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SYMMETRIC LADDERS

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In this paper we define and study ladder determinantal rings of a symmetric matrix of indeterminates. We show that they are Cohen-Macaulay domains. We give a combinatorial characterization of their h-vectors and we compute the a-invariant of the classical determinantal rings of a symmetric matrix of indeterminates.

Introduction

Let us recall the definition of ladder determinantal rings of a generic matrix of indeterminates. Let X be a generic matrix of indeterminates, K be a field and denote by $K[X]$ the polynomial ring in the set of indeterminates X_{ij}. A subset Y of X is called a ladder if whenever $X_{ij}, X_{hk} \in Y$ and $i \leq h, j \leq k$, then $X_{ik}, X_{hj} \in Y$. Given a ladder Y, one defines $I_t(Y)$ to be the ideal generated by all the t-minors of X which involve only indeterminates of Y. The ideal $I_t(Y)$ is called a *ladder determinantal ideal* and the quotient $R_t(Y) = K[Y]/I_t(Y)$ is called a *ladder determinantal ring*. This class of ideals is investigated in [1], [2], [9], [15], [17]. It turns out that the main tool in the investigation of the ladder determinantal rings is the knowledge of Gröbner bases of the classical determinantal ideals. In [8] we determined Gröbner bases of ideals generated by minors of a symmetric matrix of indeterminates. This allows us to study the ladder determinantal rings of a symmetric matrix.

Now let X be an $n \times n$ symmetric matrix of indeterminates, K be a field. Let us denote by A the set $\{(i, j) \in \mathbf{N}^2 : 1 \leq i, j \leq n\}$. A subset L of A is called a symmetric ladder if satisfies the following condition: if $(i, j) \in L$ then $(j, i) \in L$, and whenever $(i, j), (h, k) \in L$ and $i \leq h, j \leq k$, then $(i, k), (h, j) \in L$.

The set $Y = \{X_{ij} : i \leq j, (i, j) \in L\}$ is called the support of L. We say that a minor is in L if it involves only indeterminates of Y. Given a sequence of integers $\alpha = 1 \leq \alpha_1 < \cdots < \alpha_t \leq n$, we define $I_\alpha(L)$ to be the ideals generated

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by all the i-minors of the first $\alpha_i - 1$ rows of X which are in L, $i = 1, \ldots, t$, and by all the $t+1$ minors of L. Denote by $R_\alpha(L)$ the ring $K[Y]/I_\alpha(L)$. In particular if $\alpha = 1, \ldots, t-1$, then $I_\alpha(L)$ is the ideal generated by the t-minors in L.

Following the approach of Narasimhan [17], we use Gröbner bases to show that $I_\alpha(L) = I_\alpha(X) \cap K[Y]$. Since $I_\alpha(X)$ is known to be a prime ideal, see [16], it follows that $I_\alpha(L)$ is prime too. Furthermore we determine a Gröbner basis of the ideal $I_\alpha(L)$. It turns out that the ideal $\text{in}(I_\alpha(L))$ of the leading forms of $I_\alpha(L)$ is generated by square free monomials. Therefore the ring $R_\alpha(L)^* = K[Y]/\text{in}(I_\alpha(L))$ is the Stanley-Reisner ring associated with a simplicial complex $\Delta_\alpha(L)$. By a result of Stanley, the Hilbert function of $R_\alpha(L)^*$ is determinated by the f-vector of $\Delta_\alpha(L)$. We describe the facets of $\Delta_\alpha(L)$ in terms of families of non-intersecting chains in a poset, and we get a combinatorial characterization of the dimension and multiplicity of $R_\alpha(L)$. As in the case of ladders of a generic matrix, it is possible to show that $\Delta_\alpha(L)$ is shellable. Actually, we deduce this result from the analogous of [15]. The shellability is a combinatorial property of simplicial complexes which implies the Cohen-Macaulayness of the associated Stanley-Reisner rings. But it is well known that if $R_\alpha(L)^*$ is Cohen-Macaulay, then $R_\alpha(L)$ is.

In the second section we apply these results to give a combinatorial characterization of the h-vector of the rings $R_\alpha(X)$ in terms of number of families of non-intersecting paths in a poset with a fixed number of certain corners. Then we compute the a-invariant of the ring $R_t(X)$ defined by the ideal of minors of fixed size in the matrix X in the homogeneous and weighted case. The same result was obtained by Barile [3] independently and using different methods. As last application we study the determinantal ring $R_t(Z)$ associated with an $m \times n$ matrix of indeterminates Z in which an $s \times s$ submatrix is symmetric. It turns out that $R_t(Z)$ is a symmetric ladder determinantal ring. In particular $R_t(Z)$ is a Cohen-Macaulay domain, and we compute its dimension and multiplicity. If $s < m \leq n$, we prove that $R_t(Z)$ is normal and that is Gorenstein if and only if $t \geq s$ and $m = n$. In [10] we deal with the case $s = m < n$, and we show that $R_t(Z)$ is normal, and is Gorenstein if and only if $2m = n + t$. The results of this paper are part of the author's Ph. D. thesis.

1. Ladders of a symmetric matrix

Let X be an $n \times n$ symmetric matrix of indeterminates, K be a field, and denote by $K[X]$ the polynomial ring in the set of indeterminates X_{ij}, $1 \leq i \leq j \leq n$. Let τ be the term order induced by the variable order $X_{11} > \cdots > X_{1n} > X_{22} >$

$\cdots > X_{2n} > \cdots > X_{n-1n} > X_{nn}$.

Let us recall the combinatorial structure of $K[X]$ with respect to the product of minors of X. Denote by H the set of the non-empty subsets of $\{1, \dots, n\}$. Given an element a of H we will always write its elements in ascending order $1 \leq a_1 < \cdots < a_s \leq n$. On H we define the following partial order:

$$a = \{a_1, \dots, a_s\} \leq b = \{b_1, \dots, b_r\} \Leftrightarrow r \leq s \text{ and } a_i \leq b_i \text{ for } i = 1, \dots, r.$$

As usual, we denote by $[a_1, \dots, a_s \mid b_1, \dots, b_s]$ the s-minor $\det(X_{a_i b_j})$ of X, and assume that $1 \leq a_1 < \cdots < a_s \leq n$ and $1 \leq b_1 < \cdots < b_s \leq n$. The minor $[a_1, \dots, a_s \mid b_1, \dots, b_s]$ is called a *doset minor* if $a \leq b$ in H. We denote by D the set of all the doset minors of X. Let $M_1 = [a_{11}, \dots, a_{1s_1} \mid b_{11}, \dots, b_{1s_1}], \dots, M_p = [a_{p1}, \dots, a_{ps_p} \mid b_{p1}, \dots, b_{ps_p}]$ be doset minors; the product $M_1 \cdots M_p$ is called a standard monomial if $\{b_{j1}, \dots, b_{js_j}\} \leq \{a_{j+11}, \dots, a_{j+1s_{j+1}}\}$ for $j = 1, \dots, p-1$. The ring $K[X]$ is a doset algebra on D, that is, the standard monomials form a K-basis of $K[X]$ and one has a certain control on the miltiplicative table of the products of the standard monomials, see [12]. If one considers suitable ideals of minors, the same combinatorial structure is inherited by the quotient rings. Given $\alpha = \{\alpha_1, \dots, \alpha_t\} \in H$ one defines $I_\alpha(X)$ to be the ideal generated by all the minors $[a_1, \dots, a_s \mid b_1, \dots, b_s]$ with $\{\alpha_1, \dots, \alpha_s\} \not\geq \alpha$ in H. If $\alpha = \{1, \dots, t-1\}$, then the ideal $I_\alpha(X)$ is the ideal $I_t(X)$ generated by all the t-minors of X. The class of ideals $I_\alpha(X)$ is essentially the same the class of ideals defined and studied by Kutz [16].

