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Nagoya Math. J.
Vol. 130 (1993), 19–23

ON VECTOR BUNDLES ON ALGEBRAIC SURFACES AND 0-CYCLES

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Let X be an algebraic complex projective surface equipped with the euclidean topology and E a rank 2 topological vector bundle on X. It is a classical theorem of Wu ([Wu]) that E is uniquely determined by its topological Chern classes $c_1^{top}(E) \in H^2(X, \mathbf{Z})$ and $c_2^{top}(E) \in H^4(X, \mathbf{Z}) \cong \mathbf{Z}$. Viceversa, again a classical theorem of Wu ([Wu]) states that every pair $(a, b) \in (H^2(X, \mathbf{Z}), \mathbf{Z})$ arises as topological Chern classes of a rank 2 topological vector bundle. For these results the existence of an algebraic structure on X was not important; for instance it would have been sufficient to have on X a holomorphic structure. In [Sch] it was proved that for algebraic X any such topological vector bundle on X has a holomorphic structure (or, equivalently by GAGA an algebraic structure) if its determinant line bundle has a holomorphic structure. It came as a surprise when Elencwajg and Forster ([EF]) showed that sometimes this was not true if we do not assume that X has an algebraic structure but only a holomorphic one (even for some two dimensional tori (see also [BL], [BF], or [T])). In the algebraic case the proof given in [Sch] showed at once a slightly stronger statement; not only every pair $(a, b) \in (NS(X), \mathbf{Z})$ arises as topological Chern classes of algebraic bundles, but also every pair $(L, b) \in (\mathrm{Pic}(X), \mathbf{Z})$. In algebraic geometry there are finer equivalence relations on the set of 0-cycles than just the "topological" one (or "homological" one), which is simply the degree of the given 0-cycle. By far, the most important such equivalence relation is the rational equivalence relation, which gives the Chow ring $A^*(X)$ of X with $A^1(X) \cong \mathrm{Pic}(X)$ and $A^2(X)$ mapping surjectively onto $H^2(X, \mathbf{Z}) \cong \mathbf{Z}$ by the degree map. Mumford discovered that very often $A^2(X)$ is huge (see [Mu] or [B], Chapter 1). An algebraic vector bundle E has Chern classes $c_i(E) \in A^i(X)$ (with $c_1(E) = \det(E)$). Thus it seems to be natural to ask if every pair $(c, d) \in (A^1(X), A^2(X))$ arises as "algebraic" Chern classes of some rank 2 algebraic vector bundle on X. In this note we prove that the answer is YES, i.e. we prove the following result.

Received June 11, 1991.

THEOREM 0.1. *Fix a projective complex algebraic surface X. For every pair $(L, c_2) \in (\mathrm{Pic}(X), A^2(X))$, there is a rank 2 algebraic vector bundle E on X with (L, c_2) as Chern classes.*

Now fix a polarization H on X, i.e. fix $H \in \mathrm{Pic}(X)$ with H ample. There is a notion of stability (e.g. in the sense of Mumford-Takemoto) with respect to H. It is a natural question to see if the pair (L, c_2) in the statement of 0.1 arises as Chern classes of some rank 2 H-stable vector bundle on X. Even for the corresponding "numerical" problem (with c_i^{top}) there are numerical well-known restrictions on c_2^{top} (even on $\mathbf{P}^2$). By [BB], Prop. 1.2, for fixed X, H, and $L \in \mathrm{Pic}(X)$, this assertion $(\det, c_2^{\mathrm{top}}) \in (\mathrm{Pic}(X), \mathbf{Z})$ is true if the integer c_2^{top} is sufficiently large. We were unable to prove the corresponding result for all elements of $A^2(X)$ with sufficiently large degree (the construction which proves 0.1 gives very unstable vector bundles). We prove here (see 0.2) a far weaker statement replacing "rational equivalence" with the weaker "abelian equivalence" (see [Sa] or [Li], p. 127) in the following sense; fix a base point $P \in X$ so that the Albanese morphism $\alpha : X \rightarrow \mathrm{Alb}(X)$ is normalized by the condition $\alpha(\mathrm{P}) = 0$; extend by additivity (as in the case of curves) α to the set of 0-cycles of degree 0; then the Albanese class of a 0-cycle D of degree b is $\alpha(D - bP)$. Indeed the second result of this paper is the following theorem.

