ON THE STRUCTURE OF LOCAL COHOMOLOGY MODULES
FOR MONOMIAL CURVES IN P_K^3

H. BRESINSKY, F. CURTIS, M. FIORENTINI AND L. T. HOA

1. Introduction

Our setting for this paper is projective 3-space P_K^3 over an algebraically closed field K. By a curve $C \subset P_K^3$ is meant a 1-dimensional, equidimensional projective algebraic set, which is locally Cohen-Macaulay. Let $M(C) = \bigoplus_{n \in \mathbf{Z}} H^1(P_K^3, \mathscr{I}_C(n))$ be the Hartshorne-Rao module of finite length (cf. [R]). Here $\mathbf{Z}$ is the set of integers and $\mathscr{I}_C$ the ideal sheaf of C. In [GMV] it is shown that $M(C) \cong H_{\underline{m}}^1(\bar{R})$, where $\bar{R} = R/I(C) = K[x_0, \ldots, x_3]/I(C)$, $I(C)$ is the homogeneous ideal of C, $\underline{m} = (x_0, \ldots, x_3)R$ and $H_{\underline{m}}^1(M)$ is the first local cohomology module of the R-module M with respect to $\underline{m}$. Thus there exists a smallest nonnegative integer $k \in \mathbf{N}$ such that $\underline{m}^k H_{\underline{m}}^1(\bar{R}) = 0$, (see also the discussion on the 1-st local cohomology module in [GW]). Also in [GMV] it is shown that $k = 0$ if and only if C is arithmetically Cohen-Macaulay and C is arithmetically Buchsbaum if and only if $k \leq 1$. We therefore have the following natural definition.

DEFINITION 1.1. For a curve $C \subseteq P_K^3$, C is said to be strictly k-Buchsbaum if k is minimal in $\mathbf{N}$ such that $\underline{m}^k H_{\underline{m}}^1(\bar{R}) = 0$. C is said to be k-Buchsbaum if $\underline{m}^k H_{\underline{m}}^1(\bar{R}) = 0$.

If C is strictly k-Buchsbaum, then we set $k = k(C)$ and call $k(C)$ the Buchsbaum number of C.

It is our purpose in this paper to investigate for the class of monomial curves $C(n_1, n_2, n_3) \subset P_K^3$ the integer $k(C(n_1, n_2, n_3))$. These curves are defined by their generic zero $(s^{n_3}, s^{n_3-n_1}t^{n_1}, s^{n_3-n_2}t^{n_2}, t^{n_3})$, where $n_1 < n_2 < n_3$ are positive integers and g.c.d.$(n_1, n_2, n_3) = 1$. For some of these curves $k(C(n_1, n_2, n_3))$ was obtained in [FH], [H] and [FV] and we will discuss some of these results as consequences of out own investigations (see also [HV] and [MM]).

Our own main result is that $k(C) = \operatorname{diam}(M(C))$ for all monomial curves in

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P_K^3. Here $\mathrm{diam}(M)$, the diameter of a **Z**-graded module M of finite length, is explained as follows. Let $[M]_n$ be the elements in M of degree n, let $a(M) = \min\{n\,;\,[M]_n \neq 0\}$, $e(M) = \max\{n\,;\,[M]_n \neq 0\}$. Then $\mathrm{diam}(M) = e(M) - a(M) + 1$, which, since M is of finite length, is an integer. For curves $C \subseteq P_K^3$, $H_{\underline{m}}^1(\bar{R})$ is a **Z**-graded module of finite length and therefore always $k(C) \leq \mathrm{diam}(H_{\underline{m}}^1(\bar{R}))$. Thus our result implies that for all monomial curves $C(n_1, n_2, n_3) \subseteq P_K^3$, $k(C(n_1, n_2, n_3))$ is, in this sense, as large as possible. We prove this result in Section 2. We will use there mainly the language of semigroups to describe $H_{\underline{m}}^1(\bar{R})$. For this let **N** be the set of nonnegative integers, $n_0 = 0$ and $t_i = (n_3 - n_i, n_i) \in \mathbf{N}^2$, $0 \leq i \leq 3$. Let $S \subseteq \mathbf{N}^2$ be the semigroup generated by $I = \{t_i\,;\,0 \leq i \leq 3\}$ and $H = \{(\alpha, \beta) \in \mathbf{N}^2\,;\,\alpha + \beta \equiv 0 \bmod n_3\}$. Let $S_0 = \langle n_3 - n_2, n_3 - n_1, n_3 \rangle = \{z\,;\,z = \sum_{i=0}^2 z_i(n_3 - n_i),\,z_i \in \mathbf{N}\}$, $S_1 = \langle n_1, n_2, n_3 \rangle$. For $i \in \{0, 1\}$ we define $A_i(S_i, n_3) = \{0, \omega_i(1), \ldots, \omega_i(n_3 - 1)\}$, $\omega_i(j) = \min\{z\,;\,z \in S_i,\,z \equiv j \bmod n_3\}$, to be the Apery sequence of S_i with respect to n_3. We set $S' = \{e \in H\,;\,e + m_0 t_0 \in S,\,e + m_1 t_3 \in S,\,\text{for some } m_0, m_1 \in \mathbf{N}\}$. By identifying $\bar{R}$ with the graded ring $K[S]$, we then have by [B1], [TH] and [FH] the following lemma.

LEMMA 1.1.

(i) $H_{\underline{m}}^1(K[S]) = K[S' \setminus S]$, where $\underline{m}$ is identified with $K[S \setminus (0)]$.

(ii) $S \subseteq S' = H \cap (S_0 \times S_1)$.

(iii) $K[S]$ is a k-Buchsbaum ring if and only if one of the following equivalent conditions holds:

    a) $S' + k(S \setminus (0)) \subseteq S$.

    b) For $J = H \cap (A_0(S_0, n_3) \times A_1(S_1, n_3)) \setminus S$, $J + kI \subseteq S$.

(*In both* a) *and* b) k *times a set means the set added to itself* k *times.*)

In Section 3 we relate our result in Section 2 to the algorithm in [BR], which obtains a minimal generating set $\mathscr{B}(n_1, n_2, n_3)$ of binomials for $I(C(n_1, n_2, n_3))$. This enables us to calculate $k(C(n_1, n_2, n_3))$ by considering a subset of $\mathscr{B}(n_1, n_2, n_3)$. For this we need to relate a minimal generating set of $H_{\underline{m}}^1(\bar{R})$ to $\mathscr{B}(n_1, n_2, n_3)$ (see Lemma 6 in [BSS]).

In Section 4 we show that the Castelnuovo-Mumford regularity for $\bar{R} = R/I(C(n_1, n_2, n_3))$, $\mathrm{reg}(\bar{R}) = e(H_{\underline{m}}^1(\bar{R})) + 1$ if $\bar{R}$ is not Cohen-Macaulay.

Section 5 establishes that for the subset of non Cohen-Macaulay monomial curves $C(1 = n_1, n_2, n_3)$, $k(C(n_1, n_2, n_3)) = n_2 - 2$.

We conclude our paper with Section 6 which deals with liaison among monomial curves in P_K^3. It is shown that no two curves $C(1, n_2, n_3)$, which are not arithmetically Cohen-Macaulay, are in the same even linkage class. Our main re-

sult in this section is that for a fixed nonnegative integer k there are only finitely many linkage classes among strictly k-Buchsbaum monomial curves in P_K^3.

2. The equality of $k(C(n_1, n_2, n_3))$ and $\mathrm{diam}(H_{\underline{m}}^1(R/I(C(n_1, n_2, n_3)))$

We need to state some more definitions and prove some preliminary lemmas and propositions. We will use the notation and terminology introduced in 1.

DEFINITION 2.1. For $j \in S_1$ let the S_1-degree of j be

$$\delta_1(j) = \min\{a_1 + a_2 + a_3 ; j = a_1 n_1 + a_2 n_2 + a_3 n_3, a_i \in \mathbf{N}\}.$$

Analogously we define the S_0-degree $\delta_0(i)$ for $i \in S_0$.

For $(i, j) \in H$, we define its degree to be the number $\delta(i, j) = (i + j)/n_3 \geq 0$.

DEFINITION 2.2. For (i_1, j_1) and (i_2, j_2) in S' we define the following partial orders:
(i) $(i_1, j_1) \leq_S (i_2, j_2)$ if $(i_2 - i_1, j_2 - j_1) \in S$,
(ii) $(i_1, j_1) \leq_{S'} (i_2, j_2)$ if $(i_2 - i_1, j_2 - j_1) \in S'$.

$(i, j) \in S' \setminus S$ is an S-maximal (minimal) element, if (i, j) is maximal (minimal) in $S' \setminus S$ with respect to the partial order $\leq_S$. $(i, j) \in S' \setminus S$ is an S'-minimal element, if (i, j) is minimal in $S' \setminus S$ with respect to the partial order $\leq_{S'}$.

LEMMA 2.1 ([B1], [K]). *Let $(i, j) \in S'$. The following are equivalent:*
(i) $(i, j) \in S$.
(ii) $\delta(i, j) \geq \delta_1(j)$.
(iii) $\delta(i, j) \geq \delta_0(i)$.

Proof. (i) $\Rightarrow$ (ii). Assume $(i, j) \in S$, $(i, j) = \sum_{i=0}^3 a_i t_i$. Then $\delta(i, j) = \sum_{i=0}^3 a_i \geq a_1 + a_2 + a_3 \geq \delta_1(j)$.

(ii) $\Rightarrow$ (i). Assume $\delta(i, j) \geq \delta_1(j)$, let $j = a_1 n_1 + a_2 n_2 + a_3 n_3$ with $a_1 + a_2 + a_3 = \delta_1(j)$. Let $a_0 = \delta(i, j) - (a_1 + a_2 + a_3)$ and $(i_1, j_1) = \sum_{i=0}^3 a_i t_i$. Then $j_1 = j$, $\delta(i_1, j_1) = \delta(i, j)$, thus $i_1 = i$ and $(i, j) \in S$.

(i) $\Leftrightarrow$ (iii) follows by symmetry as in the preceding. (Symmetry here and

throughout means switching from one of S_0 or S_1 to the other.)

Remark 2.1. The following elementary facts will be used in the sequel.

a) If $\delta_1(a_1 n_1 + a_2 n_2 + a_3 n_3) = a_1 + a_2 + a_3$ and $b_i \leq a_i$, $1 \leq i \leq 3$, then $\delta_1(b_1 n_1 + b_2 n_2 + b_3 n_3) = b_1 + b_2 + b_3$. Otherwise $\delta_1(a_1 n_1 + a_2 n_2 + a_3 n_3) \neq a_1 + a_2 + a_3$.

b) If $n_1 < n_2 < n_3$ are positive integers, then the following are true for non-negative integers a, b, c :

(i) If $a \neq 0$ and $a n_1 = b n_2 + c n_3$ then $a > b + c$,

(ii) If $c \neq 0$ and $c n_3 = a n_1 + b n_2$ then $c < a + b$.

LEMMA 2.2. *Let* $(i, j) \in S' \setminus (0)$. *The following are equivalent*:

(i) (i, j) is an S-minimal element of $S' \setminus S$.

(ii) $i = a(n_3 - n_2)$, $j = b n_1$ and $\delta_0(i) = a$, $\delta_1(j) = b$.

Proof. (i) $\Rightarrow$ (ii). Any element in S' can, by subtracting or adding multiples of $t_0, \ldots, t_3$, be changed into an element of the form $(a(n_3 - n_2), b n_1)$ or $(a(n_3 - n_1), b n_2)$. For an element $(a(n_3 - n_1), b n_2)$, $\delta(a(n_3 - n_1), b n_2) = a + (b n_2 - a n_1)/n_3 \geq a$ if $b n_2 - a n_1 \geq 0$. For $b n_2 - a n_1 = c n_3$, $c < 0$, from $b n_2 + (-c)n_3 = a n_1$ and Remark 2.1 b), $a > b + (-c)$ or $a + c > b$. Thus in any case by Lemma 2.1 $(a(n_3 - n_1), b n_2) \in S$. Now the S-minimality of (i, j) implies (ii).

(ii) $\Rightarrow$ (i). $\delta(i, j) = \delta(a(n_3 - n_2), b n_1) = a + (b n_1 - a n_2)/n_3$, $b n_1 - a n_2 = c n_3$ and $c \geq 0$ would imply by Remark 2.1 $\delta_1(j) \neq b$. Thus $c < 0$ and $\delta(i, j) < a = \delta_0(i)$, from which $(i, j) \notin S$ by Lemma 2.1. S-minimality now follows by observing that if $(i', j') = (a'(n_3 - n_2), b' n_1)$, $a' \leq a$, $b' \leq b$, then $(i', j') \not\leq_S (i, j)$.

Remark 2.2. a) For another proof of Lemma 2.2 see Lemma 6 in [BSS]. From that Lemma 6 we also obtain that if $\{x_1^{\alpha_1} - x_0^{\alpha_{10}} x_2^{\alpha_{12}} x_3^{\alpha_{13}}, x_2^{\alpha_0} - x_0^{\alpha_{00}} x_1^{\alpha_{01}} x_3^{\alpha_{03}}\} \subseteq \mathscr{B}(n_1, n_2, n_3)$, then $a < \alpha_0$ and $b < \alpha_1$, where a and b are as in Lemma 2.2.

b) Note that if (i, j) is S'-minimal, then (i, j) is also S-minimal.

LEMMA 2.3. *Let* $(i, j) = (a(n_3 - n_2), b n_1)$ *be an* S-*minimal element of* $S' \setminus S$. *Let* $c = (b(n_3 - n_1) - a(n_3 - n_2) - n_3)/n_3$. *Then* c *is the largest nonnegative integer such that* $(i, j) + c t_0 \notin S$. *Similarly, if* $d = (a n_2 - b n_1 - n_3)/n_3$, *then* d *is the largest nonnegative integer such that* $(i, j) + d t_3 \notin S$.

Proof. $(i, j) + ct_0 = (b(n_3 - n_1) - n_3, bn_1)$, thus $\delta((i, j) + (ct_0) = b - 1 < b = \delta_1(bn_1)$ and therefore $(i, j) + ct_0 \notin S$ by Lemma 2.1. As $(i, j) + (c + 1)t_0 = bt_1$, c is maximal. Proof of the second statement is obtained by symmetry.

Remark 2.3. In the sequel integers c_i and d_i will be defined for $(a_i(n_3 - n_2), b_i n_1)$ as were c and d for $(a(n_3 - n_2), bn_1)$.

LEMMA 2.4. *Let* $(i, j) = (a(n_3 - n_3), bn_1)$ *be an* S'-*minimal element of* $S' \setminus S$ (*see Remark* 2.2 *b*), *and let* c *and* d *be as in Lemma* 2.3. *Then* $(i, j) + ct_0 + dt_3 \notin S$. *Moreover,* $\delta_1(bn_1 + dn_3) = b + d$, *and* $\delta_0(a(n_3 - n_2) + cn_3) = a + c$.

