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REMARKS ON FUJIWARA'S STATIONARY PHASE METHOD ON A SPACE OF LARGE DIMENSION WITH A PHASE FUNCTION INVOLVING ELECTROMAGNETIC FIELDS

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1. Introduction

We consider an oscillatory integral of the form

$$I(\{t_j\}, S, a, \nu)(x_L, x_0) = \prod_{j=1}^{L} \left(\frac{\nu i}{2\pi t_j}\right)^{d/2} \int_{\mathbf{R}^{d(L-1)}} e^{-i\nu S(x_L, \cdots, x_0)} a(x_L, \cdots, x_0) \prod_{j=1}^{L-1} dx_j. \tag{1.1}$$

Here each x_j, $j = 0, 1, \ldots, L$, runs in $\mathbf{R}^d$, $\nu > 1$ is a constant and t_j, $j = 1, \ldots, L$, are positive constants. Fujiwara [5] discussed this integral for L large and developed the stationary phase method with an estimate of the remainder term for the phase function $S(x_L, \ldots, x_0)$ coming from the action integral for a particle in an electric field. But his results cannot be applied to the integral which naturally arises in the discussion of quantum mechanics of a charged particle moving in a magnetic field. In this paper we extend his results to the case for the phase function involving both electric and magnetic fields.

We denote the l-th component of $x \in \mathbf{R}^d$ by $(x)_l$, and use the notations: $\partial_j^\alpha = \partial_{xj}^\alpha = \partial_{(x_j)_1}^{\alpha_1} \cdots \partial_{(x_j)_d}^{\alpha_d}$ with a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, and $\partial_j f(x_j) = \partial_{x_j} f(x_j)$ as the gradient of $f(x_j)$.

Our assumption for the phase function $S(x_L, \ldots, x_0)$ is the following:
(H.1) $S(x_L, \ldots, x_0)$ is a real-valued function of the form

$$S(x_L, \ldots, x_0) = \sum_{j=1}^{L} S_j(t_j, x_j, x_{j-1}), \tag{1.2}$$

where

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$$(1.3)\qquad S_j(t_j, x_j, x_{j-1}) = \frac{|x_j - x_{j-1}|^2}{2t_j} + \omega_j(t_j, x_j, x_{j-1}),\ j = 1,\dots, L,$$

and $\omega_j(t_j, x_j, x_{j-1})$ satisfies the following conditions:
(i) For any $m \geq 2$ there exists a constant $\kappa_m > 0$ independent of j and t_j such that

$$(1.4)\qquad \max_{2\leq|\alpha+\beta|\leq m}\ \sup_{x,y\in\mathbf{R}^d} |\partial_x^\alpha \partial_y^\beta \omega_j(t_j, x, y)| \leq \kappa_m.$$

(ii) Let $(\bar{x}_L,\dots, \bar{x}_0)$ be an arbitrary solution of the system of the equation

$$(1.5)\qquad \partial_{x_j} S_{j+1}(t_{j+1}, \bar{x}_{j+1}, \bar{x}_j) + \partial_{x_j} S_j(t_j, \bar{x}_j, \bar{x}_{j-1}) = 0,\ j = 1,\dots, L-1.$$

For any $m \geq 1$, there exists a constant B_m independent of $(\bar{x}_L,\dots, \bar{x}_0)$, L and t_j, $j = 1,\dots, L$, but dependent on d such that

$$(1.6)\qquad \sum_{j=1}^{L-1} \sum_{\substack{1\leq|\alpha|\leq m\\ |\beta|=1}} |[(\partial_{x_{j-1}} + \partial_{x_j} + \partial_{x_{j+1}})^\alpha \partial_{x_j}^\beta (\omega_j + \omega_{j+1})](\bar{x}_{j-1}, \bar{x}_j, \bar{x}_{j+1})| \leq B_m,$$

where $(\partial_{x_{j-1}} + \partial_{x_j} + \partial_{x_{j+1}})^\alpha = \Pi_{k=1}^d (\partial_{(x_{j-1})_k} + \partial_{(x_j)_k} + \partial_{(x_{j+1})_k})^{\alpha_k}$ for a multi-index $\alpha = (\alpha_1,\dots, \alpha_d)$.

Fujiwara's assumption for the phase function in [5] is strictly stronger than that of ours. He assumed that the phase function is of the form

$$S(x_L,\dots, x_0) = \sum_{j=1}^{L} S_j(t_j, x_j, x_{j-1}),$$

with

$$S_j(t_j, x_j, x_{j-1}) = \frac{|x_j - x_{j-1}|^2}{2t_j} + t_j\omega_j(t_j, x_j, x_{j-1}),\ j = 1,\dots, L,$$

where $\omega_j(t_j, x_j, x_{j-1})$ satisfies the estimate (1.4). In his case, our condition (H.1)(ii) is automatically satisfied. Let $S_j(t_j, x_j, x_{j-1})$ be the classical action of a charged particle moving in an electromagnetic field discussed in Yajima [9]. Then $S_j(t_j, x_j, x_{j-1})$ satisfies our assumption (H.1) but does not satisfy the assumption in [5]. This will be discussed at the end of §2.

When $S(x_L,\dots, x_0)$ satisfies (H.1), then if $T_L = t_1 + \cdots + t_L$ is small enough, for any x_0, $x_L \in \mathbf{R}^d$ there exists the unique critical point $(x_{L-1}^*,\dots, x_1^*)$, i.e.

$$(1.7)\qquad \partial_{x_j} S_{j+1}(t_{j+1}, x_{j+1}^*, x_j^*) + \partial_{x_j} S_j(t_j, x_j^*, x_{j-1}^*) = 0,\ j = 1,\dots, L-1,$$

where $x_L^* = x_L$, $x_0^* = x_0$ (The proof is in §3).

To state the assumption for the amplitude function, we use Fujiwara's notation:

$$a(\overbrace{x_L, x_0}) = a(x_L, x_{L-1}^*, \dots, x_1^*, x_0).$$

Similarly, for any pair of integers k, m with $k+1 < m$ let $(x_{k+1}^*, \dots, x_{m-1}^*)$ be the partial critical point, i.e.

$$\partial_{x_j} S_{j+1}(t_{j+1}, x_{j+1}^*, x_j^*) + \partial_{x_j} S_j(t_j, x_j^*, x_{j-1}^*) = 0, j = k+1, \dots, m-1,$$

where $x_k^* = x_k$, $x_m^* = x_m$. Then we set

$$a(x_L, \dots, \overbrace{x_m, x_k}, \dots, x_0) = a(x_L, \dots, x_m, x_{m-1}^*, \dots, x_{k+1}^*, x_k, \dots, x_0).$$

If $m = k+1$, we define

$$a(x_L, \dots, \overbrace{x_{k+1}, x_k}, \dots, x_0) = a(x_L, \dots, x_{k+1}, x_k, \dots x_0).$$

The assumption for the amplitude function is the following:
(H.2) $a(x_L, \dots, x_0)$ is a real-valued function in $\mathscr{B}(\mathbf{R}^{d(L+1)})$. For any $K \geq 0$ there exist constants A_K and X_K with the following properties:
For any sequence of positive integers with $j_0 = 0 < j_1 - 1 < j_1 < j_2 - 1 < \cdots < j_s \leq L$, $s = 1, \dots, L-1$,

$$\text{(1.8a)} \quad | \partial_{x_0}^{\alpha_0} \partial_{x_L}^{\alpha_L} \prod_{u=1}^{s} \partial_{x_{j_u-1}}^{\alpha_{j_u-1}} \partial_{x_{j_u}}^{\alpha_{j_u}} a(\overbrace{x_L, x_{j_s}}, \overbrace{x_{j_s-1}, x_{j_{s-1}}}, \dots, \overbrace{x_{j_1-1}, x_0}) | \leq A_K X_K^s,$$

if $|\alpha_j| \leq K$, $j = 0, j_1 - 1, j_1, \dots, j_s - 1, j_s, L$. If $j_s = L$, then we read the above inequality as

$$\text{(1.8b)} \quad | \partial_{x_0}^{\alpha_0} \prod_{u=1}^{s} \partial_{x_{j_u-1}}^{\alpha_{j_u-1}} \partial_{x_{j_u}}^{\alpha_{j_u}} a(x_L, \overbrace{x_{j_s-1}, x_{j_{s-1}}}, \dots, \overbrace{x_{j_1-1}, x_0}) | \leq A_K X_K^s.$$

Let us state our main theorems. Let H be the $d(L-1) \times d(L-1)$ matrix

$$H = \begin{pmatrix} \frac{1}{t_1} + \frac{1}{t_2} & -\frac{1}{t_2} & 0 & 0 & \cdots \\ -\frac{1}{t_2} & \frac{1}{t_2} + \frac{1}{t_3} & -\frac{1}{t_3} & 0 & \cdots \\ 0 & -\frac{1}{t_3} & \frac{1}{t_3} + \frac{1}{t_4} & -\frac{1}{t_4} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and W the Hessian matrix of $\sum_{j=1}^{L}\omega_j(t_j, x_j, x_{j-1})$ at the critical point $(x_{L-1}^*, \ldots, x_1^*)$.

THEOREM 1. *Assume* (H.1) *and* (H.2). *There exists a positive constants* δ *such that if* $T_L = t_1 + \cdots + t_L < \delta$ *then*

$$(1.9)\quad I(\{t_j\}, S, a, \nu)(x_L, x_0) = \left(\frac{\nu i}{2\pi T_L}\right)^{d/2} \exp\{-i\nu S(\overline{x_L, x_0})\} \det(I + H^{-1}W)^{-1/2}(a(\overline{x_L, x_0}) + r(x_L, x_0)),$$

and for any $K \geq 0$ *there exist positive constants* C_K *and* $M(K)$ *such that if* $|\alpha_0|, |\alpha_L| \leq K$,

$$(1.10)\qquad |\partial_{x_L}^{\alpha_L}\partial_{x_0}^{\alpha_0} r(x_L, x_0)| \leq A_{M(K)}\left(\prod_{j=1}^{L}(1 + C_K X_{M(K)}\nu^{-1}t_j) - 1\right).$$

Constants δ *and* C_K *are independent of* a, L, $\{t_j\}$, x_L, x_0 *and* ν *but depend on the dimension* d *of space* $\mathbf{R}^d$ *and* $\{\kappa_m\}$ *and* $\{B_m\}$, $M(K)$ *depends only on* K *and* d.

THEOREM 2. *Assume that* $a \equiv 1$ *and* (H.1) *and let* δ *be the constant as in Theorem* 1. *Then for any* $K \geq 0$ *there exists a constant* C_K *such that if* $|\alpha_0|, |\alpha_L| \leq K$,

$$(1.11)\qquad |\partial_{x_L}^{\alpha_L}\partial_{x_0}^{\alpha_0} r(x_L, x_0)| \leq \prod_{j=1}^{L}(1 + C_K\nu^{-1}t_j T_L) - 1.$$

We remark that our estimate of $r(x_L, x_0)$ in Theorem 1 is the same as that in Fujiwara [5], but that in Theorem 2 differs from his in the power of T_L : our power is 1 while his power is 2.

In §2 we see that the phase function coming from the action integral for a charged particle in an electromagnetic field satisfies (H.1). In the later sections we mimic the discussion of [5]. The existence of the critical point of the phase function is proved in §3. In §4 we write down a lemma about the stationary phase method on a space of large dimension. Theorems 1 and 2 are proved in §5.

2. Piecewise classical path in electromagnetic fields

We give an example of $S(x_L, \ldots, x_0)$ which satisfies the assumption (H.1). We consider a charged particle in an electromagnetic field in $\mathbf{R}^d$ which satisfies the assumption considered by Yajima [9]. In this section we denote the l-th component of $x \in \mathbf{R}^d$ by x_l. We make the following assumption for the vector and scalar potentials $A(t, x)$ and $V(x)$:

ASSUMPTION (A). For $k = 1, \dots, d$, $A_k(t, x)$ is a real-valued function of $(t, x) \in \mathbf{R} \times \mathbf{R}^d$, and for any α, $\partial_x^\alpha A_k(t, x)$ is C^1 in $(t, x) \in \mathbf{R} \times \mathbf{R}^d$. There exists $\varepsilon > 0$ such that

$$(2.1) \quad |\partial_x^\alpha A_k(t, x)| + |\partial_x^\alpha \partial_t A_k(t, x)| \le C_a, \quad |\alpha| \ge 1, \quad (t, x) \in \mathbf{R} \times \mathbf{R}^d,$$

$$(2.2) \quad |\partial_x^\alpha B(t, x)| \le C_a(1 + |x|)^{-1-\varepsilon}, \quad |\alpha| \ge 1,$$

where $B(t, x)$ is the skew symmetric matrix with (k, l)-component $B_{kl}(t, x) = (\partial A_l / \partial x_k - \partial A_k / \partial x_l)(t, x)$ and $|B|$ denotes the norm of matrix B regarded as an operator on $\mathbf{R}^d$, $V(x)$ is a real-valued C^∞ function which satisfies

$$(2.3) \quad |\partial_x^\alpha V(x)| \le C_a, \quad |\alpha| \ge 2.$$

In the form of oscillatory integrals Yajima [9] constructed the propagator for the Schrödinger evolution equation with a vector potential satisfying Assumption (A). We remark that this assumption is satisfied by constant magnetic fields.

Let $H(t, x, \xi)$ be the Hamiltonian

$$H(t, x, \xi) = 2^{-1}(\xi - A(t, x))^2 + V(x).$$

Then Hamilton's differential equation is

$$\dot{x} = \partial_\xi H(t, x, \xi), \quad \dot{\xi} = -\partial_x H(t, x, \xi)$$

with $\dot{x} = dx/dt$ and $\dot{\xi} = d\xi/dt$. When we introduce the position-velocity variables by $(q(t), v(t)) = (x(t), \xi(t) - A(t, x(t)))$, then Hamilton's differential equation is equivalent to Lagrange's differential equation:

$$(2.4) \quad \dot{q}(t) = v(t), \quad \dot{v}(t) = B(t, q(t))v(t) + F(t, q(t)),$$

where $F(t, x) = -(\partial_t A)(t, x) - (\partial_x V)(x)$. The next lemma is a result of Yajima [9].

LEMMA 2.1. *Let* $|t - s| \le 1$.
(i) *For any* α *with* $|\alpha| \ge 1$, *there exists a constant* C'_α *such that for any solution* $(q(\tau), v(\tau))$, $s \le \tau \le t$, *of* (2.4),

$$\int_s^t |(\partial_x^\alpha B)(\tau, q(\tau))| \, |v(\tau)| \, d\tau \le C'_\alpha.$$

(ii) *There exists a constant* $T > 0$ *such that if* $0 < |t - s| < T$, *then for any* $x, y \in \mathbf{R}^d$ *there exists a unique solution* $(q(\tau), v(\tau))$, $s \leq \tau \leq t$, *of* (2.4) *with* $q(s) = y$ *and* $q(t) = x$.

