M. Itoh
Nagoya Math. J.
Vol. 137 (1995), 149–181

THE WEITZENBÖCK FORMULA FOR THE BACH OPERATOR

MITSUHIRO ITOH

Dedicated to Professor Tsunero Takahashi on his sixtieth birthday

§1. Introduction

(Anti-)self-dual metrics are 4-dimensional Riemannian metrics whose Weyl conformal tensor W half vanishes. The Weyl conformal tensor W of an arbitrary metric on an oriented 4-manifold has in general the self-dual part W^+ and the anti-self-dual part W^- with respect to the Hodge star operator $*$ and one says that a metric is self-dual or anti-self-dual if $W^- = 0$ or $W^+ = 0$, respectively.

Because of the conformal invariance of the defining equations $W^{\pm} = 0$ (anti-)self-dual metrics are, as a generalization of conformally flat metrics, an object of great interest from conformal geometry.

The notion of (anti-)self-duality of metrics depends on a choice of orientation so that a self-dual metric becomes anti-self-dual when we reverse the orientation. However, we are mainly interested in anti-self-dual metrics, unless especially mentioned.

Consider the unit sphere bundle $Z_M = U(\Omega_M^+)$ over an oriented Riemannian 4-manifold M. Then the vanishing of the self-dual part of the Weyl tensor gives an integrable condition for the almost complex structure naturally defined on Z_M. So Z_M becomes a 3-dimensional complex manifold having a smooth fibration over M with fibers $\mathbf{C}P^1$ and a fixed-point free anti-holomorphic involution, called the real structure.

The Penrose twistor theories assert that elliptic differential operators geometrically arising over an anti-self-dual 4-manifold M relate to the $\bar{\partial}$-operators on certain holomorphic vector bundles over the twistor space Z_M.

Particularly, the Kodaira-Spencer complex on $Z = Z_M$;

Received March 9, 1994.

(1.1) $$\bar{\partial}: C^{\infty}(Z\,;\Omega_Z^{0,i}\otimes \mathcal{J}_Z^{1,0}) \to C^{\infty}(Z\,;\Omega_Z^{0,i+1}\otimes \mathcal{J}_Z^{1,0})$$

corresponds to the elliptic complex over M

(1.2) $$C^{\infty}(M\,;T) \xrightarrow{L} C^{\infty}(M\,;S_o^2(T^*)) \xrightarrow{D} C^{\infty}(M\,;S_o^2(\Omega^+)).$$

The Penrose twistor theories then show that the cohomology group $H^2(Z\,;\Theta_Z)$, $\Theta_Z = \mathcal{O}_Z(\mathcal{J}_Z^{1,0})$, of the complex (1.1) is isomorphic to the complexification of the second cohomology group $\mathbf{H}^2$ of (1.2) (see §3.3 in [4]).

The Kodaira-Spencer-Kuranishi theory tells us that $H^2(Z\,;\Theta_Z)$ gives the obstruction space for complex structure deformations on Z.

Similarly $\mathbf{H}^2$ represents the obstruction for local deformations of anti-self-dual metrics on M. We have in fact the following ([12], [15]).

If $\mathbf{H}^2 = 0$, then the local moduli of conformal structures represented by anti-self-dual metrics can be described as the quotient of an ε-ball in $\mathbf{H}^1$ by the conformal transformation group $C^o(M)$ whose Lie algebra is $\mathbf{H}^0$, the space of conformal Killing fields.

Another important meaning of $\mathbf{H}^2$ is observed in the following connected sum procedure ([4]).

If two anti-self-dual 4-manifolds M_1, M_2 have $\mathbf{H}^2 = 0$, then the connected sum $M_1 \# M_2$ admits an anti-self-dual metric of $\mathbf{H}^2 = 0$.

Vanishing of $H^2(Z_M\,;\Theta_Z)$ and hence of $\mathbf{H}^2$ is shown in terms of complex geometry for several typical anti-self-dual 4-manifolds, for instances the standard 4-sphere S^4 and the reversely oriented complex projective plane $\overline{\mathbf{C}P^2}$ with the Fubini-Study metric.

To determine $\mathbf{H}^2$ for an arbitrary anti-self-dual metric we will make direct use of the definition $\mathbf{H}^2 = \operatorname{Ker} D\,D^*$ (D^* is the adjoint of the operator D and we will call D^* the *Bach operator*) to treat of the elliptic operator $D\,D^*$ in purely differential geometric way.

In the complex (1.2) the operator D is defined as the minus sign of the linearization of the anti-self-dual part of W (see §3 for the precise defininion of D).

On the other hand we see that the linearization of the self-dual part W^+ at an anti-self-dual metric g turns out to be the self-dual part of the linearization of W, namely $\delta_g(W^+) = (\delta_g W)^+$, and that $\delta_g W(h)$, $h \in C^{\infty}(M\,;S_0^2(T^*))$ has the principal part represented in covariant derivatives

$$\nabla_i \nabla_k h_{jl} - \nabla_i \nabla_l h_{jk} - \nabla_j \nabla_k h_{il} + \nabla_j \nabla_l h_{ik}.$$

This principal part can be then written as $d_L \circ d_R(h) \in C^{\infty}(M\,;\Omega^2 \otimes \Omega^2)$, when we consider an infinitesimal metric deformation h as a smooth section of $\Omega^1 \otimes \Omega^1$ and introduce the left-exterior derivative d_L and the right exterior derivative d_R operating on the space $C^\infty(M\,;\Omega^i \otimes \Omega^j)$ which are thought to be a natural generalization of the ordinary exterior derivatives.

So, $d_L^+ \circ d_R^+(h) + d_R^+ \circ d_L^+(h)$ gives, up to constant factor, the principal part of $D(h)$. Here $d_L^+ : C^\infty(M\,;\Omega^1 \otimes \Omega^p) \to C^\infty(M\,;\Omega^+ \otimes \Omega^p)$, $p \geq 0$ and $d_R^+ : C^\infty(M\,;\Omega^p \otimes \Omega^1) \to C^\infty(M\,;\Omega^p \otimes \Omega^+)$, $p \geq 0$ are the self-dual part of d_L and d_R, respectively.