In order to define ladders and ladder determinantal ideals of the symmetric matrix X we introduce some notations. Let $A = \{(i, j) \in \mathbf{N}^2 : 1 \leq i \leq n$ and $1 \leq j \leq n\}$ and $B = \{(i, j) \in A : i \leq j\}$. We consider A a distributive lattice with the following partial order: $(i, j) \leq (k, h) \Leftrightarrow i \geq k$ and $j \leq h$.

In the generic case there is a one-to-one correspondence between minors and monomials which are product of elements of main diagonals of minors. When we deal with minors of a symmetric matrix we lose this correspondence. The monomial $X_{a_1 b_1} \dots X_{a_s b_s}$, with $a_i < a_{i+1}$ and $b_i < b_{i+1}$, is the product of the elements on the main diagonal of all the minors $M = [c_1, \dots, c_s \mid d_1, \dots, d_s]$ such that $\{c_i, d_i\} = \{a_i, b_i\}$ and $c_i < c_{i+1}$, $d_i < d_{i+1}$. But if we require that the minor is a doset minor then it is unique.

Therefore the natural choice for the definition of a ladder of the symmetric matrix X is the following:

DEFINITION 1.1. A subset L of A is a symmetric ladder if:
(a) L is a sublattice of A;

(b) L is symmetric, that is $(i, j) \in L$ if and only if $(j, i) \in L$.

We represent ladders as subsets of points of $\mathbf{N}^2$. An example of symmetric ladder is the following:

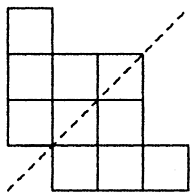


Fig. 1

Let L be a symmetric ladder, we put $L^+ = L \cap B$ and $Y = \{X_{ij} : (i, j) \in L, i \leq j\}$. The set Y is called the support of L. We say that a minor $M = [a_1, \ldots, a_s \mid b_1, \ldots, b_s]$ is in L if the following equivalent conditions are satisfied:

(1) For all $1 \leq i, j \leq s$, then $(a_i, b_j) \in L$.
(2) For all $1 \leq i \leq s$, then $(a_i, b_i) \in L$.
(3) The entries of M belong to Y.
(4) The entries of the main diagonal of M belong to Y.

Let $\alpha = \{\alpha_1, \ldots, \alpha_t\} \in H$. For systematic reasons it is convenient to set $\alpha_{t+1} = n + 1$. Following [15], we define the ideal cogenerated by α in L.

Definition 1.2. Let L be a symmetric ladder and Y its support. We denote by $I_\alpha(L)$ the ideal generated by all the minors $M = [a_1, \ldots, a_s \mid b_1, \ldots, b_s]$ of L such that $\{a_1, \ldots, a_s\} \not\geq \alpha$ and set $R_\alpha(L) = K[Y]/I_\alpha(L)$.

In particular, if $\alpha = \{1, \ldots, t-1\}$, then $I_\alpha(L)$ is the ideal generated by all the t-minors of L.

Let $J_\alpha(L)$ be the set of all the doset minors $[a_1, \ldots, a_r \mid b_1, \ldots, b_r]$ of L such that $1 \leq r \leq t+1$, $a_i \geq \alpha_i$ for $i = 1, \ldots, r-1$ and $a_r < \alpha_r$. The main result of [8] is the determination of a Gröbner basis of the ideal $I_\alpha(X)$ with respect to τ: the set $J_\alpha(X)$ is a minimal system of generators and a Gröbner basis with respect to τ of the ideal $I_\alpha(X)$, see [8, 2.7, 2.8]. From this we deduce the following:

Theorem 1.3. (a) *The ideal $I_\alpha(L)$ is prime.*
(b) *The set $J_\alpha(L)$ is a Gröbner basis of $I_\alpha(L)$ with respect to τ.*
(c) *The set $J_\alpha(L)$ is a minimal system of generators of $I_\alpha(L)$.*

Proof. (a) Since $I_\alpha(X)$ is a prime ideal, see [16, Th. 1], it is sufficient to show that $I_\alpha(L) = I_\alpha(X) \cap K[Y]$. We have $I_\alpha(L) \subset I_\alpha(X) \cap K[Y]$ since, by definition, $I_\alpha(L) \subset I_\alpha(X)$. Let $f \in I_\alpha(X) \cap K[Y]$ be an homogeneous polynomial and denote by $\mathrm{in}(f)$ its initial term with respect to τ. The set $J_\alpha(X)$ is a Gröbner basis of $I_\alpha(X)$. Therefore $\mathrm{in}(f)$ is divisible by the initial term of a doset minor M of $J_\alpha(X)$, that is, $\mathrm{in}(f) = \mathrm{in}(M)h$. Of course $\mathrm{in}(f) \in K[Y]$, and therefore the minor M is in L. Note that $M \in J_\alpha(L)$. Set $g = f - hM$; then we have $g \in I_\alpha(X) \cap K[Y]$ and $g = 0$ or $\mathrm{in}(g) < \mathrm{in}(f)$ in the term ordering. Therefore, by induction, we may suppose $g \in I_\alpha(L)$ and $f = g + hM \in I_\alpha(L)$.
(b) Let $f \in I_\alpha(L)$, since $\mathrm{in}(f) \in \mathrm{in}(I_\alpha(X)) \cap K[Y]$ we may argue as in the proof of part (a) and show that $\mathrm{in}(f)$ is divisible by the initial term of a minor of $J_\alpha(L)$.
(c) Since $J_\alpha(L)$ is a Gröbner basis of $I_\alpha(L)$, it is also a system of generators. But $J_\alpha(X)$ is a minimal system of generators of $I_\alpha(X)$ and $J_\alpha(L) \subset J_\alpha(X)$. Therefore $J_\alpha(L)$ is a minimal system of generators of $I_\alpha(L)$. □

Now we see how we may interpret the ideal $I_\alpha(L)$ as an ideal of minors associated with more general subsets of A.

DEFINITION 1.4. A subset V of A is a semi-symmetric ladder if:
(a) V is a sublattice of A.
(b) If $(i, j) \in V$ and $i \geq j$, then $(j, i) \in V$.

Given a semi-symmetric ladder V, we say that a minor $[a_1, \dots, a_s \mid b_1, \dots, b_s]$ is in V if $(a_i, b_j) \in V$ for all $1 \leq i, j \leq s$. We define $I_\alpha(V)$ to be the ideal generated of all minors in V whose sequence of row indices is not greater than or equal to α.

Remark 1.5. Let V be a semi-symmetric ladder and set $L(V) = \{(i, j) \in A : (i, j) \in V \text{ or } (j, i) \in V\}$. It is easy to see that $L(V)$ is a symmetric ladder and that $L(V)^+ \subset V$. If we consider a doset minor M in $L(V)$, then its main diagonal is in $L(V)^+$, and therefore M is in V. Hence $I_\alpha(V) = I_\alpha(L(V))$. In other words, to study ideals of minors of symmetric ladders is the same as to study ideals of minors of semi-symmetric ladders.

In the picture V is a semi-symmetric ladder and $L(V)$ is its associated symmetric ladder.

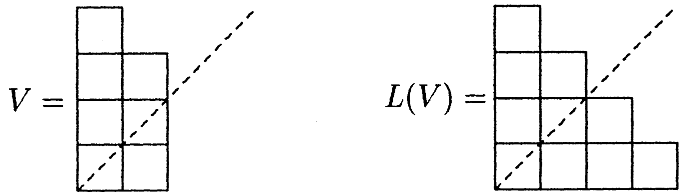


Fig. 2

The ideal $\mathrm{in}(I_\alpha(L))$ of the leading forms of $I_\alpha(L)$ is generated by the leading terms of the minors in $J_\alpha(L)$, and hence it is a square-free monomial ideal. Therefore $R_\alpha(L)^* = K[Y]/\mathrm{in}(I_\alpha(L))$ is the Stanley-Reisner ring associated with a simplicial complex. For the theory of the Stanley-Reisner ring associated with a simplicial complex we refer the reader to [18].