Proof. We refer the proof to Yajima [9, Lemma 2.1 and Proposition 2.6]. □

Let $T > 0$ be as in Lemma 2.1(ii) and $|t - s| \leq T$. We write the unique solution $q(\tau)$ of (2.4) with $q(s) = y$ and $q(t) = x$ as

$$q(\tau) = q^0(\tau) + q^1(\tau)$$

where $q^0(\tau) = \frac{\tau - s}{t - s}(x - y) + y$. Then we have

$$\ddot{q}^1(\tau) = B(\tau, q(\tau))v(\tau) + F(\tau, q(\tau)), \tag{2.5}$$

and

$$q^1(s) = q^1(t) = 0.$$

Let G be the Green operator of the Dirichlet boundary value problem:

$$-\ddot{q}(\tau) = f(\tau), \; s \leq \tau \leq t, \quad q(s) = q(t) = 0.$$

Then we have

$$(Gf)(\tau) = \int_s^t g(\tau, u) f(u) du,$$

where

$$\begin{aligned} g(\tau, u) &= \frac{(u - s)(t - \tau)}{(t - s)}, \text{ if } s \leq u \leq \tau \leq t, \\ &= \frac{(\tau - s)(t - u)}{(t - s)}, \text{ if } s \leq \tau \leq u \leq t. \end{aligned}$$

Put $\|f\|_{L^1} = \int_s^t |f(\tau)| d\tau$ and $\|f\|_{L^\infty} = \sup_{s \leq \tau \leq t} |f(\tau)|$. Then we have

$$\left\| \frac{d(Gf)}{d\tau} \right\|_{L^1} \leq |t - s| \, \|f\|_{L^1}. \tag{2.6}$$

LEMMA 2.2. *There exists a constant* $0 < T^0 < \min(T, 1)$ *such that if* $|t - s| \leq T^0$ *then for any* α, β *with* $|\alpha + \beta| \geq 1$, (2.7) $$\| \partial_x^\alpha \partial_y^\beta q^1 \|_{L^\infty} \le \| \partial_x^\alpha \partial_y^\beta \dot{q}^1 \|_{L^1} \le C_{\alpha\beta} | t - s |.$$

Proof. The first inequality is Poincare's inequality. Differentiating (2.5) and using (2.6), we have

$$\begin{aligned} \| \partial_{x_l} \dot{q}^1 \|_{L^1} &\le | t - s | \| B \cdot \partial_{x_l} (\dot{q}^0 + \dot{q}^1) \\ &\qquad + \sum_{m=1}^{d} \partial_{x_l} (q^0 + q^1)_m \partial_{x_m} B \cdot \dot{q} + \sum_{m=1}^{d} \partial_{x_l} (q^0 + q^1)_m \cdot \partial_{x_m} F \|_{L^1} \\ &\le | t - s | [C_1 (1 + \| \partial_{x_l} \dot{q}^1 \|_{L^1}) + C_1' (1 + \| \partial_{x_l} q^1 \|_{L^\infty}) \\ &\qquad + C_1 | t - s | (1 + \| \partial_{x_l} q^1 \|_{L^\infty})] \\ &\le | t - s | (C_1 + C_1' + C_1 | t - s |)(1 + \| \partial_{x_l} \dot{q}^1 \|_{L^1}), \end{aligned}$$

noting $\partial_{x_l} \dot{q}_m^0 = \frac{\delta_{lm}}{(t - s)}$, $\partial_{x_l} q_m^0 = \frac{\delta_{lm} (\tau - s)}{(t - s)}$ and Lemma 2.1(i) and using the first inequality of (2.7). Hence if $| t - s |$ is sufficiently small, we have the second inequality of (2.7). Similar arguments lead to (2.7) for general α and β. □

Let $S(t, s, x, y)$ be the action of the classical path $(q(\tau), v(\tau))$ joining (s, y) to (t, x):

(2.8) $$S(t, s, x, y) = \int_s^t L(\tau, q(\tau), v(\tau)) d\tau,$$

where $L(\tau, q, v)$ is the Lagrangian corresponding to $H(\tau, x, \xi)$:

$$L(\tau, q, v) = v\xi - H(\tau, x, \xi) = \frac{v^2}{2} + A(\tau, q)v - V(q).$$

For any sequence $0 = T_0 < T_1 < \cdots < T_L < T^0$ and any points $x^j \in \mathbf{R}^d$, $j = 0, \ldots, L$, we put

$$S_j(t_j, x^j, x^{j-1}) = S(T_j, T_{j-1}, x^j, x^{j-1}), \quad j = 1, \ldots, L,$$

where $t_j = T_j - T_{j-1}$. We denote by $q_\Delta = q_\Delta^0 + q_\Delta^1$ the piecewise classical path joining (T_j, x^j), $j = 0, \ldots, L$, i.e. q_Δ^0 is

$$q_\Delta^0(\tau) = \frac{\tau - T_{j-1}}{t_j} (x^j - x^{j-1}) + x^{j-1}, \quad T_{j-1} \le \tau \le T_j, \quad j = 1, \ldots, L,$$

and q_Δ^1 satisfies

$$\ddot{q}_\Delta^1(\tau) = B(\tau, q_\Delta(\tau)) \dot{q}_\Delta(\tau) + F(\tau, q_\Delta(\tau)), \quad T_{j-1} \le \tau \le T_j,$$

and $q_\Delta^1(T_j) = 0$, $j = 0, \dots, L$. The action along the piecewise classical path can be written as

$$S(q_\Delta) = S(x^L, \dots, x^0) = \sum_{j=1}^{L} S_j(t_j, x^j, x^{j-1}).$$

THEOREM 2.3. *Let* $T_L < T^0$. *Then* $S(x^L, \dots, x^0) = \sum_{j=1}^{L} S_j(t_j, x^j, x^{j-1})$ *satisfies Assumption* (H.1).

Proof. First we verify (H.1). Let $q(\tau) = q^0(\tau) + q^1(\tau)$ be the classical path joining (s, y) to (t, x). We have

$$\begin{aligned} S(t, s, x, y) &= \int_s^t \left(\frac{|\dot{q}^0(\tau) + \dot{q}^1(\tau)|^2}{2} + A(\tau, q(\tau))\dot{q}(\tau) - V(q(\tau))\right)d\tau \\ &= \frac{|x-y|^2}{2(t-s)} + \int_s^t \left(\frac{|\dot{q}^1(\tau)|^2}{2} + A(\tau, q(\tau))\dot{q}(\tau) - V(q(\tau))\right)d\tau \\ &= \frac{|x-y|^2}{2(t-s)} + \omega(t, s, x, y) \end{aligned}$$

where

$$\omega(t, s, x, y) = \int_s^t \left(\frac{|\dot{q}^1(\tau)|^2}{2} + A(\tau, q(\tau))\dot{q}(\tau) - V(q(\tau))\right)d\tau. \tag{2.9}$$

Since q satisfies (2.5), it follows that

$$(\partial_{y_k}\omega)(t, s, x, y) = \int_s^t \partial_{y_k} q^0 (B(\tau, q(\tau))\dot{q}(\tau) + F(\tau, q(\tau))d\tau - A_k(s, y).$$

Noting $\partial_{y_k} q_m^0 = (t-\tau)(t-s)^{-1}\delta_{km}$, we obtain

$$\begin{aligned} (\partial_{y_l}\partial_{y_k}\omega)(t, s, x, y) = \int_s^t \frac{t-\tau}{t-s}\Big(&\sum_{m=1}^{d} B_{km}\partial_{y_l}\dot{q}_m + \sum_{n,m=1}^{d} \partial_{y_l} q_n \partial_{x_n} B_{km}\dot{q}_m \\ &+ \sum_{m=1}^{d} \partial_{y_l} q_m \cdot \partial_{x_m} F_k\Big)d\tau - (\partial_{y_l}A_k)(s, y). \end{aligned} \tag{2.10}$$

So from Assumption (A), Lemma 2.1(i) and Lemma 2.2, we have

$$\begin{aligned} |\partial_{y_l}\partial_{y_h}\omega| \le C_1(1 + C|t-s|) + C_1'(1 + C|t-s|) + \\ C_1|t-s|(1 + C|t-s|) + C_1 \le \kappa_2, \end{aligned}$$

where κ_2 is independent of x, y and $t-s$. For the other higher derivatives of ω, similar arguments hold. So we have proved (H.1)(i).

Next we show (H.1)(ii). We put

$$\omega_j(x^j, x^{j-1}) = \omega(T_j, T_{j-1}, x^j, x^{j-1}). \tag{2.11}$$

In the same way as above we have

$$\begin{aligned}
&\partial_{x_l^j}\partial_{x_k^j}(\omega_{j+1}(x^{j+1}, x^j) + \omega_j(x^j, x^{j-1}))\\
&= \int_{T_j}^{T_{j+1}} \frac{T_{j+1} - \tau}{t_{j+1}} \Big(\sum_{m=1}^{d} B_{km}\partial_{x_l^j}(\dot{q}_\Delta^0 + \dot{q}_\Delta^1)_m + \sum_{n,m=1}^{d} \partial_{x_l^j}(q_\Delta)_n \partial_{x_n} B_{km}(\dot{q}_\Delta)_m\\
&\qquad + \sum_{m=1}^{d} \partial_{x_l^j}(q_\Delta)_m \cdot \partial_{x_m} F_k\Big) d\tau\\
&+ \int_{T_{j-1}}^{T_j} \frac{\tau - T_{j-1}}{t_j} \Big(\sum_{m=1}^{d} B_{km}\partial_{x_l^j}(\dot{q}_\Delta^0 + \dot{q}_\Delta^1)_m + \sum_{n,m=1}^{d} \partial_{x_l^j}(q_\Delta)_n \partial_{x_n} B_{km}(\dot{q}_\Delta)_m\\
&\qquad + \sum_{m=1}^{d} \partial_{x_l^j}(q_\Delta)_m \cdot \partial_{x_m} F_k\Big) d\tau,
\end{aligned}$$

$$\begin{aligned}
&\partial_{x_l^{j+1}}\partial_{x_k^j}\omega_{j+1}(x^{j+1}, x^j)\\
&= \int_{T_j}^{T_{j+1}} \frac{T_{j+1} - \tau}{t_{j+1}} \Big(\sum_{m=1}^{d} B_{km}\partial_{x_l^{j+1}}(\dot{q}_\Delta^0 + \dot{q}_\Delta^1)_m + \sum_{n,m=1}^{d} \partial_{x_l^{j+1}}(q_\Delta)_n \partial_{x_n} B_{km}(\dot{q}_\Delta)_m\\
&\qquad + \sum_{m=1}^{d} \partial_{x_l^{j+1}}(q_\Delta)_m \cdot \partial_{x_m} F_k\Big) d\tau
\end{aligned}$$

and

$$\begin{aligned}
&\partial_{x_l^{j-1}}\partial_{x_k^j}\omega_j(x^j, x^{j-1})\\
&= \int_{T_{j-1}}^{T_j} \frac{\tau - T_{j-1}}{t_j} \Big(\sum_{m=1}^{d} B_{km}\partial_{x_l^{j-1}}(\dot{q}_\Delta^0 + \dot{q}_\Delta^1)_m + \sum_{n,m=1}^{d} \partial_{x_l^{j-1}}(q_\Delta)_n \partial_{x_n} B_{km}(\dot{q}_\Delta)_m\\
&\qquad + \sum_{m=1}^{d} \partial_{x_l^{j-1}}(q_\Delta)_m \cdot \partial_{x_m} F_k\Big) d\tau.
\end{aligned}$$

This together with

$$-\partial_{x_l^j}(\dot{q}_\Delta^0)_m = \partial_{x_l^{j+1}}(\dot{q}_\Delta^0)_m = \frac{1}{t_{j+1}}\delta_{lm}, \quad T_j \le \tau \le T_{j+1},$$

and

$$-\partial_{x_l^{j-1}}(\dot{q}_\Delta^0)_m = \partial_{x_l^j}(\dot{q}_\Delta^0)_m = \frac{1}{t_j}\delta_{lm}, \quad T_{j-1} \le \tau \le T_j,$$

yields that

$$(\partial_{x_l^{j-1}} + \partial_{x_l^j} + \partial_{x_l^{j+1}})\partial_{x_k^j}(\omega_j + \omega_{j+1})(x^{j-1}, x^j, x^{j+1})$$

$$= \int_{T_j}^{T_{j+1}} \frac{T_{j+1} - \tau}{t_{j+1}} \Big(\sum_{m=1}^{d} B_{km}(\partial_{x_l^{j+1}} + \partial_{x_l^j})\dot{q}_{\Delta m}^1$$

$$+ \sum_{n,m=1}^{d} (\partial_{x_l^{j+1}} + \partial_{x_l^j})(q_\Delta)_n \partial_{x_n} B_{km}(\dot{q}_\Delta)_m + \sum_{m=1}^{d} (\partial_{x_l^{j+1}} + \partial_{x_l^j})(q_\Delta)_m \cdot \partial_{x_m} F_k\Big) d\tau$$

$$+ \int_{T_{j-1}}^{T_j} \frac{\tau - T_{j-1}}{t_j} \Big(\sum_{m=1}^{d} B_{km}(\partial_{x_l^j} + \partial_{x_l^{j-1}})\dot{q}_{\Delta m}^1$$

$$+ \sum_{n,m=1}^{d} (\partial_{x_l^j} + \partial_{x_l^{j-1}})(q_\Delta)_n \partial_{x_n} B_{km}(\dot{q}_\Delta)_m + \sum_{m=1}^{d} (\partial_{x_l^j} + \partial_{x_l^{j-1}})(q_\Delta)_m \cdot \partial_{x_m} F_k\Big) d\tau.$$

When $(\bar{x}^L, \dots, \bar{x}^0)$ is a critical point of $S(q_\Delta)$, the piecewise classical path $q_\Delta(\tau)$ coincides with the classical path $q(\tau)$ joining $(0, \bar{x}^0)$ and $(T_L, \bar{x}^0)$. So we have from Lemma 2.2

$$|(\partial_{x_l^{j-1}} + \partial_{x_l^j} + \partial_{x_l^{j+1}})\partial_{x_k^j}(\omega_j + \omega_{j+1})(\bar{x}^{j-1}, \bar{x}^j, x^{j+1})|$$

$$\le C(t_{j+1} + t_j) + C\int_{T_{j-1}}^{T_{j+1}} |(\partial B)(\tau, q(\tau))||v(\tau)| d\tau.$$

Therefore, we have by Lemma 2.1(i)

$$\sum_{j=1}^{L-1} |(\partial_{x_l^{j-1}} + \partial_{x_k^j} + \partial_{x_l^{j+1}})\partial_{x_k^j}(\omega_j + \omega_{j+1})(\bar{x}^{j-1}, \bar{x}^j, \bar{x}^{j+1})|$$

$$\le CT_L + C\int_0^{T_L} |(\partial B)(\tau, q(\tau))||v(\tau)| d\tau$$

$$\le B_1,$$

where B_1 is independent of $(\bar{x}^L, \dots, \bar{x}^0)$, L and T_L if $T_L < T^0$. Similar discussions hold for other differentiation $(\partial_{x^{j-1}} + \partial_{x^j} + \partial_{x^{j+1}})^\alpha$. Thus we have proved (H.1)(ii).