The operator D^* has then, up to constant, the principal part

$$\delta_L^+ \circ \delta_R^+(Z) + \delta_R^+ \circ \delta_L^+(Z)$$

where δ_L^+ and δ_R^+ are the formal adjoints of d_L^+ and d_R^+, respectively.

By the aid of the Weitzenböck formulae for the elliptic operators $d_L^+ \circ \delta_L^+$ and $d_R^+ \circ \delta_R^+$ we can write for an arbitrary anti-self-dual metric the Weitzenböck formula for $D D^*$.

In this article we focus on Einstein anti-self-dual metrics and write the formula of $D D^*$ for the Einstein case in terms of the rough Laplacian $\nabla^* \nabla = -\sum_{i=1}^4 \nabla_i \nabla_i$ in the following simple form (see Proposition 5.1, §5)

$$D D^* Z = \frac{1}{24}(3\nabla^*\nabla + 2\rho)(2\nabla^*\nabla + \rho)Z \tag{1.3}$$

(ρ is the scalar curvature of g).

From this formula we obtain

THEOREM 1. *Let M be a compact connected oriented 4-manifold and g be an Einstein anti-self-dual metric on M.*
(i) *If $\rho > 0$, then $\mathbf{H}^2 = 0$.*
(ii) *For g of $\rho = 0$ $\mathbf{H}^2$ is the space of all covariantly constant sections of $S_0^2(\Omega^+)$.*
(iii) *If $\rho < 0$, then $\mathbf{H}^2 = \mathbf{E}_{-2\rho/3} \oplus \mathbf{E}_{-\rho/2}$ where $\mathbf{E}_\lambda$ denotes the eigenspace of $\nabla^*\nabla$ of eigenvalue λ.*

It is known that Einstein, anti-self-dual compact oriented 4-manifolds of positive scalar curvature are only the standard 4-sphere S^4 and $\overline{\mathbf{C}P^2}$ with the Fubini-Study metric ([11], [6]). From our theorem we have that these manifolds have vanishing $\mathbf{H}^2$, even though this fact was already shown by the Penrose twistorial correspondence ([5], [4]).

The above Weitzenböck formula can also be applied to the case of orbifolds. An (oriented) orbifold is locally a quotient U/Γ of a neighborhood U in $\mathbf{R}^4$ by a finite group Γ (the isotropy group) which acts on U as (orientation preserving) smooth transformations and a metric on an orbifold is considered locally as a Γ invariant metric on U (for the precise definitions of these see [18], [1] and [7]) so that one can consider an anti-self-dual metric on an orbifold.

THEOREM 2. *Let* (X, g) *be a compact connected oriented* 4 *dimensional orbifold and an Einstein anti-self-dual metric of positive scalar curvature. Then the second cohomology group* $\mathbf{H}^2 = \{0\}$.

There are infinitely many compact anti-self-dual, Einstein 4-orbifolds of positive scalar curvature ([8], [9]). So, these orbifolds have $\mathbf{H}^2 = \{0\}$.

For Ricci flat anti-self-dual 4-manifolds $\dim \mathbf{H}^2$ is computed in the following theorem.

THEOREM 3. *Let* (M, g) *be a compact anti-self-dual Ricci flat* 4*-manifold. Then,*
(i) $\dim \mathbf{H}^2 = 5$ *when* (M, g) *is a Ricci flat Kähler* $K3$ *complex surface or a flat Kähler complex* 2*-torus,*
(ii) $\dim \mathbf{H}^2 = 3$ *when* (M, g) *is a Ricci flat Kähler Enriques surface or a flat Kähler hyperelliptic surface satisfying* $\mathcal{O}(K_M^2) = \mathcal{O}$,
(iii) $\dim \mathbf{H}^2 = 2$ *when* (M, g) *is a* $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$*-quotient of a Ricci flat Kähler* $K3$ *surface and*
(iv) $\dim \mathbf{H}^2 = 1$ *for a hyperelliptic surface with* $\mathcal{O}(K_M^k) = \mathcal{O}$, $k = 3,4,6$ *and* $\mathcal{O}(K_M^i) \neq \mathcal{O}$ *for all* $0 < i < k$.

Another application of the Weitzenböck formula is to ALE (asymptotically locally Euclidean) hyperkähler 4-manifolds.

For these manifolds the reader refers to [16]. They are all anti-self-dual and Ricci flat, since they are Kähler and of zero scalar curvature and further they have a triple of covariantly constant complex structures so that the holomorphically trivial canonical line bundle K_M is flat.

THEOREM 4. *Let* (M, g) *be an ALE hyperkähler* 4*-manifold. Then* $\mathbf{H}^2 = \operatorname{Ker} D^*$ *vanishes. Here* D^* *is defined over the space* $W_k^2(M; S_0^2(\Omega^+))$ *of sections of* L^2 *finite derivatives up to order* k, $k \geq 2$.

These theorems will be shown in §5.

By using the conformal compactification at infinity we get from each ALE hyperkähler 4-manifold (M, g) a compact oriented anti-self-dual orbifold $(\bar{M}, \bar{g})$ of one singular point x_∞. Notice that the conformal compactification of the Euclidean space $\mathbf{R}^4$ is just the inverse of the stereographic projection from S^4 and the stereographic projection is orientation reversing so that the above derived orbifold has the natural orientation induced via the orientation reversing conformal compactification from the orientation of M and thus we should rather say the metric is self-dual with respect to this orientation.

An ALE hyperkähler 4-manifold (M, g) has an associated finite subgroup Γ of $SU(2)$ in such a way that (M, g) is asymptotically isometric to $\mathbf{C}^2/\Gamma$.

