

In this formula, $w'' = (w_2, \ldots, w_{n-1})$ for all $w \in \mathbf{C}^n$, $f(\,\cdot\,;q)$ is a holomorphic polynomial satisfying $\frac{\partial^\gamma f(0\,;q)}{\partial w^\gamma} = 0$, whenever $\gamma_1 = 0$, $\frac{1}{2}\sum_{a=2}^{n-1}\gamma_a + \frac{1}{2k}\gamma_n \leq 1$, further, P_j and Q_j are real-valued polynomials of degree j without harmonic terms, and the g_a are complex polynomials without holomorphic or anti-holomorphic terms. The σ_i are error functions which vanish at of i.th order at 0.

3. Estimation of the coupling terms

Let us agree upon the following notation: For a homogeneous polynomial p we denote by $\| p \|$ the quantity

$$\|p\| = \max_{\theta\in[-\pi,\pi]} |p(e^{i\theta})|.$$

We have to adapt Lemma (1.5) and Proposition (1.6) from [F-S] to our situation. This is done in

LEMMA 3.1. *There exist positive constants* C_0, ρ_0, *such that for any* $2 \leq a \leq n$, *and any* $q \in \partial\Omega \cap U$ *the following all hold*

(a) *If for a radius* $0 < \rho < \rho_0$ *and any numbers* $i \in \{2,\ldots,2k\}$, $j \in \{2,\ldots,k\}$ *one has*

$$\| Q_j(\cdot;q) \| \rho^j \geq C_0 \max_{l\neq j} \| Q_l(\cdot;q) \| \rho^l,$$

and

$$\| P_j(\cdot;q) \| \rho^i \geq C_0 \max_{l\neq i} \| P_l(\cdot;q) \| \rho^l,$$

then it must be that

$$\| Q_j(\cdot;q) \| \rho^j \leq C_0^2 \rho \sqrt{\| P_l(\cdot;q) \| \rho^i}.$$

(b) *If for a radius* $0 < \rho < \rho_0$ *and any numbers* $i \in \{2,\ldots,2k\}$, $j \in \{2,\ldots,k\}$ *one has*

$$\| g_{a;j}(\cdot;q) \| \rho^j \geq C_0 \max_{l\neq j} \| g_{a;l}(\cdot;q) \| \rho^l,$$

and

$$\| P_i(\cdot;q) \| \rho^i \geq C_0 \max_{l\neq i} \| P_l(\cdot;q) \| \rho^l,$$

then

$$\| g_{a;j}(\cdot;q) \| \rho^j \leq C_0^2 \rho \sqrt{\| P_i(\cdot;q) \| \rho^i}.$$

Here we denote by $g_{a;j}$ *the homogeneous part of* g_a *of degree* j.

Proof. The proof of (a) goes in complete analogy to that of Lemma (1.5) in [F-S]. We only need to apply their arguments to the complex two-dimensional section $\{\hat{r} < 0\} \cap \{w_2 = \ldots w_{n-1} = 0\}$. (Here $\hat{r} := \hat{r}_q$). We will even obtain the following statement: There exists a radius $r_0 > 0$ with the property:

(a′) If for a radius $0 < \rho < r_0$ and a number $j \in \{2,\ldots,k\}$ one has

$$\| Q_j(\cdot\,; q) \| \rho^j \ge C_0 \max_{l \ne j} \| Q_l(\cdot\,; q) \| \rho^l,$$

then

$$\| Q_j(\cdot\,; q) \| \rho^j \le C_0^2 \rho \sqrt{\sum_{i=2}^{2k} \| P_i(\cdot\,; q) \| \rho^i}.$$

Let us now pass to (b). For $a, b \in \{2, \ldots, n\}$ we set

$$\hat{\mathscr{L}}_{a\bar b} = \hat r_{a\bar b} | \hat r_1 |^2 - \hat r_{a\bar 1} \hat r_1 \overline{\hat r_b} - \hat r_{1\bar b} \hat r_a \overline{\hat r_1} + \hat r_{1\bar 1} \hat r_a \overline{\hat r_b}. \tag{3.1}$$

For a fixed number $a \in \{2, \ldots, n\}$ we choose arbitrary complex numbers w_a, w_n close to 0 and additionally a real $\tilde w_1 = \tilde w_1(w_a, w_n)$, such that

$$\tilde q(w_a, w_n) = (\tilde w_1, 0, \ldots, w_a, 0, \ldots, w_n)$$

becomes a boundary point of Ω. Then, by the pseudoconvexity of Ω one has

$$\hat{\mathscr{L}}_{n\bar n}(\tilde q(w_a, w_n)) \ge 0. \tag{3.2}$$

Furthermore, one has

$$\begin{aligned} \hat r = {} & \operatorname{Re}(w_1 + f(w\,; q)) + \operatorname{Im} w_1 \left[\sum_{j=2}^{k} Q_j(w_n\,; q) + \sigma_1(w') \sigma_1(w_n) + \sigma_{k+1}(w_n) \right] \\ & + \sigma_2(\operatorname{Im} w_1) + \sum_{b=2}^{n-1} | w_b |^2 + 2\operatorname{Re} \sum_{a=2}^{n-1} w_b g_b(w_n\,; q) \\ & + \sum_{i=2}^{2k} P_i(w_n\,; q) + \mathscr{E}(w'), \end{aligned} \tag{3.3}$$

where $\mathscr{E}(w')$ denotes the error term

$$\mathscr{E}(w') = \sigma_3(w'') + \sigma_1(w'') \sigma_{k+1}(w_n) + \sigma_2(w'') \sigma_1(w_n) + \sigma_{2k+1}(w_n).$$

From this we can see that

$$| \tilde w_1 | \le C\left(| w_a |^2 + \sum_{i=2}^{2k} \| P_i(\cdot\,; q) \| \, | w_n |^i + | w_a | \left(1 + \frac{k}{C_0}\right) \| g_{a;j} \| (\cdot\,; q) \, | w_n |^j \right), \tag{3.5}$$

with some universal positive constant C. If we substitute (3.3) into the formula (3.1) for $\hat{\mathscr{L}}_{n\bar n}(\tilde q(w_a, w_n))$, we obtain

$$\begin{aligned} | \hat{\mathscr{L}}_{n\bar n}(\tilde q(w_a, w_n)) - 2 | \hat r_1(\tilde q(w_a, w_n)) |^2 \operatorname{Re} w_a (g_{a;j})_{n\bar n}(w_n\,; q) | \le {} & \\ 2 | \hat r_1(\tilde q(w_a, w_n)) |^2 \left| \operatorname{Re} w_a \sum_{l \ne j} (g_{a;l})_{n\bar n}(w_n\,; q) \right| & \\ + (2 | \hat r_{n\bar 1} | \, | \hat r_1 | + | \hat r_{1\bar 1} | \, | \hat r_n |) \, | \hat r_n | (\tilde q(w_a, w_n)) & \end{aligned} \tag{3.6}$$

$$+ 4k^2 \sum_{\iota=2}^{2k} \| P_\iota(\cdot\,; q) \| \, | w_n |^{\iota-2} + | \mathscr{E}_{n\bar{n}} | .$$

We can find a positive constant C_1 independent of C_0 such that

$$| \mathscr{E}_{n\bar{n}} | \leq C_1 \Big(| w_a |^2 + \sum_{\iota=2}^{2k} \| P_i(\cdot\,; q) \| \, | w_n |^{\iota} \Big),$$

and for $| w_n | = \rho$,

$$| \hat{r}_n(\tilde{q}(w_a, w_n)) | \leq C_1 \Big[| w_a | \rho_0 \Big(1 + \frac{k}{C_0} \Big) \| g_{a;j} (\cdot\,; q) \| \rho^{j-2} + \sum_{\iota=2}^{2k} \| P_\iota(\cdot\,; q) \| \rho^{\iota-1} + | w_a |^2 + \rho^{2k} \Big].$$

For small enough $\rho_0 \ll 1$ it follows from (3.2) that

$$- 2 | \hat{r}_1(\tilde{q}(w_a, w_n)) |^2 \operatorname{Re} w_a \, (g_{a;j})_{n\bar{n}}(w_n\,; q) \leq C_1^3 \Big(\Big(k \frac{1}{C_0} + 2\rho_0 \Big) | w_a | \, \| g_{a;j}(\cdot\,; q) \| \rho^{j-2} + | w_a |^2 + \sum_{\iota=2}^{2k} \| P_\iota(\cdot\,; q) \| \rho^{\iota-2} \Big). \tag{3.7}$$