□

Finally we remark that our phase function $S(t, s, x, y)$ does not satisfy Fujiwara's assumption in [5]. In fact, in the case that $V(x) \equiv 0$ and $A(t, x) = A^0 x$, where A^0 is a real constant $d \times d$ matrix, we can see from (2.10) that

$$(\partial_{y_l}\partial_{y_k}\omega)(t, s, x, y) = \int_s^t \frac{t - \tau}{t - s}\Big(\sum_{m=1}^{d} B_{km}\partial_{y_l}\dot{q}_m\Big)d\tau - (\partial_{y_l}A_k)(s, y)$$

$$= -\frac{A_{kl}^0 + A_{lk}^0}{2} + \int_s^t \frac{t - \tau}{t - s}\Big(\sum_{m=1}^{d} B_{km}\partial_{y_l}\dot{q}_m^1\Big)d\tau$$

$$= -\frac{A^0_{kl} + A^0_{lk}}{2} + O(t-s),$$

as $t - s$ goes to zero.

3. Phase functions

In this section we discuss the unique existence of the critical point of S (Lemma 3.5) and study some of its properties. The method is similar to that of Yajima [9]. In what follows, we assume (H.1) and abbreviate $S_j(t_j, x_j, x_{j-1})$ as $S_j(x_j, x_{j-1})$ and $\omega_j(t_j, x_j, x_{j-1})$ as $\omega_j(x_j, x_{j-1})$. To avoid additional complexity we put $d = 1$.

LEMMA 3.1. *Let $2t_j\kappa_2 \le 1$, $j = 1, \ldots, L$. Then for any y and $k \in \mathbf{R}$, there exists a unique $(x^{\#}_0, \ldots, x^{\#}_L) = (x^{\#}_0(y, k), \ldots, x^{\#}_L(y, k))$ which satisfies $x^{\#}_0 = y$, $\dfrac{x^{\#}_1 - y}{t_1} = k$ and*

$$\text{(3.1)} \quad \frac{x^{\#}_{j+1} - x^{\#}_j}{t_{j+1}} - \frac{x^{\#}_j - x^{\#}_{j-1}}{t_j} = \partial_j\omega_j(x^{\#}_j, x^{\#}_{j-1}) + \partial_j\omega_{j+1}(x^{\#}_{j+1}, x^{\#}_j), \quad j = 1, \ldots, L-1.$$

Proof. We have $x^{\#}_1 = x^{\#}_1(y, k) = t_1 k + y$. Put

$$\text{(3.2)} \qquad k^{\#}_j = \frac{x^{\#}_j - x^{\#}_{j-1}}{t_j}, \quad j = 1, \ldots, L.$$

Then the system of the equation (3.1) is equivalent to

$$\text{(3.3)} \qquad \begin{aligned} k^{\#}_{j+1} - k^{\#}_j &= \partial_j\omega_j(x^{\#}_{j-1} + t_j k^{\#}_j, x^{\#}_{j-1}) \\ &\quad + \partial_j\omega_{j+1}(x^{\#}_{j-1} + t_j k^{\#}_j + t_{j+1}k^{\#}_{j+1}, x^{\#}_{j-1} + t_j k^{\#}_j), \quad j = 1, \ldots, L-1. \end{aligned}$$

If $2t_2\kappa_2 \le 1$, for any y, $k \in \mathbf{R}$, the map Φ_1:

$$k_2 \mapsto \Phi_1(k_2) = k + (\partial_1\omega_1)(y + t_1 k, y) + (\partial_1\omega_2)(y + t_1 k + t_2 k_2, y + t_1 k)$$

is a contraction. So there exists a unique $k^{\#}_2 = k^{\#}_2(y, k)$ which satisfies (3.3) for $j = 1$. Hence we have $x^{\#}_2(y, k) = x^{\#}_1(y, k) + t_2 k^{\#}_2(y, k)$. Similarly we have the unique existence of $k^{\#}_3, \ldots, k^{\#}_L$ and $x^{\#}_3, \ldots, x^{\#}_L$, successively. $\square$

As in the proof of Lemma 3.1, we set $k^{\#}_j(y, k) = \dfrac{x^{\#}_j(y, k) - x^{\#}_{j-1}(y, k)}{t_j}$, $j = 1, \ldots, L$, where $k^{\#}_1 = k$ and $x^{\#}_0 = y$. Let $T_j = t_1 + \cdots + t_j$.

LEMMA 3.2. *If* $2t_j\kappa_2 \leq 1$, $j = 1, \ldots, L$, *then for* $|\alpha + \beta| \geq 1$,

$$|\partial_y^\alpha \partial_k^\beta (x_j^\#(y, k) - y - T_j k)| \leq C_{\alpha\beta} T_j^{|\beta|+1}, \tag{3.4}$$

$$|\partial_y^\alpha \partial_k^\beta (k_j^\#(y, k) - k)| \leq C_{\alpha\beta} T_j^{|\beta|}. \tag{3.5}$$

Proof. We prove this by induction on $l = |\alpha + \beta|$. We denote $x_j^\#(y, k)$ by x_j, $k_j^\#(y, k)$ by k_j, $\partial_y^\alpha \partial_k^\beta x_j^\#$ by $x_j^{\alpha\beta}$ and $\partial_y^\alpha \partial_k^\beta k_j^\#$ by $k_j^{\alpha\beta}$.

Let $l = 1$. Then we have from (3.2,3),

$$\begin{aligned} &x_j^{\alpha\beta} - x_{j-1}^{\alpha\beta} = t_j k_j^{\alpha\beta}, \quad j = 1, \ldots, L, \\ &k_{j+1}^{\alpha\beta} - k_j^{\alpha\beta} = (\partial_{j-1} + \partial_j + \partial_{j+1})\partial_j(\omega_j + \omega_{j+1})x_{j-1}^{\alpha\beta} \\ &+ (\partial_j^2(\omega_j + \omega_{j+1}) + \partial_{j+1}\partial_j\omega_{j+1})t_j k_j^{\alpha\beta} + \partial_{j+1}\partial_j\omega_{j+1}t_{j+1}k_{j+1}^{\alpha\beta}, \quad j = 1, \ldots, L-1. \end{aligned} \tag{3.6}$$

So we obtain with $\phi_j = (\partial_{j-1} + \partial_j + \partial_{j+1})\partial_j(\omega_j + \omega_{j+1})(x_{j-1}, x_j, x_{j+1})$

$$(1 - \kappa_2 t_{j+1})|k_{j+1}^{\alpha\beta}| + |x_j^{\alpha\beta}| \leq (1 + (3\kappa_2 + 1)t_j)|k_j^{\alpha\beta}| + (1 + |\phi_j|)|x_{j-1}^{\alpha\beta}|.$$

Hence if $1 - \kappa_2 t_{j+1} \geq 1/2$, then

$$|k_{j+1}^{\alpha\beta}| + |x_j^{\alpha\beta}| \leq (1 + 2|\phi_j| + 2(3\kappa_2 + 1)t_j + 2\kappa_2 t_{j+1})(|k_j^{\alpha\beta}| + |x_{j-1}^{\alpha\beta}|).$$

Here we have used $(1 + b)(1 - a)^{-1} \leq 1 + 2(a + b)$ for $0 \leq 2a \leq 1$. Since $k_1^{\alpha\beta}$, $x_0^{\alpha\beta} = 0$ or 1, it follows from Assumption (H.1)(ii) that $|k_{j+1}^{\alpha\beta}| + |x_j^{\alpha\beta}| \leq C$. So we have

$$|\partial_y(k_j - k)| \leq C \text{ and } |\partial_y(x_j - y - T_j k)| = |\sum_{l=1}^{j} t_l \partial_y k_l| \leq CT_j.$$

Moreover since we have

$$|\partial_k x_j| = |\partial_k(x_j - y)| = |\sum_{l=1}^{j} t_l \partial_y k_l| \leq CT_j,$$

we obtain by summing (3.6) for j

$$|\partial_k(k_j - k)| \leq CT_j \text{ and } |\partial_k(x_j - y - T_j k)| = |\sum_{l=1}^{j} t_l \partial_k(k_l - k)| \leq CT_j^2.$$

Next we suppose that (3.4,5) are true for $|\alpha + \beta| \leq l$ and prove them for $|\alpha + \beta| = l + 1$. We put

$$\begin{aligned} &g(x_{j-1}, k_j, k_{j+1}) \\ &= (\partial_j\omega_j)(x_{j-1} + t_j k_j, x_{j-1}) + (\partial_j\omega_{j+1})(x_{j-1} + t_j k_j + t_{j+1}k_{j+1}, x_{j-1} + t_j k_j). \end{aligned}$$

Differentiating (3.3) we have

$$\begin{aligned} k_{j+1}^{\alpha\beta} - k_j^{\alpha\beta} &= \partial_{x_{j-1}} g \cdot x_{j-1}^{\alpha\beta} + \partial_{k_j} g \cdot k_j^{\alpha\beta} + \partial_{k_{j+1}} g \cdot k_{j+1}^{\alpha\beta} \\ &+ \sum C \partial_{x_{j-1}}^{\gamma} g \cdot x_{j-1}^{\bar{\alpha}_1 \bar{\beta}_1} \cdots x_{j-1}^{\bar{\alpha}_{|\gamma|} \bar{\beta}_{|\gamma|}} \\ &+ \sum C \partial_{x_{j-1}}^{\gamma} \partial_{k_j}^{\delta} \partial_{k_{j+1}}^{\varepsilon} g \cdot x_{j-1}^{\alpha_1 \beta_1} \cdots x_{j-1}^{\alpha_{|\gamma|} \beta_{|\gamma|}} k_j^{\alpha'_1 \beta'_1} \cdots k_j^{\alpha'_{|\delta|} \beta'_{|\delta|}} k_{j+1}^{\alpha''_1 \beta''_1} \cdots k_{j+1}^{\alpha''_{|\varepsilon|} \beta''_{|\varepsilon|}}, \end{aligned}$$

where the sums are taken in the suitable manner, and

$$\begin{aligned} &(\bar{\alpha}_1, \bar{\beta}_1) + \cdots + (\bar{\alpha}_{|\gamma|}, \bar{\beta}_{|\gamma|}) = (\alpha, \beta), \quad 2 \leq |\gamma| \leq l + 1, \\ &(\alpha_1, \beta_1) + \cdots + (\alpha_{|\gamma|}, \beta_{|\gamma|}) + (\alpha'_1, \beta'_1) + \cdots + (\alpha'_{|\delta|}, \beta'_{|\delta|}) \\ &\qquad + (\alpha''_1, \beta''_1) + \cdots + (\alpha''_{|\varepsilon|}, \beta''_{|\varepsilon|}) = (\alpha, \beta), \\ &2 \leq |\gamma| + |\delta| + |\varepsilon| \leq l + 1 \text{ and } 1 \leq |\delta| + |\varepsilon|. \end{aligned}$$

It is clear that

$$\begin{aligned} &\partial_{x_{j-1}} g = (\partial_{j-1} + \partial_j + \partial_{j+1}) \partial_j (\omega_j + \omega_{j+1}), \\ &\partial_{k_j} g = t_j (\partial_j^2 (\omega_j + \omega_{j+1}) + \partial_{j+1} \partial_j \omega_{j+1}), \quad \partial_{k_{j+1}} g = t_{j+1} \partial_{j+1} \partial_j \omega_{j+1}, \\ &\partial_{x_{j-1}}^{\gamma} g = (\partial_{j-1} + \partial_j + \partial_{j+1})^{\gamma} \partial_j (\omega_j + \omega_{j+1}) \\ &\quad \text{and } |\partial_{x_{j-1}}^{\gamma} \partial_{k_j}^{\delta} \partial_{k_{j+1}}^{\varepsilon} g| \leq C_{\alpha\beta} \kappa_{l+1} t_j^{|\delta|} t_{j+1}^{|\varepsilon|}. \end{aligned}$$

By induction hypothesis (3.4) we have

$$|x_{j-1}^{\bar{\alpha}_1 \bar{\beta}_1} \cdots x_{j-1}^{\bar{\alpha}_{|\gamma|} \bar{\beta}_{|\gamma|}}| \leq C_{\alpha\beta} T_{j-1}^{|\beta|}.$$

We can show that

$$\begin{aligned} &|\partial_{x_{j-1}}^{\gamma} \partial_{k_j}^{\delta} \partial_{k_{j+1}}^{\varepsilon} g \cdot x_{j-1}^{\alpha_1 \beta_1} \cdots x_{j-1}^{\alpha_{|\gamma|} \beta_{|\gamma|}} k_j^{\alpha'_1 \beta'_1} \cdots k_j^{\alpha'_{|\delta|} \beta'_{|\delta|}} k_{j+1}^{\alpha''_1 \beta''_1} \cdots k_{j+1}^{\alpha''_{|\varepsilon|} \beta''_{|\varepsilon|}}| \\ &\qquad\qquad \leq C_{\alpha\beta} (t_j + t_{j+1}) T_{j+1}^{(|\beta|-1)_+}, \end{aligned} \tag{3.7}$$

with $(a)_+ = \max(a, 0)$. In fact, in the case $0 \leq |\beta| - |\beta_1 + \cdots + \beta_{|\gamma|}| \leq 1$, it is clear from $|\delta + \varepsilon| \geq 1$. In the case that $|\beta| - |\beta_1 + \cdots + \beta_{|\gamma|}| = s \geq 2$, if $s \leq |\delta + \varepsilon|$, then the left-hand side of (3.7) is less than or equal to

$$C_{\alpha\beta} (t_j + t_{j+1})^s T_{j-1}^{|\beta_1 + \cdots + \beta_{|\gamma|}|} \leq C_{\alpha\beta} (t_j + t_{j+1}) T_{j+1}^{|\beta|-1}.$$