In order to describe this simplicial complex and its facets we introduce some notation and terminology. Given a simplicial complex Δ, its elements are called faces and facets its maximal elements under inclusion. A face of dimension i is a face with $i+1$ elements, the dimension of Δ is the maximum of the dimensions of its faces and f_i is the number of the faces of dimension i. The sequence $f_0, \ldots, f_d$, $d = \dim(\Delta)$, is called the f-vector of Δ. The Hilbert function of the Stanley-Reisner ring $k[\Delta]$ is determined by its f-vector [18]. In particular, $\dim k[\Delta] = d+1$ and $e(k[\Delta]) = f_d$.

Let P be a finite poset and $x \in P$. We define the rank of x in P to be the maximum of the integers i such that there exists a chain $x_1 < \cdots < x_i = x$ and the rank of P to be the maximum of the ranks of its elements. A set of incomparable elements of P is called an antichain. An antichain of B is a set $\{(v_1, u_1), \ldots, (v_p, u_p)\}$ with $v_i \leq u_i$ for $i = 1, \ldots, p$ such that $v_1 < \cdots < v_p$ and $u_1 < \cdots < u_p$ and therefore it corresponds to the main diagonal of a doset minor.

For $k = 1, \ldots, t+1$, let $S_k = \{(i, j) \in A : i < \alpha_k \text{ or } j < \alpha_k\}$, $G_k = B \cap S_k$, $S'_k = A \setminus S_k$ and $G'_k = B \setminus G_k$.

We define $\Delta'_\alpha(L)$ to be the simplicial complex of all the subsets of L which, for $k = 1, \ldots, t+1$, do not contain k-antichains (antichains with k elements) of $S_k \cap L$, and let $\Delta_\alpha(L)$ be the restriction of $\Delta'_\alpha(L)$ to L^+. By construction $\Delta_\alpha(L)$ is the simplicial complex of all the subsets of L^+ which, for $k = 1, \ldots, t+1$, do not contain k-antichains of $G_k \cap L^+$. Furthermore the simplicial complex $\Delta'_\alpha(L)$ coincides with the simplicial complex $\Delta_M(L)$ defined in [15], where $M = [\alpha_1, \ldots, \alpha_t \mid \alpha_1, \ldots, \alpha_t]$.

We know that $\mathrm{in}(I_\alpha(L))$ is generated by the k-antichains of $G_k \cap L^+$ for $k = 1, \dots, t+1$. Therefore the Stanley-Reisner ring $K[\Delta_\alpha(L)]$ associated with $\Delta_\alpha(L)$ is $R_\alpha(L)^*$.

It is well known that $R_\alpha(L)$ and $R_\alpha(L)^*$ have the same Hilbert series, therefore their dimensions and multiplicities coincide. Thus the Hilbert function, the multiplicity, and the dimension of $R_\alpha(L)$ may be characterized in terms of f-vector of $\Delta_\alpha(L)$.

Let $p = (a, b)$ be an element of L, we define $R_p = \{(i, j) \in L : a < i,\ b < j\}$, and for $Z \subset L$ we set $R_Z = \cup_{p \in Z} R_p$. It is easy to see that R_Z is a sublattice of L which is symmetric if Z is. We set $L_1 = L \cap S_1$ and recursively for $i = 2, \dots, t$, we set $L_i = S_i \cap R_{L_{i-1}}$. Finally we put $L_i^+ = L_i \cap B$. Since S_1 and L are symmetric sublattices of A, L_1 is also a symmetric sublattice of A. By recursion it follows that L_t is the intersection of symmetric sublattices, and therefore it is also a symmetric sublattice.

By [15, Th. 4.6] a subset $\bar{Z}$ of L is a facet of $\Delta'_\alpha(L)$ if and only if $\bar{Z}$ is the union of disjoint maximal chains of L_i, $i = 1, \dots, t$.

LEMMA 1.6. *Let $\bar{Z}$ be a facet of $\Delta'_\alpha(L)$, then $|\bar{Z} \cap L^+| = \sum_{i=1}^t rk(L_i^+)$ where $rk(L_i^+)$ is the rank of the poset L_i^+.*

Proof. Let $p \in L_i$, we claim: $p \in L_i^+ \Leftrightarrow \mathrm{rk}(p) > [\mathrm{rk}(L_i)/2]$, where $\mathrm{rk}(p)$ is the rank of p in the lattice L_i, and $[x] = \max\{n \in \mathbf{Z} : n \le x\}$ denote the integer part of a real number x.

$\Rightarrow$: Let $p_1 < \cdots < p_s$ be a maximal chain of L_i^+ which contains p, say $p = p_k$. If we consider the sequence $q_1, \dots, q_s$ of the symmetric points (q_j is obtained from p_j by exchanging the coordinates), then $q_s < \cdots < q_2 < q_1 \le p_1 < p_2 < \cdots < p_s$ is a maximal chain of L_i. Since L_i is a distributive lattice and all the maximal chains of a distributive lattice have the same number of elements, we have $\mathrm{rk}(L_i) = 2s$ if $p_1 \ne q_1$, and $\mathrm{rk}(L_i) = 2s - 1$ if $p_1 = q_1$. In any case $\mathrm{rk}(p) \ge \mathrm{rk}(p_1) > [\mathrm{rk}(L_i)/2]$.

$\Leftarrow$: Suppose $p \notin L_i^+$ and let $q_1 < \cdots < q_k = p$ be a chain with $k = \mathrm{rk}(p)$ elements. If we consider the sequence of the symmetric points $p_1, \dots, p_k$ then $q_1 < \cdots < q_k < p_k < \dots < p_1$ is a chain of L_i with $2\mathrm{rk}(p)$ elements Therefore $\mathrm{rk}(L_i) \ge 2\mathrm{rk}(p) > 2[\mathrm{rk}(L_i)/2]$, a contradiction.

From the previous claim it follows that every maximal chain of L_i contains exactly $\mathrm{rk}(L_i) - [\mathrm{rk}(L_i)/2]$ elements of L_i^+ and $\mathrm{rk}(L_i^+) = \mathrm{rk}(L_i) - [\mathrm{rk}(L_i)/2]$. Hence the assertion of the lemma follows from the description of the facets of $\Delta'_\alpha(L)$ and the claim. □

As immediate consequence we get:

PROPOSITION 1.7. *Let Z be a face of $\Delta_\alpha(L)$. Then Z is a facet of $\Delta_\alpha(L)$ if and only if there exists a facet $\bar{Z}$ of $\Delta'_\alpha(L)$ such that $Z = \bar{Z} \cap L^+$.*

Proof. $\Rightarrow$: The simplicial complex $\Delta_\alpha(L)$ is the restriction of the simplicial complex $\Delta'_\alpha(L)$ to L^+. Therefore there exists a facet $\bar{Z}$ of $\Delta'_\alpha(L)$ such that $Z \subset \bar{Z} \cap L^+$. Since Z is a facet, $Z = \bar{Z} \cap L^+$.
$\Leftarrow$: Of course Z is contained in a facet Z_1 of $\Delta_\alpha(L)$. By 1.6 it follows that $|Z| = |Z_1|$, and hence $Z = Z_1$. □

We get the following characterization of the facets of $\Delta_\alpha(L)$:

PROPOSITION 1.8. *The set Z is a facet of $\Delta_\alpha(L)$ if and only if Z is the union of disjoint sets $Z_1, \ldots, Z_t$, where Z_i is a maximal chain of L_i^+. Furthermore the decomposition of Z as union of disjoint maximal chains of L_i^+ is unique.*

Proof. The set Z is a facet of $\Delta_\alpha(L)$ if and only if there exists a facet $\bar{Z}$ of $\Delta'_\alpha(L)$ such that $Z = \bar{Z} \cap L^+$. But $\bar{Z}$ is the union of disjoint sets $\bar{Z}_1, \ldots, \bar{Z}_t$, where $\bar{Z}_i$ is a maximal chain of L_i. If we set $Z_i = \bar{Z}_i \cap L_i^+$ then Z_i is a maximal chain of L_i^+ and Z is the union of $Z_1, \ldots, Z_t$. The uniqueness of the decomposition of Z is a consequence of the construction of the decomposition of $\bar{Z}$ as union of disjoint maximal chains, see [15, pp. 20]. □

COROLLARY 1.9. *The dimension of $R_\alpha(L)$ is $\sum_{i=1}^t \mathrm{rk}(L_i^+)$, and its multiplicity is the number of the families of disjoint sets $Z_1, \ldots, Z_t$, where Z_i is a maximal chain of L_i^+.*

Using this result we computed in [8] the dimension and the multiplicity of the ring $R_\alpha(X)$.