After enlarging the constant C_0 if necessary, we obtain from (3.7)

$$\frac{1}{2C_0} | (g_{a;j}(\cdot\,; q))_{n\bar{n}} |^2 - (1 + \rho_0 C_0) \frac{kC_1^3}{C_0^2} \| g_{a;j}(\cdot\,; q) \|^2 \rho^{2j-4} \leq \sum_{\iota=2}^{2k} \| P_\iota(\cdot\,; q) \| \rho^{\iota-2}). \tag{3.8}$$

But neither $\operatorname{Re} g_{a;j}$ nor $\operatorname{Im} g_{a;j}$ contains any harmonic terms. Therefore, with a certain constant $C(j)$, depending only on j, one has

$$\sup_{|w_n| = p} | (g_{a;j})_{n\bar{n}}(w_n\,; q) | \geq C(j) \, \| g_{a;j}(\cdot\,; q) \| \rho^{j-2} .$$

As ρ_0 we may choose $\rho_0 = 1/C_0$: here we enlarged C_0 such that for any j: $C(j) > 2kC_1^3/C_0$. This will imply

$$\| g_{a;j}(\cdot\,; q) \| \rho^{j-2} \leq \sqrt{4C_0/C(j)} \Big(\sum_{\iota=2}^{2k} \| P_\iota(\cdot\,; q) \| \rho^{\iota-2} \Big)^{\frac{1}{2}}$$

$$\leq (2k \| P_\iota(\cdot\,; q) \| \rho^{\iota-2})^{\frac{1}{2}}.$$

The proof of the lemma is now complete.

The following lemma contains the crucial estimates for the coupling terms Q_j, $g_{a;j}$:

LEMMA 3.2. *Let all the notations be as so far. Then we have*

(a) *If we write* $j_{Q,q} = \min\{\iota \leq k \mid \| Q_\iota(\cdot\,; q) \| > 0\}$, $j_{a,q} = \min\{j \mid g_{a;j}(\cdot\,; q) \neq 0\}$, *and* $i_q = \min\{\iota \mid \| P_\iota(\cdot;q) \| > 0\}$, *then* $j_{Q,q} \geq \frac{i_q}{2} + 1$, $j_{a;q} \geq \frac{i_q}{2} + 1$.

(b) *There exist positive* ρ_1, C_2, *such that for any* $|w_n| < \rho_1$

$$\sum_{j=2}^{k} \| Q_j(\cdot\,; q) \| \, |w_n|^j \leq C_2 |w_n| \left(\sum_{\iota=2}^{2k} \| P_\iota(\cdot\,; q) \| \, |w_n|^\iota\right)^{\frac{1}{2}} \tag{3.9}$$

and

$$\sum_{j=2}^{k} \| g_{a;j}(\cdot\,; q) \| \, |w_n|^j \leq C_2 |w_n| \left(\sum_{\iota=2}^{2k} \| P_\iota(\cdot\,; q) \| \, |w_n|^\iota\right)^{\frac{1}{2}}. \tag{3.10}$$

(c) *If* ρ_1 *is as in* (b) *and* $h_a(\cdot\,; q)$ *are the functions appearing in the last column of the Jacobi matrix of the mapping* $F(\cdot\,; q)$ *of Theorem* 3, *then for any* $0 < \rho < \rho_1/2$, *all* $|w_n| < \rho$, *and all positive integers* m *we have the estimate*

$$| h_a^{(m)}(w_n\,; q) | \leq m! C_2 \left(\sum_{j=2}^{2k} \| P_\iota(\cdot\,; q) \| \rho^\iota\right)^{\frac{1}{2}} \left(\frac{2}{\rho}\right)^{m+1}. \tag{3.11}$$

Proof. (a) Obviously we have for $0 < \rho \ll \rho_0$:

$$\| Q_{j_{Q,q}}(\cdot\,; q) \| \rho^{j_{Q,q}} \geq C_0 \max_{l \neq j_{Q,q}} \| Q_l(\cdot\,; q) \| \rho^l,$$

$$\| P_{\iota_q}(\cdot\,; q) \| \rho^{\iota_q} \geq C_0 \max_{l \neq j_q} \| P_l(\cdot\,; q) \| \rho^l,$$

and

$$\| g_{a;j_{a,q}}(\cdot\,; q) \| \rho^{j_{a,q}} \geq C_0 \max_{l \neq j_{a,q}} \| g_{a;l}(\cdot\,; q) \| \rho^l.$$

This, combined with Lemma (3.1) gives (a).

(b) For $n = 2$, (3.9) is just Proposition (1.6) of [F-S], which is stated there without proof. Our statements (3.9) and (3.10) are generalizations of that proposition. Therefore we give a sketch of proof for reader's convenience. Let ρ_0 be the radius from Lemma 3.1 and $0 < \rho_1 < \rho_0$. We denote by M_j one of the quantities $\| Q_j(\cdot\,; q) \|$ or $\| g_{a;j}(\cdot\,; q) \|$. Also fix a point $w_n \in \mathbf{C}$, $|w_n| \leq \rho_1$, and let $T = C_0^{-2}$. If $M_k |w_n|^k \geq C_0 \max_{l \neq k} M_l |w_n|^l$, then everything will follow from Lemma 3.1, when we choose $C_2 \geq C_0^2$. If not, let l_1 be the largest number less than k, such that $M_k |w_n|^k \leq C_0 M_{l_1} |w_n|^{l_1}$. It is easy to show that

$$M_{l_1}(T |w_n|)^{l_1} \geq C_0 \max_{l > l_1} M_l (T |w_n|)^l.$$

If now even

$$M_{l_1}(T|\,w_n\,|)^{l_1} \geq \max_{l \neq l_1} M_l(T|\,w_n\,|)^l,$$

we will be done by virtue of Lemma (3.1), otherwise let l_2 be the largest number less than l_1, such that $M_{l_1}(T|\,w_n\,|)^{l_1} \geq C_0 M_{l_2}(T|\,w_n\,|)^{l_2}$. Then we can prove

$$M_{l_2}(T^2|\,w_n\,|)^{l_2} \geq C_0 \max_{l > l_2} M_l(T^2|\,w_n\,|)^l.$$

We continue in this way and obtain after a finite number of steps a number $l_m \leq k$, $m \leq k$, for which

$$M_{l_m}(T^m|\,w_n\,|)^{l_m} \geq C_0 \max_{l \neq l_m} M_l(T^m|\,w_n\,|)^l.$$

By Lemma 3.1 the claim follows with $C_2 = C_0^{4k^2+2}$.

(c) In order to prove (3.11), we work in the coordinate system $(w_1', \ldots, w_n') = (w_1^{3k+2}, \ldots, w_n^{(3k+2)})$ of section 1. The domain Ω is described with respect to this coordinate system by a defining function r' which has the form (2.4) but the g_a-functions, which we denote here by $g_a'(\cdot\,;q)$, still contain antiholomorphic terms.

We have

$$g_a'(w_n\,;q) = \overline{\tilde{h}_a(w_n\,;q)} + \tilde{g}_a(w_n\,;q)$$

with a holomorphic polynomial $\tilde{h}_a(\cdot\,;q)$ of degree at most k, while in the second member there are no holomorphic or anti-holomorphic terms. Let for $a, b \in \{2, \ldots, n\}$:

$$\mathcal{L}'_{a\bar{b}} = r'_{a\bar{b}}|\,r_1'\,|^2 - r'_{a\bar{1}}\, r_1' \overline{r_b'} - r'_{1\bar{b}}\, r_a' \overline{r_1'} + r'_{1\bar{1}}\, r_a' \overline{r_b'}.$$

For $0 < |\,w_n\,| < \rho_1$ we choose a real $q_1'(w_n)$, such that

$$q'(w_n) := (q_1'(w_n), 0, \ldots, w_n)$$

is a boundary point of Ω. Then, given a fixed index $a \in \{2, \ldots, n\}$, we have

$$(3.12) \qquad |\,\mathcal{L}'_{a\bar{n}}\,|^2 \leq \mathcal{L}'_{a\bar{a}}\,\mathcal{L}'_{n\bar{n}}$$

at the point $q'(w_n)$. On the other hand

$$\mathcal{L}'_{a\bar{n}}(q'(w_n))\,|^2 \geq \frac{1}{2}\left|\,\frac{\partial \tilde{h}_a}{\partial w_n}(w_n\,;q)\,\right|^2 - \mathcal{F}_1.$$