The Eguchi-Hanson 4-space (M_{EG}, g) is just an ALE hyperkähler 4-manifold with $\Gamma = \mathbf{Z}_2$, the center of $SU(2)$ so that the conformal compactification $\overline{M_{EG}}$ has one singular point x_∞ with the isotropy group $\mathbf{Z}_2$ which is in the center of $SO(4) = SU(2)\cdot SU(2)$.

Finally we remark on self-dual metrics on a 4-manifold which is given by the generalized connected sum of copies of orbifolds $\overline{M_{EG}}$.

The generalized connected sum $\overline{M_{EG}} \#_{x_\infty} \overline{M_{EG}}$ at the singular points is a 4-manifold which is obtained by gluing the corresponding boundaries of $\overline{M_{EG}}\setminus D$ and its copy (D is an orbifold ball centered at x_∞ so D is a $\mathbf{Z}_2$-quotient of an ordinary ball in $\mathbf{R}^4$ and then ∂D is $S^3/\mathbf{Z}_2 = \mathbf{R}P^3$).

It is known that $\overline{M_{EG}} \# \overline{M_{EG}}$ is diffeomorphic to the ordinary connected sum $\mathbf{C}P^2 \# \mathbf{C}P^2$ as an oriented manifold.

Since from Theorem 4 $\overline{M_{EG}}$ has $\mathbf{H}^2 = 0$ and there exists an orientation reversing isometry of $\mathbf{R}^4$ commuting with the isotropy action of $\mathbf{Z}^2$, we can apply the orbifold connected sum theorem given in [14] so that we get

THEOREM 5. *The generalized connected sum* $\overline{M_{EG}} \# \overline{M_{EG}}$ *admits a self-dual metric of* $\mathbf{H}^2 = \{0\}$.

Remark. A self-dual metric thus derived on $\overline{M_{EG}} \# \overline{M_{EG}} = \mathbf{C}P^2 \# \mathbf{C}P^2$ is just one of the self-dual metrics obtained by Poon [17] (see also §1 in [4]).

§2. 4-dimensional Weyl conformal tensor

2.1. Throughout this article M will denote a compact connected oriented C^∞ 4-manifold with a Riemannian metric g and $\{e_1, \ldots, e_4\}$ will denote an orthonormal frame field with the dual frame field $\{\theta^1, \ldots, \theta^4\}$.

We denote by $R = R_{ijkl}$, $\mathrm{Ric} = R_{ik}$ and by ρ the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively.

In this section we indicate several geometric properties of the 4-dimensional Weyl conformal tensor.

We denote the Weyl conformal tensor W in the following way

$$R = W + K + G,$$

namely

$$R_{ijst} = W_{ijst} + K_{ijst} + G_{ijst} \tag{2.1}$$

where

$$K_{ijst} = \frac{1}{2}(B_{is}\delta_{jt} - B_{it}\delta_{js} + \delta_{is}B_{jt} - \delta_{it}B_{js}), \tag{2.2}$$

and

$$G_{ijst} = \frac{1}{12}\rho(\delta_{is}\delta_{jt} - \delta_{it}\delta_{js}) \tag{2.3}$$

$\left(B_{is}\right.$ is the tracefree Ricci tensor, $\left.B_{is} = R_{is} - \frac{\rho}{4}\delta_{is}\right)$.

2.2. We denote by Ω^p the bundle of p-forms on M, $p = 0,1,\dots,4$ and by $\Omega^p \otimes \Omega^q$ the tensor product of the bundles Ω^p and Ω^q. Further we denote by $S^2(\Omega^p)$ the symmetric product of Ω^p and by $\Lambda^2(\Omega^p)$ the skew symmetric product of Ω^p.

By using the metric g we identify $\Omega^p \otimes \Omega^p$ with the endomorphism bundle $\mathrm{End}(\Omega^p)$ and we have the trace operator $\mathrm{tr} : \Omega^p \otimes \Omega^p \cong \mathrm{End}(\Omega^p) \to \Omega^0$.

The bundle $S_0^2(\Omega^p)$, the tracefree symmetric product of Ω^p, is the subbundle of $S^2(\Omega^p)$ whose trace is zero.

Note that for $\phi = (\phi_{ijst}) \in \Omega^2 \otimes \Omega^2$

$$\sum_{i<j} \phi_{ijij} = \frac{1}{2}\sum_{i,j} \phi_{ijij}$$

gives the trace of ϕ.

Because of the symmetry of indices the Riemannian curvature tensor $R = (R_{ijst})$ is regarded as a self-adjoint endomorphism

$$\Omega^2 \to \Omega^2 ; \theta^s \wedge \theta^t \mapsto \frac{1}{2}\sum_{i,j} R_{ijst}\theta^i \wedge \theta^j$$

and also as a symmetric bilinear form on Ω^2

$$\Omega^2 \times \Omega^2 \to \Omega^0 \,;\, (\theta^i \wedge \theta^j, \theta^s \wedge \theta^t) \mapsto R_{ijst}.$$

If we consider (R_{ijst}) as an endomorphism of Ω^2, then the Weyl conformal tensor W is a tracefree self-adjoint endomorphism of Ω^2 and G is $\frac{1}{12}\rho\, Id_{\Omega^2}$, where Id_{Ω^2} is the identity transformation of Ω^2.

Since W satisfies $\sum_{i=1}^4 W_{isit} = 0$ and the first Bianchi identity $(W_{ijst} + W_{istj} + W_{itjs} = 0)$, we get the following formulae

$$\text{(2.4)} \qquad \begin{aligned} W_{isis} &= W_{jtjt} \\ W_{istt} + W_{jsjt} &= 0 \end{aligned}$$

for all indices i, j, s, t distinct each other.

The Hodge star operator $*$ on 2-forms is an involutive endomorphism of Ω^2. The bundle Ω^2 splits then into the eigen-subbundles $\Omega^2 = \Omega^+ \oplus \Omega^-$ of eigenvalues ± 1. We say a 2-form of $\Omega^\pm$ self-dual or anti-self-dual, respectively.

