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COMPLEXES OF COUSIN TYPE AND MODULES OF GENERALIZED FRACTIONS

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0. Introduction

Let $\mathbf{R}$ be a commutative (Noetherian) ring, $\mathbf{M}$ an $\mathbf{R}$-module and let $\mathscr{F} = (\mathbf{F}_i)_{i\geq 0}$ be a filtration of $\mathrm{Spec}(\mathbf{R})$ which admits $\mathbf{M}$.

A complex of $\mathbf{R}$-modules is said to be of Cousin type if it satisfies the four conditions of ([GO], 3.2) which are reproduced below (Definition (1.5)). In ([RSZ], 3.4), Riley, Sharp and Zakeri proved that the complex, which is constructed from a chain of special triangular subsets defined in terms of $\mathscr{F}$ (Example (1.3)(3)), is of Cousin type for $\mathbf{M}$ with respect to $\mathscr{F}$ (Corollary (3.5)(2)). Gibson and O'carroll ([GO], 3.6) showed that the complex, which is obtained by means of a chain $\mathscr{U} = (\mathbf{U}_i)_{i\geq 1}$ of saturated triangular subsets and the filtration $\mathscr{G} = (\mathbf{G}_i)_{i\geq 0}$ induced by $\mathscr{U}$ and $\mathbf{M}$, is of Cousin type for $\mathbf{M}$ with respect to $\mathscr{G}$ (Corollary (3.5)(3)).

The purpose of this paper is to show that, when the complex is defined by a chain of triangular subsets, one can give a simpler criterion, consisting of only two conditions, for being of Cousin type (Theorem (3.1) and Corollary (3.2)). In fact, we prove that, for every complex induced by a chain of triangular subsets, the first and the second conditions of the definition of Cousin type hold (Remark (2.5)).

In ([RSZ], 3.3), Riley, Sharp and Zakeri proved that every complex of Cousin type for $\mathbf{M}$ with respect to $\mathscr{F}$ is isomorphic to the Cousin complex. Hence when we investigate the structure of a complex of Cousin type, it is useful to study the complex $\mathbf{C}(\mathscr{U}, \mathbf{M})$ of Cousin type which is constructed from special modules of generalized fractions (Corollary (3.5)) whose properties are well known.

We also get a refinement of the Exactness theorem ([SZ2], 3.3 and [O], 3.1) in our Proposition (2.13).

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1. Preliminaries

Throughout this paper, $\mathbf{R}$ is a commutative ring with identity and $\mathbf{M}$ is an $\mathbf{R}$-module. We use ^{T} to denote matrix transpose and $\mathbf{D}_n(\mathbf{R})$ to denote the set of all $n \times n$ lower triangular matrices over $\mathbf{R}$. For $\mathbf{H} \in \mathbf{D}_n(\mathbf{R})$, $|\mathbf{H}|$ denotes the determinant of $\mathbf{H}$. $\mathbf{N}$ denotes the set of positive integers.

DEFINITION (1.1) ([SZ1], 2.1). Let n be a positive integer. A non-empty subset $\mathbf{U}_n$ of $\mathbf{R}^n$ is said to be *triangular* if

(i) whenever $(a_1,\ldots,a_n) \in \mathbf{U}_n$, then $(a_1^{\alpha_1},\ldots,a_n^{\alpha_n}) \in \mathbf{U}_n$ for all choices of positive integers $\alpha_1,\ldots,\alpha_n$; and

(ii) whenever $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_n) \in \mathbf{U}_n$, then there exist $(c_1,\ldots,c_n) \in \mathbf{U}_n$ and $\mathbf{H}, \mathbf{K} \in \mathbf{D}_n(\mathbf{R})$ such that $\mathbf{H}[a_1 \ldots a_n]^T = [c_1 \ldots c_1]^T = \mathbf{K}[b_1 \ldots b_n]^T$.

DEFINITION (1.2) ([S4], 1.1 and 1.2). Let $\mathbf{R}$ be a ring and $\mathbf{M}$ an $\mathbf{R}$-module. A *filtration* of $\mathrm{Spec}(\mathbf{R})$ is a descending sequence $\mathcal{F} = (\mathbf{F}_i)_{i\geq 0}$ of subsets of $\mathrm{Spec}(\mathbf{R})$, so that

$$\mathrm{Spec}(\mathbf{R}) \supset \mathbf{F}_0 \supset \mathbf{F}_1 \supset \cdots \supset \mathbf{F}_i \supset \mathbf{F}_{i+1} \supset \cdots,$$

with the property that, for each $i \geq 0$, each member of $\mathbf{F}_i \setminus \mathbf{F}_{i+1}$ is a minimal member of $\mathbf{F}_i$ with respect to inclusion. We then set $\partial\mathbf{F}_i = \mathbf{F}_i \setminus \mathbf{F}_{i+1}$. We say that the filtration $\mathcal{F}$ *admits* an $\mathbf{R}$-module $\mathbf{M}$ if $\mathrm{Supp}(\mathbf{M}) \subset \mathbf{F}_0$. Let $\mathcal{F}_M = (\mathbf{F}_{\mathbf{M}i})_{i\geq 0}$ be the $\mathbf{M}$-height filtration of $\mathrm{Spec}(\mathbf{R})$, *i.e.*, $\mathbf{F}_{\mathbf{M}i} = \{\mathfrak{p} \in \mathrm{Supp}(\mathbf{M}) : \mathrm{ht}_{\mathbf{M}}\mathfrak{p} \geq i\}$.

We say that a sequence of elements $a_1,\ldots,a_n$ of $\mathbf{R}$ is a *poor* $\mathbf{M}$-*sequence* if a_i is not a zerodivisor on $\mathbf{M}/(a_1,\ldots,a_{i-1})\mathbf{M}$ for each $i = 1,\ldots,n$; it is an $\mathbf{M}$-*sequence* if, in addition, $\mathbf{M} \neq (a_1,\ldots,a_n)\mathbf{M}$.

EXAMPLE (1.3). Let $\mathbf{R}$ be a Noetherian ring. Then the following five non-empty sets are triangular subsets of $\mathbf{R}^n$.

(1) ([SZ1], 3.10) Let $\mathbf{M}$ be a finitely generated $\mathbf{R}$-module.

$(\mathbf{U}_r)_n = \{(a_1,\ldots,a_n) \in \mathbf{R}^n : a_1,\ldots,a_n$ forms a poor $\mathbf{M}$-sequence$\}$.

(2) (cf. [SZ2], 5.2) Suppose that $\mathbf{M}$ is a finitely generated $\mathbf{R}$-module.

$(\mathbf{U}_h)_n = \{(a_1,\ldots,a_n) \in \mathbf{R}^n : \mathrm{ht}_{\mathbf{M}}(a_1,\ldots,a_i)\mathbf{R} \geq i \quad (1 \leq i \leq n)\}$.

(3) ([RSZ], 2.3) Assume that $\mathbf{M}$ is an $\mathbf{R}$-module such that $\mathrm{Ass}(\mathbf{M})$ contains only finitely many minimal members.

$(\mathbf{U}_{\bar{h}})_n = \{(a_1,\ldots,a_n) \in \mathbf{R}^n :$ for each $i = 1,\ldots,n,\quad (a_1,\ldots,a_i)\mathbf{R} \not\subset \mathfrak{p}$ for all $\mathfrak{p} \in \partial\mathbf{F}_{i-1} \cap \mathrm{Supp}(\mathbf{M})\}$.

(4) ([C], 1.1) Suppose that $\mathbf{M}$ is a finitely generated $\mathbf{R}$-module of dimension d.

$(\mathbf{U}_s)_n = \{(a_1,\ldots,a_n) \in \mathbf{R}^n : \dim \mathbf{M}/(a_1,\ldots,a_i)\mathbf{M} = d - i \quad (1 \leq i \leq n)\}.$

(5) ([C], 1.2) Suppose that $(\mathbf{R}, \mathfrak{m})$ is a local ring and $\mathbf{M}$ is a finitely generated $\mathbf{R}$-module.

$(\mathbf{U}_f)_n = \{(a_1,\ldots,a_n) \in \mathbf{R}^n : a_1,\ldots, a_n$ is an f-regular sequence (See [SV], p. 252) with respect to $\mathbf{M}\}$.

$= \{(a_1,\ldots,a_n) \in \mathbf{R}^n : \frac{a_1}{1},\ldots,\frac{a_n}{1}$ in $\mathbf{R}_{\mathfrak{p}}$ forms an $\mathbf{M}_{\mathfrak{p}}$-sequence for all $\mathfrak{p} \in \mathrm{Supp}(\mathbf{M}) \setminus \{\mathfrak{m}\}$ such that $(a_1,\ldots,a_n)\mathbf{R} \subset \mathfrak{p}\}$.

For a given triangular subset $\mathbf{U}_n$ of $\mathbf{R}^n$, let $\bar{\mathbf{U}}_n = \{(a_1,\ldots,a_i, 1,\ldots, 1) \in \mathbf{R}^n :$ for all $i\ (0 \leq i \leq n)$, $\exists\ a_{i+1},\ldots, a_n \in \mathbf{R}$ s.t. $(a_1,\ldots,a_i, a_{i+1},\ldots, a_n) \in \mathbf{U}_n\}$. This is a triangular subset of $\mathbf{R}^n$ and is called the *expansion* of $\mathbf{U}_n$ ([SZ1], p. 38). Then, by ([SZ1], 3.2), we may assume without loss of the generality that $\mathbf{U}_n$ is *expanded, i.e.,* $\mathbf{U}_n = \bar{\mathbf{U}}_n$, when we consider the module of generalized fractions for $\mathbf{M}$ with respect to $\mathbf{U}_n$. So, from now on, we assume that every triangular subset is expanded by means of the expansion of itself.

For a fixed non-negative integer n, $\mathbf{U}_{n+1}^{-n-1}\mathbf{M}$ denotes the module of generalized fractions of $\mathbf{M}$ with respect to $\mathbf{U}_{n+1}$ ([SZ1]). The other notation and terminology about the module of generalized fractions follow ([SZ1]).