Proof. $(i, j) + ct_0 + dt_3 = (b(n_3 - n_1) - n_3, bn_1 + dn_3)$, thus by Lemma 2.1, it suffices to show $\delta_1(bn_1 + dn_3) = b + d$, since the corresponding δ-degree is $b - 1 + d$. Suppose $\delta_1(bn_1 + dn_3) < b + d$, and let $bn_1 + dn_3 = \alpha_1 n_1 + \alpha_2 n_2 + \alpha_3 n_3$ with

(*) $$\alpha_1 + \alpha_2 + \alpha_3 < b + d.$$

By Lemma 2.2 $\delta_1(bn_1) = b$, hence $d > \alpha_3$, since otherwise (*) is contradicted, thus $bn_1 + (d - \alpha_3) = \alpha_1 n_1 + \alpha_2 n_2$. $\alpha_1 \geq b$ implies $d - \alpha_3 < \alpha_1 - b + \alpha_2$ by Remark 2.1 b), which contradicts (*). Thus $\alpha_1 < b$ and $\alpha_2 > 0$. As $bn_1 + dn_3 = an_2 - n_3$ by definition of d, $\alpha_2 < a$. But then by a straightforward calculation $(i, j) + dt_3 - \alpha_2 t_2 - \alpha_3 t_3 = ((a - \alpha_2)(n_3 - n_2), \alpha_1 n_1) <_{S'} (i, j)$, contrary to S'-minimality of (i, j). The proof that $\delta_0(a(n_3 - n_2) + cn_3) = a + c$ follows by symmetry.

Let $G' = \{(a_i(n_3 - n_2), b_i n_1) ; 1 \leq i \leq r\}$ be the S'-minimal elements of $S' \setminus S$, where $a_1 < a_2 < \cdots < a_r$ and $b_1 > b_2 > \cdots > b_r$. Let a_{r+1} be the smallest positive integer satisfying $\delta_0(a_{r+1}(n_3 - n_2)) < a_{r+1}$ and b_0 the smallest positive integer satisfying $\delta_1(b_0 n_1) < b_0$. For each i, $1 \leq i \leq r$, let $c_i = (b_i(n_3 - n_1) - a_i(n_3 - n_2) - n_3)/n_3$ and $d_i = (a_i n_2 - b_i n_1 - n_3)/n_3$.

Remark 2.4. Let $\bar{R} = R/I(C(n_1, n_2, n_3))$. By [BSS], $H_{\underline{m}}^1(\bar{R}) \cong (\bar{R}_{\bar{x}_0} \cap \bar{R}_{\bar{x}_3})/\bar{R}$ and the elements in G' correspond to the elements $\dfrac{\bar{x}_2^{a_i}}{\bar{x}_3^{d_i+1}} \equiv \dfrac{\bar{x}_1^{b_i}}{\bar{x}_0^{c_i+1}}$, obtained from the elements $x_0^{c_i+1} x_2^{a_i} - x_1^{b_i} x_3^{d_i+1}$ in $\mathscr{B}(n_1, n_2, n_3)$. Furthermore $a_{r+1} = \alpha_0$ and $b_0 = \alpha_1$, α_0 and α_1 defined in Remark 2.2 a). Also the S-minimal elements of $S' \setminus S$ correspond to products of elements $\dfrac{\bar{x}_2^{a_i}}{\bar{x}_3^{d_i+1}}$.

LEMMA 2.5. *For* $1 \leq i \leq r$, $\delta_1(b_i n_1 + (a_{i+1} - a_i - 1)n_2) = b_i + a_{i+1} - a_i - 1$, *and* $\delta_0(a_i(n_3 - n_2) + (b_{i-1} - b_i - 1)(n_3 - n_1)) = a_i + b_{i-1} - b_i - 1$.

Proof. Suppose we have

(*) $b_i n_1 + (a_{i+1} - a_i - 1)n_2 = \alpha_1 n_1 + \alpha_2 n_2 + \alpha_3 n_3$,

such that

$$\alpha_1 + \alpha_2 + \alpha_3 < b_i + \alpha_{i+1} - a_i - 1.$$

By Lemma 2.2 $a_{i+1} - a_i - 1 > \alpha_2$ (otherwise the minimality of b_i is contradicted).

Case 1. $\alpha_1 \geq b_i$. Starting with (*) a straightforward calculation gives $(a_{i+1} - a_i - 1 - \alpha_2)(n_3 - n_2) = (b_i + a_{i+1} - a_i - 1 - \alpha_1 - \alpha_2 - \alpha_3)n_3 + (\alpha_1 - b_i)(n_3 - n_1)$. By the definition of a_{r+1}, $\delta_0((a_{i+1} - a_i - 1 - \alpha_2)(n_3 - n_2)) = a_{i+1} - a_i - 1 - \alpha_2$, which contradicts Remark 2.1 b).

Case 2. $\alpha_1 < b_i$. Then $(a_i(n_3 - n_2), b_i n_1) + (a_{i+1} - a_i - 1)t_2 - \alpha_2 t_2 - \alpha_3 t_3 = ((a_{i+1} - 1 - \alpha_2)(n_3 - n_2), \alpha_1 n_1)$ would be an S-minimal element of $S' \setminus S$ by Lemma 2.2. But $a_i < a_{i+1} - 1 - \alpha_2 < a_{i+1}$ and $\alpha_1 < b_i$ thus such an element is neither S'-minimal nor comparable to any S'-minimal element by definition of G', a contradiction. Therefore $\delta_1(b_i n_1 + (a_{i+1} - a_i - 1)n_2) = b_i + a_{i+1} - a_i - 1$.

Proof of the other statement follows by symmetry.

LEMMA 2.6. *For* $1 \leq i \leq r$, *assume* $0 \leq e < a_{i+1} - a_i$, $0 \leq f < b_{i-1} - b_i$, *and that there is no equation* $\beta_1 n_1 + \beta_2 n_2 = \beta_3 n_3$ *with* $\beta_3 > 0$, $0 \leq \beta_1 \leq b_i + f$, $0 \leq \beta_2 \leq e$. *Then* $\delta_1((b_i + f)n_1 + en_2) = b_i + f + e$ *and* $\delta_0(f(n_3 - n_1) + (a_i + e)(n_3 - n_2)) = a_i + e + f$.

Proof. Suppose $(b_i + f)n_1 + en_2 = \alpha_1 n_1 + \alpha_2 n_2 + \alpha_3 n_3$ with $\alpha_1 + \alpha_2 + \alpha_3 < b_i + e + f$. If $\alpha_2 \geq e$, then $\delta_1((b_i + f)n_1) < b_i + f < b_{i-1}$ would follow from the inequality, a contradiction to the definition of b_0. Thus $\alpha_2 < e$. Similarly $\alpha_1 < b_i + f$ since $0 \leq e < a_{i+1} - a_i < a_{i+1} \leq a_{r+1}$. Then $(b_i + f - \alpha_1)n_1 + (e - \alpha_2)n_2 = \alpha_3 n_3$ contradicts the hypothesis. Hence $\delta_1((b_i + f)n_1 + en_2) = b_i + f + e$.

Suppose $f(n_3 - n_1) + (a_i + e)(n_3 - n_2) = \alpha_1(n_3 - n_1) + \alpha_2(n_3 - n_2) + \alpha_3 n_3$ with $\alpha_1 + \alpha_2 + \alpha_3 < a_i + e + f$. As before, we would have $f > \alpha_1$. If $e \leq \alpha_2$, then $\delta_0(f(n_3 - n_1) + a_i(n_3 - n_2)) < f + a_i$ would contradict Lemma 2.5. Thus $e > \alpha_2$. Then S' contains $(a_i(n_3 - n_2), b_i n_1) + ft_1 + et_2 - \alpha_1 t_1 - \alpha_2 t_2 = (\alpha_3 n_3, (b_i + f - \alpha_1)n_1 + (e - \alpha_2)n_2)$, which contradicts the hypothesis. Therefore

$$\delta_0(f(n_3 - n_1) + (a_i + e)(n_3 - n_2)) = a_i + e + f.$$

LEMMA 2.7. *For* $1 \le i \le r$,
(i) $\delta_1(b_i n_1 + (a_{i+1} - a_i - 1)n_2 + d_i n_3) = b_i + a_{i+1} - a_i - 1 + d_i$, *and*
(ii) $\delta_0(a_i(n_3 - n_2) + (b_{i-1} - b_i - 1)(n_3 - n_1) + c_i n_3) = a_i + b_{i-1} - b_i - 1 + c_i$.

Proof. Suppose that $b_i n_1 + (a_{i+1} - a_i - 1)n_2 + d_i n_3 = \alpha_1 n_1 + \alpha_2 n_2 + \alpha_3 n_3$ with $\alpha_1 + \alpha_2 + \alpha_3 < b_i + a_{i+1} - a_i - 1 + d_i$. By Lemma 2.5 $d_i > \alpha_3$. Suppose $\alpha_2 \le a_{i+1} - a_i - 1$. Then $(a_{i+1} - a_i - 1 - \alpha_2)n_2 + (d_i - \alpha_3)n_3 = (\alpha_1 - b_i)n_1$ and $\alpha_1 - b_i < (a_{i+1} - a_i - 1 - \alpha_2) + d_i - \alpha_3$ by the initially assumed inequality, a contradiction to Remark 2.1b). Thus $a_{i+1} - a_i - 1 < \alpha_2$ and $b_i n_1 + d_i n_3 = \alpha_1 n_1 + (\alpha_2 - a_{i+1} + a_i + 1)n_2 + \alpha_3 n_3$, which is contrary to the second part of Lemma 2.4. Therefore (i) holds and the proof of (ii) follows by symmetry.

LEMMA 2.8. *For* $1 \le i \le r$, $\delta_0((a_{i+1} - 1)(n_3 - n_2) + c_i n_3) = a_{i+1} - 1 + c_i$ *and* $\delta_1((b_{i-1} - 1)n_1 + d_i n_3) = b_{i-1} - 1 + d_i$. *Moreover each of* $(a_i(n_3 - n_2), b_i n_1) + (a_{i+1} - a_i - 1)t_2 + c_i t_0 + d_i t_3$, $(a_i(n_3 - n_2), b_i n_1) + (b_{i-1} - b_i - 1)t_1 + c_i t_0 + d_i t_3$ *is in* $S' \setminus S$.

Proof. By substituting for $d_i n_3$ we obtain $\delta((a_i(n_3 - n_2), b_i n_1) + (b_{i-1} - b_i - 1)t_1 + c_i t_0 + d_i t_3) = (a_i(n_3 - n_2) + b_i n_1 + (b_{i-1} - b_i - 1 + c_i)n_3 + a_i n_2 - b_i n_1 - n_3)/n_3 = a_i + b_{i-1} - b_i - 1 + c_i - 1$. It follows by Lemma 2.7 (ii) and Lemma 2.1 that this element is not in S. But then, substituting for $c_i n_3$, $\delta((a_i(n_3 - n_2), b_i n_1) + (b_{i-1} - b_i - 1)t_1 + c_i t_0 + d_i t_3) = (a_i(n_3 - n_2) + b_i n_1 + (b_{i-1} - b_i - 1 + d_i)n_3 + b_i(n_3 - n_1) - d_i(n_3 - n_2) - n_3)/n_3 = b_{i-1} - 2 + d_i$. Thus $\delta_1((b_{i-1} - 1)n_1 + d_i n_3) > b_{i-1} - 2 + d_i$ by Lemma 2.1. Since $\delta_1((b_{i-1} - 1)n_1 + d_i n_3) \le b_{i-1} - 1 + d_i$, equality holds.
Proof of the other two statements follows by symmetry.

PROPOSITION 2.1. *Let* $1 \le i \le r$. *Assume* $0 \le e < a_{i+1} - a_i$, $0 \le f < b_{i-1} - b_i$ *and that there is no equation* $\beta_1 n_1 + \beta_2 n_2 = \beta_3 n_3$ *with* $\beta_3 > 0$, $0 \le \beta_1 \le b_i + f$, $0 \le \beta_2 \le e$. *Then* $(a_i(n_3 - n_2), b_i n_1) + f t_1 + e t_2 + c_i t_0 + d_i t_3$ *is in* $S' \setminus S$.

Proof. Suppose $(b_i + f)n_1 + e n_2 + d_i n_3 = \alpha_1 n_1 + \alpha_2 n_2 + \alpha_3 n_3$ and $\alpha_1 + \alpha_2 + \alpha_3 < b_i + f + e + d_i$. Lemma 2.6 implies $d_i > \alpha_3$ and thus $e < \alpha_2$ by Remark 2.1 b). But then $(b_i + f)n_1 + d_i n_3 = \alpha_1 n_1 + (\alpha_2 - e)n_2 + \alpha_3 n_3$, which contradicts Lemma 2.8. Therefore $\delta_1((b_i + f)n_1 + e n_2 + d_i n_3) = b_i + f + e + d_i$. Then,

substituting for $c_i n_3$, one obtains $\delta((a_i(n_3 - n_2), b_i n_1) + f t_1 + e t_2 + c_i t_0 + d_i t_3) = (a_i(n_3 - n_2) + b_i n_1 + (f + e + d_i)n_3 + b_i(n_3 - n_2) - a_i(n_3 - n_2) - n_3)/n_3 = b_i + f + e + d_i - 1$, from which the conclusion follows by Lemma 2.1.

LEMMA 2.9. *Assume that* $(a(n_3 - n_2), bn_1)$ *is an* S-*minimal element of* $S' \setminus S$ *which is not* S'-*minimal. Let* $c = (b(n_3 - n_2) - an_1 - n_3)/n_3$ *and* $d = (an_2 - bn_1 - n_3)/n_3$. *Then either* $\delta_1(bn_1 + dn_3) < b + d$ *or* $\delta_0(a(n_3 - n_2) + cn_3) < a + c$.

Proof. By symmetry we can assume without loss of generality $a \leq b$. (This will imply the first of the two stated inequalities, the other follows from $a > b$.) By S-minimality assume $(a(n_3 - n_2), bn_1) = \sum_{j=1}^{t} (a_{i(j)}(n_3 - n_2), b_{i(j)}n_1)$, where each summand is an S'-minimal element of $S' \setminus S$ and $t \geq 2$. $\sum_{j=1}^{t} a_{i(j)} = a \leq b = \sum_{j=1}^{t} b_{i(j)}$ implies that there exists an h, $1 \leq h \leq t$, such that $a_{i(h)} \leq b_{i(h)}$ (otherwise $a > b$). W.l.o.g. let $h = 1$. Then $bn_1 + dn_3 = bn_1 + an_2 - bn_1 - n_3 = an_2 - n_3 = \sum_{j=1}^{t} a_{i(j)}n_2 - n_3 = \sum_{j=2}^{t} b_{i(j)}n_1 + (\sum_{j=1}^{t} a_{i(j)}n_2 - \sum_{j=2}^{t} b_{i(j)}n_1) - n_3 = a_{i(1)}n_2 + \sum_{j=2}^{t} b_{i(j)}n_1 + \sum_{j=2}^{t}(a_{i(j)}n_2 - b_{i(j)}n_1 - n_3) + (t-1)n_3 - n_3 = a_{i(1)}n_2 + \sum_{j=2}^{t} b_{i(j)}n_1 + (t - 2 + \sum_{j=2}^{t} d_{i(j)})n_3$. The definition of b_0 and $b_{i(j)}$, $1 \leq j \leq t$, implies $d > (t-2) + \sum_{j=2}^{t} d_{i(j)}$. From this and $b_{i(1)} \geq a_{i(1)}$ (thus $b \geq a_{i(1)} + \sum_{j=2}^{t} b_{i(j)}$), $\delta_1(bn_1 + dn_3) < b + d$ is obtained.