Combining this with (3.12) we arrive at

$$\left| \frac{\partial \tilde{h}_a}{\partial w_n} (w_n\,; q) \right|^2 \leq C_3 \mathscr{L}'_{n\bar{n}}(w_n\,; q) + \mathscr{F}_2,$$

where C_3 is a universal constant (independent of w_n), and $\mathscr{F}_1$, $\mathscr{F}_2$ are remainder terms, which can, as also $\mathscr{L}'_{n\bar{n}}(q'(w_n))$, be controlled, with some universal constant C_4, by

$$C_4 \sum_{i=2}^{2k} \| P_i(\cdot\,; q) \| \, | w_n |^{i-2}.$$

Altogether we obtain

$$\left| \frac{\partial \tilde{h}_a}{\partial w_n} (w_n\,; q) \right| \leq C_5 \sqrt{\sum_{i=2}^{2k} \| P_i(\cdot\,; q) \| \, | w_n |^i} \; \frac{1}{| w_n |}.$$

But the functions h_a in Theorem 3 are just given by

$$h_a(w_n\,; q) = \frac{\partial \tilde{h}_a}{\partial w_n} (w_n\,; q).$$

So (c) will now follow from the Cauchy inequalities.

We are going to prove the appropriate analogue of the bumping lemma, cf. Theorem A in [F-S].

PROPOSITION 3.3 (cf. [F-S], Lemma 3.3.2k). *There exist positive constants* A, B, $\rho_2 < \rho_1$ *and for each point* $q \in \partial\Omega \cap B(0, 2\rho_2)$ *a continuous function* $\tilde{P}(\cdot\,; q) : \mathbf{C} \rightarrow \mathbf{R}$ *with the following properties:*

(1) *With a positive universal constant* C_6 *one has for each* w_n, $w'_n \in \mathbf{C}$, $R > 0$, *such that*

$$| w'_n | \leq \left(1 + \frac{| w_n |}{R}\right)^{-2k} \cdot R,$$

the estimate

$$\tilde{P}(w_n + w'_n\,; q) \leq \tilde{P}(w_n\,; q) + C_6 \sum_{j=2}^{2k} \| P_i(\cdot\,; q) \| \, R^j$$

holds.

(2) *The function* $\tilde{P}(\cdot\,; q)$ *is subharmonic on the disc* $D = \{| w_n | < 4\rho_2\}$.

(3) *On* D *the estimates*

$$- B \sum_{j=2}^{2k} \| P_j(\cdot\,; q) \| \, | w_n |^j \leq \tilde{P}(w_n\,; q) - \sum_{j=2}^{2k} P_i(w_n\,; q) \leq - A \sum_{j=2}^{2k} \| P_j(\cdot\,; q) \| \, | w_n |^j$$

are satisfied.

Remark. In [F-S] the function $\tilde{P}$ is constructed only on a disc and in case $n=2$. Also, property (1) is not discussed. If we pursue step by step all the constructions made in that paper, we can see that all of them just so go through in our situation. One can also obtain property (1), which is crucial for the estimation of the Caratheodory metric, since one can show it for all the functions constructed in Lemmas (3.3.i) of [F-S]. All the constants which appear during the single steps of construction can be chosen uniformly with respect to q.

The proposition enables us to write down a precise bumping function for Ω_q at the origin of the (w)-system.

THEOREM 4. *For sufficiently large numbers K, $L \gg 1$ the function*

$$\varphi(w\,;q) = \mathrm{Re}(w_1 + Lw_1^2 + f(w\,;q)) + \frac{1}{2}|\,w''\,|^2 + \tilde{P}(w_n\,;q)$$

is plurisubharmonic on the ball $B(0, 2\rho_2)$, for $q \in \partial\Omega \cap B(0, 2\rho_2)$, and it satisfies the estimate

$$\hat{r}(w) - L^2 v_1^2 - K\,V_q(w) \le \varphi(w\,;q) \le -\frac{1}{K}V_q(w) + \frac{1}{2}\hat{r}(w), \tag{3.13}$$

where $v_1 = \mathrm{Im}\, w_1$, and $V_q(w) = |\,w''\,|^2 + \sum_{j=2}^{2k} \|P_j(\cdot\,;q)\|\,|\,w_n\,|^j$, and $\hat{r} = \hat{r}_q$.

Proof. If we write $u_1 = \mathrm{Re}\, w_1$, we get

$$u_1 = \hat{r}(w) - \mathrm{Re} f(w\,;q) - R_1(w) - |\,w''\,|^2 - R_2(w) - \sum_{j=2}^{2k} P_j(w_n\,;q), \tag{3.14}$$

where

$$R_1(w) = v_1\left(\sum_{j=2}^{k} Q_j(w_n\,;q) + \sigma_{k+1}(w_n) + \sigma_1(w'')\sigma_1(w_n)\right) + \sigma_2(v_1)$$

and

$$\begin{aligned} R_2(w) = 2\,\mathrm{Re} \sum_{a=2}^{n-1} w_a g_a(w_n\,;q) + \sigma_3(w'') + \sigma_1(w'')\sigma_{k+1}(w_n) \\ + \sigma_2(w'')\sigma_1(w_n) + \sigma_{2k+1}(w_n). \end{aligned}$$

Substitution into the definiton of φ gives us

$$\begin{aligned}(3.15)\qquad \varphi(w\,;q) &= \hat{r}(w) - R_1(w) - \frac{1}{2}|\,w''\,|^2 - R_2(w)\\ &\quad + \tilde{P}(w_n\,;q) - \sum_{j=2}^{2k} P_j(w_n\,;q) + Lu_1^2 - Lv_1^2.\end{aligned}$$

Because of Lemma 3.2 and the normalization of f we can estimate

$$|\,R_1(w)\,| \le C_7\left(v_1^2 + |\,w_n\,|^2 \sum_{j=2}^{2k} \|\,P_j(\cdot\,;q)\,\|\,|\,w_n\,|^j + |\,w_n^2\,|\,|\,w'''\,|^2\right),$$

$$|\,R_2(w)\,| \le \frac{1}{10}|\,w''\,|^2 + C_7\,|\,w_n\,|^2 \sum_{j=2}^{2k} \|\,P_j(\cdot\,;q)\,\|\,|\,w_n\,|^j,$$

and

$$|\,\mathrm{Re}\,f(w\,;q)\,|^2 \le C_7\left[u_1^4 + v_1^4 + \sum_{l=2}^{n}|\,w_l\,|^2\sum_{j=1}^{n-1}|\,w_j\,|^2 + \left(\sum_{j=2}^{2k}\|\,P_j(\cdot\,;q)\,\|\,|\,w_n\,|^j\right)^2\right],$$

with a universal positive constant C_7. Now, for large enough L, the right inequality in (3.13) is obtained by substituting these estimates into (3.15) and taking care of proposition (3.3). The left side of (3.13) follows in a similar way.

4. Estimations for the necessary domain functionals in the normalized coordinates

Throughout this section let us fix a boundary point q of $\partial\Omega$ close to 0 and a positive number t. We denote by p_t the point $(-t, 0, \ldots, 0)$. Furthermore, let $\Omega_q = F(\cdot\,;q)(\Omega) = \{\hat{r} < 0\}$. For a bounded domain $D \subset \mathbf{C}^n$ we denote by $K_D(z, \bar{z})$ the Bergman kernel function of D, by $B_D(z, X)$, $C_D(z, X)$, and $\mathrm{Kob}_D(z, X)$ its Bergman metric, Caratheodory metric, and Kobayashi metric, respectively. We also will need the functional $b_D^2(z, X) = K_D(z, \bar{z})B_D^2(z, X)$. The following relations are well-known:

$$\begin{aligned}K_D(z, \bar{z}) &= \max\{|\,f(z)\,|^2 \mid f \in H^2(D), \|\,f\,\| = 1\}\\ b_D(z, X) &= \max\{|\,(\partial f(z), X)\,| \mid f \in H^2(D), f(z) = 0, \|\,f\,\| = 1\}\\ C_D(z, X) &= \max\{|\,(\partial f(z), X)\,| \mid f \in H^\infty(D), f(z) = 0, \|\,f\,\|_\infty = 1\}\\ \frac{1}{\mathrm{Kob}_D(z, X)} &= \sup\{R > 0 \mid \exists\, f : \{|\,\tau\,| < R\} \to D,\\ &\qquad \text{holomorphic}, f(0) = z, f'(0) = X\}\end{aligned}$$

for $(z, X) \in D \times \mathbf{C}^n$. Here we abbreviate $H^j(D) = \mathcal{O}(D) \cap L^j(D)$, for $j = 2, \infty$, and $\|\;\| = \|\;\|_{L^2}$. We will at first give upper estimates for the functionals defined above. In order to do so, we introduce (analogously to [C 1]) for any $s > 0$ the radius

$$R_n(s) = \text{solution to the equation } \sum_{j=2}^{2k} \| P_j(\cdot\,; q) \| (R(s))^j = s.$$