Definition (1.4) ([RSZ], p. 52). Let $\mathbf{R}$ be a ring. A family $\mathcal{U} = (\mathbf{U}_i)_{i \geq 1}$ is called a *chain of triangular subsets* on $\mathbf{R}$ if the following conditions are satisfied:

(i) $\mathbf{U}_i$ is a triangular subset of $\mathbf{R}^i$ for all $i \in \mathbf{N}$;

(ii) $(1) \in \mathbf{U}_1$;

(iii) whenever $(a_1,\ldots,a_i) \in \mathbf{U}_i$ with $i \in \mathbf{N}$, then $(a_1,\ldots,a_i, 1) \in \mathbf{U}_{i+1}$; and

(iv) whenever $(a_1,\ldots,a_i) \in \mathbf{U}_i$ with $1 < i \in \mathbf{N}$, then $(a_1,\ldots,a_{i-1}) \in \mathbf{U}_{i-1}$.

Each $\mathbf{U}_i$ leads to a module of generalized fractions $\mathbf{U}_i^{-i}\mathbf{M}$ and we can obtain a complex

$$0 \xrightarrow{e^{-1}} \mathbf{M} \xrightarrow{e^0} \mathbf{U}_1^{-1}\mathbf{M} \xrightarrow{e^1} \mathbf{U}_2^{-2}\mathbf{M} \to \cdots \to \mathbf{U}_i^{-i}\mathbf{M} \xrightarrow{e^i} \mathbf{U}_{i+1}^{-i-1}\mathbf{M} \to \cdots,$$

denoted by $\mathbf{C}(\mathcal{U}, \mathbf{M})$, for which $e^0(m) = \frac{m}{(1)}$ for all $m \in \mathbf{M}$ and

$$e^i\left(\frac{x}{(a_1,\ldots,a_i)}\right) = \frac{x}{(a_1,\ldots,a_i, 1)}$$

for all $i \in \mathbf{N}$, $x \in \mathbf{M}$ and $(a_1,\ldots,a_i) \in \mathbf{U}_i$.

$H_U^i(\mathbf{M})$ denotes the i-th cohomology group of $\mathbf{C}(\mathcal{U}, \mathbf{M})$. That is $H_U^i(\mathbf{M}) = \mathrm{Ker}\, e^i / \mathrm{Im}\, e^{i-1}$.

DEFINITION (1.5) ([GO], 3.2). Let $\mathbf{R}$ be a Noetherian ring and $\mathbf{M}$ an $\mathbf{R}$-module. Let $\mathcal{F} = (\mathbf{F}_i)_{i\geq 0}$ be a filtration of Spec($\mathbf{R}$) that admits $\mathbf{M}$. A complex $\mathbf{X}^{\cdot} = \{\mathbf{X}^i : i \geq -2\}$ of $\mathbf{R}$-modules and $\mathbf{R}$-homomorphisms is said to be of *Cousin type for* $\mathbf{M}$ *with respect to* $\mathcal{F}$ if it has the form

$$0 \xrightarrow{d^{-2}} \mathbf{M} \xrightarrow{d^{-1}} \mathbf{X}^0 \xrightarrow{d^0} \mathbf{X}^1 \to \cdots \to \mathbf{X}^i \xrightarrow{d^i} \mathbf{X}^{i+1} \to \cdots$$

and satisfies the following, for each $n \in \mathbf{N} \cup \{0\}$,

(i) $\mathrm{Supp}(\mathbf{X}^n) \subset \mathbf{F}_n$;
(ii) $\mathrm{Supp}(\mathrm{Coker}\ d^{n-2}) \subset \mathbf{F}_n$;
(iii) $\mathrm{Supp}(\mathrm{Ker}\ d^{n-1}/\mathrm{Im}\ d^{n-2}) \subset \mathbf{F}_{n+1}$; and
(iv) The natural $\mathbf{R}$-homomorphism $\xi(\mathbf{X}^n) : \mathbf{X}^n \to \bigoplus_{\mathfrak{p}\in\partial\mathbf{F}_n} (\mathbf{X}^n)_{\mathfrak{p}}$, such that, for $x \in \mathbf{X}^n$ and $\mathfrak{p} \in \partial\mathbf{F}_n$, the component of $\xi(\mathbf{X}^n)(x)$ in the summand $(\mathbf{X}^n)_{\mathfrak{p}}$ is $x/1$, is an isomorphism.

LEMMA (1.6). *Let* $\mathbf{R}$ *be a ring and* $\mathbf{M}$ *an* $\mathbf{R}$*-module. Let* $\mathbf{U}_n$ *be an expanded triangular subset of* $\mathbf{R}^n$. *Let* $(a_1,\ldots,a_n)$ *and* $(b_1,\ldots,b_n)$ *be elements of* $\mathbf{U}_n$ *such that* $\mathbf{H}[a_1 \ldots a_n]^T = [b_1 \ldots b_n]^T$ *for some* $\mathbf{H} \in \mathbf{D}_n(\mathbf{R})$. *Then we have*

(1) ([SZ1], 2.8 *and* 3.3(i)) $\dfrac{m}{(a_1,\ldots,a_n)} = \dfrac{|\mathbf{H}|\, m}{(b_1,\ldots,b_n)}$ *and* $\dfrac{a_n m}{(a_1,\ldots,a_n)} = \dfrac{m}{(a_1,\ldots,a_{n-1}, 1)}$ *in* $\mathbf{U}_n^{-n}\mathbf{M}$.

(2) ([SZ1, 3.3(ii)] *and* [SY, 2.2]) *If* $m \in (a_1,\ldots,a_{n-1})\mathbf{M}$ *then* $\dfrac{m}{(a_1,\ldots,a_n)} = 0$ *in* $\mathbf{U}_n^{-n}\mathbf{M}$. *In particular, if each element of* $\mathbf{U}_n$ *is a poor* $\mathbf{M}$*-sequence, then the converse holds.*

(3) ([SZ2], 5.1 *and* [SZ3], 2.1) $\mathrm{Ann}_{\mathbf{R}}\left(\dfrac{m}{(a_1,\ldots,a_n)}\right) = \mathrm{Ann}_{\mathbf{R}}\left(\dfrac{m}{(a_1,\ldots,a_{n-1}, 1)}\right)$.

LEMMA (1.7) ([C], 2.4). *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Noetherian local ring and let* $\mathbf{M}$ *be a finitely generated* $\mathbf{R}$*-module of dimension* d. *Let* $(\mathbf{U}_s)_{d+1}$ *be the expansion of the triangular subset* $\{(a_1,\ldots,a_d, 1) \in \mathbf{R}^{d+1} : \dim \mathbf{M}/(a_1,\ldots,a_d)\mathbf{M} = 0\}$. *Let* $\{x_1,\ldots, x_d\}$ *be a fixed system of parameters for* $\mathbf{M}$. *Then we have*

$$(\mathbf{U}_s)_{d+1}^{-d-1}\mathbf{M} \cong \mathbf{U}(x)_d[1]^{-d-1}\mathbf{M} \cong \mathbf{H}_{\mathfrak{m}}^d(\mathbf{M}),$$

where $\mathbf{U}(x)_d[1] = \{(x_1^{\alpha_1},\ldots, x_d^{\alpha_d}, 1) \in \mathbf{R}^{n+1}$: *there is* $i\ (0 \leq i \leq d)$ *such that* $\alpha_1,\ldots, \alpha_i \in \mathbf{N}$ *and* $\alpha_{i+1} = \cdots = \alpha_d = 0\}$.

LEMMA (1.8) ([GO], 3.4). *Let* $\mathbf{R}$ *be a ring. For a positive integer* n, *suppose that* $\dfrac{m}{(a_1,\ldots,a_n, 1)} = 0$ *in* $\mathbf{U}_{n+1}^{-n-1}\mathbf{M}$. *Then there exist* $(b_1,\ldots, b_{n+1}) \in \mathbf{U}_{n+1}$ *and* $\mathbf{H} \in \mathbf{D}_n(\mathbf{R})$ *such that* $\mathbf{H}[a_1 \ldots a_n]^T = [b_1 \ldots b_n]^T$ *and* $b_{n+1} | \mathbf{H} | m \in (b_1,\ldots, b_n)\mathbf{M}$.

LEMMA (1.9) ([GO], 3.3 and [SY], 2.7). *Let* $\mathbf{R}$ *be a ring and* $\mathbf{M}$ *an* $\mathbf{R}$-*module. Let* $\mathcal{U} = (\mathbf{U}_i)_{i\geq 1}$ *be a chain of triangular subsets on* $\mathbf{R}$. *Then in* $\mathbf{C}(\mathcal{U}, \mathbf{M})$, *for all* $n \in \mathbf{N}$

$$\mathrm{Coker}\ e^{n-1} \cong \mathbf{U}_n^{-n}\mathbf{M}/\mathrm{Im}\ e^{n-1} \cong \mathbf{U}_n[1]^{-n-1}\mathbf{M},$$

where $\mathbf{U}_n[1] = \{(a_1,\ldots,a_n, 1) \in \mathbf{R}^{n+1} : (a_1,\ldots,a_n) \in \mathbf{U}_n\}$.

2. Associated prime ideals of modules of generalized fractions

LEMMA (2.1). *Let* $\mathbf{R}$ *be a ring and* $\mathbf{M}$ *an* $\mathbf{R}$-*module. Fix a positive integer* n. *Let* $\mathbf{U}_n$ *be a triangular subsets of* $\mathbf{R}^n$. *Let* $0 \neq \dfrac{m}{(a_1,\ldots,a_n)} \in \mathbf{U}_n^{-n}\mathbf{M}$. *Then we have, for all* $(b_1,\ldots,b_n) \in \mathbf{U}_n$,

$$(b_1,\ldots,b_n)\mathbf{R} \not\subset \left(0 : \frac{m}{(a_1,\ldots,a_n)}\right).$$

Proof. Suppose that for some $(b_1,\ldots, b_n) \in \mathbf{U}_n$

$$(b_1,\ldots,b_n)\mathbf{R} \subset \left(0 : \frac{m}{(a_1,\ldots,a_n)}\right).$$

Then by the definition of triangular subset there are $(c_1,\ldots, c_n) \in \mathbf{U}_n$ and $\mathbf{H}$, $\mathbf{K} \in \mathbf{D}_n(\mathbf{R})$ such that $\mathbf{H}[a_1 \ldots a_n]^T = [c_1 \ldots c_n]^T = \mathbf{K}[b_1 \ldots b_n]^T$. Hence we get $(c_1,\ldots, c_n)\mathbf{R} \subset (b_1,\ldots, b_n)\mathbf{R}$.