Remark that $\sigma_1^+ = \theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4$, $\sigma_2^+ = \theta^1 \wedge \theta^3 + \theta^4 \wedge \theta^2$ and $\sigma_3^+ = \theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3$ give a locally defined orthogonal frame field of Ω^+. Note $|\sigma_i^+|^2 = 2$, $i = 1,2,3$.

From (2.4) it is easily checked that W and $*$ commute as endomorphisms of Ω^2, namely

$$* \circ W = W \circ *.$$

On the other hand K satisfies

$$* \circ K = -K \circ *,$$

because (B_{is}) is trace zero.

Therefore W maps $\Omega^\pm$ into itself and K maps Ω^+ into Ω^- and Ω^- into Ω^+.

According to the splitting $\Omega^2 = \Omega^+ \oplus \Omega^-$ the Riemannian curvature tensor R thus has the block decomposition

$$\text{(2.5)} \qquad R = \begin{pmatrix} W^+ & 0 \\ 0 & W^- \end{pmatrix} + \begin{pmatrix} 0 & K_- \\ K_+ & 0 \end{pmatrix} + \frac{\rho}{12} \begin{pmatrix} Id_{\Omega^+} & 0 \\ 0 & Id_{\Omega^-} \end{pmatrix}.$$

DEFINITION 2.1. A Riemannian metric g is self-dual, or anti-self-dual, when W^-, or W^+ vanishes over M.

Notice that $K = 0$ if and only if $B = 0$, in other words, g is Einstein.

§3. Left- and right-exterior derivatives

3.1. We define the *left-exterior derivative* and the *right-exterior derivative*, respectively;

$$d_L : C^\infty(\Omega^p \otimes \Omega^q) \to C^\infty(\Omega^{p+1} \otimes \Omega^q), \tag{3.1}$$

$$d_R : C^\infty(\Omega^p \otimes \Omega^q) \to C^\infty(\Omega^p \otimes \Omega^{q+1}) \tag{3.2}$$

which are natural generalization of the ordinary exterior derivatives

$$d : C^\infty(\Omega^p) \to C^\infty(\Omega^{p+1}). \tag{3.3}$$

More generally, if we are given a vector bundle E with a connection ∇, then we can define the operators d_L, d_R for E-valued p-forms by using ∇ and the Levi-Civita connection of g as

$$d_L : C^\infty(\Omega^p \otimes E) \to C^\infty(\Omega^{p+1} \otimes E)$$

and

$$d_R : C^\infty(E \otimes \Omega^p) \to C^\infty(E \otimes \Omega^{p+1}).$$

The operators (3.1), (3.2) are those for the case of $E = \Omega^q$ with the connection given by the Levi-Civita connection.

Although the ordinary exterior derivative does not depend on the metric g, our exterior derivatives depend on the metric.

For example, for $\phi = \sum \phi_{ijs}(\theta^i \wedge \theta^j) \otimes \theta^s \in C^\infty(\Omega^2 \otimes \Omega^1)$

$$d_L \phi = \sum (\nabla_i \phi_{jks} + \nabla_j \phi_{kis} + \nabla_k \phi_{ijs})(\theta^i \wedge \theta^j \wedge \theta^k) \otimes \theta^s, \tag{3.4}$$

and

$$d_R \phi = \sum (\nabla_s \phi_{ijt} - \nabla_t \phi_{ijs})(\theta^i \wedge \theta^j) \otimes (\theta^s \wedge \theta^t), \tag{3.5}$$

respectively. Here $\nabla_i \phi_{jks}$ is the covariant derivative of ϕ_{jks} to the direction of e_i.

The second Bianchi identity

$$\nabla_h R_{ijst} + \nabla_i R_{jhst} + \nabla_j R_{hist} = 0 \tag{3.6}$$

can read as $d_L R = 0$ when we consider R as a section of $\Omega^2 \otimes \Omega^2$.

3.2. Denote by p^+ the projection $\Omega^2 \to \Omega^+$, $\theta \mapsto \frac{1}{2}(\theta + *\theta)$. We have then the operator

$$d^+ = p_+ \circ d : C^\infty(\Omega^1) \to C^\infty(\Omega^+),$$

and moreover the following operators

$$d_L^+ = p_+ \circ d_L : C^\infty(\Omega^1 \otimes \Omega^p) \to C^\infty(\Omega^+ \otimes \Omega^p) \tag{3.7}$$

and

$$d_R^+ = p_+ \circ d_R : C^\infty(\Omega^p \otimes \Omega^1) \to C^\infty(\Omega^p \otimes \Omega^+). \tag{3.8}$$

In subsequent sections we shall frequently encounter the operator

$$d_L^+ \circ d_R^+ : C^\infty(S_0^2\Omega^1) \to C^\infty(\Omega^+ \otimes \Omega^+)$$

and its left-right symmetric dual operator $d_R^+ \circ d_L^+$ which are both crucial in expressing the linearization of the self-dual part of the Weyl conformal tensor, as explained at §4.

LEMMA 3.1. *Let* $(p_+, p_+) : \Omega^2 \otimes \Omega^2 \to \Omega^+ \otimes \Omega^+$ *denote the natural projection. Then the operators* $(p_+, p_+) \circ (d_L \circ d_R)$ *and* $(p_+, p_+) \circ (d_R \circ d_L) : C^\infty(S_0^2\Omega^1) \to C^\infty(\Omega^+ \otimes \Omega^+)$ *coincide with the operators* $d_L^+ \circ d_R^+ = (p_+ \circ d_L) \circ (p_+ \circ d_R)$ *and* $d_R^+ \circ d_L^+ = (p_+ \circ d_R) \circ (p_+ \circ d_L)$, *respectively.*

For proving this lemma we make use of the following

OBSERVATION. *For* $\omega \in C^\infty(\Omega^+)$ *(respectively,* $\omega \in C^\infty(\Omega^-)$*) the left-covariant derivative* $\nabla\omega$ *sits in* $C^\infty(\Omega^1 \otimes \Omega^+)$ *(respectively in* $C^\infty(\Omega^1 \otimes \Omega^-)$*).*

This observation stems from the fact that the group $SO(4)$ factors through the two groups $SO(3)^+$, $SO(3)^-$ acting on the self-dual (anti-self-dual) subbundles Ω^+ and Ω^-.