LEMMA 2.10. *Let* (A, B) *be an* S-*maximal element of* $S' \setminus S$, *where* $A = A_1(n_3 - n_1) + A_2(n_3 - n_2) + A_3 n_3$, $B = B_1 n_1 + B_2 n_2 + B_3 n_3$ *with* $\delta_0(A) = \sum_{i=1}^{3} A_i$, $\delta_1(B) = \sum_{i=1}^{3} B_i$. *Then for some* i, $1 \leq i \leq r$, $A_2 - B_2 = a_i$, $B_1 - A_1 = b_i$, $A_3 = c_i$, *and* $B_3 = d_i$.

Proof. By Lemma 2.1 $\delta_1(B) > \delta(A, B)$. If $\delta_1(B) > \delta(A, B) + 1$, then $\delta_1(B) > \delta((A, B) + t_0)$, and then $(A, B) <_S (A, B) + t_0 \in S'$, contrary to S-maximality. Thus $B_1 + B_2 + B_3 = \delta_1(B) = \delta(A, B) + 1$. Then $(B_1 + B_2 + B_3 - 1)n_3 = \delta(A, B)n_3 = A_1(n_3 - n_1) + A_2(n_3 - n_2) + A_3 n_3 + B_1 n_1 + B_2 n_2 + B_3 n_3$, which implies $A_1(n_3 - n_1) + A_2(n_3 - n_2) + A_3 n_3 = B_1(n_3 - n_1) + B_2(n_3 - n_2) - n_3$. Thus $A_3 = ((B_1 - A_1)(n_3 - n_1) - (A_2 - B_2)(n_3 - n_2) - n_3)/n_3$. By symmetry, $B_3 = ((A_2 - B_2)n_2 - (B_1 - A_1)n_1 - n_3)/n_3$. Next consider the following element of $S' \setminus S$: $(A, B) - A_3 t_0 - B_3 t_3 = (A_1(n_3 - n_1) + A_2(n_3 - n_2), B_1 n_1 + B_2 n_2)$. Suppose $A_1 \geq B_1$. Then $((A_1 - B_1)(n_3 - n_1) + A_2(n_3 - n_2), B_2 n_2) \leq_S (A, B)$. $A_2 \geq B_2$ immediately implies $(A, B) \in S$, contrary to hypothesis. If $A_2 < B_2$ then $((A_1 - B_1)(n_3 - n_1), (B_2 - A_2)n_2) \in S$ by Remark 2.1 b) and Lemma 2.1 since $\delta_0(A) = A_1 + A_2 + A_3$. But then again $(A, B) \in S$ contrary to assumption. Thus $A_1 < B_1$. By symmetry $B_2 < A_2$. Hence, by Remark 2.1 a), we must

have $((A_2 - B_2)(n_3 - n_2), (B_1 - A_1)n_1) \in S'$ with $\delta((A_2 - B_2)(n_3 - n_2)) = A_2 - B_2$ and $\delta_1((B_1 - A_1)n_1) = B_1 - A_1$. By Lemma 2.2 $((A_2 - B_2)(n_3 - n_2), (B_1 - A_1)n_1)$ is an S-minimal element of $S' \setminus S$. But $\delta_0((A_2 - B_2)(n_3 - n_2) + A_3 n_3) = A_2 - B_2 + A_3$ and $\delta_1((B_1 - A_1)n_1 + B_3 n_3) = B_1 - A_1 + B_3$, thus by Lemma 2.9, $((A_2 - B_2)(n_3 - n_2), (B_1 - A_1)n_1)$ is an S'-minimal element of $S' \setminus S$. Therefore for some i, $1 \le i \le r$, $A_2 - B_2 = a_i$, and $B_1 - A_1 = b_i$. $A_3 = c_i$ and $B_3 = d_i$ now follow from the definition of c_i and d_i and the calculations for A_3 and B_3 above.

PROPOSITION 2.2. *Let* (A, B) *be an* S-*maximal element of* $S' \setminus S$. *Then* $(A, B) = (a_i(n_3 - n_2), b_i n_1) + ft_1 + et_2 + c_i t_0 + d_i t_3$ *where*:
(i) $(a_i(n_3 - n_2), b_i n_1)$ *is an* S'-*minimal element of* $S' \setminus S$.
(ii) c_i *and* d_i *are as defined before, namely* $c_i = (b_i(n_3 - n_1) - a_i(n_3 - n_2) - n_3)/n_3$, $d_i = (a_i n_2 - b_i n_1 - n_3)/n_3$.
(iii) $0 \le e < a_{i+1} - a_i$, $0 \le f < b_{i-1} - b_i$ *and there is no equation* $\beta_1 n_1 + \beta_2 n_2 = \beta_3 n_3$, $0 \le \beta_1 \le b_i + f$, $0 \le \beta_2 \le e$.
Thus every S-*maximal element is one of the elements in Proposition 2.1.*

Proof. We use the notation of Lemma 2.10. Thus for some i, $1 \le i \le r$, $(A, B) = (a_i(n_3 - n_2), b_i n_1) + A_1 t_1 + B_2 t_2 + c_i t_0 + d_i t_3$. It suffices to show that A_1 and B_2 satisfy the conditions for f and e in (iii) above. Suppose $A_1 \ge b_{i-1} - b_i$. Consider $(a_i(n_3 - n_2), b_i n_1) + (b_{i-1} - b_i)(n_3 - n_1, n_1) = (a_i(n_3 - n_2) + (b_{i-1} - b_i)(n_3 - n_1), b_{i-1} n_1) \in S' \setminus S$.

Case 1. $i = 1$. Then $\delta_1(b_{i-1} n_1) = \delta_1(b_0 n_1) < b_0$ and $\delta_1(B_1 n_1 + B_2 n_2 + B_3 n_3) < B_1 + B_2 + B_3$ since $B_1 = B_1 - A_1 + A_1 = b_1 + A_1 \ge b_0$, contrary to assumption.

Case 2. $2 \le i \le r$. Since $(a_{i-1}(n_3 - n_2), b_{i-1} n_1) \in S' \setminus S$ and $a_{i-1} < a_i$, $(a_{i-1}(n_3 - n_2), b_{i-1} n_1) <_S (a_i(n_3 - n_2) + (b_{i-1} - b_i)(n_3 - n_1), b_{i-1} n_1)$. Thus $a_i(n_3 - n_2) + (b_{i-1} - b_i)(n_3 - n_1) = a_{i-1}(n_3 - n_2) + g n_3$, $g > 0$. As $a_{i-1} < a_i$, $\delta_0(a_i(n_3 - n_2) + (b_{i-1} - b_i)(n_3 - n_3)) < a_i + b_{i-1} - b_i$ and $\delta_0(A_1(n_3 - n_1) + A_2(n_3 - n_2) + A_3 n_3) < A_1 + A_2 + A_3$ since $A_2 \ge a_i$, which is contrary to assumption. Therefore $A_1 < b_{i-1} - b_i$ and by symmetry $B_2 < a_{i+1} - a_i$. Finally if $\beta_1 n_1 + \beta_2 n_2 = \beta_3 n_3$ with $\beta_3 > 0$, $0 \le \beta_1 \le b_1 + A_1 = B_1$, $0 \le \beta_2 \le B_2$, then $\delta_1(B_1 n_1 + B_2 n_2 + B_3 n_3) < B_1 + B_2 + B_3$, a contradiction to the choice of B_i, $1 \le i \le 3$. Hence there is no such equation.

PROPOSITION 2.3. *Let* (A, B) *be an* S-*maximal element in* $S' \setminus S$ *and* (A_0, B_0) *an* S'-*minimal element of* $S' \setminus S$. *Then* $(A_0, B_0) \le_S (A, B)$.

Proof. By Proposition 2.2 let $(A, B) = (a_i(n_3 - n_2), b_i n_1) + ft_1 + et_2 + c_i t_0 + d_i t_3$ and let $(A_0, B_0) = (a_j(n_3 - n_2), b_j n_1)$. If $i = j$ there is nothing to prove. Assume $i \neq j$. By symmetry we can assume $b_j < b_i$ and $a_j > a_i$ (for $b_j > b_i$ we use $a_j < a_i$). Then $(A_0, B_0) + (b_i - b_j)(n_3 - n_1, n_1) = (a_j(n_3 - n_2) + (b_i - b_j)(n_3 - n_1), b_i n_1) \in S'$. $(a_i(n_3 - n_2), b_i n_1) \in S'$ implies $(a_j - a_i)(n_3 - n_2) + (b_i - b_j)(n_3 - n_1) = gn_3$, $g > 0$, thus $a_j(n_3 - n_2) + (b_i - b_j)(n_3 - n_1) = a_i(n_3 - n_2) + gn_3$. Thus $(a_j(n_3 - n_2) + (b_i - b_j)(n_3 - n_1), b_i n_1) = (a_i(n_3 - n_2) + gn_3, b_i n_1)$ for some $g > 0$. If $(a_i(n_3 - n_2) + gn_3, b_i n_1) \in S$, then $g > c_i$ by Lemma 2.3. Then $a_j(n_3 - n_2) + (b_i - b_j)(n_3 - n_1) \geq a_i(n_3 - n_2) + (c_i + 1)n_3 = b_i(n_3 - n_1)$ (by definition of c_i). From this $a_j(n_3 - n_2) \geq b_j(n_3 - n_1)$, from which $(a_j(n_3 - n_2), b_j n_1) \in S$ by Lemma 2.1 since then $a_j + (- a_j n_2 + b_j n_1)/n_3 \geq b_j$. But this is contrary to assumption. Hence $(a_i(n_3 - n_2) + gn_3, b_i n_1) \notin S$ and $g \leq c_i$. Therefore $(A_0, B_0) \leq_S (A_0, B_0) + (b_i - b_j)(n_3 - n_1, n_1) \leq_S (a_i(n_3 - n_2), b_i n_1) + c_i t_0 \leq_S (A, B)$, which finishes the proof.

We can now state and prove the following.

THEOREM 2.1. *For a monomial curve $C(n_1, n_2, n_3)$ in P_K^3 we have $k(C(n_1, n_2, n_3)) = \operatorname{diam}(H_m^1(\bar{R}))$. Furthermore an element of minimal degree in $H_m^1(\bar{R})$ must occur amongst the S'-minimal elements, thus by Remark 2.4, amongst the generators of $H_m^1(\bar{R})$ obtained from elements $x_0^{c_i+1} x_2^{a_i} - x_1^{b_i} x_3^{d_i+1} \in \mathcal{B}(n_1, n_2, n_3)$. An element of maximal degree is obtained by considering all possible extensions of S'-minimal elements to S-maximal elements.*

Proof. The first statement follows immediately from Proposition 2.3, since (A, B) and (A_0, B_0) are arbitrary in their respective sets. The second statement is a consequence of the correspondence between S'-minimal elements of $S' \setminus S$ and generators of $H_m^1(\bar{R})$ as specified by Remark 2.4. The third statement follows immediately from Proposition 2.3.

3. An algorithm for computing $k(C(n_1, n_2, n_3))$

This section requires a somewhat more detailed study of the minimal generating set $\mathcal{B} = \mathcal{B}(n_1, n_2, n_3)$ of $I(C)$ obtained in [BR] (see also [B2] and [BH]). For the convenience of the reader not familiar with the algorithm of [BR], we also include a brief description of it at the end of this section when we compute $k(C(n_1, n_2, n_3))$. Let

$$\mathcal{B}_2 = \{\beta_2; \beta_2 = x_0^{a_0} x_2^{a_2} - x_1^{a_1} x_3^{a_3} \in \mathcal{B}(n_1, n_2, n_3), \text{ and } a_i > 0, 0 \leq i \leq 3\}.$$

From elements in $\mathscr{B}_2$, $\dfrac{x_2^{a_2}}{x_3^{a_3}} \equiv \dfrac{x_1^{a_1}}{x_0^{a_0}} \in H_{\underline{m}}^1(\bar{R})$ are obtained and these elements correspond to the S'-minimal elements of $S' \setminus S$, S and S' defined in Section 2. (By abuse of notation we shall delete bars on variables.) Let

$$\mathscr{B}_3 = \{\beta_3 \, ; \, \beta_3 = x_0^{b_0} x_3^{b_3} - x_1^{b_1} x_2^{b_2} \in \mathscr{B}(n_1, n_2, n_3), \text{ and } b_i > 0, \, 0 \leq i \leq 3\}.$$

Then $\mathscr{B}(n_1, n_2, n_3) = \mathscr{B}_2 \cup \mathscr{B}_3 \cup \{F_1 = x_1^{\alpha_1} - x_0^{\alpha_{10}} x_2^{\alpha_{12}} x_3^{\alpha_{13}}, \ F_2 = x_2^{a_0} - x_0^{\alpha_{00}} x_1^{\alpha_{01}} x_3^{\alpha_{03}}\}$ and every binomial $\beta \in \mathscr{B}(n_1, n_2, n_3)$, except for the first two binomials in the algorithm of [BR], is obtained from binomials β' and β'' by cross-multiplying the monomial terms of β' with the monomial terms of β'' and deleting common factors of the resulting two monomials. We will write for this $\beta''(\beta') = \beta$ and say β'' acts on β' to produce β.

DEFINITION 3.1. Let $\{i, j\} = \{2, 3\}$, $\beta_j \in \mathscr{B}_j$. The set

$$B(\beta_j) = \{\beta_i \, ; \, \beta_j(\bar{\beta}_i) = \beta_i, \ \{\beta_i, \bar{\beta}_i\} \subseteq \mathscr{B}_i\} \cup \{\bar{\beta}_i \, ; \, \beta_j(\bar{\beta}_i) = \beta_i, \ \{\beta_i, \bar{\beta}_i\} \subseteq \mathscr{B}_i\},$$

will be called a block in $\mathscr{B}_i$.

Ordering the elements in $\mathscr{B}_i$, $i = 1,2$ by decreasing x_0 and x_1 exponents (and therefore increasing x_2 and x_3 exponents) establishes by [BR] a linear order on each block and on $\mathscr{B}_i$.

DEFINITION 3.2. $\beta_i \in B(\beta_j)$ of largest or smallest x_0 and x_1 exponent are called end polynomials. All other $\beta_i \in B(\beta_j)$ are called middle polynomials.