Then we have the estimates

$$\frac{1}{C_8}\frac{1}{R_n(s)} \leq \sum_{j=2}^{2k} \left[\frac{\| P_j(\cdot\,; q)\|}{s}\right]^{\frac{1}{j}} \leq C_8 \frac{1}{R_n(s)}, \tag{4.1}$$

and for any $c > 0$

$$\frac{1}{1+c} R_n(s) \leq R_n(cs) \leq (1+c) R_n(s)$$

with a positive C_8 independent of q and s.

Lemma 4.1. *There exists a constant $C_9 > 0$, such that for all $t > 0$, $Y \in \mathbf{C}^n$ the following estimates all hold*

$$K_{\Omega_q}(p_t, \bar{p}_t) \leq C_9 t^{-\frac{n}{2}} R_n(t)^{-2} \tag{4.2}$$

$$b^2_{\Omega_q}(p_t, Y) \leq C_9 t^{-\frac{n}{2}} R_n(t)^{-2} \left[\frac{|Y_1|^2}{t^2} + \sum_{j=2}^{n-1} \frac{|Y_j|^2}{t} + \frac{|Y_n|^2}{R_n(t)^2}\right] \tag{4.3}$$

$$C_{\Omega_q}(p_t, Y) \leq C_9 \left[\frac{|Y_1|}{t} + \sum_{j=2}^{n-1} \frac{|Y_j|}{\sqrt{t}} + \frac{|Y_n|}{R_n(t)}\right] \tag{4.4}$$

$$\mathrm{Kob}_{\Omega_q}(p_t, Y) \leq C_9 \left[\frac{|Y_1|}{t} + \sum_{j=2}^{n-1} \frac{|Y_j|}{\sqrt{t}} + \frac{|Y_n|}{R_n(t)}\right] \tag{4.5}$$

for any $Y \in \mathbf{C}^n$.

Proof. All the domain functionals under consideration will increase, if the Ω_q are replaced by a domain which is contained in Ω_q. For a sufficiently small ε the polydisc

$$\Delta(p_t) = \Delta(-t, \varepsilon t) \times \prod_{j=2}^{n-1} \Delta(0, \sqrt{\varepsilon t}) \times \Delta(0, R_n(\varepsilon t))$$

will be a subset of Ω_q. This is apparent from the considerations of section 3. So (4.2) through (4.5) will follow immediately, since the right-hand sides of these estimates are just the respective domain functionals for the polydisc.

Because of the well-known inequalities $C_D \leq B_D$, and $C_D \leq \mathrm{Kob}_D$ for any domain D, we only have to estimate the Bergman kernel and the Caratheodory metric of Ω_q from below. This will be done by constructing certain holomorphic functions on Ω_q using the $\bar{\partial}$-technique with plurisubharmonic weight functions.

MAIN LEMMA 4.2. *There exist holomorphic functions* $F_0 \in H^2(\Omega_q)$, $F_1, \ldots, F_n \in H^\infty(\Omega_q)$ *with the following properties*:

(1) $F_0(p_t) = t^{-\frac{n}{2}} R_n(t)^{-1}$,

(2) *For any* $l = 1, \ldots, n$:

$$\left| \frac{\partial F_l}{\partial w_l}(p_t) \right| \geq \begin{cases} t^{-1}, & \textit{for } l = 1 \\ t^{-\frac{1}{2}}, & \textit{for } 2 \leq l \leq n-1. \\ R_n(t)^{-1}, & \textit{for } l = n \end{cases}$$

For all $l, j \in \{1, \ldots, n\}$, $l \neq j$ *one has*

$$\frac{\partial F_j}{\partial w_l}(p_t) = 0.$$

(3) *There exists a constant* C_{10} *independent of* t, q *such that*

$$\| F_0 \| \leq C_{10}, \; \| F_l \|_\infty \leq C_{10}, \textit{ for any } l = 1, \ldots, n.$$

Proof. We write $\varphi'(w'; q) = \varphi(0, w'; q)$, where φ denotes the bumping function from Theorem 4. Further let G be the tube $G = \mathbf{C}^{n-2} \times \Delta(0, 4\rho_2)$, and

$$Q_t(w') = \frac{|w''|^2}{\varepsilon t} + \frac{|w_n|^2}{\varepsilon R_n(t)^2}.$$

Then the function

$$\begin{aligned} \psi_t(w') &= \log(1 + |w'|^2) + \log(1 + Q_t(w')) \\ &+ n \log Q_t(w') + \frac{1}{t} \varphi'(w'; q) \end{aligned}$$

is plurisubharmonic on G. Furthermore, we define the functions

$$g_0 \equiv t^{-\frac{n}{2}}(R_n(t))^{-1}, \text{ on } \mathbf{C}^n$$

$$g_1 \equiv 1, \text{ on } \mathbf{C}^{n-1}$$

$$g_l(w') = \frac{w_l}{\sqrt{t}}, \text{ for } l = 2, \ldots, n-1, \text{ on } \mathbf{C}^{n-1}$$

and finally

$$g_n(w') = \frac{w_n}{R_n(t)}, \text{ also on } \mathbf{C}^{n-1}.$$

We first want to construct the functions $F_1, \ldots, F_n$. For this we will solve on G the Cauchy-Riemann equation

$$\begin{aligned} \bar\partial u_l = v_l &:= \bar\partial[g_l\, \chi \circ Q_t] \\ &= g_l\, \chi \circ Q_t \cdot \bar\partial Q_t. \end{aligned} \tag{4.6}$$

Here, χ is a smooth cut-off function on the real line, such that $|\chi'| \leq 2$, $\chi(x) = 1$, for $x \leq 1/4$, $\chi(x) = 0$, if $x > 1$. If ε and t are small enough, then

$$\operatorname{supp}(v_l) \subset \left\{\frac{1}{4} \leq Q_t \leq 1\right\} \subset\subset \{Q_t \leq 2\} \subset\subset G.$$

In sections (4.2) and (4.4), in particular Lemma (4.4.1) of Hörmander's book, [Hör] the following theorem is contained

THEOREM 5. *Let N be a positive integer and $D \subset \mathbf{C}^n$ a pseudoconvex domain, Φ a plurisubharmonic function, and v be a $\bar\partial$-closed* (0.1) *form with locally square-integrable coefficients on D. Suppose we are given a strictly plurisubharmonic function Ψ of class $\mathscr{C}^2$ on D, such that $\Phi - \Psi$ is plurisubharmonic on the support of v, and the integral*

$$I(v) = \int_D |v|^2_{\partial\bar\partial\Psi} e^{-\Phi}\, d^{2N} z$$

is finite. Then there exists a solution u for the equation

$$\bar\partial u = v$$

which is locally square-integrable on D, and satisfies

$$\int_D |u|^2 e^{-\Phi} d^{2N} z \leq 2I(v).$$

In our context $D = G$, $N = n - 1$, $v = v_l$, for $l \geq 1$, $\Phi = \phi_t$, and $\Psi = \log(1 + Q_t)$. Next we estimate the integral $I(v_l)$. From

$$\partial\bar\partial\Psi \geq \frac{1}{(1 + Q_t)^2}\, \partial\bar\partial Q_t \geq \frac{\partial Q_t \overline{\partial Q_t}}{Q_t(1 + Q_t)^2}$$

it follows that

$$|v_l|^2_{\partial\bar\partial\Psi} \leq 4\xi_{\operatorname{supp} v_l},$$

where ξ_M denotes the characteristic function of a set M. The left half of (3.13) implies, that on supp v_l