For $\omega \in C^\infty(\Omega^+)$ we put $\omega_{ij} = \omega(e_i, e_j)$. Then, $\omega_{12} = \omega_{34}$, $\omega_{13} = \omega_{42}$ and $\omega_{14} = \omega_{23}$. The (ijk)-component $\nabla_i\omega_{jk}$ of $\nabla\omega$, that is, $(\nabla_{e_i}\omega)(e_j, e_k)$, is given by the covariant derivative rules as

$$\nabla_i\omega_{jk} = e_i(\omega_{jk}) - \Gamma_{ij}^a\, \omega_{ak} - \Gamma_{ik}^a\, \omega_{ja},$$

where Γ_{ij}^a is the connection coefficient of the metric g relative to the orthonormal frame field $\{e_i\}$. Then $\nabla_i\omega_{12}$ is

$$\nabla_i\omega_{12} = e_i(\omega_{12}) - \Gamma_{i1}^3\, \omega_{32} - \Gamma_{i1}^4\, \omega_{42} - \Gamma_{i2}^3\, \omega_{13} - \Gamma_{i2}^4\, \omega_{14}.$$

On the other hand

$$\nabla_i\omega_{34} = e_i(\omega_{34}) - \Gamma^1_{i3}\,\omega_{14} - \Gamma^2_{i3}\,\omega_{24} - \Gamma^1_{i4}\,\omega_{31} - \Gamma^2_{i4}\,\omega_{32}.$$

Thus, from $\Gamma^a_{ij} = -\Gamma^j_{ia}$ we have $\nabla_i\omega_{12} = \nabla_i\omega_{34}$. Similarly we have $\nabla_i\omega_{13} = \nabla_i\omega_{42}$, $\nabla_i\omega_{14} = \nabla_i\omega_{23}$ so that $\nabla\omega$ is a section of $\Omega^1 \otimes \Omega^+$.

Here we adopted the Einstein convention that repeated latin indices are summed even the summation symbol is omitted. In what follows we also follow this convention.

This observation even holds for bundles tensored with $\Omega^{\pm}$. In fact, if $\varphi \in C^{\infty}((\otimes^p \Omega^1)\otimes \Omega^{\pm})$ (respectively, $\varphi \in C^{\infty}(\Omega^p \otimes \Omega^{\pm})$), then $\nabla\varphi \in C^{\infty}(\otimes^{p+1}\Omega^1 \otimes \Omega^{\pm})$ (respectively, $d_L\varphi \in C^{\infty}(\Omega^{p+1} \otimes \Omega^{\pm})$).

The lemma is now clearly seen, since for $h \in C^{\infty}(S^2_0\,\Omega^1)$, $d_L(d_R\,h)$ is

$$d_L(d_R^+h + d_R^-h) = d_L^+d_R^+h + d_L^-d_R^+h + d_L^+d_R^-h + d_L^-d_R^-h$$

and from the above observation the last two terms are in $C^{\infty}(\Omega^2 \otimes \Omega^-)$, and $d_L^-d_R^+h \in C^{\infty}(\Omega^- \otimes \Omega^+)$. So $(p_+, p_+)\circ(d_L\circ d_R) = d_L^+\circ d_R^+$.

3.3. The bundle $\Omega^p \otimes \Omega^q$ carries the inner product inherited from the metric g. Then with respect to this inner product the operators d_L, d_R, d_L^+ and d_R^- have their formal adjoints δ_L, δ_R, δ_L^+ and δ_R^+, respectively.

For $\phi = \phi_{ijs} \in C^{\infty}(\Omega^2 \otimes \Omega^1)$ for example, we have

$$\delta_L\phi = (\delta_L\phi)_{is},\ (\delta_L\phi)_{is} = -\nabla_a\phi_{ais}$$

and

$$\delta_R\phi = (\delta_R\phi)_{ij},\ (\delta_R\phi)_{ij} = -\nabla_a\phi_{ija}.$$

Remark. The Ricci tensor $\mathrm{Ric} = (R_{is})$ satisfies from the second Bianchi identity

(3.9) $$\delta_R\mathrm{Ric} = \frac{1}{2}\,d_R\rho$$

and

(3.10) $$d_R\mathrm{Ric} = -\,\delta_L\,R.$$

Moreover, when the scalar curvature ρ is constant, by applying (2.1) and that $\nabla B = \nabla\,\mathrm{Ric}$ one has

(3.11) $$\delta_R B = 0 \quad \text{and} \quad d_R B = -\,2\delta_L W.$$

The left(right)-exterior derivatives are implicitly treated in [2] where Bourguignon actually gives the Weitzenböck formulae of the operator $d_L \delta_L + \delta_L d_L$ for sections of $\Omega^1 \otimes \Omega^1$ and of $\Omega^2 \otimes \Omega^2$.