On the other hand, by Lemma (1.6)(1)(3) we have

$$\left(0 : \frac{m}{(a_1,\ldots,a_n)}\right) = \left(0 : \frac{|\mathbf{H}|m}{(c_1,\ldots,c_n)}\right) = \left(0 : \frac{|\mathbf{H}|m}{(c_1,\ldots,c_{n-1}, 1)}\right)$$
$$\supset (b_1,\ldots,b_n)\mathbf{R} \supset (c_1,\ldots,c_n)\mathbf{R}.$$

Therefore we have the following contradiction.

$$\frac{c_n\,|\,\mathbf{H}\,|\,m}{(c_1,\ldots,c_n)} = \frac{|\,\mathbf{H}\,|\,m}{(c_1,\ldots,c_{n-1},\,1)} = 0.$$

From now on, we suppose that $\mathbf{U}_0[1]^{-1}\mathbf{M} = \mathbf{M}$, $\mathbf{U}_0^0\mathbf{M} = \mathbf{M}$ and n is a non-negative integer.

LEMMA (2.2). *Let* $\mathbf{R}$ *and* $\mathbf{M}$ *be as above. Then in* $\mathbf{C}(\mathcal{U}, \mathbf{M})$ *we have*

$$\mathrm{Supp}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) \subset \mathrm{Supp}(\mathbf{U}_n[1]^{-n-1}\mathbf{M}) \subset \mathrm{Supp}(\mathbf{U}_n^{-n}\mathbf{M}).$$

Proof. For the first half, this follows from the following short exact sequence

$$\begin{array}{ccccccc} 0 \to \mathrm{Ker}\, e^n/\mathrm{Im}\, e^{n-1} & \to & \mathbf{U}_n^{-n}\mathbf{M}/\mathrm{Im}\, e^{n-1} & \to & \mathbf{U}_n^{-n}\mathbf{M}/\mathrm{Ker}\, e^n & \to 0, \\ & & \| & & \| & \\ & & \mathbf{U}_n[1]^{-n-1}\mathbf{M} & & \mathrm{Im}\, e^n & \end{array}$$

since $\mathrm{Supp}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) = \mathrm{Supp}(\mathrm{Im}\, e^n)$ by Lemma (1.6)(3).

For the second inclusion, it follows from Lemma (1.9) that

$$\mathrm{Supp}(\mathbf{U}_n[1]^{-n-1}\mathbf{M}) = \mathrm{Supp}(\mathbf{U}_n^{-n}\mathbf{M}/\mathrm{Im}\, e^{n-1}) \subset \mathrm{Supp}(\mathbf{U}_n^{-n}\mathbf{M}).$$

EXAMPLE (2.3). In general, $\mathrm{Supp}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) \neq \mathrm{Supp}(\mathbf{U}_u[1]^{-n-1}\mathbf{M})$. Let $(\mathbf{R}, \mathfrak{m})$ be a Noetherian local ring. Suppose that $\mathbf{M}$ is an f-module (see [SZ4], 1.8(ii)) of dimension d. Then $\mathrm{Supp}((\mathbf{U}_f)_d[1]^{-d-1}\mathbf{M}) = \mathrm{Supp}((\mathbf{U}_s)_{d+1}^{-d-1}\mathbf{M}) = \{\mathfrak{m}\}$. But $\mathrm{Supp}((\mathbf{U}_f)_{d+1}^{-d-1}\mathbf{M}) = \emptyset$ by ([C], 2.3).

LEMMA (2.4). *Let* $\mathbf{R}$ *and* $\mathbf{M}$ *be as above. Then in* $\mathbf{C}(\mathcal{U}, \mathbf{M})$ *we have*

$$\mathrm{Supp}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) \subset \mathrm{Supp}(\mathbf{U}_n[1]^{-n-1}\mathbf{M}) \subset \mathbf{F}_{\mathbf{M}n} \subset \mathbf{F}_n.$$

Proof. This follows from Lemma (2.2), ([HS], 3.1) and ([C], 2.7).

Remark (2.5). Lemma (2.4) shows that, for every complex $\mathbf{C}(\mathcal{U}, \mathbf{M})$, the first and the second conditions of the definiton of Cousin type hold by Lemma (1.9).

LEMMA (2.6). *Let* $\mathbf{R}$ *and* $\mathbf{M}$ *be as above. Then in* $\mathbf{C}(\mathcal{U}, \mathbf{M})$ *we have the following.*

(1) $\partial\mathbf{F}_n \cap \mathrm{Supp}(\mathbf{M}) = (\cup_{i=0}^{n}\partial\mathbf{F}_{\mathbf{M}i}) \cap \partial\mathbf{F}_n$.

(2) (cf. [ST], 2.7) $\partial\mathbf{F}_n \cap \mathrm{Supp}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) \subset \partial\mathbf{F}_n \cap \mathrm{Supp}(\mathbf{U}_n[1]^{-n-1}\mathbf{M}) \subset \partial\mathbf{F}_n \cap \partial\mathbf{F}_{\mathbf{M}n}$.

(3) $\partial\mathbf{F}_n \cap \partial\mathbf{F}_{\mathbf{M}n} = \cup_{\mathfrak{q}\in\partial\mathbf{F}_{n-1}\cap\partial\mathbf{F}_{\mathbf{M}(n-1)}}(V(\mathfrak{q}) \cap \partial\mathbf{F}_n \cap \partial\mathbf{F}_{\mathbf{M}n})$.

Proof. (1) Let $\mathfrak{p} \in \partial\mathbf{F}_n \cap \mathrm{Supp}(\mathbf{M})\setminus \cup_{i=0}^{n} \partial\mathbf{F}_{\mathbf{M}i}$. Hence $\mathrm{ht}_{\mathbf{M}}\mathfrak{p} > n$. Therefore there is $\mathfrak{q} \in \partial\mathbf{F}_{\mathbf{M}n}(\subset \mathbf{F}_n)$ such that $\mathfrak{q} \subsetneqq \mathfrak{p}$. That is, $\mathfrak{p}$ is not minimal in $\mathbf{F}_n$.

(2) Since $\mathrm{Supp}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) \subset \mathrm{Supp}(\mathbf{U}_n[1]^{-n-1}\mathbf{M}) \subset \mathbf{F}_{\mathbf{M}n}$, we have

$$\partial\mathbf{F}_n \cap \mathrm{Supp}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) \subset \partial\mathbf{F}_n \cap \mathrm{Supp}(\mathbf{U}_n[1]^{-n-1}\mathbf{M}) \subset \partial\mathbf{F}_n \cap \mathrm{Supp}(\mathbf{M}) \cap \mathbf{F}_{\mathbf{M}n}$$
$$\subset (\bigcup_{i=0}^{n} \partial\mathbf{F}_{\mathbf{M}i}) \cap \partial\mathbf{F}_n \cap \mathbf{F}_{\mathbf{M}n} = \partial\mathbf{F}_n \cap \partial\mathbf{F}_{\mathbf{M}n}$$

by (1).

(3) Let $\mathfrak{p} \in \partial\mathbf{F}_n$ and $\mathrm{ht}_{\mathbf{M}}\mathfrak{p} = n$. Suppose that $\mathfrak{q} \notin \partial\mathbf{F}_{n-1}$ for some $\mathfrak{q} \in \mathrm{Supp}(\mathbf{M})$ such that $\mathrm{ht}_{\mathbf{M}}\mathfrak{q} = n-1$ and $\mathfrak{q} \subsetneqq \mathfrak{p}$. Hence $\mathfrak{q} \in \mathbf{F}_n$, since $\partial\mathbf{F}_{n-1} = \mathbf{F}_{n-1} \setminus \mathbf{F}_n$ and $\mathbf{F}_{\mathbf{M}(n-1)} \subset \mathbf{F}_{n-1}$. This contradicts that $\mathfrak{p}$ is a minimal element in $\mathbf{F}_n$.

LEMMA (2.7). *Let* $\mathbf{R}$ *be a ring and* $\mathbf{M}$ *an* $\mathbf{R}$*-molule. Then in* $\mathbf{C}(\mathcal{U}, \mathbf{M})$, *for each* $\dfrac{m}{(a_1,\ldots,a_n)} + \mathrm{Im}\, e^{n-1} \in H_U^n(\mathbf{M})$, *there are* $(b_1,\ldots, b_{n+1}) \in \mathbf{U}_{n+1}$ *and* $\mathbf{H} \in \mathbf{D}_n(\mathbf{R})$ *such that* $\mathbf{H}[a_1 \ldots a_n]^T = [b_1 \ldots b_n]^T$ *and*

$$(b_1,\ldots, b_{n+1})\mathbf{R} \subset \left(\mathrm{Im}\, e^{n-1} : \frac{m}{(a_1,\ldots, a_n)}\right).$$

Proof. Since $\dfrac{m}{(a_1,\ldots, a_n)} \in \mathrm{Ker}\, e^n$, we have $\dfrac{m}{(a_1,\ldots, a_n, 1)} = 0$ in $\mathbf{U}_{n+1}^{-n-1}\mathbf{M}$. Hence by Lemma (1.8) there are $(b_1,\ldots, b_{n+1}) \in \mathbf{U}_{n+1}$ and $\mathbf{H} \in \mathbf{D}_n(\mathbf{R})$ such that $\mathbf{H}[a_1 \ldots a_n]^T = [b_1 \ldots b_n]^T$ and $b_{n+1}|\mathbf{H}|m \in (b_1,\ldots, b_n)\mathbf{M}$. Therefore we have

$$(b_1,\ldots, b_{n+1})\mathbf{R} \subset \left(\mathrm{Im}\, e^{n-1} : \frac{|\mathbf{H}|m}{(b_1,\ldots, b_n)}\right) = \left(\mathrm{Im}\, e^{n-1} : \frac{m}{(a_1,\ldots, a_n)}\right).$$

LEMMA (2.8). *Let* $\mathbf{R}$ *be a ring and* $\mathbf{M}$ *an* $\mathbf{R}$*-module. Let* $\mathcal{U} = (\mathbf{U}_i)_{i\geq 1}$ *be a chain of triangular subsets on* $\mathbf{R}$. *Then in* $\mathbf{C}(\mathcal{U}, \mathbf{M})$, *for a fixed non-negative integer* n, *we have the following.*

(1) $\mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) \cap \mathrm{Supp}(\mathbf{U}_{n+2+i}^{-n-2-i}\mathbf{M}) = \emptyset$ *for all* $i \geq 0$.