THEOREM 0.2. *Fix a projective complex algebraic surface X and line bundles H, L on X with H ample. Fix a base point $P \in X$. There is an integer k_0, depending on X, H and L, such that for every $k \geq k_0$ and every $\mathbf{a} \in \mathrm{Alb}(X)$ there is a rank 2 H-stable vector bundle E on X with $c_1(E) = L$, $\deg(c_2(E)) = k$ and such that $\mathbf{a}$ is the Albanese class of the degree zero 0-cycle $c_2(E) - kP$.*

Note that if X has Kodaira dimension $\kappa(X) < 0$, then "rational equivalence" and "Albanese equivalence" coincide.

We want to thank the referee for his/her very competent and useful job. The author was partially supported by MURST and GNSAGA of CNR (Italy).

§1. The proofs

Here we prove Theorems 0.1 and 0.2.

Proof of 0.1. Fix L and c_2 (as a class in the Chow ring), with, say, c_2 represented by the cycle $A - B$ with A and B effective and disjoint. Let H be a very ample line bundle. Just to fix the notations we assume B reduced; it is easy to do the general case changing the notations in step (b) below; alternatively, it is easy to reduce the general case to the case in which B is reduced. The proof will be divided in two parts.

(a) Let F be a rank 2 vector bundle on X. For every integer m the splitting principle shows that in the Chow ring $A^*(X)$ we have $c_1(F(mH)) = c_1(F)$. $+ 2mH$ and

$$c_2(F(mH)) = c_2(F) + c_1(F)\cdot(mH) + m^2H^2. \tag{1}$$

Hence to solve our problem it is sufficient to find an integer z and a rank 2 vector bundle Q on X with $c_1(Q) = L + 2zH$ and $c_2(Q) = c_2 + zL\cdot H + z^2H^2$. We will find z and Q solving our problem and with z very negative.

(b) Set $b := \text{card(B)}$. Fix an integer $c \geq b$ and c smooth curves $C_i \in |H|$ with $\text{card}(C_i \cap B) = 1$ if $i \leq b$, $\text{card}(C_i \cap B) = 0$ if $i > b$ and $C_i \cap C_j \cap B = \emptyset$ if $i \neq j$; set $x_i := B \cap C_i$, $i = 1,\ldots,b$. We assume that $(cH - L)\cdot H > 2p_a(C_i) := (K + H)\cdot H + 2$. Hence there are reduced disjoint effective divisors $F_i \subset C_i$, $1 \leq i \leq c$, with $x_i \in F_i$ if $i \leq b$, F_i with $\mathbf{O}(cH - L) \mid C_i$ as associated line bundle on C_i $(1 \leq i \leq c)$. Let $\mathbf{Z}$ be the union of A, $F_i \setminus \{x_i\}$ for all i with $1 \leq i \leq b$, and F_j for all j with $b < j \leq c$. By construction and the fact that rational equivalence commutes with proper push-forward ([Fu], Th. 1.1.4), the rational equivalence class of Z is $c_2 - zL\cdot H + z^2H^2$ with $z = -c$. Hence to prove 0.1 it is sufficient to prove the existence of a rank 2 vector bundle Q which fits in the following exact sequence:

$$0 \to \mathbf{O}_x \to Q \to L \otimes H^{\otimes(-2c)} \otimes \mathbf{I}_z \to 0 \tag{2}$$

since $c_2(\mathbf{O}_z) = -Z$ by Riemann-Roch theorem. Furthermore, taking c large enough, we may assume $h^0(X, K_X \otimes L \otimes H^{\otimes(-2c)}) = 0$. We will fix any $c \geq b$ with this property. By the choice of c the pair $(L \otimes H^{\otimes(-2c)}, Z)$ satisfies trivially the Cayley-Bacharach property (see e.g. [Br] or [C]). Hence among the extensions of $L \otimes H^{\otimes(-2c)} \otimes \mathbf{I}_Z$ by $\mathbf{O}_X$ (i.e. like (2)) there is at least one with middle term, $\mathbf{Q}$, locally free, as wanted. $\square$

Proof of 0.2. Fix the base point $P \in X$ to define uniquely the Albanese morphism $\alpha : X \to \text{Alb}(X)$ with $0 = \alpha(P)$. Fix H and L. We may assume H very ample (taking if necessary a multiple depending only on X of the given polarization). Twisting L by mH for some $m > 0$ depending only on X and H, we may assume $h^0(K \otimes L^{-1}) = 0$ (a condition used in [BB], §1). We may assume L and $K \otimes L$ very ample (twisting again L by mH for some $m > 0$ depending only on X

and H). Set $q := \dim(\mathrm{Alb}(X)) = h^1(\mathbf{O})$. Fix the class $\mathbf{a} \in \mathrm{Alb}(X)$ as in the statement of 0.2. Fix an integer $t' > 0$ such that for every $t \geq t'$ the morphism a_t: $S^t(X) \to \mathrm{Alb}(X)$ from the t-th symmetric product of X, induced by the Albanese morphism $\alpha = a_1 : X \to A$ (with respect to P, i.e. with $a_t(D) := D - tP$ for every cycle $D \in S^t(X)$) is surjective. The proof will be divided into two steps.