3.4. It is a standard fact that for an arbitrary metric the self-dual part of the de Rham complex is elliptic:

$$0 \to C^\infty(\Omega^0) \xrightarrow{d} C^\infty(\Omega^1) \xrightarrow{d^+} C^\infty(\Omega^+) \to 0 \tag{3.12}$$

Tensoring this complex with the bundle Ω^+, we obtain the sequence composed of certain operators, for instance, left-exterior derivatives:

$$C^\infty(\Omega^+) \xrightarrow{d_L} C^\infty(\Omega^1 \otimes \Omega^+) \xrightarrow{d_L^+} C^\infty(\Omega^+ \otimes \Omega^+) \tag{3.13}$$

PROPOSITION 3.2. *Suppose that a metric g is anti-self-dual. Then the components of $d_L^+ d_L \phi$, $\phi = \phi_{st} \in C^\infty(\Omega^+)$ are*

$$\begin{aligned}(d_L^+ d_L \phi)_{1i1j} &= 0 & (i = j)\\ &= 1/12\, \rho\varepsilon_{ijk}\, \phi_{1k} & (i \neq j)\end{aligned} \tag{3.14}$$

where $\{i, j, k\}$ is a permutation of $\{2,3,4\}$. Therefore, if g is further has zero scalar curvature, the sequence (3.13) *forms an elliptic complex.*

Proof. Apply the Ricci identity to $d_L d_L \phi$. We have then

$$\nabla_i \nabla_j \phi_{st} - \nabla_j \nabla_i \phi_{st} = -(R_{asij}\phi_{at} + R_{atij}\phi_{sa}).$$

Substitute the formula (2.1) $R_{ijst} = W_{ijst} + K_{ijst} + G_{ijst}$ into the curvature terms and use the fact that $W^+ = 0$ and $B = B_{is}$ is tracefree. Then the components become $(d_L^+ d_L \phi)_{1212} = 0$, $(d_L^+ d_L \phi)_{1213} = 1/12\, \rho\phi_{14}$ and $(d_L^+ d_L \phi)_{1214} = 1/12\, \rho\phi_{13}$. Other components are similarly calculated so that the proposition is obtained.

3.5. Weitzenböck formulae for $d_L^+ \delta_L^+$. Denote by δ^+ the formal adjoint of d^+. Then

PROPOSITION 3.3. *The Weitzenböck formula for $d^+ \delta^+ ; C^\infty(\Omega^+) \to C^\infty(\Omega^+)$ is given in the form*

$$d^+ \delta^+ \phi = \frac{1}{2} \nabla^* \nabla \phi - 2W^+(\phi) + \frac{1}{6} \rho\phi,\ \phi \in C^\infty(\Omega^+). \tag{3.15}$$

Note $W^+ : \Omega^+ \to \Omega^+$ has the components $W^+(\phi)_{1a} = W^+_{1a1b}\phi_{1b}$, $a = 2,3,4$. This formula says that the positive scalar curvature implies that the space

$H^+(M) = \mathrm{Ker}\,\delta^+$ of self-dual harmonic 2-forms on M vanishes provided g is anti-self-dual.

Although the formula is well known, we shall verify it, since verifying it is useful to showing other Weitzenböck formulae.

Proof. Since $(d\delta^+\phi)_{ij} = -\nabla_i\nabla_a\phi_{aj} + \nabla_j\nabla_a\phi_{ai}$,

$$(d^+\delta^+\phi)_{12} = -\frac{1}{2}\nabla_a\nabla_a\phi_{12}$$
$$+\frac{1}{2}\{([\nabla_1,\nabla_3] + [\nabla_4,\nabla_2])\phi_{14} - ([\nabla_1,\nabla_4] + [\nabla_2,\nabla_3])\phi_{13}\}.$$

From the Ricci identity the last term, which we denote for brevity by $\mathcal{R}(\phi)_{12}$, is represented in terms of curvature terms as

$$\mathcal{R}(\phi)_{12} = -\frac{1}{2}\{(R_{a113} + R_{a142})\phi_{a4} + (R_{a413} + R_{a442})\phi_{1a} - (R_{a114} + R_{a123})\phi_{a3} - (R_{a314} + R_{a323})\phi_{1a}\}. \tag{3.16}$$

Substitute (2.1) and make use of the fact that $W_{aiaj} = 0$. Then this reduces to

$$-2(W^+_{1212}\phi_{12} + W^+_{1213}\phi_{13} + W^+_{1214}\phi_{14}) + 1/6\,\rho\phi_{12},$$

from which the formula follows.

NOTATION. Here we use the notation W^+_{ijab}, for instance, $W^+_{12ab} = \frac{1}{2}(W_{12ab} + W_{34ab})$.

Consider now the following operator $d^+_L\,\delta^+_L : C^\infty(\Omega^+\otimes\Omega^1) \to C^\infty(\Omega^+\otimes\Omega^1)$. Then,

PROPOSITION 3.4. *For $\phi \in C^\infty(\Omega^+\otimes\Omega^1)$*

$$d^+_L\,\delta^+_L\,\phi = \frac{1}{2}\nabla^*\nabla\phi + \mathcal{W}^{+,1}(\phi) + \frac{5}{2}\cdot\frac{1}{12}\rho\phi + \frac{1}{12}\rho(\phi^\vee\wedge g)^+ + \frac{1}{2}\phi\odot B + \frac{1}{2}(\phi^\vee\wedge B)^+ + \frac{1}{2}((\phi\odot B)^\vee\wedge g)^+. \tag{3.17}$$

Here $\mathcal{W}^{+,1}(\phi) = -2\mathcal{W}^+(\phi) + \mathcal{W}_1(\phi)$ is given by

$$\mathcal{W}^+(\phi)_{12s} = \sum_a W^+_{121a}\,\phi_{1as},$$

$$\mathscr{W}_1(\phi)_{12s} = - W^+_{13as}\,\phi_{14a} + W^+_{14as}\,\phi_{13a}$$

and $\phi^\vee \in C^\infty(\Omega^1)$ *is the contraction of* ϕ, *namely,* $\phi^\vee_i = \phi_{aia}$. *Further* $(\phi^\vee \wedge B)^+$ *is the* $\Omega^+ \otimes \Omega^1$*-part of* $\phi^\vee \wedge B \in C^\infty(\Omega^2 \otimes \Omega^1)$, $(\phi^\vee \wedge B)_{ijs} = \phi^\vee_i B_{js} - \phi^\vee_i B_{is}$, *and* $\phi \odot B \in C^\infty(\Omega^+ \otimes \Omega^1)$ *is given by* $(\phi \odot B)_{ijs} = \phi_{ija} B_{sa}$.