If $s > |\delta + \varepsilon|$, then the left-hand side of (3.7) is less than or equal to

$$C_{\alpha\beta} (t_j + t_{j+1})^{|\delta+\varepsilon|} T_{j-1}^{|\beta_1 + \cdots + \beta_{|\gamma|}|} T_{j+1}^{\sigma} \leq C_{\alpha\beta} (t_j + t_{j+1}) T_{j+1}^{|\beta|},$$

with $\sigma = \sum_{|\beta'_m| \geq 2} |\beta'_m| + \sum_{|\beta''_m| \geq 2} |\beta''_m|$, because $|\delta + \varepsilon| - 1 \geq \sum_{|\beta'_m|=1} |\beta'_m| + \sum_{|\beta''_m|=1} |\beta''_m|$. So we have together with $x_j^{\alpha\beta} - x_{j-1}^{\alpha\beta} = t_j k_j^{\alpha\beta}$,

$$\begin{aligned} (1 - \kappa_2 t_{j+1}) |k_{j+1}^{\alpha\beta}| + |x_j^{\alpha\beta}| &\leq (1 + (2\kappa_2 + 1) t_j) |k_j^{\alpha\beta}| + (1 + |\phi_j|) |x_{j-1}^{\alpha\beta}| \\ &+ C_{\alpha\beta} |\phi_j^{(l+1)}| T_{j+1}^{|\beta|} + C_{\alpha\beta} (t_j + t_{j+1}) T_{j+1}^{(|\beta|-1)_+}, \end{aligned}$$

where $\phi_j^{(l+1)}(x_{j-1}, x_j, x_{j+1}) = \sum_{1 \le |\alpha| \le l+1} |(\partial_{j-1} + \partial_j + \partial_{j+1})^\alpha \partial_j(\omega_j + \omega_{j+1})(x_{j-1}, x_j, x_{j+1})|$. Hence if $1 - \kappa_2 t_{j+1} \ge 1/2$, then

$$|k_{j+1}^{\alpha\beta}| + |x_j^{\alpha\beta}| \le (1 + 2|\phi_j| + 2(2\kappa_2 + 1)t_j + 2\kappa_2 t_{j+1})(|k_j^{\alpha\beta}| + |x_{j-1}^{\alpha\beta}|) + 2C_{\alpha\beta}(|\phi_j^{(l+1)}| T_{j+1}^{|\beta|} + (t_j + t_{j+1}) T_{j+1}^{(|\beta|-1)_+}).$$

It follows from Assumption (H.1)(ii) and $x_0^{\alpha\beta} = k_1^{\alpha\beta} = 0$ that

$$|k_{j+1}^{\alpha\beta}| + |x_j^{\alpha\beta}| \le C_{\alpha\beta} T_{j+1}^{|\beta|}.$$

Hence we have

$$|k_j^{\alpha\beta}| \le C_{\alpha\beta} T_j^{|\beta|} \text{ and } |x_j^{\alpha\beta}| = |\sum_{l=1}^{j} t_l k_l^{\alpha\beta}| \le C_{\alpha\beta} T_j^{|\beta|+1}.$$

The proof is completed. □

We need the inverse of the map $(y, k) \mapsto (y, x_L^\#(y, k))$. To this end we introduce the new variables:

(3.8) $\quad \tilde{x}(y, k) = x_j^\#(y, k/T_j)$ and $\tilde{k}_j(y, k) = T_j k_j^\#(y, k/T_j), \quad j = 1, \dots, L.$

LEMMA 3.3. *For any α and β, there exists $C_{\alpha\beta}$ such that*

$$|\partial_y^\alpha \partial_k^\beta(\partial_y \tilde{x}_j - 1)| + |\partial_y^\alpha \partial_k^\beta(\partial_k \tilde{x}_j - 1)| + |\partial_y^\alpha \partial_k^\beta(\partial_y \tilde{x}_j)| + |\partial_y^\alpha \partial_k^\beta(\partial_k \tilde{k}_j - 1)| \le C_{\alpha\beta} T_j.$$

Proof. This follows from Lemma 3.2. □

LEMMA 3.4. *There exists a constant $T > 0$ such that if $T_L < T$, then the map $(y, k) \mapsto (y, x) = (y, \tilde{x}_L(y, k))$ is a global diffeomorphism on* $\mathbf{R} \times \mathbf{R}$.

Proof. Let T satisfy $2C_{00}T \le 1$ with the constant C_{00} in Lemma 3.3 and $2\kappa_2 T \le 1$. Then by Lemma 3.3 the map $k \mapsto U(k) = x + k - \tilde{x}_L(y, k)$ is a contraction. So Lemma 3.4 is proved. □

Let $(y, \tilde{k}(y, x))$ be the inverse of the map $(y, k) \mapsto (y, x) = (y, \tilde{x}_L(y, k))$ in Lemma 3.4 and set $k(y, x) = \tilde{k}(y, x)/T_L$. Put

(3.9)
$$x_j^*(y, x) = x_j^\#(y, k(y, x)), \; j = 1, \dots, L-1,$$
$$k_j^*(y, x) = \frac{x_j^*(y, x) - x_{j-1}^*(y, x)}{t_j}, \quad j = 1, \dots, L,$$

where $x_0^* = y$ and $x_L^* = x$.

LEMMA 3.5. *If $T_L < T$, then $x_j^*(y, x)$, $j = 1, \ldots, L-1$ is the unique critical point of S with $x_0^* = y$ and $x_L^* = x$, i.e. it satisfies* (1.7).

Proof. Let $y, x \in \mathbf{R}$. Then by Lemma 3.1, for y, $k = k(y, x)$ there exists a unique $(x_0^\#(y, k), \ldots, x_L^\#(y, k))$ which satisfies (3.1). And we have $x_L^\#(y, k(y, x)) = x$ by Lemma 3.4. These $x_j^\#(y, k(y, x))$ are nothing but the desired $x_j^*(y, x)$. □

The next lemma gives the estimates of the critical point.

LEMMA 3.6. *We have*

$$|T_L \partial_y k_j^* + 1| + |T_L \partial_x k_j^* - 1| \le CT_L, \quad 1 \le j \le L. \tag{3.10}$$

$$|\partial_y^\alpha \partial_x^\beta k_j^*| \le C_{\alpha\beta}, \quad |\alpha + \beta| \ge 2, \quad 1 \le j \le L. \tag{3.11}$$

$$|\partial_y x_j^*| + |\partial_x x_j^*| \le C, \quad 1 \le j \le L-1. \tag{3.12}$$

$$|\partial_y^\alpha \partial_x^\beta x_j^*| \le C_{\alpha\beta} T_L, \quad |\alpha + \beta| \ge 2, \quad 1 \le j \le L-1. \tag{3.13}$$

Proof. (3.10): From the facts that $T_L k_1^*(y, x) = T_L k(y, x) = \tilde{k}(y, x)$ and

$$\tilde{x}_L(y, \tilde{k}(y, x)) = x, \tag{3.14}$$

differentiating (3.14) and using Lemma 3.3 we have (3.10) for the case $j = 1$.

(3.10) for $2 \le j \le L$ follow from Lemma 3.3, (3.10) for $j = 1$ and from the fact that

$$T_j k_j^*(y, x) = T_j k_j^\#(y, k(y, x)) = \tilde{k}_j\Big(y, \frac{T_j}{T_L} \tilde{k}(y, x)\Big). \tag{3.15}$$

(3.11): For $|\alpha + \beta| \ge 2$, differentiating (3.14) we have

$$\begin{aligned} 0 = \partial_y^\alpha \partial_x^\beta \tilde{x}_L(y, \tilde{k}(y, x)) &= \partial_k \tilde{x}_L \cdot \partial_y^\alpha \partial_x^\beta \tilde{k} \\ &\quad + \sum C \partial_y^{\alpha'} \partial_k^{\beta'} \tilde{x}_L \cdot \partial_y^{\alpha_1} \partial_x^{\beta_1} \tilde{k} \cdots \partial_y^{\alpha_l} \partial_x^{\beta_l} \tilde{k}, \end{aligned}$$

where $2 \le |\alpha' + \beta'|$, $0 \le l \le |\alpha + \beta|$, $(0,0) \ne (\alpha_m, \beta_m) < (\alpha, \beta)$, $1 \le m \le l$ and $(\alpha_1, \beta_1) + \cdots + (\alpha_l, \beta_l) \le (\alpha, \beta)$. Hence we have (3.11) for the case $j = 1$:

$$|(\partial_y^\alpha \partial_x^\beta \tilde{k})(y, x)| \le C_{\alpha\beta} T_L$$

by induction on $n = |\alpha + \beta|$, using Lemma 3.3.

Similarly, differentiating (3.15) we have

$$|(\partial_y^\alpha \partial_x^\beta T_j k_j^*)(y, x)| \leq C_{\alpha\beta} T_j$$

by induction, using Lemma 3.3 and the estimate for $\partial_y^\alpha \partial_x^\beta \tilde{k}(y, x)$.

(3.12): Since we have

$$x_j^*(y, x) = x_j^\#(y, k(y, x)) = \tilde{x}_j\Big(y, \frac{T_j}{T_L}\tilde{k}(y, x)\Big),$$

the proof is clear by (3.10) and Lemma 3.3.

(3.13): For $|\alpha + \beta| \geq 2$, we have similarly to the proof of (3.11)

$$\begin{aligned}\partial_y^\alpha \partial_x^\beta x_j^*(y, x) &= \partial_k \tilde{x}_j \cdot (T_j / T_L) \partial_y^\alpha \partial_x^\beta \tilde{k} \\ &\quad + \sum C \partial_y^{\alpha'} \partial_k^{\beta'} \tilde{x}_j \cdot (T_j / T_L) \partial_y^{\alpha_1} \partial_x^{\beta_1} \tilde{k} \cdots (T_j / T_L) \partial_y^{\alpha_l} \partial_x^{\beta_l} \tilde{k},\end{aligned}$$

where $2 \leq |\alpha' + \beta'|$, $0 \leq l \leq |\alpha + \beta|$, $(0,0) \neq (\alpha_m, \beta_m) < (\alpha, \beta)$, $1 \leq m \leq l$ and $(\alpha_1, \beta_1) + \cdots + (\alpha_l, \beta_l) \leq (\alpha, \beta)$. Therefore from (3.11) and Lemma 3.3, we have (3.13). □

We introduce the same notations as in [5]. Let m and k be two positive integers with $m > k + 1$. We define $(x_{m-1}^*, \ldots, x_{k+1}^*)$ as the partial critical point, i.e.

$$\partial_j S_{j+1}(x_{j+1}^*, x_j^*) + \partial_j S_j(x_j^*, x_{j-1}^*) = 0, \quad j = k+1, \ldots, m-1.$$

Here $x_m^* = x_m$ and $x_k^* = x_k$. We denote the critical level by $S_{m,k+1}^\#(x_m, x_k)$, i.e.

$$S_{m,k+1}^\#(x_m, x_k) = S_m(x_m, x_{m-1}^*) + \cdots + S_{k+1}(x_{k+1}^*, x_k).$$

If $k + 1 = m$, then we set $S_{m,k+1}^\#(x_m, x_k) = S_m(x_m, x_{m-1})$. For any $m > k$, we put $T(m, k) = t_m + \cdots + t_k$, and $T(k, k) = t_k$. For a sequence of integers $(j_1, \ldots, j_s)$ such as $0 = j_0 < j_1 < j_2 < \cdots < j_s < L = j_{s+1}$, we put

$$S_{j_s \cdots j_1}^\#(x_L, x_{j_s}, \ldots, x_{j_1}, x_0) = \sum_{r=1}^{s+1} S_{j_r, j_{r-1}+1}^\#(x_{j_r}, x_{j_{r-1}}).$$

LEMMA 3.7. *Let $T_L < T$. Then $S_{j_s \cdots j_1}^\#(x_L, x_{j_s}, \ldots, x_{j_1}, x_0)$ satisfies* (H.1) *with constants $\kappa_m^\#$ and $B_m^\#$ different from κ_m and B_m:*
(i′) *$S_{j_r, j_{r-1}+1}^\#(x_{j_r}, x_{j_{r-1}})$ is of the form*

$$S_{j_r, j_{r-1}+1}^\#(x_{j_r}, x_{j_{r-1}}) = \frac{|x_{j_r} - x_{j_{r-1}}|^2}{2T(j_r, j_{r-1}+1)} + \omega_{j_r, j_{r-1}+1}^\#(x_{j_r}, x_{j_{r-1}}).$$

For any $m \geq 2$, there exists $\kappa_m^\#$ such that

$$\max_{2 \leq |\alpha+\beta| \leq m} \sup_{x,y} |\partial_x^\alpha \partial_y^\beta \omega_{j_r, j_{r-1}+1}^\#(x, y)| \leq \kappa_m^\#, \tag{3.16}$$

where $\kappa_m^{\#}$ depends on $\{\kappa_l\}$ and $\{B_l\}$ but not on r and t_j.

(ii′) *Let $(\bar{x}_L, \bar{x}_{j_s}, \dots, \bar{x}_{j_1}, \bar{x}_0)$ be an arbitrary critical point of $S^{\#}_{j_s \dots j_1}$, i.e.*

$$\partial_{j_r} S^{\#}_{j_{r+1}, j_r+1}(\bar{x}_{j_{r+1}}, \bar{x}_{j_r}) + \partial_{j_r} S^{\#}_{j_r, j_{r-1}+1}(\bar{x}_{j_r}, \bar{x}_{j_{r-1}}) = 0, \quad r = 1, \dots, s. \tag{3.17}$$

Then for any $K \geq 1$ there exists $B_K^{\#}$ such that

$$\sum_{r=1}^{s} \sum_{|\beta|=1, 1 \leq |\alpha| \leq K} |\, [(\partial_{j_{r-1}} + \partial_{j_r} + \partial_{j_{r+1}})^{\alpha} \partial_{j_r}^{\beta} (\omega^{\#}_{j_r, j_{r-1}+1} + \omega^{\#}_{j_{r+1}, j_r+1})] (\bar{x}_{j_{r-1}}, \bar{x}_{j_r}, \bar{x}_{j_{r+1}}) \,| \leq B_K^{\#}, \tag{3.18}$$

where $B_K^{\#}$ depends on $\{\kappa_l\}$ and $\{B_l\}$ but not on $(\bar{x}_L, \bar{x}_{j_s}, \dots, \bar{x}_{j_1}, \bar{x}_0)$ and s.

Proof. (i′) We investigate simply $S(\overline{x_L, x_0})$ instead of $S^{\#}_{j_r, j_{r-1}+1}(x_{j_r}, x_{j_{r-1}})$, to which a similar argument applies. Since $(x^*_{L-1}, \dots, x^*_1)$ is the critical point of S, we have

$$\begin{aligned} \partial_0 S(\overline{x_L, x_0}) &= \partial_0 (S(x_L, x^*_{L-1}, \dots, x^*_1, x_0)) \\ &= (\partial_0 S_1)(x_1^*, x_0). \end{aligned}$$

Hence we have

$$\begin{aligned} \partial_0^2 S(\overline{x_L, x_0}) &= (\partial_0^2 S_1)(x_1^*, x_0) + (\partial_1 \partial_0 S_1)(x_1^*, x_0) \partial_0 x_1^* \\ &= t_1^{-1} + \partial_0^2 \omega_1 + (-t_1^{-1} + \partial_1 \partial_0 \omega_1)(1 + t_1 \partial_0 k_1^*) \\ &= \partial_0^2 \omega_1 + \partial_1 \partial_0 \omega_1 + (-1 + t_1 \partial_0 \partial_1 \omega_1) \partial_0 k_1^*, \end{aligned}$$

where we have used $\partial_0 x_1^* = 1 + t_1 \partial_0 k_1^*$ which follows from (3.9). Since by (3.10,11) of Lemma 3.6 we can write

$$\partial_0 k_1^*(x_L, x_0) = -\frac{1}{T_L} + b(x_L, x_0), \quad b(x_L, x_0) \in \mathscr{B}(\mathbf{R} \times \mathbf{R}),$$

we have

$$\partial_0^2 S(\overline{x_L, x_0}) = \frac{1}{T_L} + \partial_0^2 \omega_1 + \partial_0 \partial_1 \omega_1 - \frac{t_1}{T_L} \partial_0 \partial_1 \omega_1 + (-1 + t_1 \partial_0 \partial_1 \omega_1) b(x_L, x_0).$$

For the other derivatives of $S(\overline{x_L, x_0})$, similar arguments hold, since we have

$$\partial_0 \partial_L S(\overline{x_L, x_0}) = \partial_L k_1^* (-1 + t_1 \partial_0 \partial_1 \omega_1) = \partial_0 k_L^* (1 - t_L \partial_L \partial_{L-1} \omega_L)$$

and so on. Therefore we obtain (i′).