$$\varphi' \geq -\varepsilon^2(B+1)t$$

and in particular

$$\phi_t \geq -n\log 4 - \varepsilon^2(B+1).$$

This gives us

$$I(v_l) \leq C_{11}\mathrm{vol}(\{Q_t \leq 1\}) \leq C_{12}t^{n-1}R_n(t)^2.$$

Let now $u_l \in \mathscr{C}^\infty(G)$ be a solution to $\bar\partial u_l = v_l$, according to Theorem 5, such that

$$\int_G |u_l|^2 e^{-\phi_t} d^{2n-2}w' \leq 2I(v_l).$$

Since $e^{-\phi_t}$ becomes as singular as $|w'|^{-2n}$ near 0, all the u_l must vanish to at least second order at 0. Furthermore, the u_l are all holomorphic on $G\setminus\{Q_t \geq 1\}$. As in section 2 of [F-S] we now apply the mean value inequality in order to gain an upper estimate for the holomorphic function

$$\tilde f_l(w') = \chi(Q_t(w'))g_l(w') - u_l(w'),$$

defined on $G' = \mathbf{C}^{n-2} \times \varDelta(0, 3\rho_2)$. Let $w' \in G'$, such that $Q_t(w') \leq 5$; for small $0 < a \ll \varepsilon/n$, the polydisc $\hat P(w')$ around w' with the radii

$$\hat R_2 = \cdots = \hat R_{n-1} = \frac{a\sqrt{t}}{(1+Q_t(w'))^{2k}},$$

$$\hat R_n = \frac{aR_n(t)}{(1+Q_t(w'))^{2k}}$$

is contained in G' as a relatively compact subset. From the inequality $|a-b|^2 \geq |a|^2/2 - |b|^2$ we obtain for any $\zeta' \in \hat P(w')$, that $Q_t(\zeta') \geq \frac{1}{2}Q_t(w') - Q_t(w'-\zeta') \geq 5/2 - (a/n)^2\varepsilon \geq 2$. Thus u_l is holomorphic near $\hat P(w')$. If we denote by $\hat P$ the polydisc around 0 with the same radii as $\hat P(w')$, we obtain by the mean value inequality

$$|u_l(w')|^2 \leq (\hat R_2 \ldots \hat R_n)^{-2}\int_{\hat P(w')} |u_l(\zeta')|^2 d^2\zeta_2 \ldots d^2\zeta_n \tag{4.7}$$

$$\leq 2(\hat R_2\ldots\hat R_n)^{-2}\max_{\xi'\in\hat P} e^{\phi_t(w'+\xi')}\, I(v_1).$$

From property (1) for the bumping function φ of Theorem 4 we get for $\xi' \in \hat P$:

$$\psi_t(w' + \xi') \leq C_{13} + \psi_t(w').$$

Thus the right-hand side of (4.7) can be estimated by

$$C_{14}(1 + Q_t(w'))^{2m}\exp\left(\frac{1}{t}\,\varphi'(w' ; q)\right),$$

with universal constants C_{13}, C_{14}, and $m = k(n-1) + n + 2$. It is easy to see that

$$|\tilde{f}_l(w')| \leq C_{15}(1 + Q_t(w'))^m e^{\frac{\varphi'(w' ; q)}{2t}}. \tag{4.8}$$

Since on $\{Q_t(w') \leq 5\}$ the right-hand side is bounded from below uniformly with respect to t, this estimate is also satisfied for w' with $Q_t(w') \leq 5$. This follows from the maximum principle. The functions $\tilde{f}_l$ all vanish at 0, and

$$\frac{\partial \tilde{f}_l}{\partial w_j}(0) = \frac{\partial g_l}{\partial w_j}(0), \text{ for } 2 \leq j \leq n,\ 1 \leq l \leq n.$$

We are now ready to define near $0 \in \partial\Omega$ holomorphic functions with the properties required in the Main Lemma. Let

$$f_l(w) = \exp\left(\frac{w_1 + f(w ; q) + Lw_1^2}{2t}\right)\tilde{f}_l(w') \tag{4.9}$$

for $w \in \mathbf{C} \times G'$, $1 \leq l \leq n$. Finally we have to replace the f_l by functions $F_l \in H^\infty(\Omega_q)$, with the same behavior at the point p_t. We proceed in a similar way as Bedford-Fornæss did in section 2 of [BF], or Range in [R, proof of Theorem 2.2]. By [C 2] the domain Ω_q is regular in the sense of [D-F 1], such that we can choose a Stein neighborhood basis $(\Omega_q^s)_{s>0}$ for $\bar{\Omega}_q$ with $\Omega_q^s \searrow \bar{\Omega}_q$. Let us choose another cut-off function ξ, with values between 0 and 1, and which is zero on $[25/4\rho_2^2, \infty)$, and 1 on $(-\infty, 4\rho_2^2]$. For $l = 1, \ldots, n$ we define

$$\alpha_l = \bar{\partial}[\xi(|w|^2)f_l(w)].$$

Further, let $W_t = 2\log(1 + |w|^2) + (n+1)\log(|w_1 + t|^2 + |w'|^2)$.

Our claim is: For a sufficiently small number $s > 0$ one has

$$\int_{\Omega_q^s} |\alpha_l|^2 e^{-W_t} d^{2n}w \leq C_{16}(s) \tag{4.10}$$

where C_{16} depends only on s, but not on t, and $|\alpha_l|^2$ denotes the sum of squares of the absolute values of the coefficients of α_l.

To prove this we choose for a positive number $\delta \ll 1$ an $s(\delta) > 0$, such that

$\Omega_q^{s(\delta)} \cap B(0, 3\rho_2) \subset \{\hat{r} < \delta\}$. This implies

$$\Omega_q^{s(\delta)} \cap \operatorname{supp} \alpha_l \subset \{3\rho_2/2 < |w| < 5\rho_2/2)\} = S.$$

By means of the bumping lemma, Theorem 4, we see that on S:

$$\varphi(w ; q) \leq \frac{\delta}{2} - \beta\rho_2^{2k} - \frac{1}{2K}(|w_1|^2 + V_q(w)),$$

where β does not depend on (t, q, δ). For $\delta < \beta\rho_2^{2k}$ it now follows that

$$\Omega_q^{s(\delta)} \cap \operatorname{supp} \alpha_l \subset \{\varphi(w ; q) + \frac{1}{2K}(|w_1|^2 + V_q(w)) < 0\} = M.$$

We show that $s_1 = s(\delta)$ satisfies (4.8). On $\operatorname{supp} v_l$ we have for small enough t:

$$e^{-W_t} \leq \rho_2^{-2n-2},$$

and, by virtue of (4.8), (4.9)

$$\begin{aligned}\int_{\Omega_q^{s_1}} |\alpha_l|^2 e^{-W_t} d^{2n}w &\leq \rho_2^{-2n} \int_{\Omega_q^{s_1} \cap \operatorname{supp} \alpha_l} |f_l(w)|^2 d^{2n}w \\ &\leq \rho_2^{-2n} C_{15} \int_{M \cap B(0,3\rho_2)} (1 + Q_t(w'))^m e^{\frac{\varphi'(w';q)}{2t}} d^{2n}w.\end{aligned}$$

But the last integral is bounded uniformly in t and q by a constant C_{16}, since the integrand is less than a constant times

$$(1 + Q_t(w'))^{2m} \exp(-(Q_t(w'))^{\frac{1}{k}}).$$

By Theorem 4.4.2 of [Hör] we find a smooth solution $\tilde{u}_l$ to the equation $\bar{\partial}\tilde{u}_l = \alpha_l$, satisfying

$$\int_{\Omega_q^{s_1}} |\tilde{u}_l|^2 e^{-W_t} \leq C_{17}$$

uniformly in q, t. Obviously the functions $\tilde{u}_l$ are all uniformly (in (q, t)) bounded on Ω_q. Now let

$$\begin{aligned}F_1(w) &= \frac{1}{2}\xi(|w|^2) f_1(w) - \tilde{u}_1(w) - f_1(p_t),\\ F_l(w) &= \xi(|w|^2) f_l(w) - \tilde{u}_l(w),\ 2 \leq l \leq n.\end{aligned}$$

These functions are holomorphic on Ω_q, and behave at p_t like the f_l, and because of (4.8), (4.9) they satisfy all the requirements of the Main Lemma.