Proof. One has from a calculation $(d_L \delta^+_L\,\phi)_{ijs} = - \nabla_i \nabla_a \phi_{ajs} + \nabla_j \nabla_a \phi_{ais}$ so that $d^+_L \delta^+_L\,\phi$ has the component

$$(d^+_L \delta^+_L \phi)_{12s} = \frac{1}{2}\{- \nabla_1 \nabla_a \phi_{a2s} + \nabla_2 \nabla_a \phi_{a1s} - \nabla_3 \nabla_a \phi_{a4s} + \nabla_4 \nabla_a \phi_{a3s}\}.$$

Since $\phi = \phi_{ijs}$ is self-dual with respect to i, j, one gets

$$\begin{aligned}(d^+_L \delta^+_L \phi)_{12s} &= - \frac{1}{2}\,\nabla_a \nabla_a \phi_{12s} \\ &\quad + \frac{1}{2}([\nabla_1, \nabla_3] + [\nabla_4, \nabla_2])\phi_{14s} - \frac{1}{2}([\nabla_1, \nabla_4] + [\nabla_2, \nabla_3])\phi_{13s}\end{aligned}$$

and further from the Ricci identity

$$(d^+_L \delta^+_L \phi)_{12s} = - \frac{1}{2}\,\nabla_a \nabla_a \phi_{12s} + \mathscr{R}(\phi)_{12s} + \mathscr{R}^1(\phi)_{12s} \tag{3.18}$$

where

$$\begin{aligned}\mathscr{R}(\phi)_{12s} = - \frac{1}{2}\{&(R_{a113} + R_{a142})\phi_{a4s} + (R_{a413} + R_{a442})\phi_{1as} \\ &- (R_{a114} + R_{a123})\phi_{a3s} - (R_{a314} + R_{a323})\phi_{1as}\}\end{aligned}$$

and

$$\mathscr{R}^1(\phi)_{12s} = - \frac{1}{2}\{(R_{as13} + R_{as42})\phi_{14a} - (R_{as14} + R_{as23})\phi_{13a}\}.$$

Here $\mathscr{R}(\phi)$ reduces to $- 2\mathscr{W}^+(\phi) + \frac{1}{6}\rho\phi$, similarly as before.

To get (3.17) one decomposes $\mathscr{R}^1(\phi)$ into terms involving W_{ijkl}, K_{ijkl} and G_{ijkl}, respectively: $\mathscr{R}^1(\phi) = \mathscr{W}^1(\phi) + \mathscr{K}^1(\phi) + \mathscr{G}^1(\phi)$.

From a straight computation, $\mathscr{G}^1(\phi) = \frac{1}{24}\rho\phi + \frac{1}{12}\rho(\phi^\vee \wedge g)^+$.

The term involving B_{ij} is similarly computed as

$$\mathscr{K}^1(\phi) = \frac{1}{2}(\phi \odot B) + \frac{1}{2}(\phi^\vee \wedge B)^+ + \frac{1}{2}((\phi \odot B)^\vee \wedge g)^+.$$

Putting these terms together one obtains Proposition 3.4.

COROLLARY 3.5. *If the metric g is anti-self-dual, then for $\phi \in C^\infty(\Omega^+ \otimes \Omega^1)$ having $\phi^\vee = 0$*

$$d_L^+ \delta_L^+ \phi = \frac{1}{2}\nabla^* \nabla \phi + \frac{5}{24}\rho\phi + \frac{1}{2}\phi \odot B + \frac{1}{2}((\phi \odot B)^\vee \wedge g)^+. \tag{3.19}$$

If g is further Einstein, i.e., $B = 0$, (3.19) reduces to

$$d_L^+ \delta_L^+ \phi = \frac{1}{2}\nabla^* \nabla \phi + \frac{5}{24}\rho\phi.$$

Next we consider the Weitzenböck formula for $d_L^+ \delta_L^+$ operating on $C^\infty(\Omega^+ \otimes \Omega^+)$.

To state the formula we prepare the Nomizu-Kulkarni product

$$\begin{aligned} &\owedge : (\Omega^1 \otimes \Omega^1) \times (\Omega^1 \otimes \Omega^1) \to \Omega^2 \otimes \Omega^2; \\ &(h \owedge q)_{ijst} = h_{is}q_{jt} - h_{it}q_{js} - h_{jt}q_{is} - h_{js}q_{it},\ h,\ q \in \Omega^1 \otimes \Omega^1. \end{aligned} \tag{3.20}$$

Notice that $h \owedge q \in S^2(\Omega^2)$ when h and q are in $S^2(\Omega^1)$.

PROPOSITION 3.6. *For $Z \in C^\infty(\Omega^+ \otimes \Omega^+)$*

$$\begin{aligned} d_L^+ \delta_L^+ Z &= \frac{1}{2}\nabla^* \nabla Z + \mathscr{W}(Z) + \frac{1}{4}\rho Z \\ &- \frac{1}{12}\rho(Z^\vee \owedge g)^+ - \frac{1}{2}(\mathscr{B}^*(Z) \owedge g)^+ - \frac{1}{2}(Z^\vee \owedge B)^+. \end{aligned} \tag{3.21}$$

Here $\mathscr{W}(Z) = -2\mathscr{W}^+(Z) + \mathscr{W}_2^+(Z)$, $\mathscr{W}^+(Z)_{12st} = \sum_a W^+_{121a} Z_{1ast}$, and

$$\begin{aligned} \mathscr{W}_2^+(Z)_{12st} = -(&W^+_{13as} Z_{14at} - W^+_{14as} Z_{13at} \\ &+ W^+_{13at} Z_{14sa} - W^+_{14at} Z_{13sa}). \end{aligned}$$

Moreover we denote by $Z^\vee$, $\mathscr{B}^*(Z) \in C^\infty(\Omega^1 \otimes \Omega^1)$ as $Z^\vee_{ts} = Z_{aias}$ and $\mathscr{B}^*(Z)_{is} = B_{ab}Z_{aibs}$.