DEFINITION 3.3. Let $x_0^{a_0} x_2^{a_2} - x_1^{a_1} x_3^{a_3} \in \mathscr{B}_2$, X a monomial term in R, $0 \neq X \dfrac{x_2^{a_2}}{x_3^{a_3}} \in H_{\underline{m}}^1(\bar{R})$. Then $d\left(\dfrac{x_2^{a_2}}{x_3^{a_3}}\right) = d(X) + a_2 + a_3$, where $d(X)$ is the ordinary total degree of a monomial term.

Remark 3.1. Note that $d\left(\dfrac{x_2^{a_2}}{x_3^{a_3}}\right) = \delta(a_2(n_3 - n_2), a_1 n_1)$ and if t_i and x_i correspond, $0 \leq i \leq 3$, then d and δ agree.

DEFINITION 3.4. For $x_0^{a_0} x_2^{a_2} - x_1^{a_1} x_3^{a_3} \in \mathscr{B}_2$ and X a monomial term in R, $0 \neq X \dfrac{x_2^{a_2}}{x_3^{a_3}} \in H_{\underline{m}}^1(\bar{R})$ is a maximal multiple of $\dfrac{x_2^{a_2}}{x_3^{a_3}}$ if $x_i\left(X \dfrac{x_2^{a_2}}{x_3^{a_3}}\right) = 0$ in $H_{\underline{m}}^1(\bar{R})$, $0 \leq i \leq 3$.

Remark 3.2. Note that if $\beta_2 = x_0^{a_0} x_2^{a_2} - x_1^{a_1} x_3^{a_3}$ is a middle polynomial in $\mathscr{B}_2$, then there exists a uniquely determined polynomial $x_0^{b_0} x_3^{b_3} - x_1^{b_1} x_2^{b_2} \in \mathscr{B}_3$, which acts on β_2 to produce the successor of β_2 in $\mathscr{B}_2$. If β_2 is an end polynomial in $\mathscr{B}_2$, but neither the first nor the last polynomial in $\mathscr{B}_2$, then $B(\beta_2) \neq \emptyset$ and if $B(\beta_2) = \{f_1 = x_0^{b_0} x_3^{b_3} - x_1^{b_1} x_2^{b_2}, \cdots, f_l = x_0^{c_0} x_3^{c_3} - x_1^{c_1} x_2^{c_2}\}$ with f_1 the first and f_l the last polynomial in $B(\beta_2)$, then f_1 acts on the predecessor of β_2 to produce β_2 and f_l acts on β_2 to produce the successor of β_2 in $\mathscr{B}_2$. If β_2 is the first or last polynomial in $\mathscr{B}_2$, then $B(\beta_2)$ may be empty. If for this case $B(\beta_2)$ is empty, then let $x_0^{b_0} x_3^{b_3} - x_1^{b_1} x_2^{b_2} = x_0^{c_0} x_3^{c_3} - x_1^{c_1} x_2^{c_2}$ be the first or the last polynomial in $\mathscr{B}_3$, respectively.

Now let $m = x_0^{\gamma_0} x_3^{\gamma_3} \dfrac{x_2^{a_2}}{x_3^{a_3}} x_1^{\gamma_1} x_2^{\gamma_2}$ be an arbitrary maximal multiple. Assume $\dfrac{x_2^{a_2}}{x_3^{a_3}}$ corresponds to a middle polynomial. If $\gamma_1 \geq b_1$, b_1 defined in Remark 3.2, then, since by the algorithm of [BR] $a_2 > b_2$,

$$m = x_0^{\gamma_0} x_3^{\gamma_3} \frac{x_2^{a_2}}{x_3^{a_3}} x_1^{\gamma_1} x_2^{\gamma_2} = x_0^{\gamma_0 + b_0} x_3^{\gamma_3} \frac{x_2^{a_2 - b_2}}{x_3^{a_3 - b_3}} x_1^{\gamma_1 - b_1} x_2^{\gamma_2},$$

where $\dfrac{x_2^{a_2 - b_2}}{x_3^{a_3 - b_3}}$ corresponds to the predecessor of $\dfrac{x_2^{a_2}}{x_3^{a_3}}$. Note that the x_2-exponent is unchanged. If $\gamma_2 \geq b_2$, then by the exactly analogous procedure, with $\dfrac{x_2^{a_2}}{x_3^{a_3}}$ represented by $\dfrac{x_1^{a_1}}{x_0^{a_0}}$, we obtain a reduction of γ_2 by b_2 and a shift to the successor of $\dfrac{x_2^{a_2}}{x_3^{a_3}}$. Now let $\dfrac{x_2^{a_2}}{x_3^{a_3}}$ correspond to an end polynomial. If $\gamma_1 \geq b_1$, again b_1 defined in Remark 3.2, then as above $a_2 > b_2$, and we have a reduction in m of the x_1-exponent by b_1 and a shift to the predecessor. Note that if $\dfrac{x_2^{a_2}}{x_3^{a_3}}$ cprresponds to the first element in $\mathscr{B}_2$, then $\gamma_1 \geq b_1$ is not possible, since this would imply $m = 0$. If $\gamma_2 \geq c_2$, c_2 defined in Remark 3.2 then as before we obtain a reduction of γ_2 in m by c_2 and a shift to the successor. Note again that if $\dfrac{x_2^{a_2}}{x_3^{a_3}}$ corresponds to the last polynomial in $\mathscr{B}_2$, then this is not possible.

Since this procedure leaves one of the two exponents γ_i, $i = 1,2$, invariant, while reducing the other we get a reduction of maximal multiples to maximal multiples to one of two possible cases:

I. $m^* = x_0^{\sigma_0} x_3^{\sigma_3} \dfrac{x_2^{a_2}}{x_3^{a_3}} x_1^{\sigma_1} x_2^{\sigma_2}, \dfrac{x_2^{a_2}}{x_3^{a_3}}$ corresponds to a middle polynomial and

$\sigma_1 \le b_1 - 1$, $\sigma_2 \le b_2 - 1$, b_1, b_2 defined in Remark 3.2.

II. $m^* = x_0^{\sigma_0} x_3^{\sigma_3} \dfrac{x_2^{a_2}}{x_3^{a_3}} x_1^{\sigma_1} x_2^{\sigma_2}, \dfrac{x_2^{a_2}}{x_3^{a_3}}$ corresponds to an end polynomial and

$\sigma_1 \le b_1 - 1$, $\sigma_2 \le c_2 - 1$, b_1, c_2 defined in Remark 3.2.

We will deal with the two cases separately.

LEMMA 3.1. *Let m^* be as in I. above. Then $\sigma_0 = a_0 - 1$, $\sigma_3 = a_3 - 1$, $\sigma_1 = b_1 - 1$, $\sigma_2 = b_2 - 1$.*

Proof. Clearly $\sigma_0 \le a_0 - 1$ and $\sigma_3 \le a_3 - 1$. Suppose $\sigma_3 < a_3 - 1$ or $0 < a_3 - \sigma_3 - 1$. Then $\dfrac{x_0^{\sigma_0} x_2^{a_2 + \sigma_2} x_1^{\sigma_1}}{x_3^{a_3 - \sigma_3 - 1}} = 0$, thus there must exist a monomial of a binomial in $\mathscr{B}_2$ dividing $x_0^{\sigma_0} x_2^{a_2 + \sigma_2}$ or a monomial of a binomial in $\mathscr{B}_3$ dividing $x_2^{a_2 + \sigma_2} x_1^{\sigma_1}$. In the first case since $\sigma_0 \le a_0 - 1$, this would produce an x_3-exponent $> a_3$ (since the x_3-exponents are monotonically increasing with decreasing x_0-exponents), thus $m^* = 0$, which it is not. Hence this is not possible. In the second case, since $\sigma_1 \le b_1 - 1$, by the algorithm in [BR], again the x_3-exponent produced is $> a_3$, thus $m^* = 0$, a contradiction. Hence $\sigma_3 = a_3 - 1$. Changing to the representation $\dfrac{x_1^{a_1}}{x_0^{a_0}}$, we obtain $\sigma_0 = a_0 - 1$ by an analogous argument. We consider next $x_0^{a_0 - 1} \dfrac{x_2^{a_2}}{x_3}$ $x_1^{\sigma_1} x_2^{\sigma_2}$, $\sigma_1 \le b_1 - 1$, $\sigma_2 \le b_2 - 1$. Then by the algorithm of [BR] and the middle polynomial assumption, there does not exist a binomial in $\mathscr{B}_3$ with monomial term dividing $x_1^{\sigma_1} x_2^{\sigma_2 + a_2 + 1}$, from which $\sigma_2 = b_2 - 1$. By an analogous argument with the representation $\dfrac{x_1^{a_1}}{x_0^{a_0}}$, $\sigma_1 = b_1 - 1$.

For the next three lemmata $\dfrac{x_2^{a_2}}{x_3^{a_3}}$ will correspond to an end polynomial.

LEMMA 3.2. *Let $\beta_2 = x_0^{a_0} x_2^{a_2} - x_1^{a_1} x_3^{a_3} \in \mathscr{B}_2$ be an end polynomial. If $B(\beta_2) \ne \emptyset$ let $f_1 = x_0^{b_0} x_3^{b_3} - x_1^{b_1} x_2^{b_2} \ne f_l = x_0^{c_0} x_3^{c_3} - x_1^{c_1} x_2^{c_2}$ as defined in Remark 3.2. If $B(\beta_2) = \emptyset$ (i.e. if β_2 is the first or last polynomial in $\mathscr{B}_2$ and the algorithm in [BR] proceeds such that $B(\beta_2) = \emptyset$), then let $f_1 = f_l$ be either the first or last polynomial in $\mathscr{B}_3$ (depending upon if β_2 is the first or last polynomial in $\mathscr{B}_2$). Then $m_1 = x_0^{a_0 - 1} x_3^{a_3 - 1}$.*

$$\frac{x_2^{a_2}}{x_3^{a_3}} x_1^{b_1-1} x_2^{b_2-1} \quad \text{and} \quad m_l = x_0^{a_0-1} x_3^{a_3-1} \frac{x_2^{a_2}}{x_3^{a_3}} x_1^{c_1-1} x_2^{c_2-1} \quad \text{are maximal multiples of } \frac{x_2^{a_2}}{x_3^{a_3}}.$$

Proof. Since there does not exist $\beta_i \in \mathcal{B}_i$, $i = 1, 2$, with one of its monomial terms dividing $x_0^{a_0-1} x_2^{a_2+c_2-1}$ or $x_1^{c_1-1} x_2^{a_2+c_2-1}$, $x_0^{a_0-1} \dfrac{x_2^{a_2}}{x_3} x_1^{c_1-1} x_2^{c_2-1} = x_3^{a_3-1} \dfrac{x_1^{a_1}}{x_0} \cdot x_1^{c_1-1} x_2^{c_2-1} \neq 0$ in $H_{\underline{m}}^1(\bar{R})$. Clearly x_0 and x_3 annihilate this multiple. That this is also the case for x_1 and x_2 follows from $x_0^{c_0} x_3^{c_3} - x_1^{c_1} x_2^{c_2} \in \mathcal{B}_3$. The proof for the other multiple is analogous.

LEMMA 3.3. *Let* $m^* = x_0^{\sigma_0} x_3^{\sigma_3} \dfrac{x_2^{a_2}}{x_3^{a_3}} x_1^{\sigma_1} x_2^{\sigma_2}$ *be as in II., i.e.* $\dfrac{x_2^{a_2}}{x_3^{a_3}}$ *corresponds to an end polynomial,* $\sigma_1 \leq b_1 - 1$, $\sigma_2 \leq c_2 - 1$, b_1, c_2 *as in Remark 3.2. Then* $\sigma_0 = a_0 - 1$ *and* $\sigma_3 = a_3 - 1$.

Proof. As always $\sigma_0 \leq a_0 - 1$ and $\sigma_3 \leq a_3 - 1$. Suppose $\sigma_3 < a_3 - 1$ or $0 < a_3 - \sigma_3 - 1$. Then $x_0^{\sigma_0} \dfrac{x_2^{a_2+\sigma_2}}{x_3^{a_3-\sigma_3-1}} x_1^{\sigma_1} = 0$. Since $\sigma_1 \leq b_1 - 1$, if $x_2^{a_2+\sigma_2} x_1^{\sigma_1}$ is to be divisible by a monomial term of a binomial in $\mathcal{B}_3$, its x_3-exponent is $> a_3$. But then $m^* = 0$, a contradiction. If $x_0^{\sigma_0} x_2^{a_2+\sigma_2}$ is to be divisible by a monomial term of a binomial in $\mathcal{B}_2$, then since $\sigma_0 \leq a_0 - 1$, its x_3-exponent again is $> a_3$. Thus again $m^* = 0$, a contradiction. Hence $\sigma_3 = a_3 - 1$. By replacing the representation $\dfrac{x_2^{a_2}}{x_3^{a_3}}$ by $\dfrac{x_1^{a_1}}{x_0^{a_0}}$ and an analogous argument, we obtain $\sigma_0 = a_0 - 1$.

LEMMA 3.4. *Let* $\dfrac{x_2^{a_2}}{x_3^{a_3}}$ *correspond to an end polynomial* $\beta_2 = x_0^{a_0} x_2^{a_2} - x_1^{a_1} x_3^{a_3}$. *Let*

$m = x_0^{a_0-1} x_3^{a_3-1} \dfrac{x_2^{a_2}}{x_3^{a_3}} x_1^{\sigma_1} x_2^{\sigma_2}$ *be a maximal multiple of largest possible degree. (Note that the degree of m here need not necessarily be $e(H_{\underline{m}}^1(\bar{R}))$. If $a_1 > a_2$, then $\sigma_1 = b_1 - 1$, $\sigma_2 = b_2 - 1$, if $a_1 < a_2$, then $\sigma_1 = c_1 - 1$, $\sigma_2 = c_2 - 1$ and if $a_1 = a_2$, then either of the preceding two choices is possible.*

Proof. Let $m = x_0^{a_0-1} \dfrac{x_2^{a_2+\sigma_2}}{x_3} x_1^{\sigma_1}$. We first determine the possibilities for σ_1 and σ_2. We consider two cases:

a) No monomial term of a binomial in $\mathcal{B}_3$ divides $x_2^{a_2+\sigma_2+1} x_1^{\sigma_1}$. Since $a_0 - 1 < a_0$ and $a_2 + \sigma_2 + 1 > a_2$, there must exist $x_0^{a_0-c_0} x_2^{a_2+c_2} - x_1^{a_1-c_1} x_3^{a_3+c_3} \in \mathcal{B}_2$,

$x_0^{c_0} x_3^{c_3} - x_1^{c_1} x_2^{c_2} \in \mathscr{B}_3$, and $a_2 + \sigma_2 + 1 = a_2 + c_2$, where c_i, $0 \le i \le 3$ are as defined in Remark 3.2. Hence $\sigma_2 = c_2 - 1$. By Lemma 3.2 $\sigma_1 = c_1 - 1$ now follows immediately.

b) There are monomial terms of binomials in $\mathscr{B}_3$, which divide $x_2^{a_2+\sigma_2+1} x_1^{\sigma_1}$. Since m is a maximal multiple and since the x_0 and x_1 variables do not appear in any monomial of $\mathscr{B}(n_1, n_2, n_3)$, we also must have that $x_2^{a_2+\sigma_2} x_1^{\sigma_1+1}$ is divisible by a monomial term of a binomial in $\mathscr{B}_3$. For $B(\beta_2) = \emptyset$, we revert back to case a). Let

$\emptyset \ne B(\beta_2) = \{x_0^{b_0} x_3^{b_3} - x_1^{b_1} x_2^{b_2}, \ldots, x_0^{c_0+a_0} x_3^{c_3-a_3} - x_1^{c_1+a_1} x_2^{c_2-a_2}, x_0^{c_0} x_3^{c_3} - x_1^{c_1} x_2^{c_2}\}$. In order for $x_2^{a_2+\sigma_2} x_1^{\sigma_1}$ to be as required, σ_1 has to be by one less than the x_1-exponent of an element in $\mathscr{B}_3$ and $a_2 + \sigma_2$ by one less than the x_2-exponent of its successor. From this we get $a_2 + \sigma_2 < c_2$ (otherwise $m = 0$), and by symmetry $a_1 + \sigma_1 < b_1$. Thus the maximal multiples to consider are

$$m_1 = x_0^{a_0-1} x_3^{a_3-1} \frac{x_2^{a_2}}{x_3^{a_3}} x_1^{b_1-1} x_2^{b_2-1},$$

$$m_2 = x_0^{a_0-1} x_3^{a_3-1} \frac{x_2^{a_2}}{x_3^{a_3}} x_1^{(b_1-1)-a_1} x_2^{(b_2-1)+a_2}, \ldots, m_{l-1} = x_0^{a_0-1} x_3^{a_3-1} \frac{x_2^{a_2}}{x_3^{a_3}} x_1^{(c_1-1)+a_1} x_2^{(c_2-1)-a_2}.$$

The statement about the degree now is immediate.