We now construct the function F_0. To do this we apply Theorem 5 with

$N = n$, $D = \Omega_q \cap (\mathbf{C} \times G)$ to the $\bar{\partial}$-data

$$v_0 = \bar{\partial}\chi \left(\frac{|w_1 + t|^2}{\varepsilon t^2} + Q_t\right) g_0.$$

Next we choose the right plurisubharmonic weight functions. First let j_0 be an index for which

$$R_n(t) \geq \left(\frac{\varepsilon t}{\| P_{j_0}(\cdot\, ; q) \|}\right)^{1/j_0},$$

and

$$W_t(w) = \frac{1}{t} | w'' |^2 + \left(\frac{\| P_{j_0}(\cdot\, ; q) \|}{\varepsilon t}\right)^{2/j_0} | w_n |^2.$$

Then the function

$$\lambda_t'(w) = (1 + W_t(w)) \exp\left(\frac{1}{t}\, \varphi(w\, ; q)\right)$$

is plurisubharmonic and bounded on D, and, for small enough $\varepsilon_0 \ll 1$ also the function

$$\lambda_t' - \varepsilon_0 Q_t$$

is plurisubharmonic on the polydisc $\Delta(p_t)$ used in the proof of Lemma 4.1. From the properties of φ it follows that λ'_t is bounded from above uniformly in (q, t). By the results of [D-F 2] we can choose a small number $b > 0$ and a large number M, such that $\tau := -(-\hat{r} \exp(-M | w |^2))^b$ becomes a strictly plurisubharmonic function on D. We set

$$\lambda_t'' = \exp\left(\frac{\tau}{t^b}\right).$$

Then we have, with a small positive constant c:

$$\partial\bar{\partial}\lambda_t'' \geq c \left(\frac{| dw_1 |^2}{t^2} - \frac{1}{c}\, \partial\bar{\partial} Q_t\right).$$

So Theorem 5 applies with $\Psi = c^4\left(\frac{| w_1 + t |^2}{t^2} + Q_t\right)$ and

$$\Phi = c^{-1}(c^2\lambda_t'' + \lambda_t')\, 2n \log \xi(c^{-4}\, \Psi)$$

with a cut-off function ξ, such that $\xi(x) = x$, for $x < 7/8$, and $\xi(x) = 1$, for $x > 1$. What we obtain, is a smooth function u_0, such that

$$F_0 = \chi\left(\frac{|w_1 + t|^2}{\varepsilon t^2} + Q_t\right) g_0 + u_0$$

lies in $H^2(\Omega_q)$ and has all the desired properties. The proof of the Main Lemma is complete.

As a corollary we get, using the defining formulas for the domain functionals K_{Ω_q}, b_{Ω_q}, C_{Ω_q} and Kob_{Ω_q}.

THEOREM 6. *With a universal positive constant* C_{17} *the following estimates all hold:*

$$\frac{1}{C_{17}} \le t^n (R_n(t))^2 K_{\Omega_q}(p_t, \bar{p}_t) \le C_{17} \tag{4.11}$$

$$\frac{1}{C_{17}}\left[\frac{|Y_1|^2}{t^2} + \sum_{j=2}^{n-1}\frac{|Y_j|^2}{t} + \frac{|Y_n|^2}{R_n(t)^2}\right] \le (C_{\Omega_q}(p_t\,;Y))^2,\ B^2_{\Omega_q}(p_t\,;Y), \tag{4.12}$$

$$(\mathrm{Kob}_{\Omega_q})^2(p_t\,;Y) \le C_{17}\left[\frac{|Y_1|^2}{t^2} + \sum_{j=2}^{n-1}\frac{|Y_j|^2}{t} + \frac{|Y_n|^2}{R_n(t)^2}\right]$$

for all vectors $Y \in \mathbf{C}^n$.

Proof. By means of the function F_0 we can estimate the Bergman kernel function of $D = \Omega_q \cap (\mathbf{C} \times G)$ in the desired way from below. Replacing D by Ω_q is allowed because of the localization lemma in [Oh].

5. Transformation to the original coordinates

Suppose $z \in \Omega \cap B_1$, where B_1 is a small ball centered at the origin, which is contained in the ball B which was introduced at the beginning of this paper. After shrinking B_1 we can find a boundary point $q \in \Omega \cap B$ and a positive number t, such that $z = q - te_1$. Here $t \approx |r(z)|$. We will have finished the proof of Theorems 1 and 2, if we have shown

$$\sum_{l=2}^{2k}\left(\frac{\|P_l(\cdot\,;q)\|}{t}\right)^{\frac{2}{l}} \approx \mathscr{C}_{2k}(z)^2. \tag{5.1}$$

(Here we write $f \approx g$ for two functions f, g, to indicate that there is a uniform constant $c > 0$, satisfying $\frac{1}{c}f \le g \le cf$). Because of the coupling between the weakly pseudoconvex direction L_n and the strongly pseudoconvex ones, which is reflected in the appearence of the functions $\frac{\partial F_a}{\partial z_n}$ in Theorem 3, $2 \leq a \leq n$, it is quite tedious to convert from the normalized coordinates $w_1, \ldots, w_n$ to the initial ones. We agree upon the following

Notations. By $\tilde{L}_n$ we will denote the vector field

$$\tilde{L}_n = F_*(L_n),$$

where we abbreviate $F = F(\cdot\,; q)$. For $2 \leq a \leq n$ we set

$$\tilde{L}_a = \frac{\partial}{\partial w_a} - \frac{\partial \hat{r}/\partial w_a}{\partial \hat{r}/\partial w_1} \frac{\partial}{\partial w_1}.$$

Then we obtain, with the functions $h_a(\cdot\,; q)$ from Theorem 3

$$\tilde{L}_n = \sum_{a=2}^{n} h_a(w_n\,; q) \tilde{L}_a, \tag{5.2}$$

where we define $h_n \equiv 1$. Furthermore,

$$\mathscr{L}_{a\bar{b}} = |\, r_1(q)\,|^2 \sum_{l,m=2}^{n} \hat{\mathscr{L}}_{l\bar{m}} \circ F \frac{\partial F_l}{\partial z_a} \overline{\frac{\partial F_m}{\partial z_b}}. \tag{5.3}$$

Here, $\hat{\mathscr{L}}_{l\bar{m}} = \partial \hat{r}([\hat{L}_l, \overline{\hat{L}_m}])$.

LEMMA 5.1. *Let α be a positive integer. We denote by $\mathscr{S}_\alpha$ the set of all p-tuples $A = (a_1, \ldots, a_p)$, with $p \leq \alpha$, such that $2 \leq a_i \leq n$ for all entries a_j of A, and not all a_i are equal to n. Then we have*

$$\tilde{L}_n^\alpha = \hat{L}_n^\alpha + \sum_{A \in \mathscr{S}_\alpha} \phi_A \tilde{L}_A.$$

Here $\hat{L}_A = \hat{L}_{a_1} \ldots \hat{L}_{a_1}$ for $A = (a_1, \ldots, a_p)$, and

$$\phi_A = c_A h_{i_1}^{\mu_1 - 1} \ldots h_{i_{p'}}^{\mu_{p'} - 1}$$

with integers c_A, $p' \leq p$, $i_1, \ldots, i_{p'} \in \{2, \ldots, n-1\}$, $\mu_1, \ldots, \mu_{p'} \geq 1$, $\sum_{j=1}^{p'} \mu_j = \alpha - \#\{i \mid a_i = n\}$.

Proof. The proof can be given by induction on α, using (5.2). It consists in a somewhat long but elementary computation. So we omit the details here.

LEMMA 5.2. *If we set* $\hat{\lambda} = \det(\hat{\mathscr{L}}_{j,\bar{l}})_{j,l=2}^{n}$, *then for any* $a, b \geq 1$:

$$L_n^{a-1}\bar{L}_n^{b-1}\lambda_{\partial\Omega} - (\hat{L}^{a-1}\bar{\hat{L}}_n^{b-1}\hat{\lambda})\circ F$$

is a sum of products of the form

$$h_{i_1}^{\mu_1-1}\dots h_{i_p}^{\mu_p-1}\bar{h}_{\bar{i}_1}^{\bar{\mu}_1-1}\dots\bar{h}_{\bar{i}_{\bar{p}}}^{\bar{\mu}_{\bar{p}}-1}\cdot g,$$

where g *is a smooth function,* μ_i, $\bar{\mu}_i$, *and* $p + \bar{p} \geq 2$. *Furthermore,* $\sum_1^p \mu_i \leq a - 1$, $\sum_1^{\bar{p}} \bar{\mu}_j \leq b - 1$.