(ii′) To simplify the notation we put $l = j_{r-1}$, $m = j_r$ and $n = j_{r-1}$. We have

$$
\begin{aligned}
(3.19)\qquad &(\partial_l + \partial_m + \partial_n)\partial_m(\omega^{\#}_{m,l+1}(x_m, x_l) + \omega^{\#}_{n,m+1}(x_n, x_m))\\
&= (\partial_l + \partial_m + \partial_n)\partial_m(S^{\#}_{m,l+1}(x_m, x_l) + S^{\#}_{n,m+1}(x_n, x_m))\\
&= \partial_m k^*_{l+1}(t_{l+1}\partial_l\partial_{l+1}\omega_{l+1} - 1) + \partial_m k^*_m(1 - t_m\partial_m\partial_{m-1}\omega_m)\\
&\quad + \partial^2_m\omega_m + \partial_m\partial_{m-1}\omega_m + \partial^2_m\omega_{m+1} + \partial_m\partial_{m+1}\omega_{m+1}\\
&\quad + \partial_m k^*_{m+1}(t_{m+1}\partial_m\partial_{m+1}\omega_{m+1} - 1) + \partial_m k^*_n(1 - t_n\partial_n\partial_{n-1}\omega_n),
\end{aligned}
$$

where k^*_{l+1} and k^*_m are functions of (x_l, x_m) and k^*_{m+1} and k^*_n are of (x_m, x_n). We can show that

$$
\begin{aligned}
&\partial_m k^*_{l+1}(t_{l+1}\partial_l\partial_{l+1}\omega_{l+1} - 1) + \partial_m k^*_m(1 - t_m\partial_m\partial_{m-1}\omega_m)\\
&= \sum_{j=l+1}^{m-1} \phi_j(x^*_{j-1}, x^*_j, x^*_{j+1})\partial_m x^*_j(x_l, x_m),
\end{aligned}
$$

where $\phi_j(x_{j-1}, x_j, x_{j+1}) = [(\partial_{j-1} + \partial_j + \partial_{j+1})\partial_j(\omega_j + \omega_{j+1})](x_{j-1}, x_j, x_{j-1})$. In fact we have

$$
\begin{aligned}
t_{l+1}\partial_m k^*_{l+1} &= \partial_m x^*_{l+1}.\\
t_m\partial_m k^*_m &= 1 - \partial_m x^*_{m-1},\\
\partial_m k^*_m - \partial_m k^*_{l+1} &= (1,\dots, 1)W(l+1, m\,; X^*_{l,m})\partial_m X^*_{l,m} + \partial_m\partial_{m-1}\omega_m,
\end{aligned}
$$

where ${}^tX^*_{l,m} = (x^*_{l+1},\dots, x^*_{m-1})$ and $W(l+1, m\,; X^*_{l,m})$ is the Hessian matrix of $\sum_{j=l+1}^m \omega_j$ with respect to $(x_{l+1},\dots, x_{m-1})$ at $X^*_{l,m}$:

$$
W(l+1, m\,; X^*_{l,m}) = \begin{pmatrix}
\partial^2_{l+1}(\omega_{l+1} + \omega_{l+2}) & \partial_{l+1}\partial_{l+2}\omega_{l+2} & 0 & \cdots\\
\partial_{l+1}\partial_{l+2}\omega_{l+2} & \partial^2_{l+2}(\omega_{l+2} + \omega_{l+3}) & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots\\
0 & \cdots & \cdots & \partial^2_{m-1}(\omega_{m-1} + \omega_m)
\end{pmatrix}.
$$

Therefore we have

$$
\begin{aligned}
(3.20)\qquad &(\partial_l + \partial_m + \partial_n)\partial_m(\omega^{\#}_{m,l+1}(x_m, x_l) + \omega^{\#}_{n,m+1}(x_n, x_m))\\
&= \sum_{j=l+1}^{m-1} \phi_j(x^*_{j-1}, x^*_j, x^*_{j+1})\partial_m x^*_j(x_l, x_m) + \phi_m(x^*_{m-1}, x^*_m, x^*_{m+1})\\
&\qquad + \sum_{j=m+1}^{n-1} \phi_j(x^*_{j-1}, x^*_j, x^*_{j+1})\partial_m x^*_j(x_m, x_n).
\end{aligned}
$$

When $(\bar{x}_L, \bar{x}_{j_s},\dots, \bar{x}_{j_1}, \bar{x}_0)$ is a solution of (3.17), $(\bar{x}_L, x^*_{L-1}(\bar{x}_L, \bar{x}_{j_s}),\dots, x^*_1(\bar{x}_{j_1}, \bar{x}_0), \bar{x}_0)$ is a solution of (1.5). So summing the absolute value of (3.20) over r (because $l = j_{r-1}$, $m = j_r$ and $n = j_{r+1}$) and substituting $(\bar{x}_L, \bar{x}_{j_s},\dots, \bar{x}_{j_1}, \bar{x}_0)$, we have

(3.18) for $K=1$ by (3.12) and (H.1)(ii).

Next we show (3.18) for the case $K=2$. We can rewrite (3.20) as

$$(3.21)\qquad \begin{aligned}&(\partial_l+\partial_m+\partial_n)\partial_m(\omega^{\#}_{m,l+1}(x_m,x_l)+\omega^{\#}_{n,m+1}(x_n,x_m))\\ &=\sum_{j=l+1}^{n-1}\phi_j(x_j^*-t_jk_j^*,x_j^*,x_j^*+t_{j+1}k_{j+1}^*)p_j(x_l,x_m,x_n),\end{aligned}$$

where p_j are bounded in $\mathscr{B}$ by (3.12,13). Differentiating (3.21) by $(\partial_l+\partial_m+\partial_n)$, we have

$$(3.22)\qquad \begin{aligned}&(\partial_l+\partial_m+\partial_n)^2\partial_m(\omega^{\#}_{m,l+1}(x_m,x_l)+\omega^{\#}_{n,m+1}(x_n,x_m))\\ &=\sum_{j=l+1}^{n-1}[\phi_j^{(2)}(\partial_l+\partial_m+\partial_n)x_j^*p_j-t_j\partial_{j-1}\phi_j(\partial_l+\partial_m+\partial_n)k_j^*p_j\\ &\quad+t_{j+1}\partial_{j+1}\phi_j(\partial_l+\partial_m+\partial_n)k_{j+1}^*p_j+\phi_j(\partial_l+\partial_m+\partial_n)p_j],\end{aligned}$$

where $\phi_j^{(2)}=(\partial_{j-1}+\partial_j+\partial_{j+1})^2\partial_j(\omega_j+\omega_{j+1})$. On the other hand by (3.10,11) we have

$$\begin{aligned}(\partial_l+\partial_m+\partial_n)k_j^*&=(\partial_l+\partial_m)k_j^*=q_j,\quad l+1\le j\le m,\\ (\partial_l+\partial_m+\partial_n)k_j^*&=(\partial_m+\partial_n)k_j^*=q_j,\quad m+1\le j\le n,\end{aligned}$$

where q_j are bounded in $\mathscr{B}$. So from (H.1)(i) and (3.12,13) the right-hand side of (3.22) is of the form

$$(3.23)\qquad \sum_{j=l+1}^{n-1}[\phi_j^{(2)}p_j'+(t_j+t_{j+1})q_j'+\phi_jp_j''],$$

where p_j', p_j'' and q_j' are bounded in $\mathscr{B}$. Summing the absolute value of (3.23) over r and substituting $(\bar{x}_L,\bar{x}_{j_s},\dots,\bar{x}_{j_1},\bar{x}_0)$, by (H.1)(ii) we have (3.15) for $K=2$. For the other higher derivatives similar arguments hold. So (ii′) is proved. □

Next we consider the Hessian matrix at the critical point. The Hessian matrix of S is equal to $H(L)+W(1,L;x)$, where

$$H(L)=\begin{pmatrix}\frac{1}{t_1}+\frac{1}{t_2} & -\frac{1}{t_2} & 0 & \cdots & \cdots & \\ -\frac{1}{t_2} & \frac{1}{t_2}+\frac{1}{t_3} & -\frac{1}{t_3} & 0 & \cdots & \\ 0 & -\frac{1}{t_3} & \frac{1}{t_3}+\frac{1}{t_4} & \cdots & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \cdots & \\ 0 & \cdots & \cdots & \cdots & -\frac{1}{t_{L-1}} & \frac{1}{t_{L-1}}+\frac{1}{t_L}\end{pmatrix}$$

and for $x = (x_1, \ldots, x_{L-1})$,

$$W(1, L; x) = \begin{pmatrix} \partial_1^2(\omega_1 + \omega_2) & \partial_1\partial_2\omega_2 & 0 & \cdots & \\ \partial_1\partial_2\omega_2 & \partial_2^2(\omega_2 + \omega_3) & \partial_2\partial_3\omega_3 & 0 & \\ \vdots & \vdots & \vdots & \ddots & \partial_{L-1}\partial_{L-2}\omega_{L-1} \\ 0 & \cdots & \cdots & \cdots & \partial_{L-1}^2(\omega_{L-1} + \omega_L) \end{pmatrix}.$$

We have

$$\det H(L) = \frac{T_L}{t_1 \cdots t_L}.$$

Let $G(L)$ be the inverse of $H(L)$. Then its (ij) entry is

$$\begin{aligned} (G(L))_{ij} &= \frac{T_i(T_L - T_j)}{T_L}, \quad \text{if } 1 \le i \le j \le L-1, \\ &= \frac{T_j(T_L - T_i)}{T_L}, \quad \text{if } 1 \le j \le i \le L-1. \end{aligned}$$

We set

$$G_1(L) = \frac{1}{T_L}\begin{pmatrix} t_1 & t_1 & t_1 & \cdots & t_1 \\ -(t_3 + \cdots + t_L) & T_2 & T_2 & \cdots & T_2 \\ -(t_4 + \cdots + t_L) & -(t_4 + \cdots + t_L) & T_3 & \cdots & T_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -t_L & -t_L & \cdots & -t_L & T_{L-1} \end{pmatrix}$$

and

$$G_2(L) = \begin{pmatrix} t_3 & 0 & \cdots & 0 \\ t_3 & t_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ t_L & t_L & \cdots & t_L \end{pmatrix}.$$

Then we have $G(L) = G_1(L)G_2(L)$.

We use two norms $\|x\|_\infty = \max_{1 \le j \le L-1} |x_j|$ and $\|x\|_1 = \sum_{j=1}^{L-1} |x_j|$ for any $x \in \mathbf{R}^{L-1}$.

LEMMA 3.8. *Let $x^* = (x_1^*, \dots, x_{L-1}^*) = (x_1^*(x_0, x_L), \dots, x_{L-1}^*(x_0, x_L))$ be the critical point. Then we have for any $u \in \mathbf{R}^{L-1}$,*

$$\| G_1(L)u \|_\infty \leq \| u \|_1,$$
$$\| G_2(L) W(1, L; x^*)u \|_1 \leq (9\kappa_2 + B_1) T_L \| u \|_\infty \text{ and}$$
$$\| G(L) W(1, L; x^*)u \|_\infty \leq (9\kappa_2 + B_1) T_L \| u \|_\infty.$$

Proof. For the proof we have only to sum the magnitudes of all component of the matrix $G_2(L) W(1, L; x^*)$. Since the first column of $G_2(L) W(1, L; x^*)$ is

$$h_1 = \begin{pmatrix} t_2\partial_1^2(\omega_1 + \omega_2) \\ t_3(\partial_1^2(\omega_1 + \omega_2) + \partial_1\partial_2\omega_2) \\ t_4(\partial_1^2(\omega_1 + \omega_2) + \partial_1\partial_2\omega_2) \\ \cdots \\ t_L(\partial_1^2(\omega_1 + \omega_2) + \partial_1\partial_2\omega_2) \end{pmatrix},$$

we have $\| h_1 \|_1 \leq 3\kappa_2 T_L$. For $2 \leq j \leq L-1$, the j-th column of $G_2(L) W(1, L; x^*)$ is

$$h_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ t_j\partial_{j-1}\partial_j\omega_j \\ t_{j+1}(\partial_{j-1}\partial_j\omega_j + \partial_j^2(\omega_j + \omega_{j+1})) \\ t_{j+2}\phi_j \\ \vdots \\ t_L\phi_j \end{pmatrix},$$

where $\phi_j = (\partial_{j-1} + \partial_j + \partial_{j+1})\partial_j(\omega_j + \omega_{j+1})$. So we have $\| h_j \|_1 \leq 3\kappa_2(t_j + t_{j+1}) + T_L | \phi_j |$. Therefore by (H.1)(ii), we have $\sum_{j=1}^{L-1} \| h_j \|_1 \leq (9\kappa_2 + B_1) T_L$. □

Let x_1^* be the critical point of $S_2(x_2, x_1) + S_1(x_1, x_0)$ with respect to x_1. We define a function $D(S_2 + S_1; x_2, x_0)$ through the Hessian determinant at x_1^* in the following way:

$$\det \mathrm{Hess}_{x_1^*}(S_2 + S_1) = \frac{t_1 + t_2}{t_1 t_2} D(S_2 + S_1; x_2, x_0).$$

For $m > k+1$ we define $D(x_m, x_k)$ by

$$\det(\mathrm{Hess}_{(x^*_{m-1},\dots,x^*_{k+1})}(S_m + \cdots + S_{k+1})) = \frac{t_{k+1} + \cdots + t_m}{t_{k+1} \cdots t_m} D(x_m, x_k).$$

LEMMA 3.9. *Let* $0 < T^1 < T$ *with* $2(9\kappa_2 + B_1)T^1 < 1$. *If* $T_L < T^1$, *then we have*

$$D(x_L, x_0) = \prod_{k=2}^{L} D(S_k + S^{\#}_{k-1,1}\,; x_k, x_0)\,|_{(x_{L-1},\dots,x_1)=(x^*_{L-1},\dots,x^*_1)}.$$

Proof. When $2(9\kappa_2 + B_1)T_L < 1$, we have that

$$\begin{aligned}&\det \mathrm{Hess}_{(x^*_{k-1},\dots,x^*_1)}(S_k(x_k, x^*_{k-1}) + \cdots + S_1(x^*_1, x_0))\\ &= \frac{t_k + \cdots + t_1}{t_k \cdots t_1} \det(I + G(k)W(1, k\,; x^*)) \neq 0, \quad k = 2,\dots, L,\end{aligned}$$

because $I + G(k)W(1, k\,; x^*)$ has the inverse matrix by Lemma 3.8. So applying [5, Proposition 2.6], we can prove the lemma by induction on L, similarly to [5, Proposition 2.8]. □

LEMMA 3.10. *If* $T_L < T^1$, *we can write*

$$D(x_L, x_0) = 1 + T_L\, g(x_L, x_0), \tag{3.24}$$

where $g(x_L, x_0)$ *remains bounded in* $\mathcal{B}(\mathbf{R} \times \mathbf{R})$ *uniformly with respect to* $t_1,\dots, t_L$.