(2) $\mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) \cap \mathrm{Supp}(\mathbf{U}_{n+1+i}[1]^{-n-2-i}\mathbf{M}) = \emptyset$ *for all* $i \geq 0$.

(3) $\mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) = \mathrm{Ass}(\mathrm{Im}\, e^n) = \mathrm{Ass}(\mathrm{Ker}\, e^{n+1})$.

(4) $\mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) \cap \mathrm{Supp}(H_U^{n+i}(\mathbf{M})) = \emptyset$ *for all* $i \geq 0$.

(5) $\mathrm{Ass}(H_U^n(\mathbf{M})) \subset \mathrm{Ass}(\mathbf{U}_n[1]^{-n-1}\mathbf{M}) \subset \mathrm{Ass}(H_U^n(\mathbf{M})) \cup \mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M})$.

(6) *If* $\mathbf{R}$ *is Noetherian, then*

$$\partial\mathbf{F}_n \cap \mathrm{Ass}(\mathbf{U}_n[1]^{-n-1}\mathbf{M}) = (\partial\mathbf{F}_n \cap \mathrm{Ass}(H_U^n(\mathbf{M}))) \cup (\partial\mathbf{F}_n \cap \mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M})).$$

(7) $\mathrm{Ass}(\mathbf{U}_n[1]^{-n-1}\mathbf{M}) \cap \mathrm{Ass}(\mathbf{U}_{n+1}[1]^{-n-2}\mathbf{M}) \subset \mathrm{Ass}(H_U^n(\mathbf{M}))$.

Proof. (1) and (2) follow from Lemma (2.1) and Lemma (1.6)(2).

(3) Since $\mathrm{Im}\, e^{n} \subset \mathrm{Ker}\, e^{n+1} \subset \mathbf{U}_{n+1}^{-n-1}\mathbf{M}$, this follows from Lemma (1.6)(3).

(4) This follows from Lemma (2.1), Lemma (2.7) and Lemma. (1.6)(2).

(5) The following short exact sequence and (3) complete the proof.

$$(*)\qquad \begin{array}{ccccccccc} 0 \to & \mathrm{Ker}\, e^{n}/\mathrm{Im}\, e^{n-1} & \to & \mathbf{U}_n^{-n}\mathbf{M}/\mathrm{Im}\, e^{n-1} & \to & \mathbf{U}_n^{-n}\mathbf{M}/\mathrm{Ker}\, e^{n} & \to 0. \\ & \| & & \| & & \| & \\ & H_U^n(\mathbf{M}) & & \mathbf{U}_n[1]^{-n-1}\mathbf{M} & & \mathrm{Im}\, e^{n} & \end{array}$$

(6) By Lemma (2.4), we have

$$\partial\mathbf{F}_n \cap \mathrm{Supp}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) = \partial\mathbf{F}_n \cap \mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) \subset \partial\mathbf{F}_n \cap \mathrm{Ass}(\mathbf{U}_n[1]^{-n-1}\mathbf{M}).$$

Hence the assertion follows from (5).

(7) This follows from (1), (4) and (5).

Remark (2.9). If we also change associated prime to weakly associated in the sense of ([B], p. 289 ex. 17), then we can omit the Noetherian condition of Proposition (2.8)(6).

PROPOSITION (2.10). *Let* $\mathbf{R}$ *and* $\mathbf{M}$ *be as above. Assume that* $\mathfrak{p} \in \mathrm{Spec}(\mathbf{R})$. *In* $\mathbf{C}(\mathcal{U}, \mathbf{M})$, *consider the following statements*:

(i) *For all* $(a_1,\ldots,a_{n+1}) \in \mathbf{U}_{n+1}$, $(a_1,\ldots,a_{n+1})\mathbf{R} \not\subset \mathfrak{p}$;

(ii) $(\mathbf{U}_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}} \cong (\mathbf{U}_n[1]^{-n-1}\mathbf{M})_{\mathfrak{p}}$;

(ii′) $(H_U^n(\mathbf{M}))_{\mathfrak{p}} = 0$ *and* $(\mathbf{U}_{n+1}[1]^{-n-2}\mathbf{M})_{\mathfrak{p}} = 0$;

(iii) $(\mathbf{U}_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}} \cong (\mathrm{Im}\, e^{n})_{\mathfrak{p}}$;

(iii′) $(\mathbf{U}_{n+1}[1]^{-n-2}\mathbf{M})_{\mathfrak{p}} = 0$;

(iii″) $(H_U^{n+1}(\mathbf{M}))_{\mathfrak{p}} = 0$ *and* $(\mathbf{U}_{n+2}^{-n-2}\mathbf{M})_{\mathfrak{p}} = 0$;

(iv) $(\mathrm{Ker}\, e^{n+1})_{\mathfrak{p}} \cong (\mathrm{Im}\, e^{n})_{\mathfrak{p}}$;

(iv′) $(\mathbf{U}_{n+1}[1]^{-n-2}\mathbf{M})_{\mathfrak{p}} \cong (\mathrm{Im}\, e^{n+1})_{\mathfrak{p}}$.

Then we have the following.

(1) (ii) $\Leftrightarrow$ (ii′).

(2) (iii) $\Leftrightarrow$ (iii′) $\Leftrightarrow$ (iii″).

(3) (iv) $\Leftrightarrow$ (iv′).

(4) (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). *That is, if* (i) *holds, then*

$$(\mathbf{U}_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}} \cong (\mathbf{U}_n[1]^{-n-1}\mathbf{M})_{\mathfrak{p}} \cong (\mathrm{Im}\, e^{n})_{\mathfrak{p}} \cong (\mathrm{Ker}\, e^{n+1})_{\mathfrak{p}}.$$

(5) *If* $\mathfrak{p} \in \mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M})$, *then the above four modules are isomorphic.*

(6) *If* $\mathfrak{p} \notin \mathrm{Supp}(\mathbf{U}_{n+2}^{-n-2}\mathbf{M})$, *then* (iv) $\Rightarrow$ (iii).

Proof. (1) Using the short exact exact sequence ($*$), we prove as follows.
($\Rightarrow$) Assume that $(\mathbf{U}_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}} \cong (\mathbf{U}_n[1]^{-n-1}\mathbf{M})_{\mathfrak{p}}$. Then, from the following short exact sequence

$$0 \to \operatorname{Im} e^{n-1} \to \mathbf{U}_n^{-n}\mathbf{M} \to \mathbf{U}_n^{-n}\mathbf{M}/\operatorname{Im} e^{n-1} \to 0, \quad \mathbf{U}_n^{-n}\mathbf{M}/\operatorname{Im} e^{n-1} = \mathbf{U}_n[1]^{-n-1}\mathbf{M}$$

we have a commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 \to & (\operatorname{Im} e^{n-1})_{\mathfrak{p}} & \to & (\mathbf{U}_n^{-n}\mathbf{M})_{\mathfrak{p}} & \to & (\mathbf{U}_n[1]^{-n-1}\mathbf{M})_{\mathfrak{p}} & \to 0 \\ & \| & & \| & & \| & \\ 0 \to & (\operatorname{Im} e^{n-1})_{\mathfrak{p}} & \to & (\mathbf{U}_n^{-n}\mathbf{M})_{\mathfrak{p}} & \xrightarrow{e^n_{\mathfrak{p}}} & (\mathbf{U}_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}} & \to 0. \end{array}$$

Therefore we get

$$(\operatorname{Ker} e^n)_{\mathfrak{p}} = (\operatorname{Im} e^{n-1})_{\mathfrak{p}}.$$

Hence, from the following short exact sequence

$$\begin{array}{ccccccc} 0 \to (H_U^n(\mathbf{M}))_{\mathfrak{p}} & \to & (\mathbf{U}_n[1]^{-n-1}\mathbf{M})_{\mathfrak{p}} & \to & (\operatorname{Im} e^n)_{\mathfrak{p}} \to 0 \\ \| & & \| & & \\ 0 & & (\mathbf{U}_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}} & & \end{array}$$

induced from the short exact sequence ($*$), we have

$$(\mathbf{U}_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}} \cong (\mathbf{U}_n[1]^{-n-1}\mathbf{M})_{\mathfrak{p}} \cong (\operatorname{Im} e^n)_{\mathfrak{p}}.$$

Therefore from the following short exact sequence

$$(**) \qquad 0 \to \operatorname{Im} e^n \to \mathbf{U}_{n+1}^{-n-1}\mathbf{M} \to \mathbf{U}_{n+1}[1]^{-n-2}\mathbf{M} \to 0$$

we have

$$(\mathbf{U}_{n+1}[1]^{-n-2}\mathbf{M})_{\mathfrak{p}} = 0.$$

($\Leftarrow$) By the assumption and the short exact sequences ($*$) ($**$), we have

$$(\mathbf{U}_n[1]^{-n-1}\mathbf{M})_{\mathfrak{p}} \cong (\operatorname{Im} e^n)_{\mathfrak{p}} \cong (\mathbf{U}_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}}.$$

(2) The first equivalence follows immediately from the above short exact sequence ($**$). For the second half, this follows from

$$\operatorname{Supp}(\mathbf{U}_{n+1}[1]^{-n-2}\mathbf{M}) = \operatorname{Supp}(\mathbf{H}_U^{n+1}(\mathbf{M})) \cup \operatorname{Supp}(\mathbf{U}_{n+2}^{-n-2}\mathbf{M})$$

induced by the short exact sequence ($*$) with $n+1$ instead of n and Lemma (2.8) (3).

(3) This follows similarly from the short exact sequence (∗) with n replaced by $n+1$.