(a) In this step we will show the existence of an integer $t'' \geq t'$ such that for every $t \geq t''$ there is a reduced $D \in S^t(X)$ such that for every $x \in D$ we have $h^0((K \otimes L) \otimes I_{D\setminus\{x\}}) = 0$ and such that $a_t(D)$ is the given class $\mathbf{a} \in \mathrm{Alb}(X)$. Fix any integer $z \geq t'$ with $z > h^0(K \otimes L)$ and a general $D \in S^z(X)$; in particular D is reduced, $p \notin D$ and for every $x \in D$ we have $h^0((K \otimes L) \otimes I_{D\setminus\{x\}}) = 0$. Fix z distinct smooth $C_i \in |H|$, $1 \leq i \leq z$, with $P \in C_i$, $\mathrm{card}(D \cap C_i) = 1$ for every i and such that $C_i \cap C_j \cap D = \emptyset$ if $i \neq j$; set $x_i := D \cap C_i$. Set $g := p_a(C_i)$. Note that by Lefschetz theorem and the universal property of Albanese varieties the natural map $\mathrm{Alb}(C_i) \to \mathrm{Alb}(X)$ is surjective. We want to show that we may take $t'' := (2g+1)z$ (with $z := \max(t', h^0(K \otimes L) + 1)$ if we want). We fix a reduced cycle D_i with $\deg(D_i) = 2g+1$, $x_i \in D_i$, $P \notin D_i$, $D_i - (2g+1)P$ linearly equivalent to zero in C_i if $i < z$ (hence with $a_{2g+1}(D_i) = 0 \in \mathrm{Alb}(X)$) and with $D_z - (2g+1)P$ a class in $\mathrm{Alb}(C_i)$ mapped under the surjection $\mathrm{Alb}(C_i) \to \mathrm{Alb}(X)$ into the class $\mathbf{a}$. By construction we may take as D the union of all D_i's, $1 \leq i \leq z$.

(b) Fix an integer $k \geq t''$ (with t'' described in step (a)). Set $\mathbf{S} := \{D \subset S^k(X) : D$ is reduced and for every $x \in D$, $h^0((K \otimes L) \otimes I_{D\setminus\{x\}}) = 0\}$. For any $\mathbf{b} \in \mathrm{Alb}(X)$, let $\mathbf{S}(\mathbf{b}) := \{D \in \mathbf{S} : a_k(D) = \mathbf{b}\}$. Note that $\dim(\mathbf{S}) = 2k$ and that for every $\mathbf{b}$ every irreducible component of $\mathbf{S}(\mathbf{b})$ has codimension at most q in $\mathbf{S}$. Note that every $D \in \mathbf{S}$ satisfies the Cayley-Bacharach property, hence define an extension (2) with Q locally free with $c_1(Q) = L$ and $c_2(Q) = k$ (in $H^4(X, \mathbf{Z})$, i.e. $\deg(c_2(Q)) = k$); if $D \in \mathbf{S}(\mathbf{b})$, then $c_2(Q) - (k)P = \mathbf{b}$ in $\mathrm{Alb}(X)$. Hence it is sufficient to show that the set $\mathbf{S}^{\mathrm{un}} \subseteq \mathbf{S}$ giving unstable bundles has codimension at least $q+1$ in $\mathbf{S}$. Lemma 1.1 of [BB] states exactly the existence of a constant C depending only on X, H and L but not k, such that every irreducible component of $\mathbf{S}^{\mathrm{un}}$ has dimension at most $C + q + k$. Thus it is sufficient to take $k > 2q + C$. □

We repeat that if X has Kodaira dimension $\kappa(X) < 0$, then rational equivalence and abelian equivalence coincide. The proof of 0.2 works verbatim in positive characteristic $\neq 2$ ($\neq 2$ just for the quotation of [BB]).

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