Proof. By Lemmas 3.6, 3.7 and 3.9, we can write

$$\begin{aligned} D(x_L, x_0) &= \prod_{k=2}^{L} D(S_k + S^{\#}_{k-1,1}\,; x_k, x_0)\,|_{(x_{L-1},\dots,x_1)=(x^*_{L-1},\dots,x^*_1)}\\ &= \prod_{k=2}^{L}\left(1 + \frac{t_k T_{k-1}}{T_k}\,\partial^2_{k-1}(\omega_k + \omega^{\#}_{k-1,1})\right)\Big|_{(x_{L-1},\dots,x_1)=(x^*_{L-1},\dots,x^*_1)}\\ &= \prod_{k=2}^{L}(1 + t_k p_k(x_L, x_0)),\end{aligned}$$

where $p_k(x_L, x_0)$ are bounded in $\mathcal{B}(\mathbf{R} \times \mathbf{R})$. So the lemma is proved. □

It is noted that Lemma 3.10 differs from Fujiwara [5, Proposition 2.10] in the power of T_L; our power is 1 while his is 2.

4. Key lemma

In this section we write down key lemmas to prove Theorems 1 and 2. Their assertions are the same as those of [5] except for the form of the phase function. Let $S_j(t_j, x, y) = \dfrac{|x-y|^2}{2t_j} + \omega_j(t_j, x, y)$, $i = 1,2$ be phase functions satisfying (H.1)(i), and $a(x, z, y)$ an amplitude function in $\mathscr{B}(\mathbf{R} \times \mathbf{R} \times \mathbf{R})$. We set $\tau = t_1 t_2/(t_1 + t_2)$ and $E = \nu i/(2\pi)$. The notation $D(S_2 + S_1 ; x, y)$ is given in §3.

LEMMA 4.1. *Assume that* $8\tau\kappa_2 \leq 1$. *Then*

$$\left(\frac{E}{t_1}\right)^{1/2}\left(\frac{E}{t_2}\right)^{1/2}\int_{\mathbf{R}} e^{-i\nu(S_1(t_1,x,z)+S_2(t_2,z,y))}a(x, z, y)dz$$
$$= \left(\frac{E}{t_1+t_2}\right)^{1/2} e^{-i\nu S^{\#}_{2,1}(x,y)} D(S_2 + S_1 ; x , y)^{-1/2} b(x, y),$$

with

$$b(x, y) = a(x, z^*, y) + \left(\frac{\tau}{i\nu}\right)D(S_2 + S_1 ; x, y)^{-1}\left[\frac{1}{2}(\Delta_z a)(x, z^*, y)\right.$$
$$\left. + \tau D(S_2 + S_1 ; x, y)^{-1} r_1(x, y)\right] + \left(\frac{\tau}{i\nu}\right)^2 D(S_2 + S_1 ; x, y)^{-2} r_2(x, y),$$

where Δ_z *is the Laplacian with respect to z. For any* $m \geq 0$ *there exist* C_m *and* $M(m)$ *such that if* $|\alpha|, |\beta| \leq m$,

$$|\partial_x^\alpha \partial_y^\beta r_1(x, y)| + |\partial_x^\alpha \partial_y^\beta r_2(x, y)| \leq C_m \max \sup_z |\partial_x^{\alpha'} \partial_y^{\beta'} \partial_z^{\gamma'} a(x, z, y)|,$$

where max *is taken for* $\alpha' \leq \alpha$, $\beta' \leq \beta$ *and* $\gamma' \leq M(m)$. $M(m)$ *can be chosen as* $2m + 4d + 2$.

Proof. We have only to apply the stationary phase method (cf. [1, Theorem 4.1].) □

The next lemma plays an important role.

LEMMA 4.2. *For the phase function we assume* (H.1). *Let* $a(x_L, \dots, x_0)$ *be an amplitude function in* $\mathscr{B}(\mathbf{R}^{d(L+1)})$. *Then there exists a constant* $\delta > 0$ *such that if* $T_L < \delta$ *then*

$$I(\{t_j\}, S, a, \nu)(x_L, x_0) = \left(\frac{\nu i}{2\pi T_L}\right)^{1/2} \exp(-i\nu S(\overline{x_L, x_0}))b(x_L, x_0).$$

For any $m \geq 0$ *there exist constants* C_m *and* $K(m)$ *such that if* $|\alpha_0|, |\alpha_L| \leq m$,

$$|\partial_L^{\alpha_L}\partial_0^{\alpha_0}b(x_L, x_0)| \leq C_m^L \max_{\beta} \sup_{x_{L-1},\dots,x_1} |\partial_L^{\beta_L}\partial_{L-1}^{\beta_{L-1}} \cdots \partial_0^{\beta_0}a(x_L,\dots, x_0)|,$$

where max *is taken for* $(\beta_L,\dots, \beta_0)$ *satisfying* $\beta_0 \leq \alpha_0$, $\beta_L \leq \alpha_L$ *and* $|\beta_j| \leq K(m)$, $j = 1,\dots, L-1$. C_m *and* $K(m)$ *do not depend on* L, ν *and* a. *We can choose* $K(m) = 12m + 48d + 21$.

For the proof of this lemma we refer to §3 of Fujiwara [5]. Though the assumption here for the phase function is more general than that of [5], the arguments there apply to our case word by word.

5. Proof of Theorems 1 and 2

The arguments in the proof of Theorems 1 and 2 will be the same as those in [5] except for taking (1.8b) in (H.2) into consideration.

For any $l > k$ we put $T(l, k) = t_l + \cdots + t_k$ and $T(k, k) = t_k$. We set $E = \nu i/(2\pi)$. Let δ be as in Lemma 4.2 and let T^1 be as in Lemma 3.9. Put $\delta' = \min(\delta, T^1)$. When $T_L < \delta'$, we consider the oscillatory integral

$$\text{(5.1)} \quad I(\{t_j\}, S, \alpha, \nu) = \prod_{j=1}^{L}\left(\frac{E}{t_j}\right)^{1/2} \int_{\mathbf{R}^{(L-1)}} \exp\left(-i\nu \sum_{j=1}^{L} S_j(x_j, x_{j-1})\right) a(x_L,\dots,x_0) \prod_{j=1}^{L-1} dx_j.$$

First we perform integration over x_1 space. Applying the stationary phase method, we have

$$\text{(5.2)} \quad \left(\frac{E}{t_2}\right)^{1/2}\left(\frac{E}{t_1}\right)^{1/2} \int_{\mathbf{R}} e^{-i\nu(S_2(x_2,x_1)+S_1(x_1,x_0))} a(x_L,\dots, x_0)\, dx_1$$
$$= \left(\frac{E}{T(2,1)}\right)^{1/2} e^{-i\nu S^{\#}_{2,1}(x_2,x_0)}((S_1a)(x_L,\dots, x_2, x_0) + (R_1a)(x_L,\dots,x_2, x_0)),$$

where $S_1 a$ is the main term

$$\text{(5.3)} \quad (S_1a)(x_L,\dots, x_2, x_0) = a(x_L,\dots, \overset{\ulcorner}{x_2}, \overset{\urcorner}{x_0}) D(S_2 + S_1 ; x_2, x_0)^{-1/2},$$

and $R_1 a$ is the remainder term.

Next, we integrate $S_1 a$ over x_2 space and apply the stationary phase method, then we have

$$(5.4)\quad \left(\frac{E}{t_3}\right)^{1/2}\left(\frac{E}{T(2,1)}\right)^{1/2}\int_{\mathbf{R}} e^{-i\nu(S_3(x_3,x_2)+S^{\#}_{2,1}(x_2,x_0))}(S_1a)(x_L,\ldots,x_2,x_0)\,dx_2$$

$$=\left(\frac{E}{T(3,1)}\right)^{1/2} e^{-i\nu S^{\#}_{3,1}(x_3,x_0)}((S_2S_1a)(x_L,\ldots,x_3,x_0)+(R_2S_1a)(x_L,\ldots,x_3,x_0)).$$

Here S_2S_1a is the main term and R_2S_1a is the remainder term, i.e.

$$(5.5)\quad (S_2S_1a)(x_L,\ldots,x_3,x_0)=(S_1a)(x_L,\ldots,x_3,x_2^*,x_0)D(S_3+S^{\#}_{2,1};x_3,x_0)^{-1/2},$$

where x_2^* is the critical point of $S_3+S^{\#}_{2,1}$ with respect to x_2.

Repeating this process $L-1$ times, by Lemma 3.9 we have the main term of Theorems 1 and 2:

$$(5.6)\quad \left(\frac{E}{T_L}\right)^{1/2} e^{-i\nu S^{\#}_{L,1}(x_L,x_0)}(S_{L-1}S_{L-2}\cdots S_1a)(x_L,x_0)$$

$$=\left(\frac{E}{T_L}\right)^{1/2} e^{-i\nu S^{\#}_{L,1}(x_L,x_0)}D(x_L,x_0)^{-1/2}a(\overline{x_L,x_0}).$$

Next we treat the remainder term. Since $(R_1a)(x_L,\ldots,x_2,x_0)$ has complicated structure as a function of x_2, we postpone integration over x_2 space of the term including $(R_1a)(x_L,\ldots,x_2,x_0)$ and perform integration over x_3 space beforehand. The stationary phase method gives

$$(5.7)\quad \left(\frac{E}{t_4}\right)^{1/2}\left(\frac{E}{t_3}\right)^{1/2}\left(\frac{E}{T(2,1)}\right)^{1/2}\int_{\mathbf{R}} e^{-i\nu(S_4(x_4,x_3)+S_3(x_3,x_2)+S^{\#}_{2,1}(x_2,x_0))}$$

$$\times(R_1a)(x_L,\ldots,x_2,x_0)\,dx_3$$

$$=\left(\frac{E}{T(4,3)}\right)^{1/2}\left(\frac{E}{T(2,1)}\right)^{1/2} e^{-i\nu(S^{\#}_{4,3}(x_4,x_2)+S^{\#}_{2,1}(x_2,x_0))}$$

$$\times((S_3R_1a)(x_L,\ldots,x_4,x_2,x_0)+(R_3R_1a)(x_L,\ldots,x_4,x_2,x_0)),$$

where S_3R_1a is the main term and R_3R_1a is the remainder i.e.

$$(5.8)\quad (S_3R_1a)(x_L,\ldots,x_4,x_2,x_0)=(R_1a)(x_L,\ldots,\overline{x_4,x_2},x_0)D(S_4+S_3;x_4,x_2)^{-1/2}.$$

Similarly, we skip integration over x_3 space of the term including $(R_2S_1a)(x_L,\ldots,x_3,x_0)$ and integrate it over x_4 space.

We continue this process: if R_k appears we skip integration over x_{k+1} space. Thus we can write $I(\{t_j\},S,a,\nu)$ as

$$(5.9)\quad I(\{t_j\},S,a,\nu)(x_L,x_0)=A_0(x_L,x_0)+\sum{}' A_{j_sj_{s-1}\cdots j_1}(x_L,x_0).$$

Here the main term is

$$A_0(x_L, x_0) = \left(\frac{E}{T_L}\right)^{1/2} e^{-i\nu S^{\#}_{L,1}(x_L,x_0)} D(x_L, x_0)^{-1/2} a(\overline{x_L, x_0}). \tag{5.10}$$

The sum $\sum'$ is taken over the sequences of integers $(j_s, j_{s-1}, \dots, j_1)$ with the property

$$0 = j_0 < j_1 - 1 < j_1 < j_2 - 1 < \cdots < j_s - 1 < j_s \leq L = j_{s+1}.$$

The summand is

$$\begin{aligned} (5.11)\quad & A_{j_s j_{s-1} \cdots j_1}(x_L, x_0) \\ &= \prod_{u=1}^{s+1} \left(\frac{E}{T(j_u, j_{u-1}+1)}\right)^{1/2} \int_{\mathbf{R}^s} \exp(-i\nu S^{\#}_{j_s j_{s-1} \cdots j_1}(x_L, x_{j_s}, \dots, x_{j_1}, x_0)) \\ &\quad \times b_{j_s j_{s-1} \cdots j_1}(x_L, x_{j_s}, \dots, x_{j_1}, x_0) \prod_{u=1}^{s} dx_{j_u}. \end{aligned}$$

The amplitude of this is

$$b_{j_s j_{s-1} \cdots j_1}(x_L, x_{j_s}, \dots, x_{j_1}, x_0) = (Q_{L-1} Q_{L-2} \cdots Q_1 a)(x_L, x_{j_s}, \dots, x_{j_1}, x_0), \tag{5.12}$$

where

$$\begin{aligned} Q_j &= \mathrm{Id}, && \text{if } j = j_s, j_{s-1}, \dots, j_1, \\ &= \mathrm{R}_j, && \text{if } j = j_s - 1, j_{s-1} - 1, \dots, j_1 - 1, \\ &= S_j, && \text{otherwise.} \end{aligned}$$

The phase is

$$S^{\#}_{j_s j_{s-1} \cdots j_1}(x_L, x_{j_s}, \dots, x_{j_1}, x_0) = \sum_{u=1}^{s+1} S^{\#}_{j_u, j_{u-1}+1}(x_{j_u}, x_{j_{u-1}}), \tag{5.13}$$

where we understand $S^{\#}_{j_{s+1}, j_s+1} = 0$ when $j_s = L$, and $S^{\#}_{j_{s+1}, j_s+1} = S^{\#}_{L,L} = S_L(x_L, x_{L-1})$ when $j_s = L - 1$. In (5.11), when $j_s = L$ then the integration over x_{j_s} is not performed. Moreover we understand $\dfrac{E}{T(j_{s+1}, j_s + 1)} = 1$ when $j_s = L$, and $T(j_{s+1}, j_s + 1) = T(L, L) = t_L$ when $j_s = L - 1$.