COROLLARY 3.1. *There exists an element of maximal degree amongst the maximal multiples of Lemma 3.1 and Lemma 3.2.*

Proof. This follows from the preceding, the fact that the initial maximal multiple $m = x_0^{r_0} x_3^{r_3} \frac{x_2^{a_2}}{x_3^{a_3}} x_1^{r_1} x_2^{r_2}$ was arbitrary and only homogeneous polynomials were used in all of the above (thus no change in degree), and by the structure theorem (Theorem 2.1) in Section 2.

LEMMA 3.5. *Assume that m_1 and m_2 are maximal multiples as in Lemma 3.1 and Lemma 3.2 which correspond to succesive binomials $b_1 = x_0^{a_0} x_2^{a_2} - x_1^{a_1} x_3^{a_3}$ and $b(b_1) = b_2$, $b = x_0^{d_0} x_3^{d_3} - x_1^{d_1} x_2^{d_2}$. Then $d(m_2) - d(m_1) = \Delta = d_2 - d_0$.*

Proof. The degree difference between $m_1 = x_0^{a_0-1} x_3^{a_3-1} \frac{x_2^{a_2}}{x_3^{a_3}} x_1^{d_1-1} x_2^{d_2-1}$ and m_2

$= x_0^{a_0-d_0-1} x_3^{a_3+d_3-1} \frac{x_2^{a_2+d_2}}{x_3^{a_3+d_3}} x_1^{d_1-1} x_2^{d_2-1}$ is $d_2 - d_1$.

THEOREM 3.1. *For a maximal multiple of maximal degree only the maximal multiples in Lemma 3.2 need be considered. If $\dfrac{x_2^{a_2}}{x_3^{a_3}}$ corresponds to the end polynomial $x_0^{a_0} x_2^{a_2} - x_1^{a_1} x_3^{a_3}$ and $a_1 > a_2$ then m_1 of Lemma 3.2 needs to be considered, if $a_1 < a_2$ then m_l of Lemma 3.2 will do and if $a_1 = a_2$, then $d(m_1) = d(m_l)$.*

Proof. This follows from Corollary 3.1, Lemma 3.5 and Lemma 3.4.

Remark 3.3. (i) It follows readily that an element of minimal degree in $H_m^1(\bar{R})$ is also obtained from an element $\dfrac{x_2^{a_2}}{x_3^{a_3}}$, which corresponds to an end polynomial in $\mathcal{B}_2$.

(ii) We note that our algorithmic procedure in this section calculates also the socle of $H_m^1(\bar{R})$.

We conclude this section with an algorithmic computation of $k(C(n_1, n_2, n_3))$.

1. Let α_i be minimal positive integers such that $\alpha_i n_i = \alpha_{ij} n_j + a_{ik} n_k \in \langle n_j, n_k \rangle$, $\{i, j, k\} = \{1,2,3\}$. These equations define polynomials

$$f_1 = x_1^{\alpha_1} - x_2^{\alpha_{12}} x_3^{\alpha_{13}}, \quad f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}}, \quad f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}}.$$

Assume that $0 < \alpha_2 < \alpha_{21} + \alpha_{23}$, $\alpha_{21} > 0$, and $0 < \alpha_3 < \alpha_{31} + \alpha_{32}$, with $\alpha_{31} > 0$. (If one of these conditions is not satisfied, then $C(n_1, n_2, n_3)$ is Cohen-Macaulay.) Let $\{i, j\} = \{2, 3\}$ and $\alpha_{i1} \leq \alpha_{j1}$. Then from f_2 and f_3 one obtains either $f_2(f_3)$ or $f_3(f_2)$ equal to:

$$f_{12} = x_j^{\alpha_j + \alpha_{ij}} - x_1^{\alpha_{j1} - \alpha_{i1}} x_i^{\alpha_{ji} + \alpha_i}.$$

We repeat the process as in the Euclidean algorithm for α_{21} and α_{31} until we obtain a polynomial $x_2^{\beta_2} - x_1^{\beta_{21}} x_3^{\beta_{23}}$ with $\beta_2 \geq \beta_{21} + \beta_{23}$.

Assume $f_2 \neq - f_1$ and $f_3 \neq - f_1$. Then we obtain:

$$\mathcal{B}^* = \{f_1, h_1 \in \{f_2, f_3\}, f_{11} \in \{f_2, f_3\}, f_{12}, \ldots, f_{1l_1} = h_2; \ldots; f_{k1}, \ldots,$$

$$f_{k,l_k} = h_{k+1}; \ldots; f_{r-1,1}, \ldots, f_{r-1,l_{r-1}} = h_r; f_{r1}, \ldots, f_{rl_r}\},$$

where $h_1(f_{11}) = f_{12}, \ldots, h_1(f_{1,l_1-1}) = h_2, h_2(h_1) = f_{21}, \ldots$

If $f_2 = - f_1$ or $f_3 = - f_1$ we delete f_2 or f_3 and obtain:

$$\mathcal{B}^* = \{f_1, h_1 \in \{f_2, f_3\}, f_{11}, \ldots, f_{1e_1} = h_2; \ldots; f_{r-1,1}, \ldots,$$

$$f_{r-1,e_{r-1}} = h_r; f_{r1}, \ldots, f_{re_r}\},$$

where $h_1(-f_1) = f_{11}, \ldots, h_1(f_{1,e_1-1}) = h_2, h_2(h_1) = f_{21}, \ldots$

In both cases, with the homogenized binomials of $\mathscr{B}^*$ denoted by capital letters and with corresponding subscripts, we have:

$$\mathscr{B}(n_1, n_2, n_3) = \{F_1, H_1, F_{11}, \ldots, F_{1n_1} = H_2; \ldots; F_{k1}, \ldots, F_{k,n_k} = H_{k+1}; \ldots;$$
$$F_{r-1,1}, \ldots, F_{r-1,n_{r-1}} = H_r; F_{r1}, \ldots, F_{rn_r}\}.$$

and

$$\mathscr{B}_2 = \begin{cases} \{F_{11}, \ldots, F_{1n_1}; F_{31}; \ldots, F_{3n_3}; \ldots; F_{r1}, \ldots, F_{r,n_r-1}\} & \text{if } r \text{ is odd,} \\ \{H_1, F_{21}, \ldots, F_{2n_2}; F_{41}, \ldots, F_{4n_4}; \ldots; F_{r1}, \ldots, F_{r,n_r-1}\} & \text{if } r \text{ is even.} \end{cases}$$

2. Let $d_0 = \deg(F_{11})$, $d_1 = \deg(H_1), \ldots, d_r = \deg(H_r)$, and

$$d_{r+1} = \begin{cases} \deg(F_{r,n_r-1}) & \text{if } n_r > 1, \\ d_{r-1} & \text{if } n_r = 1, \end{cases}$$

Let $\mathcal{M} = \max\{d_i + d_{i+1}; i = 0, \ldots, r\}$, and $m = \min\{a_2 - a_3; x_0^{a_0}x_2^{a_2} - x_1^{a_1}x_3^{a_3} \in \mathscr{B}_2\}$.

Then $k(C(n_1, n_2, n_3)) = \mathcal{M} - m - 3$.

4. Castelnuovo–Mumford regularity for monomial curves in P_K^3

Let $A = \bigoplus_{i \geq 0} A_i$ be a Noetherian graded standard K-algebra, i.e. $A = R/I$, where I is a homogeneous ideal of a polynomial ring $R = K[x_0, \ldots, x_n]$ in $n+1$ indeterminates. Let $\underline{m} = \bigoplus_{i>0} A_i$ be the homogeneous maximal ideal of A. Recall that the Castelnuovo-Mumford regularity of A is defined as follows:

$$\operatorname{reg} A = \max\{e(H_{\underline{m}}^i(A) + i; i \leq \dim A\},$$

where $e(H_{\underline{m}}^i(A)) = \max\{j; [H_{\underline{m}}^i(A)]_j \neq 0\}$. This is an important invariant of A. If A has the following minimal graded free resolution:

$$0 \to \bigoplus_{i=1}^{n_p} R(-e_{pi}) \to \cdots \to \bigoplus_{i=1}^{n_1} R(-e_{1i}) \to R \to A \to 0,$$

it is well-known that (see [M], [EG])

$$\operatorname{reg} A = \max\{e_{ji} - j; j = 1, \ldots, p \text{ and } 1 \leq i \leq n_j\}.$$

In this section we will prove:

THEOREM 4.1. *Assume that* $C = C(n_1, n_2, n_3)$ *is not Cohen-Macaulay* (C-M). *Then*

$$\mathrm{reg}(K[S]) = e(H_{\underline{m}}^1(K[S])) + 1.$$

As an immediate consequence of this theorem and Theorem 2.1 we have

COROLLARY 4.1. *If $C(n_1, n_2, n_3)$ is not C-M, then*

$$k(C(n_1, n_2, n_3)) = \mathrm{reg}(K[S]) - a(H_{\underline{m}}^1(K[S])),$$

where $a(H_{\underline{m}}^1(K[S]))$ is defined in Section 1.

The proof of Theorem 4.1 is divided into several lemmata. First, as in Section 1, for $i = 0,1$, the set $A_i(S_i, n_3) = \{0, \omega_i(1),\ldots, \omega_i(n_3 - 1)\}$, where

$$\omega_i(j) = \min\{a \in S_i\,;\, a \equiv j \bmod n_3\},\, j = 1,\ldots, n_3 - 1,$$

$H \cap (A_0(S_0, n_3) \times A_1(S_1, n_3)) = \{(\omega_0(i), \omega_1(n_3 - i))\,;\, i = 0,\ldots, n_3 - 1\} = \tilde{I}$, and

$$J = \tilde{I} \setminus S.$$

It is clear that if $\omega_1(i) = an_1 + bn_2$ then $\omega_0(n_3 - i) \leq a(n_3 - n_1) + b(n_3 - n_2)$ and the equality holds if and only if $a(n_3 - n_1) + b(n_3 - n_2) \notin n_3 + S_0$. Note that J generates $K[S' \setminus S]$ as $K[S]$-module (see [FH], Lemma 2.4). $C(n_1, n_2, n_3)$ is not C-M if and only if $J \neq \emptyset$.

Next we define α_1 (resp. β_1) to be the least number such that $\delta_1(\alpha_1 n_1) < \alpha_1$ (resp. $\beta_1 n_2 \in n_3 + S_1$). Analogously we define the corresponding numbers α_0, β_0 for the generators $n_3 - n_2 < n_3 - n_1 < n_3$ of S_0.

Remark 4.1. From the definition of α_i, β_i, $i = 0,1$, we get $\beta_0 \leq \alpha_1$ and $\beta_1 \leq \alpha_0$. These relations are also immediate consequences of the algorithm in [BR].

In the next lemma we use the following partial order $\leq$ on the set $\mathbf{N}^2$: $(a, b) \leq (a', b')$ if $a \leq a'$ and $b \leq b'$.

LEMMA 4.1. *If the set $\{(a, b)\,;\, 0 \leq a < \alpha_1,\, 0 < b \leq \beta_1$ and $an_1 + bn_2 = mn_3$ for some $m > 0\} \neq \emptyset$ then it has a unique minimal element.*

Proof. Assume that there are two equations: $m_1'n_1 + m_2'n_2 = m'n_3$ and $m_1 n_1 + m_2 n_2 = mn_3$ with $m_1' > m_1$ and $m_2' < m_2$ or $m_1' < m_1$ and $m_2' > m_2$. Subtracting one equation from the other we get $an_1 = bn_2 + cn_3$ or $bn_2 = an_1 + cn_3$, where $0 \leq a < \alpha_1,\, 0 \leq b \leq \beta_1$. By the minimality of α_1 and β_1, this can happen only if

$b = \beta_1$, i.e. one of the two numbers m_2 and m_2' is 0. But this contradicts the minimality of α_1.

If the set defined in Lemma 4.1 is not empty, we denote its minimal element by (m_1, m_2). Otherwise we set $m_1 = \alpha_1$ and $m_2 = \beta_1$. Let us introduce the following set:

$$B_1 = \{(a, b)\,;\, 0 \leq a < m_1,\ 0 \leq b < \beta_1\} \cup \{(a, b)\,;\, m_1 \leq a < \alpha_1,\ 0 \leq b < m_2\}.$$

Then we have:

LEMMA 4.2. *The set* $B_1' = \{an_1 + bn_2\,;\, (a, b) \in B_1\}$ *consists of distinct elements and coincides with the Apery sequence* $A_1(S_1, n_3)$ *of* S_1 *(up to permutations). Moreover, for each element* $an_1 + bn_2 \in B_1'$ *we have* $\delta_1(an_1 + bn_2) = a + b$.