Proof. By the definition of $\tilde{L}_n$ we have

$$L_n^{a-1}\bar{L}_n^{b-1}(\hat{\mathscr{L}}_{l\bar{m}}\circ F) = (\tilde{L}_n^{a-1}\bar{\tilde{L}}_n^{b-1}\hat{\mathscr{L}}_{l\bar{m}})\circ F.$$

The definition of $\lambda_{\partial\Omega}$ gives us immediately

$$L_n^{a-1}\bar{L}_n^{b-1}\lambda_{\partial\Omega} = \sum_{(A),(B)}\det\begin{pmatrix} L_n^{a_2-1}\bar{L}_n^{b_2-1}\mathscr{L}_{2\bar{2}}, & \cdots & L_n^{a_n-1}\bar{L}_n^{b_n-1}\mathscr{L}_{n\bar{2}} \\ \vdots & \ddots & \vdots \\ L_n^{a_2-1}\bar{L}_n^{b_2-1}\mathscr{L}_{n\bar{2}}, & \cdots & L_n^{a_n-1}\bar{L}_n^{b_n-1}\mathscr{L}_{n\bar{n}}, \end{pmatrix}.$$

Here, the sum is extended over all multiindices $A = (a_2, \dots, a_n)$ of length a and $B = (b_2, \dots, b_n)$ of length b. Next we substitute (5.3) and (5.4) into this and apply the Leibniz rule.

LEMMA 5.3. *For any positive integers* a, b *one has*

$$\hat{L}_n^{a-1}\bar{\hat{L}}_n^{b-1} = \frac{\partial^{a+b-2}}{\partial w_n^{a-1}\,\partial\bar{w}_n^{b-1}} +$$

a sum of terms of the form $A_{v\mu\rho\sigma}\dfrac{\partial^{v+\mu+\rho+\sigma}}{\partial w_1^v\partial\bar{w}_1^\mu\partial w_n^\rho\partial\bar{w}_n^\sigma}$, *where*

(1) $v + \mu + \rho + \sigma < a + b - 2$, *and*

(2) *each of the functions* $A_{v\mu\rho\sigma}$ *is a product of derivatives of* $\hat{r}_n/\hat{r}_1$ *with respect to* (w_1, w_n) *which contains at least one factor*

$$\frac{\partial^{(c+d)}}{\partial w_n^c\partial\bar{w}_n^d}\frac{\hat{r}_n}{\hat{r}_1}$$

with $c + d \leq a + b - 3$.

Proof. Induction over $a+b$, cf. [K 2], or formula (1.20) in [C 1]. Finally we will need also

LEMMA 5.4. *The function $\hat{\lambda}$ can be represented as*

$$\hat{\lambda} = \hat{\mathscr{L}}_{n\bar{n}} \det \begin{pmatrix} \hat{\mathscr{L}}_{2\bar{2}}, & \cdots, & \hat{\mathscr{L}}_{2\overline{n-1}} \\ \vdots & \ddots & \cdots \\ \hat{\mathscr{L}}_{n-1\bar{2}}, & \cdots & \hat{\mathscr{L}}_{n-1\overline{n-1}} \end{pmatrix}$$

$$+ \sum_{\nu,\mu=2}^{n-1} \varepsilon_{\nu\mu} \hat{\mathscr{L}}_{\nu\bar{n}} \overline{\hat{\mathscr{L}}}_{\mu\bar{n}} D_{\nu\mu},$$

where $\varepsilon_{\nu\mu} \in \{-1, 1\}$, and $D_{\nu\mu}$ denotes the determinant of the matrix which arises from $(\hat{\mathscr{L}}_{a\bar{b}} \circ F)$ by deleting the ν^{th} row and the μ^{th} column.

Proof. Apply the Laplace expansion theorem.

We are now ready for the

Proof of estimate (5.1). Let a, b be positive integers and $l = a+b$. Then

$$L_n^{a-1} \bar{L}_n^{b-1} \lambda_{\partial\Omega} = (\hat{L}_n^{a-1} \overline{\hat{L}}_n^{b-1} \hat{\lambda}) \circ F + \mathscr{F}_{(5.2)}$$

$$= \frac{\partial^{l-2}\hat{\lambda}}{\partial w_n^{a-1}\, \partial \bar{w}_n^{b-1}} \circ F + \mathscr{F}_{(5.2)} + \mathscr{F}_{(5.3)}$$

$$= \frac{\partial^{l-2}\hat{\mathscr{L}}_{n\bar{n}}}{\partial w_n^{a-1}\, \partial \bar{w}_n^{b-1}} \circ F + \mathscr{F}_{(5.2)} + \mathscr{F}_{(5.3)} + \mathscr{F}_{(5.4)}$$

$$= |\, r_1(q)\,|^2 \frac{\partial^l P_l}{\partial w_n^a\, \partial \bar{w}_n^b}(w_n\,;q) + \mathscr{F}_{(5.2)} + \mathscr{F}_{(5.3)} + \mathscr{F}_{(5.4)} + \mathscr{F}'_{(5.4)}.$$

Here $\mathscr{F}_{(5.2)}$, $\mathscr{F}_{(5.3)}$, and $\mathscr{F}_{(5.4)}$ are the error terms described in Lemmas (5.2) through (5.4), and the error term $\mathscr{F}'_{(5.4)}$ can be estimated by $\mathscr{C}_{2k}(z)^{1-1/l} \leq \text{const} \cdot t^{1/2k}\mathscr{C}_{2k}(z)$. By the choice of the redii we have on $\Delta(p_t)$:

$$\frac{\partial^{|\alpha+\beta|}\hat{r}}{\partial w^\alpha \partial \bar{w}^\beta}(w) \leq \frac{t^{1-\frac{1}{2}\sum_{j=2}^{n-1}\alpha_j+\beta_j}}{R_n(t)^{\alpha_n+\beta_n}}$$

for all multiindices α, β such that $\alpha_1 = \beta_1 = 0$, and $\frac{1}{2}(\sum_{j=2}^{n-1} \alpha_j + \beta_j) + \frac{\alpha_n+\beta_n}{2k} \leq 1$. In order to estimate the derivatives of the functions $h_a(w_n\,;q)$, we apply part

(c) of Lemma 3.2 with $\rho = R_n(t)$. This gives

$$| h_a^{(m-1)}(w_n ; q) | \leq \text{const} \frac{t}{R_n(t)^m}$$

for $m \leq 2k$. This enables us to control all the error terms $\mathscr{F}_{(5.2)}, \ldots, \mathscr{F}_{(5.4)}$ by $\mathcal{O}\left(\frac{t}{R_n(t)^{l-1}}\right)$. The above estimate can now be completed by

(5.4)

$$\begin{aligned} \left| \frac{L_n^{a-1} \bar{L}_n^{b-1} \lambda_{\partial\Omega}(q)}{t} \right|^{\frac{1}{l}} &\leq \text{const} \left| \frac{\frac{\partial^l P_l}{\partial w_n^a \partial \bar{w}_n^b}(0 ; q)}{t} \right|^{\frac{1}{l}} + t^{1/2k} C_{2k}(z) + \mathcal{O}(R_n(t)^{\frac{1}{l}-1}) \\ &\leq \text{const} \left[\left(\frac{\| P_l(\cdot ; q) \|}{t} \right)^{\frac{1}{l}} + t^{1/2k} C_{2k}(z) \right] + \mathcal{O}(R_n(t)^{\frac{1}{l}-1}) \\ &\leq C_{18} \left| \frac{1}{R_n(t)} + t^{1/2k} C_{2k}(z) \right]. \end{aligned}$$

In this estimate we used

LEMMA 5.6 *If P is a real-valued homogeneous polynomial of degree N in the plane, then there exists a constant c_N, depending only on N, such that*

$$\frac{1}{c_N} \| P \| \leq \sum | \textit{coefficients of } P | \leq c_N \| P \|.$$

We take the maximum over all a, b, satisfying $a + b = l$ in (5.4) and sum over all $l = 2, \ldots, 2k$. This yields

$$\sum_{l=2}^{2k} \left(\frac{\| P_l(\cdot ; q) \|}{t} \right)^{\frac{2}{l}} \geq \text{const } C_{2k}(z)^2.$$

The inequality is obtained in a similar way.