(4) Suppose that (i) holds. By the hypothesis and Lemma (1.6)(2) we have $(\mathbf{U}_{n+1}[1]^{-n-2}\mathbf{M})_{\mathfrak{p}}=0$. On the other hand, from the assumption and Lemma (2.7), we have $(H_U^n(\mathbf{M}))_{\mathfrak{p}}=0$.

The other assertions are obvious.

(5) This follows from the hypothesis, Lemma (2.1) and (4).

(6) This follows easily from (2), since $(H_U^{n+1}(\mathbf{M}))_{\mathfrak{p}}=0$.

EXAMPLE (2.11). (1) In Proposition (2.10), (ii) dose not imply (i). Let $\mathbf{R}=k[[X, Y]]$. Let $\mathbf{M}$ be the quotient field of $\mathbf{R}$. Let $\mathbf{U}_1=\mathbf{R}\setminus(X)$ and $\mathfrak{p}=(X, Y)$. Then $(\mathbf{U}_1^{-1}\mathbf{M})_{\mathfrak{p}}=\mathbf{M}=(\mathbf{U}_0[1]^{-1}\mathbf{M})_{\mathfrak{p}}=(\operatorname{Im} e^0)_{\mathfrak{p}}$ but $\mathbf{U}_1\cap\mathfrak{p}\neq\emptyset$.

(2) ((iii)$\Rightarrow$(ii)) is not the case. See Example (2.3) and note that $(\mathbf{U}_f)_{d+1}[1]^{-d-2}\mathbf{M}=0$. When $\mathfrak{p}\in\operatorname{Supp}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M})$, we don't know whether this holds or not.

(3) If $\mathfrak{p}\in\operatorname{Supp}(\mathbf{U}_{n+2}^{-n-2}\mathbf{M})$, then ((iv)$\Rightarrow$(iii)) does not hold. Let $(\mathbf{R}, \mathfrak{m})$ be a Buchsbaum ring of dimension $d\geq 3$ such that $\mathbf{H}_{\mathfrak{m}}^1(\mathbf{R})\neq 0$ and $\mathbf{H}_{\mathfrak{m}}^n(\mathbf{R})=0$ for $n\neq 1, d$. Let $\mathcal{U}_f=((\mathbf{U}_f)_i)_{i\geq 1}$ be the chain of triangular subsets on $\mathbf{R}$ in the following Proposition (2.15) (when $\mathbf{M}=\mathbf{R}$). Then by Proposition (2.15) we have $\operatorname{Ker} f^1/\operatorname{Im} f^0=\mathbf{H}_{\mathfrak{m}}^1(\mathbf{R})\neq 0$ and $\operatorname{Ker} f^n/\operatorname{Im} f^{n-1}=\mathbf{H}_{\mathfrak{m}}^n(\mathbf{R})=0$ for $n\neq 1, d$. Hence by the short exact sequence (∗) we have

$$(\mathbf{U}_f)_{n+1}[1]^{-n-2}\mathbf{R}\cong\operatorname{Im} f^{n+1}$$

for $n\neq 0, d-1$. Let $\mathfrak{p}\in\operatorname{Spec}(\mathbf{R})$ such that $\operatorname{ht}\mathfrak{p}=n+1$ for $n=1,\dots, d-2$. Then $(\operatorname{Im} f^{n+1})_{\mathfrak{p}}\neq 0$ since $\operatorname{Supp}(\operatorname{Im} f^{n+1})=\operatorname{Supp}((\mathbf{U}_f)_{n+2}^{-n-2}\mathbf{R})=\mathbf{F}_{\mathbf{R}(n+1)}$ by Lemma (2.8)(3) and ([C], 2.15). Therefore $\mathfrak{p}\in\operatorname{Supp}((\mathbf{U}_f)_{n+1}[1]^{-n-2}\mathbf{R})$.

(4) In general, the converse of Proposition (2.10)(5) is not true. Let $\mathbf{R}=k[[X, Y, Z]]/(X)\cap(Y, Z)=k[[x, y, z]]$. Then $\operatorname{Ass}(\mathbf{R})=\{(x), (y, z)\}$. Put $\mathfrak{p}=(x, y, z)$ and $\mathbf{U}_1=\mathbf{R}\setminus\mathfrak{p}$. Hence $\operatorname{Ass}(\mathbf{R}_{\mathfrak{p}})=\{(x), (y, z)\}$. Let $\mathfrak{q}=(x, y)$. Then $(\mathbf{U}_1^{-1}\mathbf{R})_{\mathfrak{q}}=(\mathbf{R}_{\mathfrak{p}})_{\mathfrak{q}}=\mathbf{R}_{\mathfrak{q}}=(\mathbf{U}_0[1]^{-1}\mathbf{R})_{\mathfrak{q}}=(\operatorname{Im} e^0)_{\mathfrak{q}}\neq 0$ and $\mathbf{U}_1\cap\mathfrak{q}=\emptyset$. But $\mathfrak{q}\notin\operatorname{Ass}(\mathbf{R}_{\mathfrak{p}})$.

COROLLARY (2.12). *Let* $\mathbf{R}$ *be a Noetherian ring and* $\mathbf{M}$ *an* $\mathbf{R}$*-module. Then we have the following.*

(1) $\operatorname{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M})\subset\operatorname{Ass}(\mathbf{U}_n[1]^{-n-1}\mathbf{M})$.

(2) $\operatorname{Ass}(\mathbf{U}_n[1]^{-n-1}\mathbf{M})=\operatorname{Ass}(H_U^n(\mathbf{M}))\cup\operatorname{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M})$.

Proof. (1) Let $\mathfrak{p}\in\operatorname{Ass}_{\mathbf{R}}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M})$. Then $\mathfrak{p}\mathbf{R}_{\mathfrak{p}}\in\operatorname{Ass}_{\mathbf{R}_{\mathfrak{p}}}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}}$ by ([M], p. 38 Corollary). Hence $\mathfrak{p}\mathbf{R}_{\mathfrak{p}}\in\operatorname{Ass}_{\mathbf{R}_{\mathfrak{p}}}(\mathbf{U}_n[1]^{-n-1}\mathbf{M})_{\mathfrak{p}}$ by Proposition (2.10)(5).

Therefore $\mathfrak{p} \in \mathrm{Ass}_{\mathbf{R}}(\mathbf{U}_n[1]^{-n-1}\mathbf{M})$ again by ([M], p. 38 Corollary).

(2) This follows from (1) and Lemma (2.8)(5).

PROPOSITION (2.13). *Let* $\mathbf{R}$ *be a ring and* $\mathbf{M}$ *an* $\mathbf{R}$*-module. Fix a non-negative integer* t*. Then in* $\mathbf{C}(\mathcal{U}, \mathbf{M})$*, the following four conditions are equivalent.*

(1) $H_U^n(\mathbf{M}) = 0$ *for all* $n = 0, \ldots, t$.

(2) $\mathbf{U}_n[1]^{-n-1}\mathbf{M} \cong \mathrm{Im}\, e^n$ *for all* $n = 0, \ldots, t$.

(3) *For all* $n = 0, \ldots, t$, *for each* $\dfrac{m}{(a_1, \ldots, a_{n+1})} \in \mathbf{U}_{n+1}^{-n-1}\mathbf{M}$,

$$\left(0 : \frac{m}{(a_1, \ldots, a_{n+1})}\right) = \left(0 : \frac{m}{(a_1, \ldots, a_n, 1)}\right) \textit{ where } \frac{m}{(a_1, \ldots, a_n, 1)} \in \mathbf{U}_n[1]^{-n-1}\mathbf{M}.$$

(4) *For all* $n = 0, \ldots, t$, *each element of* $\mathbf{U}_{n+1}$ *forms a poor* $\mathbf{M}$*-sequence.*

In particular, let $\mathbf{R}$ *be a Noetherian local ring and let* $\mathbf{M}$ *be a finitely generated* $\mathbf{R}$*-module of dimension* d*. Assume that the above conditions hold for* $t = d - 1$ *and* $\mathbf{U}_d[1]^{-d-1}\mathbf{M} \neq 0$*. Then* $\mathbf{M}$ *is a Cohen-Macaulay module.*

Proof. (1) $\Leftrightarrow$ (2) From the short exact sequence (*) this is clear.

(2) $\Rightarrow$ (3) By Lemma (1.6)(3) this is obvious.

(3) $\Rightarrow$ (4) We proceed by induction on n. In the case $n = 0$, assume that $a_1 m = 0$ for some $0 \neq m \in \mathbf{M}$ and $(a_1) \in \mathbf{U}_1$. Then we have $a_1 \in (0 : m) = \left(0 : \dfrac{m}{(b_1)}\right)$ for some $\dfrac{m}{(b_1)} \in \mathbf{U}_1^{-1}\mathbf{M}$ by the hypothesis. This contradicts Lemma (2.1).

Now suppose that each element of $\mathbf{U}_n$ is a poor $\mathbf{M}$-sequence. Assume that $a_{n+1}m \in (a_1, \ldots, a_n)\mathbf{M}$ for some $(a_1, \ldots, a_{n+1}) \in \mathbf{U}_{n+1}$ and $m \in \mathbf{M}$. Then by Lemma (1.6)(2) we have $\dfrac{a_{n+1}m}{(a_1, \ldots, a_{n+1})} = 0$. That is, by ([SZ3], 2.1), we have

$$\frac{m}{(a_1, \ldots, a_{n+1})} = 0 \text{ in } \mathbf{U}_{n+1}^{-n-1}\mathbf{M}.$$

Hence by the hypothesis we have

$$\frac{m}{(a_1, \ldots, a_n, 1)} = 0 \text{ in } \mathbf{U}_n[1]^{-n-1}\mathbf{M}.$$

Then, by the definition of module of generalized fractions, there are $(b_1, \ldots, b_n, 1) \in \mathbf{U}_n[1]$ and $\mathbf{H} \in \mathbf{D}_{n+1}(\mathbf{R})$ such that $\mathbf{H}[a_1 \ldots a_n\ 1]^T = [b_1 \ldots b_n\ 1]^T$ and $|\mathbf{H}|\, m \in (b_1, \ldots, b_n)\mathbf{M}$.