Recall that a simplicial complex Δ is said to be *shellable* if its facets have the same dimension and they can be given a linear order called a shelling in such a way that if $Z < Z_1$ are facets of Δ, then there exists a facet $Z_2 < Z_1$ of Δ and an element $x \in Z_1$ such that $Z \cap Z_1 \subseteq Z_2 \cap Z_1 = Z_1 \setminus \{x\}$.

By [15, Th. 4.9] the simplicial complex $\Delta'_\alpha(L)$ is shellable. Now we shall see how shellability passes from a simplicial complex to a subcomplex when a condition as 1.6 is fulfilled.

LEMMA 1.10. *Let $\varDelta$ be a shellable simplicial complex over a vertices set V, and W a subset of V. Suppose that for all the facets $\bar{Z}$ of $\varDelta$ the number $|\bar{Z} \cap W|$ does not depend on $\bar{Z}$. Then the restriction of $\varDelta$ to W is a shellable simplicial complex.*

Proof. We denote by $\varDelta_1$ the restriction of $\varDelta$ to W, $F(\varDelta)$ the set of the facets of $\varDelta$, $F(\varDelta_1)$ the set of the facets of $\varDelta_1$, $n = |\bar{Z} \cap W|$ for all $\bar{Z} \in F(\varDelta)$.

From the hypotheses follows, as in 1.7, that $\varDelta_1$ is a pure simplicial complex of dimension $n-1$ and that a subset Z of W is in $F(\varDelta_1)$ if and only if there exists $\bar{Z} \in F(\varDelta)$ such that $\bar{Z} \cap W = Z$. If $Z \in F(\varDelta_1)$, we define $Z' = \min\{\bar{Z} \in F(\varDelta) : \bar{Z} \cap W = Z\}$, where the minimum is taken with respect to the total order of $F(\varDelta)$. We define a total order on $F(\varDelta_1)$ setting: $Z < Z_1 \Leftrightarrow Z' < Z_1'$ in $F(\varDelta)$, and show that this order gives the desired shelling.

Let $Z, Z_1 \in F(\varDelta_1)$ with $Z < Z_1$. By definition $Z' < Z_1'$ in $F(\varDelta)$. Since the total order on $F(\varDelta)$ is a shelling, there exists $H \in F(\varDelta)$ and $x \in Z_1'$ such that $H < Z_1'$, $\{x\} = Z_1' \setminus H$ and $Z_1' \cap Z' \subset Z_1' \cap H$. We note that $x \in Z_1$ since otherwise $H \cap W = Z_1$ and $H < Z_1'$, a contradiction with the definition of Z_1'. Let $Z_2 = H \cap W$; $Z_2 \in F(\varDelta_1)$ since $H \in F(\varDelta)$, $\{x\} = Z_1 \setminus Z_2$ and $Z_1 \cap Z \subset Z_1 \cap Z_2$. By definition, $Z_2' \leq H < Z_1'$ and therefore $Z_2 < Z_1$. □

Let $H_S(t)$ be the Hilbert series of a homogeneous K-algebra S (here the degrees of the generators are all 1). It is well-known that $H_S(t) = \sum_{i=0}^{s} h_i t^i/(1-t)^d$, where d is the dimension of S, $h_i \in \mathbf{Z}$, and $h_s \neq 0$. The vector $(h_0, \ldots, h_s)$ is called the h-vector of S. The McMullen-Walkup formula, see [5], is a combinatorial interpretation of the h-vector of the Stanley-Reisner ring associated with a shellable simplicial complex. Given a facet Z_1 of a shellable simplicial complex $\varDelta$, we set

$$C(Z_1) = \{x \in V : \text{there exists a facet } Z \text{ of } \varDelta \text{ such that } Z < Z_1 \text{ and } Z_1 \setminus Z = \{x\}\}.$$

Let $(h_0, \ldots, h_s)$ be the h-vector of the Stanley-Reisner ring associated with $\varDelta$. The McMullen-Walkup formula is:

$$h_i = |\{Z \text{ facet of } \varDelta : |C(Z)| = i\}|.$$

Under the assumption the previous lemma and with the notation introduced in the proof, we get:

LEMMA 1.11. *Let $Z_1 \in F(\varDelta_1)$, then $C(Z_1) = C(Z_1')$.*

Proof. Let $x \in C(Z_1)$, and $Z \in F(\Delta_i)$ such that $Z < Z_1$ and $Z_1 \setminus Z = \{x\}$. Then $Z' < Z_1'$; there exist $H \in F(\Delta)$ and $y \in V$ such that $H < Z_1'$, $Z' \cap Z_1' \subset H \cap Z_1' = Z_1' \setminus \{y\}$. By definition of Z_1', the restriction of H to W is not Z_1. Therefore we get $y = x$, and $C(Z_1) \subset C(Z_1')$.

Conversely, let $y \in C(Z_1')$, and $H \in F(\Delta)$ such that $H < Z_1'$ and $Z_1' \setminus H = \{y\}$. Again the restriction of H to W is not Z_1, and therefore $y \in W$. Let $Z = H \cap W$; $Z \in F(\Delta_1)$, and $Z < Z_1$ since $Z' \leq H < Z_1'$. Furthermore $Z_1 \setminus Z = \{y\}$, and we are done. □

PROPOSITION 1.12. *The simplicial complex $\Delta_\alpha(L)$ is shellable.*

Proof. Straightforward by 1.6 and 1.10. □

The Stanley-Reisner ring associated with a shellable simplicial complex is Cohen-Macaulay, [4]. It is well-known that if $R_\alpha(L)^*$ is Cohen-Macaulay, then $R_\alpha(L)$ Cohen-Macaulay too, see for instance [14] or [6]. Therefore from the shellability of $\Delta_\alpha(L)$ we deduce the Cohen-Macaylayness of $R_\alpha(L)$. By 1.3, $R_\alpha(L)$ is a domain, and we get the main theorem of this section:

THEOREM 1.13. *The ring $R_\alpha(L)$ is a Cohen-Macaulay domain.*

In particular the previous theorem gives an alternative proof of the Cohen-Macaulayness of the ring $R_\alpha(X)$, see [16].

2. Some applications

We present some applications of the results of the first section. First, following the approach of [5] and [11], we give a combinatorial interpretation of the h-vector of the determinantal rings $R_\alpha(X)$ in terms of families of non-intersecting paths. Secondly, we compute the a-invariant of the determinantal rings $R_t(X)$ in the homogeneous and weighted case. The same formula was obtained, independently and using different methods, by Barile, see [3]. Finally we study, as an interesting class of symmetric ladder determinantal rings, the determinantal ring associated with a matrix of indeterminates in which a submatrix is symmetric.