Proof. Since $A_1(S_1, n_3) \subseteq (\mathbf{N}n_1 + \mathbf{N}n_2)\backslash(n_3 + S_1)$, we must have $A_1(S_1, n_3) \subseteq B_1'$. Let $(a, b) \in B_1$. Choose an element $a'n_1 + b'n_2 \in A_1(S_1, n_3)$ such that $(a', b') \in B_1$ and $an_1 + bn_2 = cn_3 + a'n_1 + b'n_2$. From the minimality of α_1 and β_1 it follows that $a \geq a'$, $b \geq b'$. Then we have $(a - a')n_1 + (b - b')n_2 = cn_3$. By Lemma 4.1, $(a - a', b - b') \geq (m_1, m_2)$ unless $a - a' = b - b' = c = 0$. Since $a < m_1$ or $b < m_2$, we must have $a = a'$, $b = b'$ and $c = 0$. This proves that the element $an_1 + bn_2 \notin (n_3 + S_1)$, i.e. $B_1' = A_1(S_1, n_3)$ and the elements in B_1' are distinct. From this it also follows that if $an_1 + bn_2 = cn_3 + a'n_1 + b'n_2$ for some $c, a', b' \in \mathbf{N}$, then $c = 0$ and $a + b \leq a' + b'$. (Note that $a + b < a' + b'$ is possible.) Hence $\delta_1(an_1 + bn_2) = a + b$, as required.

Remark 4.2. An equivalent formula to the one in Lemma 4.2 was given by Rödseth (see [Ro], pp. 175).

LEMMA 4.3. *Assume that* $(a, b) \in B_1$ *and* $a < \beta_0$. *Then* $a(n_3 - n_1) + b(n_3 - n_2) \in A_0(S_0, n_3)$.

Proof. Assume that $a(n_3 - n_1) + b(n_3 - n_2) = a'(n_3 - n_1) + b'(n_3 - n_2) + c'n_3$, where $c' > 0$ and $a', b' \in \mathbf{N}$. By Remark 2.1 (or [BR]), $b < \beta_1 \leq \alpha_0$, and by assumption $a < \beta_0$. From the minimality of α_0 and β_0 it follows that $(a, b) \geq (a', b')$. We then get: $(a - a')n_1 + (b - b')n_2 = (a - a' + b - b' - c')n_3$. By Lemma 4.1 we must have $(a - a', b - b') \geq (m_1, m_2)$ which implies that $(a, b) \geq (m_1, m_2)$, a contradiction.

LEMMA 4.4.   *If $C(n_1, n_2, n_3)$ is not C-M, then*

$$\max\{\delta_1(a) \; ; \; a \in A_1(S_1, n_3)\} \leq \max\{\delta_0(e_0), \; \delta_1(e_1) \; ; \; (e_0, e_1) \in J\}.$$

Proof.   First assume that $\beta_0 = \alpha_1$. Let $e = (\omega_0(n_3 - i), \omega_1(i))$ be an arbitrary element of $\tilde{I}$. We write $\omega_1(i) = an_1 + bn_2$ with $(a, b) \in B_1$. By Lemma 4.3, $a(n_3 - n_1) + b(n_3 - n_2) \in A_0(S_0, n_3)$. Hence $\omega_0(n_3 - i) = a(n_3 - n_1) + b(n_3 - n_2)$, which implies that $e \in S$. This shows that $\tilde{I} \subset S$, i.e. $C(n_1, n_2, n_3)$ is C-M, a contradiction. So, from Remark 4.1, we must have $\beta_0 < \alpha_1$, and, by symmetry, $\beta_1 < \alpha_0$.

Now we consider two cases separately, noting that $\delta(i, j)$ equals the number of steps between $-i$ and j in the congruence class n_3.

Case 1: $(m_1, m_2) = (\alpha_1, \beta_1)$.
By Lemma 4.2 there exists j such that $\omega_1(j) = (\alpha_1 - 1)n_1 + (\beta_1 - 1)n_2 \in A_1(S_1, n_3)$. Since $\alpha_1 - 1 \geq \beta_0$,

$$(\alpha_1 - 1)(n_3 - n_1) + (\beta_1 - 1)(n_3 - n_2) \in n_3 + S_0.$$

By the definition of Apery sequence, we then get $\omega_0(n_3 - j) < (\alpha_1 - 1)(n_3 - n_1) + (\beta_1 - 1)(n_3 - n_2)$. Therefore $\delta((\omega_0(n_3 - j), \omega_1(j))) < \delta((\alpha_1 - 1)(n_3 - n_1) + (\beta_1 - 1)(n_3 - n_2), \; (\alpha_1 - 1)n_1 + (\beta_1 - 1)n_2) = (\alpha_1 - 1) + (\beta_1 - 1) = \delta_1(\omega_1(j))$. Thus by Lemma 2.1, $(\omega_0(n_3 - j), \omega_1(j)) \notin S$. Hence

$$\max\{\delta_1(a) \; ; \; a \in A_1(S_1, n_3)\} \leq \alpha_1 - 1 + \beta_1 - 1 =$$
$$\delta_1(\omega_1(j)) \leq \max\{\delta_1(e_1) \; ; \; (e_0, e_1) \in J\}.$$

Case 2: $m_1 < \alpha_1$ and $m_2 \leq \beta_1$.
Similarly as in Case 1, $(\alpha_1 - 1)n_1 + (m_2 - 1)n_2$ is the second component of an element of J. Hence

$$(1) \qquad\qquad \max\{\delta_1(e_1) \; ; \; (e_0, e_1) \in J\} \geq (\alpha_1 - 1) + (m_2 - 1).$$

If $m_1 > \beta_0$ then considering $(m_1 - 1)n_1 + (\beta_1 - 1)n_2$ we get similarly as above that

$$(2) \qquad\qquad \max\{\delta_1(e_1) \; ; \; (e_0, e_1) \in J\} \geq (m_1 - 1) + (\beta_1 - 1).$$

(1), (2) and the construction of B_1 imply the claim.

Finally let $m_1 \leq \beta_0$. By Remark 2.1, $m_2 \leq \beta_1 < \alpha_0$. Note that $m_1 n_1 + m_2 n_2 = mn_3$ ($m > 0$) if and only if $m_1(n_3 - n_1) + m_2(n_3 - n_2) = m'n_3$ ($m' = m_1 + m_2 - m > 0$). This implies that in our case (m_2, m_1) is just the unique minimal element defined by Lemma 4.1 for $n_3 - n_2 < n_3 - n_1 < n_3$. Applying Lemma 4.2 for the

Apery sequence $A_0(S_0, n_3)$ we get that $A_0(S_0, n_3)$ is in one-to-one correspondence with the set

$$B_0 = \{(a, b) \ ; 0 \le a < m_2, \ 0 \le b < \beta_0\} \cup \{(a, b) \ ; m_2 \le a < \alpha_0, \ 0 \le b < m_1\}.$$

Since $\beta_1 < \alpha_0$, similarly as in Case 1 we can show that $(\alpha_0 - 1)(n_3 - n_2) + (m_1 - 1)(n_3 - n_1)$ is the first component of an element of J. Hence

$$(3) \qquad \max\{\delta_0(e_0) \ ; (e_0, e_1) \in J\} \ge \alpha_0 - 1 + m_1 - 1 > \beta_1 + m_1 - 2.$$

Combining (1) and (3) we get

$$\max\{\delta_0(e_0), \ \delta_1(e_1) \ ; (e_0, e_1) \in J\} \ge \max\{\alpha_1 + m_2 - 2, \ m_1 + \beta_1 - 2\}$$
$$= \max\{\delta_1(a) \ ; a \in B_1'\} = \max\{\delta_1(a) \ ; a \in A_1(S_1, n_3)\}, \text{ as required.}$$

Now we are able to prove Theorem 4.1.

Proof of Theorem 4.1. It suffices to prove that $e(H_{\underline{m}}^2(K[S])) \le e(H_{\underline{m}}^1(K[S])) - 1$. By [TH], Corollary 3.8, $H_{\underline{m}}^2(K[S]) \cong K[\mathbf{Z}(S) \backslash S_0 \cup S_1]$, where $\mathbf{Z}(S)$ is the additive subgroup generated by S. Hence

$$e(H_{\underline{m}}^2(K[S])) = \max\{\delta(\omega_0(n_3 - i) - n_3, \ \omega_1(i) - n_3)) \ ; 0 \le i < n_3\} = \max_{e \in \tilde{I}} \delta(e) - 2.$$

(See also Proposition 1 and Corollary 5 in [BSS].) Let $\omega_1(i) = an_1 + bn_2$, where $(a, b) \in B_1$. Since $\omega_0(n_3 - i) \le a(n_3 - n_1) + b(n_3 - n_2)$, it follows that $\delta(\omega_0(n_3 - i), \ \omega_1(i)) \le \delta(a(n_3 - n_1) + b(n_3 - n_2), \ \omega_1(i)) = a + b = \delta_1(\omega_1(i))$. Therefore

$$(4) \qquad e(H_{\underline{m}}^2(K[S])) = \max_{e \in \tilde{I}} \delta(e) - 2 \le \max_{a \in A_1(S_1, n_3)} \delta(a) - 2.$$

On the other hand, since $H_{\underline{m}}^1(K[S]) \cong K[S' \backslash S]$, $e(H_{\underline{m}}^1(K[S])) = \max_{e \in S' \backslash S} \delta(e)$. Let $e = (e_0, e_1) \in S' \backslash S$ be an arbitrary element. By Lemma 2.1 we get that $\delta_0(e_0) > \delta(e)$, $\delta_1(e_1) > \delta(e)$ and $e + (\delta_0(e_0) - \delta(e) - 1)t_0 \in S' \backslash S$, and $e + (\delta_1(e_1) - \delta(e) - 1)t_3 \in S' \backslash S$. As a consequence, $\max_{f \in S' \backslash S} \delta(f) \ge \max\{\delta_0(e_0), \delta_1(e_1)\} - 1$. Hence

$$(5) \qquad e(H_{\underline{m}}^1(K[S])) \ge \max_{e \in J} \{\delta_0(e_0), \delta_1(e_1)\} - 1.$$

Combining (4) (5) and Lemma 4.4 completes the proof of the theorem.

Analyzing the proof of Lemma 4.4 we get the following criterion for the Cohen-Macaulay property of $C(n_1, n_2, n_3)$.

COROLLARY 4.1.   *The following are equivalent*:
(i) $C(n_1, n_2, n_3)$ is *C-M*,
(ii) $\alpha_1 = \beta_0$,
(iii) $\alpha_0 = \beta_1$.

Proof.   We can use [B2] in order to give another proof as follows. By [B2] $C(n_1, n_2, n_3)$ is C-M precisely when $\mu(I(C(n_1, n_2, n_3))) \leq 3$, where μ denotes the minimum number of generators. Now from the algorithm in [BR] the equivalence of (i), (ii), and (iii) follows immediately.

In the next section we need the following.

EXAMPLE 4.1.   Let us consider the curve $C(1, a, d)$, i.e. $n_1 = 1$, $n_2 = a$ and $n_3 = d$. Let $d = pa + q$, where $0 \leq q < a$. Then $\beta_1 = p + 1$ if $q > 0$ and $\beta_1 = p$ if $q = 0$. From the equation $\alpha_0(d - a) = y(d - 1) + zd$, $y, z \in \mathbf{N}$, one can easily check that $\alpha_0 = \beta_1$ if and only if $a < p + q + 1$ or $q = 0$. Hence $C(1, a, d)$ is C-M if and only if $a < p + q + 1$ or $q = 0$.

5. Monomial curves with at most one singular point

In this section for notational convenience we denote n_2 by a and n_3 by d. We will consider the class of curves $C = C(1, a, d)$ and we compute $k(C(1, a, d))$ explicitly. From this it will become apparent, that to calculate $k(C)$ for a class of curves still is a formidable task. We always assume that C is not C-M. Let $d = pa + q$, where $0 \leq q < a$. By Example 4.1 our assumption on C says that $a \geq p + q + 1$ and $q > 0$.

LEMMA 5.1.   *Assume that $C(1, a, d)$ is not C-M. Then*
(i) *Every S-minimal element e of $S' \setminus S$ has the form $e = ((\alpha + \beta p)(d - a),$ $\alpha a - \beta q)$ for some positive integers $\alpha \leq \beta$ such that $1 < \alpha a - \beta q < a$.* *Moreover. $\delta(e) = \alpha + \beta(p - 1)$.*
(ii) $((p + 1)(d - a), a - q)$ *is an S-minimal element of $S' \setminus S$.*
(iii) *([FH], Lemma 5.3) $a(H_m^1(K[S])) = p$.*

Proof.   (i) From Lemma 2.2 we know that $e = (m(d - a), n)$ for some positive m, n with $n < a$ and $n - ma = -ld$, where $l > 0$. Hence $n = ma - l(pa + q) = (m - lp)a - lq$. Setting $\alpha = m - lp$ and $\beta = l$ we can easily get the assertions.

(ii) is immediate from Lemma 2.1 and the fact that $p + q + 1 \leq a$.

(iii) Since S-minimal elements of $S' \setminus S$ generate $K[S' \setminus S]$ as $K[S]$-module, $a(H_m^1(K[S])) = \min \delta(e)$, where e runs over the set of S-minimal elements of $S' \setminus S$. Hence (iii) follows from (i) and (ii).

The main result of this section is the following:

THEOREM 5.1. *Assume that $C(1, a, d)$ is not C-M. Then $k(C(1, a, d)) = \operatorname{diam}(M(C)) = a - 2$.*

The proof of this theorem is based on estimating degrees of elements in the numerical semigroup $S_1 = \langle 1, a, d \rangle = \mathbf{N}$ with respect to the generators $1, a, d$. Namely we need the following technical lemma.

LEMMA 5.2. *Assume that $p, q \geq 1$ and $p + q + 1 \leq a$. Then*

$$\delta_1((a - q) + l + na) \leq a + p - 2,$$

for any nonnegative integers l, n such that $l + n \leq a - 2$.

Having this lemma we can prove the main Theorem 5.1 as follows: By Lemma 5.1, $e = ((p + 1)(d - a), a - q)$ is an S'-minimal element of $S' \setminus S$ and $\delta(e) = p$. Let $f = kt_0 + lt_1 + nt_2 + mt_3$ be an arbitrary element of S with $\delta(f) = a - 2$. We have $e + t_3 = (p + 1)t_2 \in S$. Hence, if $m > 0$, $e + f \in S$. If $m = 0$, then the second component of $e + f$ is $[e + f]_1 = a - q + l + na$. By Lemma 5.2, $\delta_1([e + f]_1) \leq a + p - 2 = \delta(e) + \delta(f) = \delta(e + f)$. By Lemma 2.1 it follows that $e + f \in S$ too. Therefore $e + f \in S$ for any $f \in S$ with $\delta(f) \geq a - 2$. Let g be now an element of $S' \setminus S$ having the maximal degree. g is, of course, an S-maximal element of $S' \setminus S$. By Proposition 2.3, $g = e + h$ for some element $h \in S$. Since $g \notin S$, it follows from the above consideration that $\delta(h) < a - 2$. Hence $\delta(g) \leq a + p - 3$, and so $e(H_m^1(K[S])) \leq a + p - 3$. By Lemma 5.1 (iii) we then get $k(C(1, a, d)) \leq \operatorname{diam}(M(C)) \leq a - 2$. On the other hand, by [FH], Lemma 5.2, $k(C(1, a, d)) \geq a - 2$. Hence $k(C(1, a, d)) = \operatorname{diam}(M(C)) = a - 2$, as required.