Note. Fujiwara [5] did not take the case $j_s = L$ into consideration in the sum of (5.9).

By Lemma 3.7 we know that (5.13) satisfies (H.1). So we can apply Lemma 4.2 to $A_{j_s j_{s-1} \cdots j_1}$ and obtain

$$A_{j_s j_{s-1} \cdots j_1}(x_L, x_0) = \left(\frac{E}{T_L}\right)^{1/2} e^{-i\nu S^{\#}_{L,1}(x_L,x_0)} a_{j_s j_{s-1} \cdots j_1}(x_L, x_0),$$

where $a_{j_s j_{s-1}\cdots j_1}$ satisfies the estimate: For any $m \geq 0$ there exist C_m and $K(m)$ such that if $|\alpha_L|, |a_0| \leq m$,

(a) when $j_s < L$,

(5.14a) $$|\partial_L^{\alpha_L}\partial_0^{\alpha_0} a_{j_s j_{s-1}\cdots j_1}(x_L, x_0)| \leq C_m^s \max \sup_{x_{j_u}, u=1,\cdots,s} |\partial_L^{\beta_L}\partial_{j_s}^{\beta_{j_s}} \cdots \partial_{j_1}^{\beta_{j_1}}\partial_0^{\beta_0} b_{j_s j_{s-1}\cdots j_1}(x_L, x_{j_s}, \ldots, x_{j_1}, x_0)|,$$

where max is taken for $\beta_L \leq a_L$, $\beta_0 \leq \alpha_0$ and $\beta_{j_u} \leq K(m) = 12m + 48 + 21$, $u = 1, \ldots, s$,

(b) when $j_s = L$,

(5.14b) $$|\partial_L^{\alpha_L}\partial_0^{\alpha_0} a_{j_s j_{s-1}\cdots j_1}(x_L, x_0)| \leq C_m^s \max \sup_{x_{j_u}, u=1,\cdots,s-1} |\partial_L^{\beta_L}\partial_{j_{s-1}}^{\beta_{j_{s-1}}} \cdots \partial_{j_1}^{\beta_{j_1}}\partial_0^{\beta_0} b_{j_s j_{s-1}\cdots j_1}(x_L, x_{j_{s-1}}, \ldots, x_{j_1}, x_0)|,$$

where max is taken for $\beta_L \leq \alpha_L$, $\beta_0 \leq \alpha_0$ and $\beta_{j_u} \leq K(m) = 12m + 48 + 21$, $u = 1, \ldots, s-1$. So we have

(5.15) $$I(\{t_j\}, S, a, \nu) = \left(\frac{E}{T_L}\right)^{1/2} e^{-i\nu S^{\#}_{L,1}(x_L, x_0)} D(x_L, x_0)^{-1/2}(a(\overline{x_L, x_0}) + r(x_L, x_0)),$$
$$r(x_L, x_0) = D(x_L, x_0)^{1/2} \sum{}' a_{j_s j_{s-1}\cdots j_1}(x_L, x_0).$$

Therefore from (5.14a, b, 15) we see that we have only to estimate $b_{j_s j_{s-1}\cdots j_1}$ to prove Theorems 1 and 2.

Proof of Theorem 1. Assume (H.2).

LEMMA 5.1. *Let $T_L < \delta'$. Then for any $m \geq 0$ there exist constants $C_{m,1}$ and $M(m)$ such that for any $\alpha_0, \alpha_L, \alpha_{j_u} \leq m$, $1 \leq u \leq s$,*

(5.16) $$|\partial_L^{\alpha_L}\partial_0^{\alpha_0} \prod_{u=1}^{s} \partial_{j_u}^{\alpha_{j_u}} b_{j_s j_{s-1}\cdots j_1}(x_L, x_{j_s}, \ldots, x_{j_1}, x_0)| \leq C_{m,1}^s \left(\prod_{u=1}^{s} \nu^{-1} t_{j_u}\right) \times \max \sup |\partial_L^{\beta_L}\partial_0^{\beta_0} \prod_{u=1}^{s} \partial_{j_u-1}^{\beta_{j_u-1}}\partial_{j_u}^{\beta_{j_u}} a(\overline{x_L, x_{j_s}}, \overline{x_{j_s-1}, x_{j_{s-1}}}, \ldots, \overline{x_{j_1-1}, x_0})|,$$

where max *is taken for $\beta_L \leq \alpha_L$, $\beta_0 \leq \alpha_0$, $\beta_{j_u} \leq \alpha_{j_u}$ and $\beta_{j_u-1} \leq M(m)$ and* sup *is taken for x_{j_u-1}, $1 \leq u \leq s$. Here when $j_s = L$, the notation $\partial_L^{\alpha_L}$ appears only once and we understand $\overline{x_L, x_{j_s}} = x_{j_s}$ on both the sides of the inequality* (5.16). *We can choose* $M(m) = 2m + 4 + 2$.

We assume Lemma 5.1 for the moment and prove Theorem 1. From (H.2) the right-hand side of (5.16) is majorized by $C^s_{m,1}(\Pi^s_{u=1}\nu^{-1}t_{j_u})A_{M(m)}X^s_{M(m)}$. So combining (5.14a, b) with Lemma 5.1, we have with $m' = K(m)$

$$|\partial_L^{\alpha_L}\partial_0^{\alpha_0}a_{j_sj_{s-1}\cdots j_1}(x_L, x_0)| \le C^s_m C^s_{m',1}\Big(\prod_{u=1}^{s}\nu^{-1}t_{j_u}\Big)A_{M(m')}X^s_{M(m')}.$$

It follows with (5.15) that

$$\begin{aligned}|\partial_L^{\alpha_L}\partial_0^{\alpha_0}r(x_L, x_0)| &\le \Big|\Big({\sum}' C^s_m C^s_{m',1}X^s_{M(m')}\prod_{u=1}^{s}(\nu^{-1}t_{j_u})\Big)\Big| A_{M(m')}\\ &\le \Big(\prod_{j=1}^{L}(1 + C_m C_{m',1}X_{M(m')}\nu^{-1}t_{j_u}) - 1\Big)A_{M(m')}.\end{aligned}$$

This is the estimate (1.10) in Theorem 1 with $M(m') = M(K(m)) = 2(12m + 48 + 21) + 4 + 2$.

Lemma 5.1 follows immediately from the next lemma. For any sequence of integers $0 < k_1 - 1 < k_1 < k_2 - 1 < \cdots < k_r - 1 < k_r \le L$, we set

$$\begin{aligned}(5.17)\qquad & p_{k_rk_{r-1}\cdots k_1}(x_L, x_{L-1}, \dots, x_{k_r+1}, x_{k_r}, x_{k_{r-1}}, \dots, x_{k_1}, x_0)\\ &= (Q_{k_r}Q_{k_r-1}\cdots Q_1a)(x_L, x_{L-1}, \dots, x_{k_r+1}, x_{k_r}, x_{k_{r-1}}, \dots, x_{k_1}, x_0),\end{aligned}$$

where

$$\begin{aligned}Q_j &= \mathrm{Id}, && \text{if } j = k_r, k_{r-1}, \dots, k_1,\\ &= R_j, && \text{if } j = k_r - 1, k_{r-1} - 1, \dots, k_1 - 1,\\ &= S_j, && \text{otherwise.}\end{aligned}$$

LEMMA 5.2. *For any $m \ge 0$ there exist constants $C_{m,2}$ and $M(m)$ such that for arbitrary α_L, if $\alpha_0, \alpha_{k_j} \le m$, $1 \le j \le r$, then*

$$\begin{aligned}(5.18)\qquad & \Big|\partial_L^{\alpha_L}\partial_0^{\alpha_0}\prod_{j=1}^{r}\partial_{k_j}^{\alpha_{k_j}}p_{k_rk_{r-1}\cdots k_1}(\overline{x_L, x_{k_r}}, x_{k_{r-1}}, \dots, x_{k_1}, x_0)\Big|\\ &\le C^r_{m,2}\prod_{j=1}^{r}\Big(\frac{t_{k_j}T(k_j - 1, k_{j-1} + 1)}{\nu T(k_j, k_{j-1} + 1)}\Big)\\ &\quad\times \max\sup\Big|\partial_L^{\alpha_L}\partial_0^{\beta_0}\prod_{j=1}^{r}\partial_{k_j-1}^{\beta_{k_j-1}}\partial_{k_j}^{\beta_{k_j}}a(\overline{x_L, x_{k_r}}, \overline{x_{k_r-1}, x_{k_{r-1}}}, \dots, \overline{x_{k_1-1}, x_0})\Big|,\end{aligned}$$

where max *is taken for $\beta_0 \le \alpha_0$, $\beta_{k_j} \le \alpha_{k_j}$, $\beta_{k_j-1} \le M(m)$, $1 \le j \le r$, and* sup *is taken for x_{k_j-1}, $1 \le j \le r$. Moreover, for any sequence of integers $k_r < l_1 - 1 < l_1 < l_2 - 1 < \cdots < l_q \le L$, and for arbitrary multi-indices $\alpha_L, \alpha_{l_u}, \alpha_{l_u-1}$, $1 \le u \le q$, if $\alpha_0, \alpha_{k_j} \le m$, $1 \le j \le r$, then*

$$(5.19)\qquad |\partial_L^{\alpha_L}\partial_0^{\alpha_0}\prod_{u=1}^{q}(\partial_{l_u}^{\alpha_{l_u}}\partial_{l_u-1}^{\alpha_{l_u-1}})\prod_{j=1}^{r}\partial_{k_j}^{\alpha_{k_j}}$$

$$\times\, p_{k_r k_{r-1} \cdots k_1}(\overline{x_L, x_{l_q}}, \overline{x_{l_q-1}, x_{l_{q-1}}}, \dots, \overline{x_{l_1-1}, x_{k_r}}, x_{k_{r-1}}, \dots, x_{k_1}, x_0) \,|$$
$$\leq C^r_{m,2} \prod_{j=1}^{r} \left(\frac{t_{k_j} T(k_j - 1, k_{j-1} + 1)}{\nu T(k_j, k_{j-1} + 1)} \right)$$
$$\times \max \sup |\, \partial_L^{\alpha_L} \partial_0^{\beta_0} \prod_{u=1}^{q} (\partial_{l_u-1}^{\alpha_{l_u-1}} \partial_{l_u}^{\alpha_{l_u}} \prod_{j=1}^{r} \partial_{k_j-1}^{\beta_{k_j-1}} \partial_{k_j}^{\beta_{k_j}})$$
$$\times\, a(\overline{x_L, x_{l_q}}, \overline{x_{l_q-1}, x_{l_{q-1}}}, \dots, \overline{x_{l_1-1}, x_{k_r}}, \dots, \overline{x_{k_1-1}, x_0}) \,|,$$

where max *is taken for* $\beta_0 \leq \alpha_0$, $\beta_{k_j} \leq \alpha_{k_j}$, $\beta_{k_j-1} \leq M(m)$, $1 \leq j \leq r$, *and* sup *is taken for* x_{k_j-1}, $1 \leq j \leq r$. *Here when* $k_r = L$ *and* $l_q = L$ *respectively, the notation* $\partial_L^{\alpha_L}$ *appears only once and we understand* $\overline{x_L, x_{k_r}} = x_{k_r}$ *and* $\overline{x_L, x_{l_q}} = x_{l_q}$ *on both the sides of the inequalities* (5.18) *and* (5.19) *respectively. We can choose* $M(m) = 2m + 4 + 2$.

Proof. We prove only (5.19) by induction on r. (5.18) will be shown similarly. To prove the case for $r = 1$, we abbreviate k_1 as k. We have

$$p_k(x_L, x_{L-1}, \dots, x_{k+1}, x_k, x_0) = (R_{k-1} S_{k-2} \cdots S_1 a)(x_L, x_{L-1}, \dots, x_k, x_0), \quad k \geq 3,$$
$$= (R_1 a)(x_L, \dots, x_2, x_0), \quad k = 2.$$

We set

$$(5.20)\quad q(x_L, \dots, x_k, x_{k-1}, x_0) = (S_{k-2} \cdots S_1 a)(x_L, x_{L-1}, \dots, x_k, x_{k-1}, x_0)$$
$$= D(x_{k-1}, x_0)^{-1/2} a(x_L, x_{L-1}, \dots, x_k, \overline{x_{k-1}, x_0}), \text{ if } k \geq 3,$$
$$= a(x_L, \dots, x_2, x_1, x_0), \text{ if } k = 2.$$

Let $S^\#_{1,1}(x_1, x_0) = S_1(x_1, x_0)$. Then we have

$$(5.21)\quad \left(\frac{E}{t_k}\right)^{1/2} \left(\frac{E}{T(k-1,1)}\right)^{1/2} \int_{\mathbf{R}} e^{-i\nu(S_k(x_k, x_{k-1}) + S^\#_{k-1,1}(x_{k-1}, x_0))}$$
$$\times\, q(x_L, x_{L-1}, \dots, x_k, x_{k-1}, x_0)\, dx_{k-1}$$
$$= \left(\frac{E}{T(k, 1)}\right)^{1/2} e^{-i\nu S^\#_{k,1}(x_k, x_0)}$$
$$\times\, (D(S_k + S^\#_{k-1,1}; x_k, x_0)^{-1/2} q(x_L, x_{L-1}, \dots, x_k, x_0) + p_k(x_L, \dots, x_k, x_0)).$$

Therefore, if $k < l_1 - 1 < l_1 < l_2 - 1 < \cdots < l_q \leq L$, then

$$(5.22)\quad \left(\frac{E}{t_k}\right)^{1/2} \left(\frac{E}{T(k-1,1)}\right)^{1/2} \int_{\mathbf{R}} e^{-i\nu(S_k(x_k, x_{k-1}) + S^\#_{k-1,1}(x_{k-1}, x_0))}$$
$$\times\, q(\overline{x_L, x_{l_q}}, \overline{x_{l_q-1}, x_{l_{q-1}}}, \dots, \overline{x_{l_1-1}, x_k}, x_{k-1}, x_0)\, dx_{k-1}$$

$$
\begin{aligned}
&= \left(\frac{E}{T(k,1)}\right)^{1/2} e^{-i\nu S^{\#}_{k,1}(x_k,x_0)} \\
&\quad \times \big(D(S_k + S^{\#}_{k-1,1}\,;\, x_k, x_0)^{-1/2} q(\overbracket{x_L, x_{l_q}}, \overbracket{x_{l_q-1}, x_{l_{q-1}}}, \dots, \overbracket{x_{l_1-1}, x_k}, \overbracket{x_0}) \\
&\quad + p_k(\overbracket{x_L, x_{l_q}}, \overbracket{x_{l_q-1}, x_{l_{q-1}}}, \dots, \overbracket{x_{l_1-1}, x_k}, x_0)\big).
\end{aligned}
$$