2.1. Characterization of the h-vector

We keep the notation of the first section. The h-vector of $R_\alpha(X)$ coincides with that of $R_\alpha(X)^* = K[X]/\mathrm{in}(I_\alpha(X))$ which is the Stanley-Reisner ring associated with the simplicial complex $\Delta_\alpha(X)$. We know that $\Delta_\alpha(X)$ is a shellable simplicial complex. Therefore, we may give a combinatorial interpretation of the h-vector of $R_\alpha(X)$ via the McMullen-Walkup formula. We need only to understand the set $C(Z) = \{x \in B :$ there exists a facet F of $\Delta_\alpha(X)$ such that $F < Z$ and $Z \setminus F = \{x\}\}$. We have seen that a facet Z of the simplicial complex $\Delta_\alpha(X)$ is the union of disjoint sets $Z_1, \ldots, Z_t$ where Z_k is a maximal chain of $X_k^+ = \{(i, j) \in B : \alpha_k \le i \le j\}$. We may interpret Z_k as a *path* from a point of the set $\{(\alpha_k, \alpha_k), (\alpha_k + 1, \alpha_k + 1), \ldots, (n, n)\}$ to the point (α_k, n). Therefore the facets of $\Delta_\alpha(X)$ are families of non-intersecting paths. The following picture represents a facet of $\Delta_\alpha(X)$ where $\alpha = \{1,3\}$ and $n = 5$.

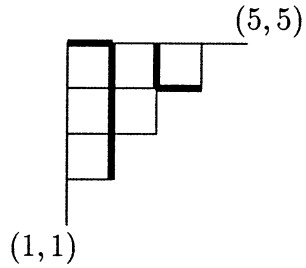


Fig. 3

By 1.11, we have $C(Z) = C(Z')$, where by definition $Z' = \min\{H : H$ is a facet of $\Delta'_\alpha(X)$, $H \cap B = Z\}$, and the minimum is taken with respect to the shelling of the facets of $\Delta'_\alpha(X)$. Suppose that Z is the family of non-intersecting paths $Z_1, \ldots, Z_t$ where Z_i is a path from (a_i, a_i) to (a_i, n) with $\alpha_i \le a_i$. Define H_i to be the path from (n, α_i) to (α_i, n) obtaining from Z_i by adding the set of points $\{(n, \alpha_i), (n-1, \alpha_i), \ldots, (a_i, \alpha_i), (a_i, \alpha_i + 1), \ldots, (a_i, a_i)\}$. Then, from the definition of the shelling of $\Delta'_\alpha(X)$, see [15, Th. 4.9], it is clear that Z' is the union of $H_1, \ldots, H_t$. In the following picture is represented the corresponding Z' of the facet in Fig. 3.

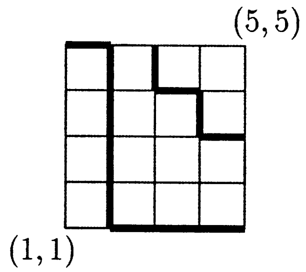


Fig. 4

Given a path P in A, a *corner* of P is an element $(i, j) \in P$ for which $(i-1, j)$ and $(i, j-1)$ belong to P as well. Let us denote by $c(P)$ the set of the corners of P. If H is a facet of $\Delta'_\alpha(X)$ and $H_1, \ldots, H_t$ is its decomposition as union of non-intersecting paths, then by, [5, 2.4], $C(H) = c(H_1) \cup \ldots \cup c(H_t)$. Thus, if Z is a facet of $\Delta_\alpha(X)$, then $C(Z)$ is the set of the corners of Z'. In our example of Fig. 3 and Fig. 4 we have $C(Z) = C(Z') = \{(2,5), (4,4)\}$.

Let P be a path from (b, b) to (a, n) in the poset B, and let (i, j) be a point of P. We define (i, j) to be an *s-corner* of P if $i < j$ and $(i-1, j)$, $(i, j-1)$ belong to P, or $i = j$ (in this case $i = b$) and $(i-1, j)$ belongs to P. Let us denote by $\mathrm{sc}(P)$ the set of the s-corners of the path P, and if Z is the family of non-intersecting paths $Z_1, \ldots, Z_t$ in B, define $\mathrm{sc}(Z) = \mathrm{sc}(Z_1) \cup \ldots \mathrm{sc}(Z_t)$. It is clear that the corners of Z' are exactly the s-corners of Z. Therefore we have:

LEMMA 2.1. *Let Z be a facet of $\Delta_\alpha(X)$, then $C(Z) = sc(Z)$.*

Using the McMullen-Walkup formula, we obtain the following characterization of the h-vector of the ring $R_\alpha(X)$:

PROPOSITION 2.2. *Let $(h_0, \ldots, h_s)$ be the h-vector of the ring $R_\alpha(X)$. Then h_i is the number of families of non-intersecting paths $Z_1, \ldots, Z_t$ in B with exactly i s-corners, where Z_k is a path from a point of the set $\{(\alpha_k, \alpha_k), \ldots, (n, n)\}$ to (α_k, n).*

EXAMPLES 2.3. (a) Let $\alpha = 1,3$ and $n = 4$. In this case $I_\alpha(X)$ is the ideal generated by the 2-minors of the first 2 rows and by all the 3-minors of a 4×4 symmetric matrix of indeterminates. The non-intersecting paths are the following:

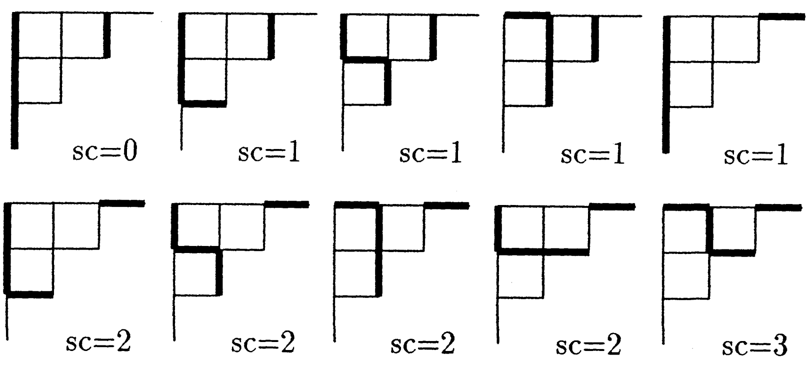


Fig. 5

Hence the h-vector of $R_\alpha(X)$ is (1, 4, 4, 1).
(b) Consider the ring $R_2(X)$, and denote by $h_0(n),\dots, h_s(n)$ its h-vector, where n is the size of the matrix X. Then $h_i(n)$ is the number of paths from one point of the set $\{(1,1),\dots, (n, n)\}$ to $(1, n)$ with i s-corners.

The number of the paths with i s-corners and which contain $(1, n-1)$ is $h_i(n-1)$. The number of those which contain $(3, n)$ is $h_i(n-1) - h_i(n-2)$. Finally, the number of those which contain $(2, n)$, $(2, n-1)$ is $h_{i-1}(n-2)$. Thus we get $h_i(n) = 2h_i(n-1) - h_i(n-2) + h_{i-1}(n-2)$. By induction on n, $h_i(n) = \binom{n}{2i}$.

(c) Now consider the ring $R_{n-1}(X)$ and denote by $h_0(n),\dots, h_s(n)$ its h-vector. By simple arguments as before one shows that $h_i(n) = 2h_{i-1}(n-1) - h_{i-1}(n-2) + c(n)$, with $c(n) = 1$ if $i \le n-2$ and $c(n) = 0$ otherwise. Then by induction, $h_i(n) = \binom{i+2}{2}$ if $i \le n-2$ and $h_i(n) = 0$ otherwise.

2.2. The a-invariant of $R_t(X)$

The a-invariant $a(S)$ of a positively graded Cohen-Macaulay K-algebra S is the negative of the least degree of a generator of its graded canonical module. It can be read off from the Hilbert series $H_S(t)$ of S; more precisely $a(S)$ is the pole order of the rational function $H_S(t)$ at infinity.