On the other hand, since $h_{n+1,n+1} = 1 - (h_{n+1,1}a_1 + \cdots + h_{n+1,n}a_n)$, by ([SZ1], 2.2) we have

$$h_{11} \cdots h_{nn} m \in (b_1, \ldots, b_n)\mathbf{M}.$$

Note that by the inductive hypothesis $b_1, \ldots, b_n$ is a poor $\mathbf{M}$-sequence and $\mathbf{H}'[a_1 \ldots a_n]^T = [b_1 \ldots b_n]^T$ where $\mathbf{H}'$ is the top left $n \times n$ submatrix of $\mathbf{H}$. Hence by ([O], 3.2) we get

$$m \in (a_1, \ldots, a_n)\mathbf{M}.$$

(4)$\Rightarrow$(1) Let $\dfrac{m}{(a_1, \ldots, a_n)} \in \operatorname{Ker} e^n$ with $\dfrac{m}{(a_0)} = m$. Then $\dfrac{m}{(a_1, \ldots, a_n, 1)} = 0$ in $\mathbf{U}_{n+1}^{-n-1}\mathbf{M}$. Hence by Lemma (1.6)(2), we have

$$m \in (a_1, \ldots, a_n)\mathbf{M}.$$

Therefore we have $\dfrac{m}{(a_1, \ldots, a_n)} \in \operatorname{Im} e^{n-1}$.

For the last assertion, since $\mathbf{U}_d[1]^{-d-1}\mathbf{M} \neq 0$, there is $(a_1, \ldots, a_d) \in \mathbf{U}_d$ such that $a_1, \ldots, a_d$ is an $\mathbf{M}$-sequence.

Remark (2.14). In Proposition (2.13), if $\mathbf{R}$ is Noetherian, then we can change the condition (3) for $\operatorname{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) = \operatorname{Ass}(\mathbf{U}_n[1]^{-n-1}\mathbf{M})$ for all $n = 0, \ldots, t$.

Let $(\mathbf{R}, \mathfrak{m})$ be a Noetherian local ring and let $\mathbf{M}$ be a finitely generated $\mathbf{R}$-module of dimension d. Let $\mathcal{U}_f = ((\mathbf{U}_f)_i)_{i \geq 1}$ be the chain of the expansions of triangular subsets (Example (1.3)(5)) on $\mathbf{R}$. Then we have the following complex

$$0 \to \mathbf{M} \xrightarrow{f^0} (\mathbf{U}_f)_1^{-1}\mathbf{M} \xrightarrow{f^1} (\mathbf{U}_f)_2^{-2}\mathbf{M} \to \cdots \to (\mathbf{U}_f)_{d-1}^{-d+1}\mathbf{M} \xrightarrow{f^{d-1}} (\mathbf{U}_f)_d^{-d}\mathbf{M} \xrightarrow{f^d} 0,$$

since $(\mathbf{U}_f)_{d+i}^{-d-i}\mathbf{M} = 0$ for all $i \geq 1$ by ([C], 2.3).

PROPOSITION (2.15). *Let $\mathbf{R}$, $\mathbf{M}$ and $\mathbf{U}_f$ be as above. Then the following four conditions are equivalent.*

(1) $\mathbf{M}$ *is an f-module* (*see* [SZ4], 1.8 (ii)).

(2) $\operatorname{Ker} f^n / \operatorname{Im} f^{n-1} \cong \mathbf{H}_{\mathfrak{m}}^n(\mathbf{M})$ *for all* $n = 0, \ldots, d$.

(3) $\operatorname{Ass}((\mathbf{U}_f)_n[1]^{-n-1}\mathbf{M}) \subset \{\mathfrak{m}\} \cup \operatorname{Ass}((\mathbf{U}_f)_{n+1}^{-n-1}\mathbf{M})$ *for all* $n = 0, \ldots, d$.

(4) $\operatorname{Supp}(\operatorname{Ker} f^n / \operatorname{Im} f^{n-1}) \subset \{\mathfrak{m}\}$ *for all* $n = 0, \ldots, d$.

In particular, if $\mathbf{M}$ *is a Cohen-Macauly module, then*

$$\begin{cases} \operatorname{Ass}((\mathbf{U}_f)_n[1]^{-n-1}\mathbf{M}) = \operatorname{Ass}((\mathbf{U}_f)_{n+1}^{-n-1}\mathbf{M}) = \mathbf{F}_{\mathbf{M}n} & \text{for all } n < d, \\ \operatorname{Ass}((\mathbf{U}_f)_d[1]^{-d-1}\mathbf{M}) = \{\mathfrak{m}\}. \end{cases}$$

Proof. (1) ⇒ (2) In the case $n = 0, \dots, d-1$, this follows from ([SZ4], 2.4), since $(\mathbf{U}_f)_n = (\mathbf{U}_s)_n$. In the case $n = d$, we have

$$\operatorname{Ker} f^d / \operatorname{Im} f^{d-1} \cong \mathbf{U}_d^{-d}\mathbf{M}/\operatorname{Im} f^{d-1} \cong \mathbf{U}_d[1]^{-d-1}\mathbf{M} \cong (\mathbf{U}_s)_{d+1}^{-d-1}\mathbf{M} \cong \mathbf{H}_{\mathfrak{m}}^d \mathbf{M}$$

by Lemma (1.9) and Lemma (1.7).

(2) ⇒ (3) ⇔ (4) These follow from Corollary (2.12)(2) and Lemma (2.8)(4).

(4) ⇒ (1)This follows from ([SZ4], 2.3).

The last assertion follows from (2), Corollary (2.12)(2) and ([C], 2.15).

3. Modules of generalized fractions and complexes of Cousin type

In this section, suppose that $\mathbf{R}$ is a Noetherian ring.

THEOREM (3.1). *Let* $\mathbf{R}$ *be a Noetherian ring and* $\mathbf{M}$ *an* $\mathbf{R}$*-module. Let* $\mathcal{U} = (\mathbf{U}_i)_{i \geq 1}$ *be a chain of triangular subsets on* $\mathbf{R}$. *Let* $\mathcal{F} = (\mathbf{F}_i)_{i \geq 0}$ *be a filtration of* $\operatorname{Spec}(\mathbf{R})$ *which admits* $\mathbf{M}$. *Then*

the complex $\mathbf{C}(\mathcal{U}, \mathbf{M})$ *is of Cousin type for* $\mathbf{M}$ *with respect to* $\mathcal{F}$

$$\Updownarrow$$

$$\operatorname{Ass}(\mathbf{U}_n[1]^{-n-1}\mathbf{M}) \cap \partial \mathbf{F}_n = \operatorname{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) \quad \textit{for all } n \geq 0 \textit{ and}$$

$$\mathbf{U}_{n+1}^{-n-1}\mathbf{M} \cong \bigoplus_{\mathfrak{p} \in \partial \mathbf{F}_n} (\mathbf{U}_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}} \; \textit{for all } n \geq 0.$$

Proof. (↑) We must verify the properties (i)-(iii) of the definition of Cousin type (see (1.4)).

(i) and (ii) By Remark (2.5) these always hold for arbitrary complexes $\mathbf{C}(\mathcal{U}, \mathbf{M})$.

(iii) We must show that $\operatorname{Supp}(H_U^n(\mathbf{M})) \subset \mathbf{F}_{n+1}$. Note that $\operatorname{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) = \operatorname{Ass}\left(\bigoplus_{\mathfrak{p} \in \partial \mathbf{F}_n} (\mathbf{U}_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}}\right) \subset \partial \mathbf{F}_n$ by Lemma (2.4). By Lemma (2.8)(5) and Lemma (2.4), we have $\operatorname{Supp}(H_U^n(\mathbf{M})) \subset \operatorname{Supp}(\mathbf{U}_n[1]^{-n-1}\mathbf{M}) \subset \mathbf{F}_n$. But it follows from the hypothesis and Lemma (2.8)(4)(6) that $\partial \mathbf{F}_n \cap \operatorname{Supp}(H_U^n(\mathbf{M})) = \emptyset$.

(↓) It is enough to show that the first condition of Theorem holds. By the third and the fourth conditions of the definition of Cousin type, we have $\partial \mathbf{F}_n \cap \operatorname{Supp}(H_U^n(\mathbf{M})) = \emptyset$ and $\operatorname{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) \subset \partial \mathbf{F}_n$. Hence Lemma (2.8)(6) completes the proof of Theorem.

COROLLARY (3.2). *With the same notation and assumption as in Theorem* (3.1), *we have the following.*

(1) *Suppose that* $\partial \mathbf{F}_{\mathrm{M}n} \cap \partial \mathbf{F}_n = \mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M})$ *for all* $n \geq 0$ *and*

$$\mathbf{U}_{n+1}^{-n-1}\mathbf{M} \cong \bigoplus_{\mathfrak{p} \in \partial \mathbf{F}_n} (\mathbf{U}_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}} \text{ for all } n \geq 0.$$

Then the complex $\mathbf{C}(\mathcal{U}, \mathbf{M})$ *is of Cousin type for* $\mathbf{M}$ *with respect to* $\mathcal{F}$.

(2) *In particular, assume that* $\partial \mathbf{F}_{\mathrm{M}n} \cap \partial \mathbf{F}_n \subset \mathrm{Supp}(\mathbf{U}_n[1]^{-n-1}\mathbf{M})$ *for all* $n \geq 0$. *Then the converse of* (1) *is true.*

Proof. (1) This follows from Theorem (3.1), since $\mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) \subset \mathrm{Ass}(\mathbf{U}_n[1]^{-n-1}\mathbf{M}) \cap \partial \mathbf{F}_n \subset \partial \mathbf{F}_{\mathrm{M}n} \cap \partial \mathbf{F}_n$ by Corollary (2.12)(1) and Lemma (2.6)(2).

(2) It is sufficient to show that $\partial \mathbf{F}_{\mathrm{M}n} \cap \partial \mathbf{F}_n = \mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M})$, since the second isomorphisms hold by the definition of Cousin type.

($\supset$) Since $\mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) \subset \partial \mathbf{F}_n$, it follows from Lemma (2.6)(2) that $\mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) \subset \partial \mathbf{F}_{\mathrm{M}n} \cap \partial \mathbf{F}_n$.