From Theorem 5.1 and Theorem 4.1 we get

COROLLARY 5.1. $\operatorname{reg}(K[S]) = a + p - 2$.

The rest of this section is devoted to the proof of Lemma 5.2. The main idea is the following: Let $E = a - q + l + na$. We consider various representations of E and try to locate one with a small sum of coefficients. Here, under a representation of E we mean a sum $E = x \cdot 1 + y \cdot a + z \cdot d$ with nonnegative integral coefficients x, y, z.

We first define some nonnegative integers as follows:

$$a = \alpha q + q_1, \; 0 \leq q_1 < q, \text{ and}$$
$$\gamma q_1 = q + q_2, \; 0 \leq q_2 < q_1 \text{ (if } q_1 > 0).$$

Note that $\gamma \geq 2$ if $q_1 > 0$. We need to prove that

$$\delta_1(E) \leq a + p - 2.$$

CLAIM 1. *One can assume that*
i) $l \leq q - 1$,
ii) $n > (\alpha - 1)p$, *and*
iii) $q_1 > 0$.

Proof. i) If $l \geq q$, then $\delta_1(E) \leq n + l \leq a - 2$, since $E = (l - q) + (n + 1)a$ and all the coefficients in this equation are nonnegative.

ii) If $n \leq (\alpha - 1)p$, let $n = xp + y$, where $0 \leq x \leq \alpha - 1$ and $0 \leq y < p$. Then $E = xd + [(\alpha - 1 - x)q + q_1 + l] + ya$. Hence

$$\delta_1(E) \leq x + (\alpha - 1 - x)q + q_1 + l + y$$
$$\leq x + (\alpha - 1 - x)q + q_1 + q - 1 + p - 1 \text{ (by(i))}$$
$$\leq \alpha - 1 + p - 1.$$

iii) Assume that $q_1 = 0$, i.e., $a = \alpha q$. By ii) above we have $n' = n - (\alpha - 1)p > 0$. Then $E = (\alpha - 1)q + l + (\alpha - 1)pa + n'a = (\alpha - 1)d + l + n'a$. Hence $\delta_1(E) \leq l + n' + \alpha - 1 \leq l + n \leq a - 2$. Note that this case also follows easily from Section 3.

We now divide into cases.

Case 1. $n \leq \alpha p + (\gamma - 1)(1 + \alpha p) - 1$.

Case 1a: $\alpha \geq 2$ and $q_1 \geq 2p + 1$
Looking at the representation

(1) $$E = (\alpha - 1)d + (l + q_1) + [n - (\alpha - 1)p]a,$$

we get

$$(2) \qquad \delta_1(E) \leq q_1 + l + n - (\alpha - 1)(p - 1)$$
$$\leq q_1 + q - 1 + \alpha p + (\gamma - 1)(1 + \alpha p) - 1 - (\alpha - 1)(p - 1)$$
$$= a + p - 2 - \alpha[q - (\gamma - 1)p - 1] + (\gamma - 1) + q - 1.$$

Since $\alpha \geq 2$ it follows easily that it suffices to show that $2q \geq (2p + 1)(\gamma - 1) + q + 1$. But this is immediate from the following inequalities: $q - 1 = \gamma q_1 - q_2 - 1 \geq (\gamma - 1)q_1 \geq (\gamma - 1)(2p + 1)$ (here we use our hypothesis).

Case 1b: $\alpha = 1$, i.e. $a = q + q_1$.

Let i be the smallest integer such that

$$n - i(p + 1) + 1 < 0.$$

By the assumption of this case and by Claim 1(ii), $0 < i \leq \gamma$. Then the coefficient of a in the following equation is nonnegative:

$$E = (i - 1)d + [l + iq_1 - a] + [n - (i - 1)(p + 1) + 1]a.$$

As in the proof of Claim 1(i) from this equation we obtain $\delta_1(E) \leq a + p - 2$ if $l + iq_1 - a \geq 0$. Let $l + iq_1 - a < 0$. Using the representation (1)(with $\alpha = 1$) and the bounds on n and l we get $\delta_1(E) \leq l + n + q_1 \leq a - iq_1 - 1 + i(p + 1) - 2 + q_1 = a + p - 2 + (i - 1)(p + 1 - q_1) \leq a + p - 2$ (since $a = q + q_1 \geq p + q + 1$ and $i \geq 1$).

Case 1c: $\alpha \geq 2$ and $q_1 \leq 2p$.

Looking again at (2) it suffices to consider the case $q_1 \geq (\alpha - 1)(p - 1) + p + 1 \geq 2p$. Hence by combining with Case 1a we have still to consider the case $\alpha = 2$ and $q_1 = 2p$. We have

$$(3) \qquad E = 2id + [l + 2ip - q] + [n + 1 - i(1 + 2p)]a,$$

for arbitrary i. Similarly to Case 1b, we choose i to be the smallest integer such that

$$n + 1 - (i + 1)(1 + 2p) < 0.$$

By Claim 1(ii) and the assumption of our Case 1, $0 \leq i < \gamma \leq q$. After using the above equation for E in (3), we have still to consider the following case, namely

$$l + 2ip - q < 0.$$

Using the representation $E = 1 \cdot d + (l + 2p) + (n - p)a$ and the above bounds

for n, l, i we get $\delta_1(E) \leq n + l + p + 1 \leq (i + 1)(1 + 2p) - 2 + q - 2ip - 1 + p + 1 = a + p - 2 - q + i + 1 \leq a + p + 2$ (since $i < q$). This completes the proof of Case 1.

Case 2. $n \geq \alpha p + (\gamma - 1)(1 + \alpha p)$.

We have

$$
\begin{aligned}
E &= (\alpha - 1)d + (l + q_1) + [n - (\alpha - 1)p]a \\
&= (\alpha - 1)d + l + q_1 + \alpha q + q_1 + [n - (\alpha - 1)p - 1]a \\
&= [\alpha - 1 + \alpha]d + (l + 2q_1) + [n - (\alpha - 1)p - (1 + \alpha p)]a \\
&\qquad\qquad \cdots\cdots \\
&= [(\alpha - 1) + (\gamma - 2)\alpha]d + [l + (\gamma - 1)q_1] + \\
&\qquad\qquad\qquad\qquad [n - (\alpha - 1)p - (\gamma - 2)(1 + \alpha p)]a \\
&= [\alpha + (\gamma - 2)\alpha]d + [l - (q_1 - q_2)] + [n - \alpha p - (\gamma - 2)(1 + \alpha p)]a.
\end{aligned}
$$

Looking at the last expression of E, similarly as in the proof of Claim 1(i), we get

CLAIM 2. *One can assume that $l < q_1 - q_2$.*

Next we define a new sequence of representations of E. Continuing the above procedure, we get the following expression of E :

$$
E = [\alpha + (\gamma - 1)\alpha]d + (l + q_2) + [n - \alpha p - (\gamma - 1)(1 + \alpha p)]a.
$$

We set $A_1 = \alpha + (\gamma - 1)\alpha$, $B_1 = q_2$ and $C_1 = n - \alpha p - (\gamma - 1)(1 + \alpha p)$. Assume that we have found an i-th expression:

$$
E = A_i d + l + B_i + C_i a.
$$

We use again the above procedure in order to find a new one.

If $l + B_i - (q_1 - q_2) \geq 0$ then the $(i + 1)$-st expression of E, obtained by iterating

$$
\begin{aligned}
E &= A_i d + l + B_i + C_i a = A_i d + l + B_i + \alpha q + q_1 + (C_i - 1)a \\
&= (A_i + \alpha)d + (l + B_i + q_1) + (C_i - 1 - \alpha p)a = \cdots,
\end{aligned}
$$

is

$$
\begin{aligned}
E &= [A_i + (\gamma - 1)\alpha]d + l + B_i + (\gamma - 1)q_1 + [C_i - (\gamma - 1)(1 + \alpha p)]a \\
&= [A_i + 1 + (\gamma - 1)\alpha]d + [l + B_i - (q_1 - q_2)] + [C_i - p - (\gamma - 1)(1 + \alpha p)]a.
\end{aligned}
$$

That means, in this case we set $A_{i+1} = A_i + 1 + (\gamma - 1)\alpha$, $B_{i+1} = B_i - (q_1 - q_2)$ and $C_{i+1} = C_i - p - (\gamma - 1)(1 + \alpha p)$.

If $l + B_i < q_1 - q_2$ then we consider the following $(i + 1)$-st expression of E :

$$E = [A_i + \gamma\alpha]d + l + B_i + \gamma q_1 + [C_i - \gamma(1 + \alpha p)]a$$
$$= [A_i + 1 + \gamma\alpha]d + [l + B_i + q_2] + [C_i - p - \gamma(1 + \alpha p)]a.$$

So, in this case we set $A_{i+1} = A_i + 1 + \gamma\alpha$, $B_{i+1} = B_i + q_2$ and $C_{i+1} = C_i - p - \gamma(1 + \alpha p)$.

From the above procedure and by Claim 2 we immediately get:

CLAIM 3. *For all $i \geq 1$ we have $E = A_i d + l + B_i + C_i a$ with A_i, B_i, C_i being defined as follows*:
i)

$$A_i = \alpha - 1 + i + (\beta_1 + \cdots + \beta_i)\alpha, \text{ and}$$
$$C_i = n - (\alpha - 1 + i)p - (\beta_1 + \cdots + \beta_i)(1 + \alpha p),$$

where $\beta_1 = \gamma - 1$, and for $i \geq 1$

$$\beta_{i+1} = \begin{cases} \gamma & \text{if } l + B_i < q_1 - q_2, \\ \gamma - 1 & \text{otherwise.} \end{cases}$$

ii) $B_1 = q_2$ *and*

$$\beta_{i+1} = \begin{cases} B_i + q_2 & \text{if } B_{i+1} = \gamma, \\ B_i - (q_1 - q_2) & \text{if } B_{i+1} = \gamma - 1. \end{cases}$$

iii) $0 \leq l + B_i < q_1$.

Let s be the smallest integer such that $C_{s+1} < 0$. By the hypothesis of Case 2, $s \geq 1$. Then all above expressions of E with $i \leq s$ are indeed *representations of E*. Hence for $1 \leq i \leq s$ we get from Claim 3 (i) that

(4) $\delta_1(E) \leq l + B_i + A_i + C_i$
$$= B_i + l + n - (\alpha - 1 + i)(p - 1) - (\beta_1 + \cdots + \beta_i)(1 + \alpha p - \alpha).$$

Since $l + n \leq a - 2$, we conclude (otherwise there is nothing left to prove):

CLAIM 4. *One can assume that for $1 \leq i \leq s$,*

$$B_i \geq (\alpha - 1 + i)(p - 1) + (\beta_1 + \cdots + \beta_i)(1 + \alpha p - \alpha) + p + 1.$$

We now consider two subcases.

Case 2a: $(p, \gamma, \beta_{s+1}) \neq (1,2,1)$.

For short, let $\beta = \beta_1 + \cdots + \beta_s$. The condition $C_{s+1} < 0$ means that

(5) $$n \le (\alpha + s)p + (\beta + \beta_{s+1})(1 + \alpha p) - 1.$$

Hence, from (4) with $i = s$ and Claim 3 (iii) we get

$$\begin{aligned}
\delta_1(E) &\le (l + B_s) + n - (\alpha - 1 + s)(p - 1) - \beta(1 + \alpha p - \alpha) \\
&\le q_1 - 1 + (\alpha + s)p + (\beta + \beta_{s+1})(1 + \alpha p) - 1 \\
&\quad - (\alpha - 1 + s)(p - 1) - \beta(1 + \alpha p - \alpha) \\
&= q_1 + p - 2 + \alpha + s + \beta_{s+1}(1 + \alpha p) + \beta\alpha - 1 \\
&= a + p - 2 - \alpha q + \alpha + s + \beta_{s+1}(1 + \alpha p) + \beta\alpha - 1.
\end{aligned}$$

We need to show that $\alpha q \ge \alpha + s + \beta_{s+1}(1 + \alpha p) + \beta\alpha - 1$, or, equivalently,

(6) $$\alpha(q - 1 - p\beta_{s+1} - \beta) \ge \beta_{s+1} + s - 1.$$

If $\beta_{s+1} = \gamma$ then, by Claim 3 (i), $B_s < q_1 - q_2$. Hence, by Claim 4, we have

$$\begin{aligned}
q = \gamma q_1 - q_2 &\ge \gamma(q_1 - q_2) \ge \gamma(B_s + 1) \\
&\ge \gamma[(\alpha - 1 + s)(p - 1) + \beta(1 + \alpha p - \alpha) + p + 2] \\
&\ge \gamma(\beta p + p + 2) \\
&\ge \gamma(p + 1) + 1 + 2\beta \text{ (because } \gamma \ge 2).
\end{aligned}$$

Since $\beta = \beta_1 + \cdots + \beta_s \ge s$, we then get

$$\begin{aligned}
\alpha(q - 1 - p\beta_{s+1} - \beta) = \alpha(q - 1 - p\gamma - \beta) &\ge q - 1 - \gamma p - \beta \\
&\ge \gamma + \beta = \beta_{s+1} + \beta \ge \beta_{s+1} + s > \beta_{s+1} + s - 1,
\end{aligned}$$

which gives (6).

If $\beta_{s+1} = \gamma - 1$, then by Claim 3 (iii) and Claim 4 we have

$$\begin{aligned}
q - 1 = \gamma q_1 - q_2 - 1 &\ge (\gamma - 1)q_1 \ge (\gamma - 1)(B_s + 1) \\
&\ge (\gamma - 1)[(\alpha - 1 + s)(p - 1) + \beta(1 + \alpha p - \alpha) + p + 2] \\
&\ge (p + 1)(\gamma - 1) + 1 + s(p - 1) + (\gamma - 2)\beta + \beta \\
&\ge \beta + (p + 1)(\gamma - 1) + s(p - 1 + \gamma - 2) \text{ (because } \beta \ge s). \\
&= \beta + (p + 1)\beta_{s+1} + s(p - 1 + \gamma - 2).
\end{aligned}$$

From this it follows, since $p > 1$ or $\gamma > 2$, that (6) holds.

Case 2b: $p = 1$, $\gamma = 2$ and $\beta_{s+1} = 1$.

If $\beta_1 = \cdots = \beta_{s+1} = 1$, then by Claim 3 (ii) $B_s = B_1 - (s - 1)(q_1 - q_2) = q_2 - (s - 1)(q_1 - q_2) \le q_1 - s$. Hence, by Claim 4, we get $q > q_1 \ge s + B_s \ge s + \beta + 2$, where β is defined as in Case 2a. This shows the inequality (6) for

$p = 1$, $\beta_{s+1} = 1$ and we are done.