For the computation of the a-invariant we restrict our attention to the ring $R_t(X) = K[X]/I_t(X)$, and we consider the weighted case too. Suppose there are given degrees to the indeterminates, say $\deg X_{ij} = v_{ij}$, such that the minors of X are homogeneous. Then one has $2v_{ij} = v_{ii} + v_{jj}$. Therefore essentially there are two possible degree types:
Type (a): There exist $e_1,\dots, e_n \in \mathbf{N}\setminus\{0\}$ such that $\deg X_{ij} = e_i + e_j$ for all $1 \le i \le j \le n$.
Type (b): There exist $e_1,\dots, e_n \in \mathbf{N}$ such that $X_{ij} = e_i + e_j + 1$ for all $1 \le i \le j \le n$.
Since the ideals under consideration are invariant under rows and columns permutations we may always assume $e_1 \le \cdots \le e_n$.

Let us denote by Δ_t the simplicial complex $\Delta_\alpha(X)$, with $\alpha = \{1,\dots, t-1\}$. The Hilbert function of $R_t(X)$ and $K[\Delta_t] = K[X]/\mathrm{in}(I_t(X))$ coincide, thus we may as well compute the a-invariant of $K[\Delta_t]$. Since Δ_t is a shellable simplicial complex, Bruns-Herzog's proposition [5, 2.1] applies and we get:

THEOREM 2.4. *Let* $R = R_t(X)$. *In the case of degree type* (a):

$$a(R) = -(t-1)\left(\sum_{i=1}^{n} e_i\right) \qquad \text{if } n \equiv t \bmod (2)$$

$$a(R) = -(t-1)\left(\sum_{i=1}^{n} e_i\right) - \sum_{i=1}^{t-1} e_i \quad \text{if } n \not\equiv t \bmod (2)$$

And in the case of degree type (b):

$$a(R) = -(t-1)\left(\sum_{i=1}^{n} e_i + \frac{n}{2}\right) \qquad \text{if } n \equiv t \bmod (2)$$

$$a(R) = -(t-1)\left(\sum_{i=1}^{n} e_i + \frac{n+1}{2}\right) - \sum_{i=1}^{t-1} e_i \quad \text{if } n \not\equiv t \bmod (2)$$

Proof. By [5, 2.1], $a(R) = -\min\{\rho(Z) : Z \text{ is a facet of } \Delta_t\}$, where

$$\rho(Z) = \sum_{(i,j)\in Z\setminus C(Z)} \deg X_{ij}.$$

We define a facet F of Δ_t and prove that $\rho(F) \leq \rho(Z)$ for all the facets Z of Δ_t. Then the desired result will follow from the computation of $\rho(F)$.

For $i = 1, \dots, t-2$, let D_i be the set $\{(i, n), (i, n-1), \dots, (i, n-t+i+2)\}$, and set $D_{t-1} = \emptyset$.

If $n \equiv t \bmod(2)$, we define F_i to be the path from (i, n) to $((n-t)/2+i+1, (n-t)/2+i+1)$ which is obtained from D_i by adding the points $(i, n-t+i+1)$, $(i+1, n-t+i+1), \dots, (i+j, n-t+i-j+1)$, $(i+j+1, n-t+i-j+1), \dots, (i+(n-t)/2, (n-t)/2+i+1)$, $((n-t)/2+i+1, (n-t)/2+i+1)$.

If $n \not\equiv t \bmod(2)$, we define F_i to be the path from (i, n) to $((n-t+1)/2+i, (n-t+1)/2+i)$ which is obtained from D_i by adding the points $(i, n-t+i+1)$, $(i, n-t+1)$, $(i+1, n-t+i), \dots, (i+j, n-t+i-j)$, $(i+j+1, n-t+i-j), \dots, (i+(n-t-1)/2, (n-t+1)/2+i)$, $((n-t+1)/2+i, (n-t+1)/2+i)$.

Finally we define F to be the family of non-intersecting paths $F_1, \dots, F_{t-1}$.

The following picture illustrates F when $t = 4$ and $n = 8,9$.

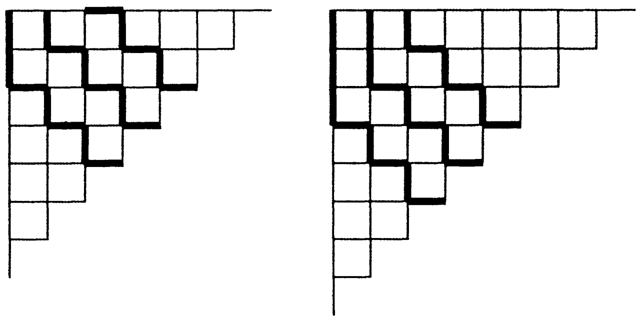


Fig. 6

We start considering $t = 2$ and n even. In this case $C(F) = \{(2, n), (3, n-1), \dots, (n/2+1, n/2+1)\}$, and therefore $F \setminus C(F) = \{(1, n), (2, n-1), \dots, (n/2, n/2+1)\}$. One has $\rho(F) = \sum_{i=1}^{n} e_i$ or $\rho(F) = \sum_{i=1}^{n} e_i + n/2$ if the degree is of type (a) or (b), respectively. Given Z a path from $(1, n)$ to (p, p) we claim that for all $i < p$ there exists j such that $(i, j) \in Z \setminus C(Z)$, and that for all the $i \geq p$ there exists j such that $(j, i) \in Z \setminus C(Z)$. From the claim it follows easily that $\rho(Z) \geq \rho(F)$. To prove the claim observe that if $i < p$ (resp. $i \geq p$) then there exists j such that $(i, j) \in Z$ (resp. $(j, i) \in Z$), and if $(i, j) \in C(Z)$, then $(i, j-1) \in Z \setminus C(Z)$ (resp. if $(j, i) \in C(Z)$, then $(j-1, i) \in Z \setminus C(Z)$).

If $t = 2$ and n is odd, we have $\rho(F) = \sum_{i=1}^{n} e_i + e_1$ or $\rho(F) = \sum_{i=1}^{n} e_i + e_1 + (n+1)/2$. Let Z be a path from $(1, n)$ to (p, p). Since n is odd we deduce from the previous claim that $|Z \setminus C(Z)| \geq (n+1)/2$, and that there exists i which appears twice as a coordinate of some elements in $Z \setminus C(Z)$. By assumption $e_1 \leq e_2 \leq \dots \leq e_n$, therefore $\rho(F) \leq \rho(Z)$.

Now let $t \geq 2$ and let Z be a facet of Δ_t, that is a family of non-intersecting paths $Z_1, \dots, Z_{t-1}$. Since the paths are non-intersecting, $D_k \subset Z_k$ for all $k = 1, \dots, t-1$. We may think of F_k and Z_k as paths starting from $(i, n-t+i+1)$, and argue as before to show that:

$$\sum_{(i,j) \in F_k \setminus SC(F_k)} \deg X_{ij} \leq \sum_{(i,j) \in Z_k \setminus SC(Z_k)} \deg X_{ij}$$

for all $k = 1, \dots, t-1$. Therefore we get:

$$\rho(F) = \sum_{k=1}^{t-1} \sum_{(i,j) \in F_k \setminus SC(F_k)} \deg X_{ij} \leq \sum_{k=1}^{t-1} \sum_{(i,j) \in Z_k \setminus SC(Z_k)} \deg X_{ij} = \rho(Z)$$

and we are done. □

The homogeneous case (all the indeterminates have degree 1) arises from a degree type (b) with $e_i = 0$ for all i. Therefore

$$a(R_t(X)) = \begin{cases} -(t-1)\dfrac{n}{2} & \text{if } n \equiv t \operatorname{mod}(2) \\ -(t-1)\dfrac{n+1}{2} & \text{if } n \not\equiv t \operatorname{mod}(2). \end{cases}$$