Differentiating (5.22) with respect to x_L, x_{l_u}, x_{l_u-1} and applying the stationary phase method Lemma 4.1, we have the estimate: For any $m \ge 0$ there exists C_m such that for arbitrary α_{l_u}, α_{l_u-1}, α_L, if α_0, $\alpha_k \le m$,

$$
\begin{aligned}
&| \partial_k^{\alpha_k}\partial_0^{\alpha_0} \prod_{u=1}^{q} (\partial_{l_u}^{\alpha_{l_u}}\partial_{l_u-1}^{\alpha_{l_u-1}})\partial_L^{\alpha_L} p_k(\overbracket{x_L, x_{l_q}}, \overbracket{x_{l_q-1}, \dots}, \overbracket{x_{l_1-1}, x_k}, x_0) | \\
&\le C_m \left(\frac{t_k T(k-1,1)}{\nu T(k,1)}\right) \max \sup_{x_{k-1}} | \partial_k^{\beta_k}\partial_{k-1}^{\beta_{k-1}}\partial_0^{\beta_0} \prod_{u=1}^{q} (\partial_{l_u}^{\alpha_{l_u}}\partial_{l_u-1}^{\alpha_{l_u-1}})\partial_L^{\alpha_L} \\
&\quad \times q(\overbracket{x_L, x_{l_q}}, \dots, \overbracket{x_{l_1-1}, x_k}, x_{k-1}, x_0) |,
\end{aligned}
$$

where max is taken for $\beta_0 \le \alpha_0$, $\beta_k \le \alpha_k$, $\beta_{k-1} \le K(m) = 2m + 4 + 2$. When $l_q = L$, the notation $\partial_L^{\alpha_L}$ appears only once on both the sides of this inequality. From (5.20) Leibnitz' rule gives

$$
\begin{aligned}
&| \partial_k^{\alpha_k}\partial_0^{\alpha_0} \prod_{u=1}^{q} (\partial_{l_u}^{\alpha_{l_u}}\partial_{l_u-1}^{\alpha_{l_u-1}})\partial_L^{\alpha_L} p_k(\overbracket{x_L, x_{l_q}}, \overbracket{x_{l_q-1}, \dots}, \overbracket{x_{l_1-1}, x_k}, x_0) | \\
&\le C_m C'_m \left(\frac{t_k T(k-1,1)}{\nu T(k,1)}\right) \max \sup_{x_{k-1}} | \partial_k^{\beta_k}\partial_{k-1}^{\beta_{k-1}}\partial_0^{\beta_0} \prod_{u=1}^{q} (\partial_{l_u}^{\alpha_{l_u}}\partial_{l_u-1}^{\alpha_{l_u-1}})\partial_L^{\alpha_L} \\
&\quad \times a(\overbracket{x_L, x_{l_q}}, \dots, \overbracket{x_{l_1-1}, x_k}, \overbracket{x_{k-1}, x_0}) |,
\end{aligned}
$$

where max is taken for $\beta_0 \le \alpha_0$, $\beta_k \le \alpha_k$, $\beta_{k-1} \le K(m) = 2m + 4 + 2$. We choose $C_{m,2} \ge C_m C'_m$. This proves (5.19) for $r = 1$.

Next we suppose (5.19) for r and prove it for $r+1$. Let $k_r < k_{r+1} - 1 < k_{r+1} < l_1 - 1 < l_1 < \cdots < l_q \le L$. We set

$$
\begin{aligned}
(5.23)\quad & q(x_L, \dots, x_{k_{r+1}}, x_{k_{r+1}-1}, x_{k_r}, \dots, x_{k_1}, x_0) \\
&= (S_{k_{r+1}-2} \cdots S_{k_r+1} p_{k_r \cdots k_1})(x_L, \dots, x_{k_{r+1}}, x_{k_{r+1}-1}, x_{k_r}, \dots, x_{k_1}, x_0) \\
&= D(x_{k_{r+1}-1}, x_{k_r})^{-1/2} p_{k_r \cdots k_1}(x_L, \dots, x_{k_{r+1}}, \overbracket{x_{k_{r+1}-1}, x_{k_r}}, \dots, x_{k_1}, x_0).
\end{aligned}
$$

Then we have from (5.23)

$$
\begin{aligned}
(5.24)\quad & \left(\frac{E}{t_{k_{r+1}}}\right)^{1/2} \left(\frac{E}{T(k_{r+1}-1, k_r+1)}\right)^{1/2} \\
&\times \int_{\mathbf{R}} e^{-i\nu(S_{k_{r+1}}(x_{k_{r+1}}, x_{k_{r+1}-1}) + S^{\#}_{k_{r+1}-1, k_r+1}(x_{k_{r+1}-1}, x_{k_r}))}
\end{aligned}
$$

$$\times\, q(x_L, x_{l_q}, \dots, x_{l_1-1}, x_{k_{r+1}}, x_{k_{r+1}-1}, x_{k_r}, \dots, x_{k_1}, x_0)\, dx_{k_{r+1}-1}$$

$$= \left(\frac{E}{T(k_{r+1}, k_r + 1)}\right)^{1/2} e^{-\iota\nu S^{\#}_{k_{r+1}, k_r+1}(x_{k_{r+1}}, x_{k_r})}$$

$$\times\, [D(S_{k_{r+1}} + S^{\#}_{k_{r+1}-1, k_r+1}\,; x_{k_{r+1}}, x_{k_r})^{-1/2} q(x_L, x_{l_q}, \dots, x_{l_1-1}, x_{k_{r+1}}, x_{k_r}, \dots, x_0)$$

$$\times\, p_{k_{r+1}\cdots k_1}(x_L, x_{l_q}, \dots, x_{l_1-1}, x_{k_{r+1}}, x_{k_r}, \dots, x_0)].$$

We apply Lemma 4.1 to (5.24). Then we have from (5.23) for any $m \ge 0$ if α_{k_r}, $\alpha_{k_{r+1}} \le m$,

$$(5.25)\quad \Big|\, \partial_L^{\alpha_L} \partial_0^{\alpha_0} \prod_{u=1}^{q} (\partial_{l_u}^{\alpha_{l_u}} \partial_{l_u-1}^{\alpha_{l_u-1}}) \prod_{u=1}^{r+1} (\partial_{k_u}^{\alpha_{k_u}}) p_{k_{r+1}\cdots k_1}(x_L, x_{l_q}, \dots, x_{l_1-1}, x_{k_{r+1}}, x_{k_r}, \dots, x_0)\, \Big|$$

$$\le C_m \left(\frac{t_{k_{r+1}} T(k_{r+1} - 1, k_r + 1)}{\nu T(k_{r+1}, k_r + 1)}\right)$$

$$\times \max \sup_{x_{k_{r+1}-1}} \Big|\, \partial_L^{\alpha_L} \partial_0^{\alpha_0} \prod_{u=1}^{q} (\partial_{l_u}^{\alpha_{l_u}} \partial_{l_u-1}^{\alpha_{l_u-1}}) \prod_{u=1}^{r-1} (\partial_{k_u}^{\alpha_{k_u}}) \partial_{k_{r+1}}^{\beta_{k_{r+1}}} \partial_{k_{r+1}-1}^{\beta_{k_{r+1}-1}} \partial_{k_r}^{\beta_{k_r}}$$

$$\times\, q(x_L, x_{l_q}, \dots, x_{l_1-1}, x_{k_{r+1}}, x_{k_{r+1}-1}, x_{k_r}, \dots, x_{k_1}, x_0)\, \Big|,$$

$$\le C_m C'_m \left(\frac{t_{k_{r+1}} T(k_{r+1} - 1, k_r + 1)}{\nu T(k_{r+1}, k_r + 1)}\right)$$

$$\times \max \sup_{x_{k_{r+1}-1}} \Big|\, \partial_L^{\alpha_L} \partial_0^{\alpha_0} \prod_{u=1}^{q} (\partial_{l_u}^{\alpha_{l_u}} \partial_{l_u-1}^{\alpha_{l_u-1}}) \prod_{u=1}^{r-1} (\partial_{k_u}^{\alpha_{k_u}}) \partial_{k_{r+1}}^{\beta_{k_{r+1}}} \partial_{k_{r+1}-1}^{\beta_{k_{r+1}-1}} \partial_{k_r}^{\beta_{k_r}}$$

$$\times\, p_{k_r\cdots k_1}(x_L, x_{l_q}, \dots, x_{l_1-1}, x_{k_{r+1}}, x_{k_{r+1}-1}, x_{k_r}, \dots, x_{k_1}, x_0)\, \Big|,$$

where max is taken for $\beta_{k_r} \le \alpha_{k_r}$, $\beta_{k_{r+1}} \le \alpha_{k_{r+1}}$, $\beta_{k_{r+1}-1} \le K(m) = 2m + 4 + 2$. If $l_q = L$, then $\partial_L^{\alpha_L}$ appears only once in any of the three members of (5.25). When we assume that α_0, $\alpha_{k_u} \le m$, $1 \le u \le r + 1$ as in Lemma 5.2, we can estimate for any α_L, α_{l_u}, α_{l_u-1}, $1 \le u \le r + 1$ the last member of (5.25) by the induction hypothesis for r where q is replaced by $q + 1$ and $(l_1, \dots, l_q)$ is replaced by $(k_{r+1}, l_1, \dots, l_q)$. Hence we have

$$\Big|\, \partial_L^{\alpha_L} \partial_0^{\alpha_0} \prod_{u=1}^{q} (\partial_{l_u}^{\alpha_{l_u}} \partial_{l_u-1}^{\alpha_{l_u-1}}) \prod_{u=1}^{r+1} (\partial_{k_u}^{\alpha_{k_u}}) p_{k_{r+1}\cdots k_1}(x_L, x_{l_q}, \dots, x_{l_1-1}, x_{k_{r+1}}, x_{k_r}, \dots, x_0)\, \Big|$$

$$\le C_m C'_m C^r_{m,2} \prod_{u=1}^{r+1} \left(\frac{t_{k_u} T(k_u - 1, k_{u-1} + 1)}{\nu T(k_u, k_{u-1} + 1)}\right)$$

$$\times \max \sup \Big|\, \partial_L^{\alpha_L} \partial_0^{\alpha_0} \prod_{u=1}^{q} (\partial_{l_u}^{\alpha_{l_u}} \partial_{l_u-1}^{\alpha_{l_u-1}}) \prod_{u=1}^{r+1} (\partial_{k_u}^{\beta_{k_u}} \partial_{k_u-1}^{\beta_{k_u-1}})$$

$$\times\, a(x_L, x_{l_q}, \dots, x_{l_1-1}, x_{k_{r+1}}, x_{k_{r+1}-1}, x_{k_r}, \dots, x_{k_1-1}, x_0)\, \Big|,$$

where max is taken for $\beta_0 \leq \alpha_0$, $\beta_{k_u} \leq \alpha_{k_u}$, $\beta_{k_u-1} \leq K(m) = 2m+4+2$, $1 \leq u \leq r+1$ and sup is taken for x_{k_u-1}. Thus Lemma 5.2 has been proved.

Proof of Theorem 2. Let $a \equiv 1$ and $p_{j_s j_{s-1} \cdots j_1}$ be a function defined by (5.17) with $(j_s, \ldots, j_1)$ in place of $(k_r, \ldots, k_1)$.

LEMMA 5.3. *Let* $T_L < \delta'$. *Then* $p_{j_s j_{s-1} \cdots j_1}(x_L, x_{L-1}, \ldots, x_{j_s+1}, x_{j_s}, x_{j_{s-1}}, \ldots, x_{j_1}, x_0)$ *is a function of only* $(x_{j_s}, x_{j_{s-1}}, \ldots, x_{j_1}, x_0)$, *i.e.*, $p_{j_s j_{s-1} \cdots j_1}$ *is independent of* x_k, $k \geq j_s + 1$. *It is of the form*

$$p_{j_s j_{s-1} \cdots j_1}(x_L, x_{L-1}, \ldots, x_{j_s+1}, x_{j_s}, x_{j_{s-1}}, \ldots, x_{j_1}, x_0) = \prod_{r=1}^{s} \nu^{-1} t_{j_r} T(j_r - 1, j_{r-1}+1) p'_{j_r}(x_{j_r}, x_{j_{r-1}}) \tag{5.26}$$

where for any α, β,

$$|\partial_{j_r}^{\alpha} \partial_{j_{r-1}}^{\beta} p'_{j_r}(x_{j_r}, x_{j_{r-1}})| \leq C_{\alpha\beta}.$$

Here the constants $C_{\alpha\beta}$ *depend only on* α, β.

We note here that Lemma 5.3 differs from Fujiwara [5, Lemma 5.1] in the power of $T(j_r - 1, j_{r-1}+1)$; our power is 1 while his is 2. However, we shall be able to prove Lemma 5.3 in the same way as there. We only indicate here one different point. Namely, we have by Lemma 3.10

$$D(x_{j-1}, x_0)^{-1/2} = 1 + T(j-1, 1) q_{j-1}(x_{j-1}, x_0),$$

for some $q_{j-1}(x_{j-1}, x_0) \in \mathscr{B}(\mathbf{R} \times \mathbf{R})$, where the power of $T(j-1, 1)$ is 1, not 2.

The proof of Theorem 2 will also proceed in the same way as in [5, §5]. We have

$$\begin{aligned} b_{j_s j_{s-1} \cdots j_1} &= S_{L-1} S_{L-1} \cdots S_{j_s+1} p_{j_s j_{s-1} \cdots j_1} \\ &= D(x_L, x_{j_s})^{-1/2} p_{j_s j_{s-1} \cdots j_1}, \end{aligned}$$

where if $j_s = L$ or $L-1$, then $D(x_L, x_{j_s}) = 1$. So we combine Lemma 5.3 with (5.14a, b) to obtain that if α_0, $\alpha_L \leq m$,

$$\begin{aligned} &|\partial_L^{\alpha_L} \partial_0^{\alpha_0} a_{j_s j_{s-1} \cdots j_1}(x_L, x_0)| \\ &\leq C_m^s \max \sup_{x_{j_u}, u=1, \cdots, s} |\partial_L^{\beta_L} \prod_{r=1}^{s} \partial_{j_r}^{\beta_{j_r}} \partial_0^{\beta_0} \\ &\quad \times D(x_L, x_{j_s})^{-1/2} \prod_{r=1}^{s} \nu^{-1} t_{j_r} T(j_r - 1, j_{r-1}+1) p'_{j_r}(x_{j_r}, x_{j_{r-1}})| \end{aligned}$$

$$\leq C_{m,1}^{s} \prod_{r=1}^{s} (\nu^{-1} t_{j_r} T(j_r - 1, j_{r-1} + 1)).$$

Therefore, from (5.15) we have

$$| \partial_L^{\alpha_L} \partial_0^{\alpha_0} r(x_L, x_0) | \leq \Sigma' \prod_{r=1}^{s} (C_{m,2} \nu^{-1} T_L) t_{j_r}$$
$$\leq \prod_{j=1}^{L} (1 + C_{m,2} \nu^{-1} T_L t_j) - 1.$$

This is the estimate (1.11) of Theorem 2.

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