Finally, assume that not all β_i are equal to 1. Let j be the largest integer such that $\beta_{j+1} = 2$. Thus $1 \leq j \leq s - 1$ and $\beta_{j+2} = \cdots = \beta_{s+1} = 1$. By Claim 3 (i), $l + B_j < q_1 - q_2$. By Claim 4 it then implies that $q_1 - q_2 > 2$. Using (4) with $i = j$ and (5) we get:

$$\begin{aligned}
\delta_1(E) &\leq q_1 - q_2 - 1 + (s + \alpha) + (\alpha + 1)(\beta_1 + \cdots + \beta_j + \cdots + B_{s+1}) - 1 - \\
&\quad (\beta_1 + \cdots + \beta_j) \\
&= q_1 - q_2 - 1 + (\alpha + s) + \alpha(\beta_1 + \cdots + \beta_j) - 1 + (\alpha + 1)(s + 2 - j) \\
&= a - 1 - \alpha q - q_2 + \alpha(\beta_1 + \cdots + \beta_j) + \alpha(s + 3 - j) + 2s + 1 - j.
\end{aligned}$$

We need to show that

$$(7) \qquad \alpha[q - (\beta_1 + \cdots + \beta_j) - (s + 3 - j)] + q_2 \geq 2s + 1 - j.$$

Since $\beta_{j+2} = \cdots = \beta_{s+1} = 1 \ (= \gamma - 1)$, Claim 3 (ii) and Claim 4 give us that

$$B_s = B_{j+1} - (s - j - 1)(q_1 - q_2) \geq \beta_1 + \cdots + \beta_s + 2.$$

Hence, by Claim 3 (iii),

$$\begin{aligned}
(8) \qquad q_1 &\geq B_{j+1} + 1 \geq (s - j - 1)(q_1 - q_2) + \beta_1 + \cdots + \beta_s + 3 \\
&\geq 2(s - j - 1) + s + 3 = 3s - 2j + 1.
\end{aligned}$$

So,

$$(9) \quad 2q_1 \geq 3s - 2j + 1 + \beta_1 + \cdots + \beta_j + 3 = \beta_1 + \cdots + \beta_j + 3s - 2j + 4.$$

Since $q \geq q_1 + 1$ and $q_1 - q_2 \geq 1$, from (8) it follows that $q \geq (\beta_1 + \cdots + \beta_s) + s + 3 - j \geq (\beta_1 + \cdots + \beta_j) + (s + 3 - j)$. That means $q - (\beta_1 + \cdots + \beta_j) - (s + 3 - j) \geq 0$. Hence

$$\begin{aligned}
\alpha[q - (\beta_1 + \cdots + \beta_j) - (s + 3 - j)] + q_2 &\geq q - (\beta_1 + \cdots + \beta_j) - \\
&\qquad\qquad (s + 3 - j) + q_2 \\
&= 2q_1 - (\beta_1 + \cdots + \beta_j) - (s + 3 - j) \ (\text{since } \gamma = 2) \\
&\geq 2s + 1 - j \quad (\text{by (9)}),
\end{aligned}$$

which shows (7). This completes the proof of Case 2b and therefore the proof of Lemma 5.2.

6. Liaison among monomial curves in P_K^3

In [BH] liaison amongst monomial curves in P_K^3 is investigated. There it is shown that some linkage classes contain only few monomial curves, while in other

linkage classes there are infinitely many monomial curves. Recall that two curves C, C' of P^3 are said to be linked if there is an R-regular sequence $\{f, g\} \in I(C) \cap I(C')$ such that $I(C) = (f, g) : I(C')$ and $I(C') = (f, g) : I(C)$. In this case one writes $C \sim C'$. C and C' are in the same linkage class (the same even linkage class) if there exists a sequence of links $C = C_0 \sim C_1 \sim \cdots \sim C_m = C'$ (with m even, resp.).

Using the notion of k-Buchsbaum curves we can get some new insight on liaison among monomial curves. An easy consequence of [R] is that two curves in the same linkage class have the same Buchsbaum nunber (see [FH], Section 5b). For $k \leq 2$, one can define all linkage classes of strictly k-Buchsbaum monomial curves (see [BSV] and [H]). It is natural to ask whether for a given k there are only finitely many linkage classes among strictly k-Buchsbaum monomial curves. Using Theorem 2.1 we answer it in the affirmative.

PROPOSITION 6.1. *Assume* $C(n_1, n_2, n_3)$ *is k-Buchsbaum for a nonnegative integer k. Then*

(i) *The number of* S'-*minimal elements in* $S' \setminus S$ *is k.*

(ii) $\mu(I(C(n_1, n_2, n_3))) \leq \begin{cases} 3 & \text{if } k = 0, \\ 2k + 2 & \text{if } k \geq 1. \end{cases}$

Proof. By Theorem 2.1 we have $H_m^1(K[S]) = M_{a+1} \oplus \cdots \oplus M_{a+k}$, where $a(H_m^1(K[S])) = a + 1$. Assume $k \geq 1$ (the case $k = 0$ being true vacuously for (i)). Let, in the notation of Section 3, $\dfrac{x_2^{a_2}}{x_3^{a_3}} \in H_m^1(\bar{R})$ correspond to the first element in $\mathcal{B}_2$ (recall that $\bar{R} \cong K[S]$). Then $d\left(\dfrac{x_2^{a_2}}{x_3^{a_3}}\right) \geq a + 1$. As in Section 3, $0 \neq x_0^{\alpha_0-1} x_3^{\alpha_3-1} \dfrac{x_2^{a_2}}{x_3^{a_3}} x_1^{b_1-1} x_2^{b_2-1} \in H_m^1(\bar{R})$. Hence $a_0 - 1 + a + 1 \leq d\left(x_0^{\alpha_0-1} \dfrac{x_2^{a_2}}{x_3^{a_3}}\right) \leq a + k$, from which $a_0 \leq k$. From this we have immediately $|\mathcal{B}_2| \leq k$, since the x_0-exponent decreases by at least one in every step of the algorithm in [BR], which proves (i). If $k = 0$, then (ii) follows from [B2]. Assume $k \geq 1$. Then $\mathcal{B}_3 \neq \emptyset$ with $x_0^{b_0} x_3^{b_3} - x_1^{b_1} x_2^{b_2}$ as its first element. As before we have $b_1 - 1 + a + 1 \leq a + k$ or $b_1 \leq k$, from which $|\mathcal{B}_3| \leq k$, since the x_1-exponent decreases by at least one in every step of the algorithm in [BR]. Thus $\mu(I(C(n_1, n_2, n_3))) \leq 2k + 2$, which proves (ii).

Remark 6.1. The upper bound in (i) and (ii) is sharp, it is attained already

for $k = 0, 1, 2$ (see [B2], [BSV] and [H]).

THEOREM 6.1.   *Let k be a fixed nonnegative integer. Then there are only finitely many linkage classes for strictly k-Buchsbaum monomial curves in P_K^3.*

Proof.   Since $k = \operatorname{diam} M(C)$, by Theorem 2.1 $M(C)$ has the following form:

$$M(C) \cong H_{\underline{m}}^1(K[S]) \cong k[S' \setminus S] = M_{a+1} \oplus \cdots \oplus M_{a+k}$$

We may consider modules over $K[x_0, x_1, x_2, x_3] = R$ or $K[S]$ via the ring homomorphism: $x_0 \mapsto s^{n_3}$, $x_1 \mapsto s^{n_3-n_1}t^{n_1}$, $x_2 \mapsto s^{n_3-n_2}t^{n_2}$, $x_3 \mapsto t^{n_3}$. Since the Hartshorne-Rao module $M(C)$ is invariant under linkage (up to K-duality and a shift in grading), we have $M(C) \cong M(C')(n)$ or $M(C) \cong \operatorname{Ext}_R^4(M(C'), R)(-n)$. Thus it is enough to show that there are only finitely many different possible R-modules of the above form. This is equivalent to showing that the dimension $|M_i|$ of K-vector space M_i, is finite for all $a + 1 \le i \le a + k$. But an element of $S' \setminus S$ of degree i is of the form $f_1 + \cdots + f_g$ (with possible repetition), where f_j is an S'-minimal element of $S' \setminus S$ or $f_j \in \{t_0, t_1, t_2, t_3\}$, and $1 \le g \le i - a$. Hence, by Proposition 6.1, $|M_i| \le (k+4)^{i-a} \le (k+4)^k$, as required.

In [BH] we have an algorithm for determining whether two monomial curves are in the same even linkage class. Here we give a particular result in the "opposite" direction.

PROPOSITION 6.2.   *No two non C-M curves of the type $C(1, a, d)$ are in the same even linkage class.*

Proof.   Assume that $C = C(1, a, d)$ and $C^* = C(1, a^*, d^*)$ are in the same even linkage class. Then $k(C) = k(C^*) = k > 0$. By Theorem 5.1 we get $a = a^* = k + 2$. One needs to show that $d = d^*$. Let $d = pa + q$ and $d^* = p^*a + q^*$, where p, q, p^*, q^* are positive integers such that $p + q \le a - 1$ and $p^* + q^* \le a - 1$ (by Example 4.1 and our assumption that C and C^* are not C-M). Consider the element $e = ((p + 1)(d - a), a - q)$. By Lemma 5.1 it follows that e has the minimal degree (which equals to p) among the elements of $S' \setminus S$. Moreover it is unique with this property if $p > 1$. If $p = 1$ there are maybe some elements of $S' \setminus S$ having the minimal degree. But e is still the unique element in $S' \setminus S$ which satisfies the following two properties:

    i) $\delta(e)$ is minimal among $\delta(e')$, $e' \in S' \setminus S$,

    ii) $e + t_3 \in S$.

The last property is true for any p. For the R-module $M(C)$ these properties

mean that:

i ') e has the minimal degree among homogeneous generators of $M(C)$ (we identify e with the corresponding element in $M(C) \cong H_m^1(K[S]) \cong K[S' \setminus S])$,

ii ') $x_3 \cdot e = 0$.

Hence, for any curve C there is unique generator e defined by the above two properties (up to a scalar). For C^*, let $e^* = ((p^* + 1)(d^* - a), a - q^*) \in (S^*)' \setminus S^*$. From the above observation and by [R], $M(C) \cong M(C^*)(n)$, we deduce that $e \mapsto e^*$.

By Lemma 2.1 it follows that $e + qt_1 \in S$, but $e + (q - 1)t_1 \in S' \setminus S$ (and an analogous statement for e^* and q^*). This means that q (resp. q^*) is the smallest integer such that $x_1^q e = 0$ in $M(C)$ ($x_1^{q^*} e^* = 0$ in $M(C^*)$, resp.). Hence $q = q^*$. Analogously, $m = a - p - q$ (resp. $m^* = a - p^* - q$) is the smallest integer such that $x_0^m e = 0$ in $M(C)$ (resp. $x_0^{m^*} e^* = 0$ in $M(C^*)$, resp.). Therefore $a - p - q = a - p^* - q$, so $p = p^*$, as required.

The following corollary examines even linkage between curves $C = C(1, a, d)$ and $C^* = C(d^* - a^*, d^* - 1, d^*)$.

COROLLARY 6.1. *Let* $C(1, a, b)$ *be a not C-M curve such that* $d > a + 1$. *If* C *and* $C^* = C(d^* - a^*, d^* - 1, d^*)$ *are in the same even linkage class, then* $a = a^* = p + q + 1 = p^* + q^* + 1$, *where* $d = pa + q$, $0 < q < a$, *and* $d^* = p^* a^* + q^*$, $0 < q^* < a^*$.

Proof. By Theorem 5.1, $a - 2 = k(C) = k(C^*) = a^* - 2$, thus $a = a^*$. To prove $p + q + 1 = a = a^* = p^* + q^* + 1$ let S^* denote the associated semigroup of C^*. As in the proof of Proposition 6.2, there is a uniquely defined element $f \in (S^*)' \setminus S^*$ such that f has the minimal degree and $x_0 f = 0$. (Note that for $I(C^*)$ one needs to permute the variables $x_0 \leftrightarrow x_3$, $x_1 \leftrightarrow x_2$.) Let $e \in S' \setminus S$ be the image of f in the isomorphism $M(C) \cong M(C^*)(n)$. Then $e \in [S' \setminus S]_p$, the p-degree component, and $x_0 e = 0$. If $[S' \setminus S]_p$ has only one element, then by the proof of Proposition 6.2, the smallest integer m such that $x_0^m e = 0 \in M(C)$ is $a - p - q$. Thus $1 = a - p - q$. By symmetry $a^* = p^* + q^* + 1 = a$. We will show next, that the above is the only possible case. Suppose therefore that $[S' \setminus S]_p$ and $[(S^*)' \setminus (S^*)]_{p*}$ have at least two elements. Then, by Lemma 5.1 (or by the algorithm [BR]), $p = p^* = 1$. Since $x_0 e = 0$, i.e. $t_0 + e \in S$, from Lemma 5.1 and Lemma 2.1 we get that $e = ((1 + \beta)(d - a), a - \beta q)$ and $a - \beta q = 2$. Note that $C(1, a, d^*)$ and $C(d - a, d - 1, d)$ are also in the same even linkage class. Hence, as before, $a - \beta^* q^* = 2$. Again by Lemma 5.1, β and β^* equal the

number of elements in $[S' \setminus S]_1$ and $[(S^*)' \setminus (S^*)]_1$, respectively. Therefore $\beta = \beta^* \geq 2$, which implies $q = q^*$. By the proof of Proposition 6.2, $x_3^{a-p^*-q^*-1} f \neq 0$, hence $x_3^{a-q-2} e \neq 0$, i.e. $(a - q - 2)t_3 + e \notin S$. By Lemma 2.1 this implies $\beta + 1 > a - q - 1$, or equivalently $0 \geq (\beta - 1)(q - 1)$ (since $a = \beta q + 2$). But this is impossible since $\beta \geq 2$ and $q > 1$ by our initial assumption $d > a + 1$. This contradiction finishes the proof.

Remark 6.2. (i) An easy example shows that it is possible to have $d = d^*$ or $d \neq d^*$ and $C(1, a, d)$ and $C(d^* - a^*, d^* - 1, d^*)$ be in the same even linkage class.

(ii) It is shown in [BSV] and [H] that for $k \leq 2$ each linkage class of strictly k-Buchsbaum monomial curves contains a curve $C(1, a, d)$. That this is not true for all nonnegative integers k was shown by M. Morales in [Mo]. We therefore conclude our paper with the following open problem: Determine all linkage classes of monomial curves in P_K^3 which contain a representative $C(1, a, d)$ or $C(d - a, d - 1, d)$ and determine the integer a for these linkage classes.

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Henrik Bresinsky
Frank Curtis
414 and 336 Neville Hall
University of Maine
Orono, Maine 04469-5752 (U.S.A.)

Mario Fiorentini
Dipartimento di Matematica
Universita di Ferrara
Via Machiavelli 35
Ferrara 44100 (Italy)

Lê Tuân Hoa
Institute of Mathematics
P. O. Box 631, Bo ho,
Hanoi (Vietnam)