By a result of Goto [3], $R_t(X)$ is Gorenstein if and only if $n \equiv t \operatorname{mod}(2)$. If $n \not\equiv t \operatorname{mod}(2)$, the canonical module of $R_t(X)$ is the prime ideal P generated by all the $t-1$ minors of the first $t-1$ rows of X. It is not difficult to see that, up to shift, P is also the graded canonical module of $R_t(X)$. Hence the graded canonical ω_t module of $R_t(X)$ is:

$$\omega_t = \begin{cases} R_t(X)\left(-(t-1)\dfrac{n}{2}\right) & n \equiv t \operatorname{mod}(2) \\ P\left(-(t-1)\dfrac{n-1}{2}\right) & n \not\equiv t \operatorname{mod}(2) \end{cases}$$

2.3. Determinantal rings associated with a matrix in which a submatrix is symmetric

Let $Z = (Z_{ij})$ be an $m \times n$ matrix, $m \leq n$, whose entries are indeterminates such that the submatrix of the last s rows and of the first s columns is symmetric, with $s > 1$. Using the blocks notation, we write:

$$Z = \begin{pmatrix} M & N \\ S & P \end{pmatrix}$$

where $M = (M_{ij})$, $N = (N_{ij})$, $P = (P_{ij})$ are generic matrices of indeterminates of size $(m-s) \times s$, $(m-s) \times (n-s)$, $s \times (n-s)$, respectively, and $S = (S_{ij})$ is an $s \times s$ symmetric matrix of indeterminates. Denote by $K[Z]$ the polynomial ring over the field K whose indeterminates are the entries of Z.

Let $I_t(Z)$ be the ideal generated by all the t-minors of Z and denote by $R_t(Z)$ the ring $K[Z]/I_t(Z)$. If $s = m$, then Z is called a partially symetric matrix. When Z is partially symmetric, $R_t(Z)$ is essentially a ring of the class $R_\alpha(X)$, see [8, 2.5].

Next we will interpret $R_t(Z)$ as a ladder determinantal ring. To do this, we take two symmetric matrices of distinct indeterminates E_1, E_2, of size $(m-s) \times (m-s)$, $(n-s) \times (n-s)$. We construct an $(m+n-s) \times (m+n-s)$ symmetric matrix of indeterminates in the following way:

$$X = \begin{pmatrix} E_1 & M & N \\ M^t & S & P \\ N^t & P^t & E_2 \end{pmatrix}$$

Denote $A = \{(i, j) \in \mathbf{N}^2 : 1 \leq i, j \leq m+n-s\}$, and $V = \{(i, j) \in A : i \leq m$ and $j > m-s\}$. V is the semi-symmetric ladder of X corresponding to Z. The set $L = \{(i, j) \in A : i > m-s$ or $j < n-s\}$ is the symmetric ladder associated with V. Let $\alpha = \{1, \ldots, t-1\}$, then by construction and by 1.5 we have $I_\alpha(L) = I_\alpha(V) = I_t(Z)$ and $R_t(Z) = R_\alpha(L)$. Let us denote by $\Delta_t(Z)$ the simplicial complex $\Delta_\alpha(L)$.

Let τ' be the lexicographic term order on the monomials of $K[Z]$ induced by the variable order which is obtained listing the entries of Z as they appear row by row. Let J be the set of all the minors $[a_1, \ldots, a_t \mid b_1, \ldots, b_t]$ of Z (the indices refer to Z and not to X) such that $b_i - a_t \geq -m+s$. In other words J is the set of the t-minors of Z whose main diagonal does not lie under the main diagonal of S. By 1.3 and 1.13, it follows immediately:

PROPOSITION 2.5. (a) *The ring $R_t(Z)$ is a Cohen-Macaulay domain.*
(b) *J is a minimal system of generators of $I_t(Z)$ and a Gröbner basis with respect to τ'.*

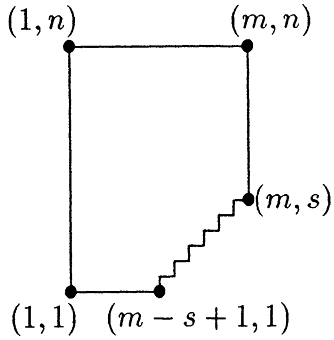


Fig. 7

In order to compute the dimension and multiplicity of $R_t(Z)$, we describe the simplicial complex $\Delta_t(Z)$. It seems more natural to use the labelling of Z instead of that of X, so that we can identify L^+ with the set $\{(i, j) \in \mathbf{N}^2 : 1 \leq i \leq m,\ 1 \leq j$

$\leq n, j - i \geq s - m\}$, see FIg. 7. Note that, in this case, L_i^+ is obtained from L_{i-1}^+ by deleting the lower border. Thus, if $i \leq s$, then $\mathrm{rk}(L_i^+) = (n + m - s + 1 - i)$, and if $i \geq s$, then $\mathrm{rk}(L_i^+) = (n + m - 2i + 1)$. Therefore, from 1.9, we get:

If $t \geq s$, then

$$\dim R_t(Z) = (n + m + 1 - t)(t - 1) - \frac{s(s-1)}{2}.$$

The dimension of the determinantal ring $R_t(X_1)$ associated with the ideal of the t-minors of an $m \times n$ generic matrix of indeterminates X_1 is $(n + m + 1 - t)$ $(t - 1)$, see [7, Cor. 5.12]. Therefore $R_t(Z)$ is nothing but a specialization of $R_t(X_1)$, that is $R_t(Z)$ is isomorphic to $R_t(X_1)/I$ where I is the ideal generated by the regular sequence of the $s(s-1)/2$ linear forms which give the symmetry relations on Z. Moreover, $R_t(Z)$ and $R_t(X_1)$ have the same multiplicity and the same h-vector.

If $t < s$, then:

$$\dim R_t(Z) = \left(n + m + 1 - s - \frac{1}{2}\right)(t - 1).$$

In this case we can interpret a facet of $\Delta_t(Z)$ as a family of non-intersecting paths $H_1, \ldots, H_{t-1}$ where H_i is a path from one point of the set $\{(m - s + 1, 1)(m - s + 2, 2), \ldots, (m, s)\}$ to (i, n). Let us denote by $P_i = (i, n)$ and $Q_j = (m - s + j, j)$. Given $1 \leq j_1 < \ldots, j_{t-1} \leq s$, according to [19, Sect. 2.7], the number of families of non-intersecting paths from $Q_{j_1}, \ldots, Q_{j_{t-1}}$ to $P_1, \ldots, P_{t-1}$ is $\det(W(P_h, Q_{j_k}))_{1 \leq h,k \leq t-1}$ where $W(P_h, Q_{j_k})$ is the number of paths from P_h to Q_{j_k}. But it is easy to see that

$$W(P_h, Q_{j_k}) = \binom{n + m - s - h}{n - j_k}.$$

Hence we get the following formula for the multiplicity of $R_t(Z)$:

$$e(R_t(Z)) = \sum_{1 \leq j_1 < \ldots < j_{t-1} \leq s} \det \left[\binom{n + m - s - h}{n - j_k} \right]_{1 \leq h,k \leq t-1}.$$

As we did for the ring $R_\alpha(X)$, we may give a combinatorial interpretation of the h-vector $R_t(Z)$ in terms of number of non-intersecting paths with a fixed number of certain corners. The case $t \geq s$, by the above discussion, is solved in [5].