($\subset$) We proceed by induction on n. In the case $n = 0$, let $\mathfrak{p} \in \partial \mathbf{F}_{\mathrm{M}0} \cap \partial \mathbf{F}_0$. Consider the following complex

$$0 \to \mathbf{M} \xrightarrow{e^0} \mathbf{U}_1^{-1}\mathbf{M} \xrightarrow{e^1} \mathbf{U}_2^{-2}\mathbf{M} \to \cdots.$$

Then by the definition of Cousin type, we have the following exact sequence

$$0 \to \mathbf{M}_{\mathfrak{p}} \xrightarrow{\cong} (\mathbf{U}_1^{-1}\mathbf{M})_{\mathfrak{p}} \to 0.$$

Since $\mathfrak{p} \in \mathrm{Ass}(\mathbf{M})$, we have $\mathfrak{p} \in \mathrm{Ass}(\mathbf{U}_1^{-1}\mathbf{M})$ by ([M], p. 38 Corollary).

Suppose that $n \geq 1$. Let $\mathfrak{p} \in \partial \mathbf{F}_{\mathrm{M}n} \cap \partial \mathbf{F}_n$. Consider the following complex

$$\cdots \to \mathbf{U}_{n-1}^{-n+1} \xrightarrow{e^{n-1}} \mathbf{U}_n^{-n}\mathbf{M} \xrightarrow{e^n} \mathbf{U}_{n+1}^{-n-1}\mathbf{M} \to \cdots.$$

It follows from the definition of Cousin type that we have the following exact sequence

$$0 \to (\mathrm{Im}\, e^{n-1})_{\mathfrak{p}} \to (\mathbf{U}_n^{-n}\mathbf{M})_{\mathfrak{p}} \to (\mathbf{U}_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}} \to 0,$$

since $(\mathrm{Ker}\, e^n)_{\mathfrak{p}} \cong (\mathrm{Im}\, e^{n-1})_{\mathfrak{p}}$. Hence by the inductive hypothesis and Lemma (2.6)(3), we have $(\mathbf{U}_n^{-n}\mathbf{M})_{\mathfrak{p}} \neq 0$. On the other hand, by Proposition (2.10)(2) and the assumption $\partial \mathbf{F}_{\mathrm{M}n} \cap \partial \mathbf{F}_n \subset \mathrm{Supp}(\mathbf{U}_n[1]^{-n-1}\mathbf{M})$, we get

$$(\mathrm{Im}\, e^{n-1})_{\mathfrak{p}} \ncong (\mathbf{U}_n^{-n}\mathbf{M})_{\mathfrak{p}}.$$

That is $(\mathbf{U}_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}} \neq 0$. Hence we conclude that $\mathfrak{p} \in \mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M})$ by Lemma (2.4).

Remark (3.3). Using Lemma (2.6)(2), Lemma (2.8)(6), the third and the fourth conditions of the definition of Cousin type, we have another proof of Corollary (3.2)(2) as follows:

$$\begin{aligned}\partial\mathbf{F}_{\mathrm{M}n} \cap \partial\mathbf{F}_n &= \partial\mathbf{F}_n \cap \mathrm{Supp}(\mathbf{U}_n[1]^{-n-1}\mathbf{M}) = \partial\mathbf{F}_n \cap \mathrm{Ass}(\mathbf{U}_n[1]^{-n-1}\mathbf{M})\\ &= (\partial\mathbf{F}_n \cap (H_U^n(\mathbf{M}))) \cup (\partial\mathbf{F}_n \cap \mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}))\\ &= \partial\mathbf{F}_n \cap \mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) = \mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}).\end{aligned}$$

Remark (3.4). If $\mathbf{M}$ is a finitely generated $\mathbf{R}$-module and a complex $\mathbf{C}(\mathcal{U}, \mathbf{M})$ is of Cousin type for $\mathbf{M}$ with respect to $\mathcal{F}_{\mathrm{M}}$, then $\mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) = \{\mathfrak{p} \in \mathrm{Supp}(\mathbf{M}) : \mathrm{ht}_{\mathrm{M}}\mathfrak{p} = n\}$ by ([RSZ], 3.3), ([C], 2.11) and the following Corollary (3.5)(1).

COROLLARY (3.5). *Let $\mathbf{M}$ be a finitely generated $\mathbf{R}$-module of dimension d. Let $\mathcal{F} = (\mathbf{F}_i)_{i\geq 0}$ be a filtration of $\mathrm{Spec}(\mathbf{R})$ which admits $\mathbf{M}$. Let $\mathcal{F}_{\mathbf{M}} = (\mathbf{F}_{\mathrm{M}i})_{i\geq 0}$ be the $\mathbf{M}$-height filtration.*

(1) (cf. [SY], 3.9) $\mathbf{C}(\mathcal{U}_h, \mathbf{M})$ *is of Cousin type for $\mathbf{M}$ w. r. t. $\mathcal{F}_{\mathrm{M}}$, where $\mathcal{U}_h = ((\mathbf{U}_h)_i)_{i\geq 0}$.*

(2) ([RSZ], 3.4) $\mathbf{C}(\mathcal{U}_{\bar{h}}, \mathbf{M})$ *is of Cousin type for $\mathbf{M}$ w. r. t. $\mathcal{F}$, where $\mathcal{U}_{\bar{h}} = ((\mathbf{U}_{\bar{h}})_i)_{i\geq 0}$.*

(3) ([GO], 3.6) *Let $\mathcal{U} = (\mathbf{U}_i)_{i\geq 0}$ be a chain of saturated triangular subsets on $\mathbf{R}$. Put $\mathbf{G}_0 = \mathrm{Supp}(\mathbf{M})$ and for $i \in \mathbf{N}$, define $\mathbf{G}_i = \{\mathfrak{p} \in \mathrm{Supp}(\mathbf{M}) :$ there exists $(a_1,\ldots,a_i) \in \mathbf{U}_i$ with $(a_1,\ldots,a_i)\mathbf{R} \subset \mathfrak{p}\}$. Assume that $\mathcal{G} = (\mathbf{G}_i)_{i\geq 0}$, induced by $\mathcal{U}$ and $\mathbf{M}$, is a filtration of $\mathrm{Spec}(\mathbf{R})$ which admits $\mathbf{M}$. Then $\mathbf{C}(\mathcal{U}, \mathbf{M})$ is of Cousin type for $\mathbf{M}$ w. r. t. $\mathcal{G}$.*

(4) *If $\dim \mathbf{M} = \mathrm{ht}_{\mathrm{M}}\mathfrak{q} + \dim \mathbf{M}/\mathfrak{q}\mathbf{M}$ for all $\mathfrak{q} \in \mathrm{Supp}(\mathbf{M})$, then $\mathbf{C}(\mathcal{U}_s, \mathbf{M})$ is of Cousin type for $\mathbf{M}$ w. r. t. $\mathcal{F}_{\mathrm{M}}$, where $\mathcal{U}_s = ((\mathbf{U}_s)_i)_{i\geq 0}$.*

(5) *Let $\mathcal{U}_r = ((\mathbf{U}_r)_i)_{i\geq 0}$. Then we have the following equivalent conditions.*

$$\begin{aligned}&\mathbf{M}\ \textit{is a Cohen-Macaulay module}\\ \Leftrightarrow\ &\mathbf{C}(\mathcal{U}_r, \mathbf{M})\ \textit{is of Cousin type for}\ \mathbf{M}\ \textit{w. r. t.}\ \mathcal{F}_{\mathrm{M}}\\ \Leftrightarrow\ &(\mathbf{U}_r)_{n+1}^{-n-1}\mathbf{M} \cong \bigoplus_{\mathrm{ht}_{\mathbf{M}}\mathfrak{p}=n} ((\mathbf{U}_r)_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}}\ \textit{for all}\ n \geq 0.\end{aligned}$$

(6) *Let $\mathbf{R}$ be a Noetherian local ring. Then*

$$\begin{aligned}&\mathbf{M}\ \textit{is a Gorenstein module}\\ \Leftrightarrow\ &\begin{cases}\mathbf{C}(\mathcal{U}_r, \mathbf{M})\ \textit{is of Cousin type for}\ \mathbf{M}\ \textit{w. r. t.}\ \mathcal{F}_{\mathrm{M}}\ \textit{and}\\ (\mathbf{U}_r)_{d+1}^{-d-1}\mathbf{M}\ \textit{is an injective}\ \mathbf{R}\textit{-module}\end{cases}\end{aligned}$$

$$\Leftrightarrow \begin{cases} (\mathbf{U}_r)_{n+1}^{-n-1}\mathbf{M} \cong \bigoplus_{\mathrm{ht}_{\mathbf{M}}\mathfrak{p}=n} ((\mathbf{U}_r)_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}} \textit{ for all } n \geq 0, \textit{ and} \\ (\mathbf{U}_r)_{d+1}^{-d-1}\mathbf{M} \textit{ is an injective } \mathbf{R}\textit{-module.} \end{cases}$$

Proof. (1) This follows from ([C], 2.11 and 3.3(2)) and Corollary (3.2).
(2) By ([RSZ], 2.6 or [C], 3.3(1)), we have for all $n \in \mathbf{N} \cup \{0\}$

$$(\mathbf{U}_{\bar{h}})_{n+1}^{-n-1}\mathbf{M} \cong \bigoplus_{\mathfrak{p}\in\partial\mathbf{F}_n} ((\mathbf{U}_{\bar{h}})_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}}.$$

Hence by Lemma (2.4) we get

$$\begin{aligned}\mathrm{Ass}((\mathbf{U}_{\bar{h}})_{n+1}^{-n-1}\mathbf{M}) &= \mathrm{Ass}\Big(\bigoplus_{\mathfrak{p}\in\partial\mathbf{F}_n} ((\mathbf{U}_{\bar{h}})_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}}\Big) \\ &= \bigcup_{\mathfrak{p}\in\partial\mathbf{F}_n} \mathrm{Ass}\Big(((\mathbf{U}_{\bar{h}})_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}}\Big) \subset \partial\mathbf{F}_n.\end{aligned}$$

By Lemma (2.7) and the definition of $(\mathbf{U}_{\bar{h}})_{n+1}$, we have, for all $\mathfrak{p} \in \partial\mathbf{F}_n \cap \mathrm{Supp}(\mathbf{M})$,

$$(H_U^n(\mathbf{M}))_{\mathfrak{p}} = 0.$$

Therefore we have $\partial\mathbf{F}_n \cap \mathrm{Ass}(H_U^n(\mathbf{M})) = \emptyset$, since $\mathrm{Ass}(H_U^n(\mathbf{M})) \subset \mathrm{Supp}(\mathbf{M})$. Hence we obtain

$$\partial\mathbf{F}_n \cap \mathrm{Ass}((\mathbf{U}_{\bar{h}})_n[1]^{-n-1}\mathbf{M}) = \partial\mathbf{F}_n \cap \mathrm{Ass}((\mathbf{U}_{\bar{h}})_{n+1}^{-n-1}\mathbf{M}) = \mathrm{Ass}((\mathbf{U}_{\bar{h}})_{n+1}^{-n-1}\mathbf{M}),$$

by Lemma (2.8) (6). Then Theorem (3.1) completes the proof.