The expression for the invariant metrics. Finally we check that the pseudometric $M_\Omega(z ; X)$ introduced before the statement of Theorem 2 satisfies

$$M_\Omega(z ; X) \approx \frac{| [F'(q)X]_1 |^2}{t^2} + \sum_{j=2}^{n-1} \frac{| [F'(q)X]_j |^2}{t} + \frac{| [F'(q)X]_n |^2}{R_n(t)^2}.$$

This will conclude the proof of Theorem 2, since if F_Ω denotes one of the invariant metrics under consideration, then

$$F_{\Omega}(z\,;X) = F_{\Omega_q}(p_t\,;F'(q)X).$$

Theorem 2 therefore follows from Theorem 6. Now we have

$$F'(q)X = \begin{pmatrix} (\partial r(q),\, X) \\ B(q)^{-1}X'' + c\cdot X_n \\ X_n \end{pmatrix}. \tag{5.5}$$

Here, the matrices $A = (\hat{\mathscr{L}}_{a\bar{b}}(q))^{n}_{a,b=2}$ and $B(q)\,(\in GL(n-2,\mathbf{C}))$, and the vector $\mathbf{C}^{n-2}$ are related by

$$A = \begin{pmatrix} B^T & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} E_{n-2} & c \\ \bar{c}^T & a_{n\bar{n}} \end{pmatrix} \begin{pmatrix} \bar{B} & 0 \\ 0 & 1 \end{pmatrix}^{-1}.$$

From (5.5) we see that for $0 < \eta \ll \frac{1}{3}$:

$$\begin{aligned}\sum_{j=2}^{n-1} |[F'(q)X]_j|^2 &= (X'')^T(B^{-1})^T\bar{B}^{-1}\bar{X}'' + 2\mathrm{Re}\,\bar{c}^T B^{-1}X'' + |c|^2|X_n|^2 \\ &\geq \eta(X'')^T(B^{-1})^T\bar{B}^{-1}\bar{X}'' - 2\eta|c|^2|X_n|^2.\end{aligned}$$

Now the functions $s_i(X)$ satisfy the relation

$$s_i(X) = X_i - s_1(X)r_i(q) \tag{5.6}$$

for $i = 2,\dots,n$. Furthermore

$$(\bar{B}B^T)^{-1} = (\hat{\mathscr{L}}_{a\bar{b}}(q))^{n-1}_{a,b=2}$$

and

$$[F'(q)X]_n = X_n = s_n(X) + s_1(X)r_n(q).$$

This implies

$$\sum_{j=2}^{n-1}\frac{|[F'(q)X]_j|^2}{t} = \frac{\sum_{a,b=2}^{n-1}\mathscr{L}_{a\bar{b}}(z)s_a(X)\overline{s_b(X)}}{t} - 2\eta\frac{|c|^2}{t}|s_n(X)|^2 - C_{19}\frac{|s_1(X)|^2}{t}.$$

Keeping in mind that for $a = 2,\dots,n-1$ one has $c_a = h_a(0\,;q)$, we can estimate $|c|^2 \leq \mathrm{const}\,\frac{t}{(R_n(t))^2}$. Together with (5.5) we now obtain

$$\frac{|[F'(q)X]_1|^2}{t^2} + \sum_{j=2}^{n-1}\frac{|[F'(q)X]_j|^2}{t} + \frac{|[F'(q)X]_n|^2}{R_n(t)^2} \geq \mathrm{const}\,M_{\Omega}(z\,;X).$$

The opposite estimate is shown in similar way. Obviously we may replace

$(\mathcal{L}_{a\bar{b}}(q))_{a,b=2}^{n-1}$ by $(\mathcal{L}_{a\bar{b}}(z))_{a,b=2}^{n-1}$. The proof of Theorem 2 is now complete.

REFERENCES

[A] J. P. d'Angelo, Real hypersurfaces, orders of contact, and applications, Ann. of Math., **115** (1982), 615–637.

[Be] S. Bergman, Über die Kernfunktion eines Bereiches und ihr Verhalten am Rande, I.,Crelle's J. Reine Angew. Math., **169** (1933), 1–42.

[B-F] E. Bedford-J. E. Fornæss, Biholomorphic maps of weakly pseudoconvex donains, Duke Math., J., **45,4** (1978), 711–719.

[B-S] L. Boutet de Monvel-J. Sjöstrand, Sur la singularité des noyaux de Bergman et de Szegö, Soc. Math. de France, Astérisque, **34—35** (1976), 123–164.

[C 1] D. W. Catlin, Estimation of invariant metrics on pseudoconvex domains of dimension two, Math. Z., **200,3** (1989), 429–466.

[C 2] ——, Global Regularity of the $\bar{\partial}$-Neumann Problem, Proceedings of Symposia in Pure Mathematics, vol. **41,** 1984, pp. 39–49.

[Di 1] K. Diederich, Das Randverhalten der Bergmanschen Kernfunktion und Metrik in streng pseudokonvexen Gebieten, Math. Ann., **189** (1970), 9–36.

[Di 2] ——, Über die 1. und 2. Ableitungen der Bergmanschen Kernfunktion und ihr Randverhalten, Math. Ann., **203** (1973), 129–170.

[D-F 1] K. Diederich-J. E-Fornæss, Pseudoconvex domains: Existence of Stein neighborhoods, Duke Math. J., **44,3** (1977), 641–662.

[D-F 2] ——, Pseudoconvex domains: Bounded strictly plurisubharmonic exhaustion functions, Invent. Math., **39** (1977), 129–141.

[D-H-O] K. Diederich-G. Herbort-T. Ohsawa, The Bergman kernel on uniformly extendable pseudoconvex domains, Math. Ann., **273** (1986), 471–478.

[F] Ch. L. Fefferman, The Bergman kernel and biholomorphic mappings of pseudoconvex domains, Invent. Math., **26** (1974), 1–65.

[F-S] J. E. Fornæss-N. Sibony, Construction of p. l. s. functions on weakly pseudoconvex domains, Duke Math. J., **58,3** (1989), 633–655.

[Gr] I. Graham, Boundary behavior of the Caratheodory and Kobayashi metrics on strongly pseudoconvex domains in $\mathbf{C}^n$ with smooth boundary, Trans. Amer. Math. Soc, **207** (1975), 219–239.

[H] G. M. Henkin, An analytic polyhedron is not holomorphically equivalent to a strictly pseudoconvex domain, Soviet Math. Dokl., **14** (1973), 858–862.

[He 1] G. Herbort, Über das Randverhalten der Bergmanschen Kernfunktion einer speziellen Klasse schwach pseudokonexer Gebiete des $\mathbf{C}^n$, Math. Z., **184** (1983), 193–202.

[He 2] ——, Logarithmic growth of the Bergman kernel for weakly pseudoconvex domains in $\mathbf{C}^3$ of finite type, Manuscripta Math., **45** (1983), 69–76.

[He 3] Invariant metrics and peak functions on pseudoconvex domains of homogeneous finite diagonal type, Math. Z., **209** (1992) 223–243.

[Hör] L. Hörmander, An Introduction to Complex Analysis in Several Variables, Van Nostrand, North Holland Publication Company, 1973.

[K 1] J. J. Kohn, Harmonic Integrals on Strongly Pseudoconvex Manifolds, I., Ann. of Math., **78** (1963), 112–148.

[K 2] ——, Boundary behavior of $\bar{\partial}$ on weakly pseudoconvex manifolds of dimension two, J. Differential Geom., **6** (1972), 523–542.

[K 3] ——, Global regularity for $\bar{\partial}$ on weakly pseudoconvex manifolds, Trans. Amer. Math. Soc., **181** (1973), 273–292.

[N-S-W] A. Nagel-E. Stein-S. Wainger, Balls and metrics defined by vector fields I, Acta Math., **155** (1985), 103–147.

[Oh] T. Ohsawa, Boundary Behavior of the Bergman Kernel Function on Pseudoconvex Domains, Publications of the R. I. M.S., Kyoto University, **20** (1984), 897–902.

[R] M. Range, The Caratheodory metric and holomorphic maps on a class of weakly pseudoconvex domains, Pacific Journal Math., **78,1** (1978), 173–189.

Bergische Universitaet-Gesamthochschule Wuppertal
Fachbereich Mathematik
Gaussstrasse 20 D-56 Wuppertal 1