Suppose $t < s$. A facet H of $\Delta_t(Z)$ is a family of non-intersecting paths $H_1, \ldots, H_{t-1}$, where H_i is a path from one point of set $\{(m - s + i, i), (m - s +$

$i+1, i+1), \dots, (m, s)\}$ to (i, n). We distinguish two cases:

If $s = m$, then $C(H) = \mathrm{sc}(H_1) \cup \dots \cup \mathrm{sc}(H_{t-1})$. This follows from the fact that when we consider $\Delta_t(Z)$ as a sub-complex of $\Delta_t(X)$, it has the following property: if H is a facet of $\Delta_t(Z)$ and $H_1 \in \Delta_t(X)$ with $H_1 < H$ in the shelling of $\Delta_t(X)$ and $H \setminus H_1 = \{(a, b)\}$, then $H_1 \in \Delta_t(Z)$. Therefore, if we denote by h_i the number of families of non-intersecting paths with exactly i s-corners, $(h_0, \dots, h_s)$ is the h-vector of $R_t(Z)$.

If $s < m$, then $C(H) = (\mathrm{sc}(H_1) \setminus \{T_1\}) \cup \dots \cup (\mathrm{sc}(H_{t-1}) \setminus \{T_{t-1}\})$, where T_i is the point $(m - s + i, i)$. This follows from the fact that when we consider $\Delta_t(Z)$ as a subcomplex of $\Delta_t(Z)$, if H_1 is a facet of $\Delta_t(X)$ such that $H_1 < H$ and $H \setminus H_1 = \{(a, b)\}$ then H_1 is in $\Delta_1(Z)$ unless $(a, b) = T_i$ for some i andT_i belongs to H_i.

For instance, consider the case in which $t = 4$, $s = 5$, $m = n = 6$. The two facets H and K in the following picture have s-corners respectively in $\{(2,1), (3,2), (5,6), (6,5)\}$, and $\{(2,3), (3,2), (5,4), (5,6), (6,5)\}$. It is clear from the picture that it is not possible to find a family of paths which differs from H only in (3.2) and that is earlier in the shelling. The point $T_1 = (2,1)$ (resp. $T_2 = (3,2)$) is not in $C(H)$ since it is an s-corner of H_1 (resp. H_2). The point (3,2) is in $C(K)$ since it is an s-corner but not of K_2. Hence $C(H) = \{(5,6), (6,5)\}$, and $C(K) = \{(2,3), (3,2), (5,4), (5,6), (6,5)\}$.

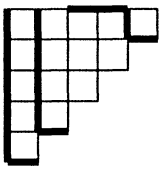
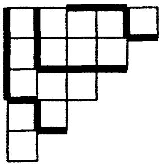
$H =$  $K =$ 

Fig. 8

Therefore, if we denote by h_i the number of families of non-intersecting paths H with $|(\mathrm{sc}(H_1) \setminus \{T_1\}) \cup \dots (\mathrm{sc}(H_{t-1}) \setminus \{T_{t-1}\})| = i$, then $(h_0, \dots, h_s)$ is the h-vector of $R_t(Z)$.

EXAMPLES 2.6. From the computation of the h-vector of $R_2(X)$ it follows immetiately: (a) If $s = m$ and $n = m + 1$, then $h_i(R_2(Z)) = \binom{n}{2i}$ if $i \neq 1$, and $h_1(R_2(Z)) = \binom{n}{2} - 1$.

(b) If $s+1=m=n$, then $h_i(R_2(Z))=\binom{n+1}{2i}$ if $i\neq 1$, and $h_1(R_2(Z))=\binom{n+1}{2}-2$.

If $s=m<n$, then the ring $R_t(Z)$ is essentially one of the class $R_\alpha(X)$, and in [10] we proved that it is always normal and that is Gorenstein if and only if $2m=n+t$. We now show:

THEOREM 2.7. *Let $s<m$, then* (a) *$R_t(Z)$ is a normal domain.*
(b) *$R_t(Z)$ is Gorenstein if and only if $t\geq s$ and $m=n$.*

Proof. (a) Let us consider the following two symmetric ladders of $X: L_1=\{(i,j)\in A: i>m-s \text{ or } j>m-s\}$, $L_2=\{(i,j)\,A: i<n-s \text{ or } j<n-s\}$. The ladder determinantal rings $R_t(L_1)$, $R_t(L_2)$ are the determinantal rings associated with the partially symmetric matrices Z_1 and Z_2, where:

$$Z_1=\begin{pmatrix} M & N\\ S & P\\ P^t & E_2\end{pmatrix}\qquad Z_2=\begin{pmatrix} E_1 & M & N\\ M^t & S & P\end{pmatrix}$$

Denote by Y_i the support of L_i. The set of the doset t-minors is a Gröbner basis of $I_t(X)$. Then the set $B(X)$ of the monomials in the set of indeterminates X_{ij}, $1\leq i\leq j\leq n+m-s$, which are not divisible by leading terms of t-minors form a K-basis of the ring $R_t(X)$. For the same reason the subset $B(Y_i)$ of $B(X)$ of the monomials in the set Y_i not divisible by leading terms of doset t-minors form a K-basis of the ring $R_t(L_i)$. A K-basis of $R_t(L_1)\cap R_t(L_2)$ is $B(Y_1)\cap B(L_2)$, but the last is also a K-basis of $R_t(Z)$. Hence $R_t(Z)=R_t(L_1)\cap R_t(L_2)$, and we conclude that $R_t(Z)$ is normal since $R_t(L_1)$ and $R_t(L_2)$ are.

(b) If $t\geq s$ and $m=n$, then $R_t(Z)$ is Gorenstein since it is a specialization of a Gorenstein ring, [7, 8.9].

To prove the converse we argue by induction on t. Let $t=2$; consider the residue class x of N_{1n-s} in $R_2(Z)$, and denote by D the set of the residue classes of the indeterminates in the first row and last column of Z, that is $M_{11},\dots,M_{1s}$, $N_{11},\dots,N_{1n-s},\dots,N_{m-sn-s}$, $P_{1n-s},\dots,P_{sn-s}$. Let $K[D]$ be the K-subalgebra of $R_2(Z)$ generated by D.

It is clear that $K[D][x^{-1}]=R_2(Z)[x^{-1}]$. Furthermore, we have the following relations $M_{1i}P_{jn-s}=S_{ji}N_{1n-s}=S_{ij}N_{1n-s}=M_{1j}P_{in-s} \bmod I_2(Z)$, for all $1\leq i,j\leq s$. By dimension considerations, $K[D][x^{-1}]$ is isomorphic to the polynomial ring

$$R[N_{11},\dots,N_{1n-s},\dots,N_{m-sn-s}][N_{1n-s}^{-1}]$$

over the ring R, where

$$R=K[M_{11},\dots,M_{1s},P_{1n-s},\dots,P_{sn-s}]/I,$$

and I is the ideal generated by the 2 minors of the matrix

$$\begin{pmatrix} M_{11} & \dots & M_{1s} \\ P_{1n-s} & \dots & P_{sn-s} \end{pmatrix}.$$

By assumption $R_2(Z)$ is Gorenstein. Therefore $R_2(Z)[x^{-1}]$ is Gorenstein and R is Gorenstein too. But this is possible only if $s=2$, [7, 8.9]. Then $R_2(Z)$ is a specialization of the determinantal ring associated with the ideal of the 2-minors of a generic $m \times n$ matrix. Therefore, by [7, 8.9], $m=n$. If $t>2$, we apply the usual inversion trick. After inversion of s_{11} the residue class of S_{11}, we get an isomorphism between $R_t(Z)[s_{11}^{-1}]$ and $R_{t-1}(Z_1)[T_1,\dots,T_{m+n-s}][T_1^{-1}]$, where the T_i are indeterminates and Z_1 is an $m-1 \times n-1$ matrix of indeterminates such that the submatrix of the last $s-1$ rows and first $s-1$ columns is symmetric (when $s=2$, Z_1 is generic). Since $R_t(Z)$ is Gorenstein, $R_{t-1}(Z_1)$ is Gorenstein and, by induction, $s-1 \le t-1$ and $m-1=n-1$. Therefore $s \le t$ and $n=m$. □

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