(3) By ([GO], 3.6), we have for all $n \in \mathbf{N} \cup \{0\}$

$$\mathbf{U}_{n+1}^{-n-1}\mathbf{M} \cong \bigoplus_{\mathfrak{p}\in\partial\mathbf{G}_n} (\mathbf{U}_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}}.$$

Hence we get $\mathrm{Ass}\,(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}) \subset \partial\mathbf{G}_n$.

Next for all $\mathfrak{p} \in \partial\mathbf{G}_n$ we have

$$(H_U^n(\mathbf{M}))_{\mathfrak{p}} = 0.$$

In fact, if $(H_U^n(\mathbf{M}))_{\mathfrak{p}} \neq 0$, then there is $x \in H_U^n(\mathbf{M})$ such that $(0 : x) \subset \mathfrak{p}$. But by Lemma (2.7), we have $(a_1,\ldots,a_{n+1})\mathbf{R} \subset (0 : x) \subset \mathfrak{p}$ for some $(a_1,\ldots,a_{n+1}) \in \mathbf{U}_{n+1}$. Hence from the definition of $\mathbf{G}_{n+1}$ we have $\mathfrak{p} \in \mathbf{G}_{n+1}$. This contradicts $\mathfrak{p} \in \partial\mathbf{G}_n$.

Therefore we have $\partial\mathbf{G}_n \cap \mathrm{Ass}(H_U^n(\mathbf{M})) = \emptyset$.

Then by Lemma (2.8)(6) we get

$$\begin{aligned}\partial\mathbf{G}_n \cap \mathrm{Ass}(\mathbf{U}_n[1]^{-n-1}\mathbf{M}) &= (\partial\mathbf{G}_n \cap \mathrm{Ass}(H_U^n(\mathbf{M}))) \cup (\partial\mathbf{G}_n \cap \mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M})) \\ &= \mathrm{Ass}(\mathbf{U}_{n+1}^{-n-1}\mathbf{M}).\end{aligned}$$

The result follows from Theorem (3.1).

(4) This follows from ([C], 2.12 and 3.3(3)) and Corollary (3.2).

(5) Since $\mathbf{C}(\mathcal{U}_r, \mathbf{M})$ is an exact sequence by Proposition (2.13), the first equivalence follows from ([S2], 2.4). From Proposition (2.13)(3) and Theorem (3.1), we have the second equivalence.

(6) This follows from (5) and ([S2], 3.11).

Remark (3.6). Let $(\mathbf{R}, \mathfrak{m})$ be a Noetherian ring and let $\mathbf{M}$ be a finitely generated f-module of dimension d. Then $\mathbf{C}(\mathcal{U}_s, \mathbf{M})$ is of Cousin type for $\mathbf{M}$ with respect to $\mathcal{F}_{\mathbf{M}}$ (Corollary (3.5)(4)) but $\mathbf{C}(\mathcal{U}_f, \mathbf{M})$ is not, even though $(\mathbf{U}_f)_{n+1}^{-n-1}\mathbf{M} \cong \bigoplus_{\mathrm{ht}_{\mathbf{M}}\mathfrak{p}=n}((\mathbf{U}_f)_{n+1}^{-n-1}\mathbf{M})_{\mathfrak{p}}$ for all $n \geq 0$ ([C], 3.3(5)). For, by ([C], 2.15), we have $\mathrm{Ass}((\mathbf{U}_f)_{d+1}^{-d-1}\mathbf{M}) = \emptyset$ but $\mathrm{Ass}((\mathbf{U}_f)_d[1]^{-d-1}\mathbf{M}) \cap \partial\mathbf{F}_{\mathbf{M}d} = \mathrm{Ass}((\mathbf{U}_s)_{d+1}^{-d-1}\mathbf{M}) \cap \partial\mathbf{F}_{\mathbf{M}d} = \{\mathfrak{m}\}$. Hence we have

$$\mathrm{Ass}((\mathbf{U}_f)_d[1]^{-d-1}\mathbf{M}) \cap \partial\mathbf{F}_{\mathbf{M}d} \neq \mathrm{Ass}((\mathbf{U}_f)_{d+1}^{-d-1}\mathbf{M}).$$

Therefore the result follows from Theorem (3.1).

EXAMPLE (3.7). Let $\mathbf{R} = k[[x, y, z]]$. Let $\mathbf{U}_1 = \{(tx^{\alpha}) \in \mathbf{R}^1 : 0 \neq t \in k$ and $\alpha \in \mathbf{N} \cup \{0\}\}$. Let $\mathbf{U}_i = \mathbf{U}_{i-1}[1]$ for $i = 2, 3, \ldots$. Then $\mathcal{U} = (\mathbf{U}_i)_{i\geq 1}$ is a chain of saturated triangular subsets on $\mathbf{R}$. Put $\mathbf{G}_0 = \mathrm{Spec}(\mathbf{R})$, $\mathbf{G}_1 = \{\mathfrak{p} \in \mathrm{Spec}(\mathbf{R}) : x \in \mathfrak{p}\}$ and $\mathbf{G}_i = \emptyset$ for $i \geq 2$. Then $\mathcal{G} = (\mathbf{G}_i)_{i\geq 0}$ is induced by $\mathcal{U}$ and $\mathbf{M}$ as in (3) of Corollary (3.5), but is not a filtration of $\mathrm{Spec}(\mathbf{R})$. For, $\partial\mathbf{G}_0 = \mathbf{G}_0 \setminus \mathbf{G}_1 \supset \{(y), (y, z)\}$.

EXAMPLE (3.8). Let $\mathbf{R} = k[[X, Y, Z]]/(X) \cap (Y, Z) = k[[x, y, z]]$. Then $\mathbf{R}$ is not equidimensional and $\{(x), (y, z)\} = \partial\mathbf{F}_{\mathbf{R}0} \cap \mathrm{Spec}((\mathbf{U}_s)_0[1]^{-1}\mathbf{R}) \not\subset \mathrm{Ass}((\mathbf{U}_s)_1^{-1}\mathbf{R}) = \{(x)\}$. Hence $\mathbf{C}(\mathcal{U}_s, \mathbf{R})$ is not of Cousin type for $\mathbf{R}$ w. r. t. $\mathcal{F}_{\mathbf{R}}$. In fact, $k\langle\langle y, z\rangle\rangle \times k\langle\langle x\rangle\rangle \cong (\mathbf{U}_h)_1^{-1}\mathbf{R} \not\cong (\mathbf{U}_s)_1^{-1}\mathbf{R} \cong k\langle\langle y, z\rangle\rangle$ (cf. Corollary (3.5)(1) (4)).

EXAMPLE (3.9). Let $\mathbf{R} = k[[x, y]]$. Let $\mathbf{U}_1 = \{(x^{\alpha}) \in \mathbf{R}^1 : \alpha \in \mathbf{N} \cup \{0\}\}$ and $\mathbf{U}_n = \{(x^{\alpha}, 1, \ldots, 1) \in \mathbf{R}^n : \alpha \in \mathbf{N} \cup \{0\}\}$ for $n \geq 2$. Then we have $\mathrm{Ass}(\mathbf{U}_1^{-1}\mathbf{R}) = \{(0)\} = \partial\mathbf{F}_{\mathbf{R}0} \cap \mathrm{Supp}(\mathbf{U}_0[1]^{-1}\mathbf{R})$, $\mathrm{Ass}(\mathbf{U}_2^{-2}\mathbf{R}) = \{(x)\} = \partial\mathbf{F}_{\mathbf{R}1} \cap \mathrm{Supp}(\mathbf{U}_2^{-2}\mathbf{R}) = \partial\mathbf{F}_{\mathbf{R}1} \cap \mathrm{Ass}(\mathbf{U}_1[1]^{-2}\mathbf{R})$ and $\mathbf{U}_i^{-i}\mathbf{R} = 0$ for all $i \geq 3$. But $\mathbf{U}_2^{-2}\mathbf{R} \not\cong (\mathbf{U}_2^{-2}\mathbf{R})_{(x)}$.

EXAMPLE (3.10). Let $\mathbf{R} = k[[X, Y, Z]]$ and $\mathbf{M} = k[[X, Y, Z]]/(X) \cap (X^2, Y) = k[[x, y, z]]$. Let $\mathbf{U}_1 = \{(Y^n) \in \mathbf{R}^1 : n \geq 0\}$. Let $\mathbf{F}_i = \{\mathfrak{p} \in \mathrm{Spec}(\mathbf{R}) : \mathrm{ht}\,\mathfrak{p} \geq$

$i+1\}$ for $i \geq 0$. Then $\mathrm{Ass}(\mathbf{U}_1^{-1}\mathbf{M}) = \{(X)\} = \partial \mathbf{F}_0 \cap \mathrm{Ass}(\mathbf{M}) = \partial \mathbf{F}_0 \cap \mathrm{Ass}(\mathbf{U}_0[1]^{-1}\mathbf{M})$ but $\mathbf{M}_Y \cong \mathbf{U}_1^{-1}\mathbf{M} \ncong (\mathbf{U}_1^{-1}\mathbf{M})_{(X)} \cong \mathbf{M}_{(X)}$.

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