Proof. We calculate $d_L^+\delta_L^+ Z$ for $(i, j) = (1,2)$.

$$(d_L^+\delta_L^+ Z)_{12st} = \frac{1}{2}\{-\nabla_a\nabla_a Z_{12st} - ([\nabla_1, \nabla_4] + [\nabla_2, \nabla_3])Z_{13st} + ([\nabla_1, \nabla_3] + [\nabla_4, \nabla_2])Z_{14st}\}$$

From the Ricci identity we can write this as

$$(d_L^+\delta_L^+ Z)_{12st} = \frac{1}{2}(-\nabla_a\nabla_a Z_{12st}) + \mathcal{R}(Z)_{12st} + \mathcal{R}^+(Z)_{12st} \tag{3.22}$$

where

$$\mathcal{R}(Z)_{12st} = -(R^+_{13a1} Z_{a4st} + R^+_{13a4} Z_{1ast} - R^+_{14a1} Z_{a3st} - R^+_{14a3} Z_{1ast})$$

and

$$\mathcal{R}^+(Z)_{12st} = -(R^+_{13as} Z_{14at} - R^+_{14as} Z_{13at} + R^+_{13at} Z_{14sa} - R^+_{14at} Z_{13sa}).$$

For the term $\mathcal{R}(Z)$ we get similarly as before the following

$$\mathcal{R}(Z) = -2\mathcal{W}^+(Z) + \frac{1}{6}\rho Z.$$

Now decompose $\mathcal{R}^+(Z)$ into terms $\mathcal{R}^+(Z) = \mathcal{W}_2^+(Z) + \mathcal{K}_2^+(Z) + \mathcal{G}_2^+(Z)$ involving W_{ijst}, K_{ijst} and G_{ijst}, respectively.

From a simple computation the third term $\mathcal{G}_2^+(Z)$ becomes

$$\mathcal{G}_2^+(Z) = \frac{1}{12}\rho Z - \frac{1}{12}\rho(Z^\vee \owedge g)^+.$$

Here $(h \owedge q)^+ = (p_+, p_+)(h \owedge q)$ for $h, q \in C^\infty(\Omega^1 \otimes \Omega^1)$.

To compute $\mathcal{K}_2^+(Z)$ we make use of the fact that B_{is} is tracefree and Z is self-dual in s, t to get

$$\mathcal{K}_2^+(Z) = -\frac{1}{2}\{(Z^\vee \owedge B)^+ + (\mathcal{B}^*(Z) \owedge g)^+\}.$$

Note that the term $\mathcal{W}(Z)$ in Proposition 3.6 contains only the self-dual Weyl conformal tensor.

The tensor product $\Omega^+ \otimes \Omega^+$ has the natural decomposition

$$\Omega^+ \otimes \Omega^+ = S_0^2(\Omega^+) \oplus \Lambda^2(\Omega^+) \oplus \mathbf{R}\Phi$$

where Φ is the global section of $S^2(\Omega^+)$ defined locally by $\sum_a \sigma_a^+ \otimes \sigma_a^+$ associated to the canonical orthogonal frame $\{\sigma_1^+, \sigma_2^+, \sigma_3^+\}$ of Ω^+

COROLLARY 3.7. *For* $Z \in C^\infty(S_0^2(\Omega^+))$

$$d_L^+\delta_L^+ Z = 1/2\nabla^*\nabla Z + \mathscr{W}(Z) + \frac{1}{4}\rho Z. \tag{3.23}$$

If the metric g is anti-self-dual and the scalar curvature ρ is positive, then

$$\mathrm{Ker}\{\delta_L^+ : C^\infty(S_0^2(\Omega^+)) \to C^\infty(\Omega^1 \otimes \Omega^+)\} = \{0\}.$$

The proof is obviously seen, since $\mathscr{B}^*(Z)$ is tracefree and symmetric for $Z \in S_0^2(\Omega^+)$ and from the following lemma.

LEMMA 3.8. *The metric tensor g satisfies*

$$(h \owedge g)^+ = 0$$

for each $h \in S_0^2(\Omega^1)$.

Proof. The component $(h \owedge g)^+_{1212}$ is

$$\begin{aligned}&\frac{1}{4}\{(h \owedge g)_{1212} + 2(h \owedge g)_{1234} + (h \owedge g)_{3434}\}\\&= \frac{1}{4}\{(h \owedge g)_{1212} + (h \owedge g)_{3434}\},\end{aligned}$$

and hence $(h \owedge g)^+_{1212} = 0$, since h is tracefree.

Moreover,

$$\begin{aligned}(h \owedge g)^+_{1213} &= \frac{1}{4}\{(h \owedge g)_{1213} + (h \owedge g)_{1242} + (h \owedge g)_{3413} + (h \owedge g)_{3442}\}\\&= \frac{1}{4}(h_{23} - h_{32} + h_{14} - h_{41}) = 0.\end{aligned}$$

These computations complete the proof of the lemma.

COROLLARY 3.9. *For* $Z \in C^\infty(\Lambda^2(\Omega^+))$

$$d_L^+\delta_L^+ Z = \frac{1}{2}\nabla^*\nabla Z + \mathscr{W}(Z) + \frac{1}{6}\rho Z. \tag{3.24}$$

Therefore,

$$\mathrm{Ker}\{\delta_L^+ : C^\infty(\Lambda^2(\Omega^+)) \to C^\infty(\Omega^1 \otimes \Omega^+)\} = \{0\},$$

when the metric is anti-self-dual and of positive scalar curvature.

Proof. It suffices from (3.21) and the above lemma to show

$$(Z^\vee \owedge g)^+ = 2Z,\ (\mathscr{B}^*(Z) \owedge g)^+ = 0 \text{ and } (Z^\vee \owedge B)^+ = 0.$$

The first two of these are obtained as follows.

We have for $q \in \Lambda^2(\Omega^1)$

$$(q \owedge g)^+_{1212} = \frac{1}{4}\sum_a q_{aa} = 0 \tag{3.25}$$

and

$$(q \owedge g)^+_{1213} = \frac{1}{2}(q_{14} + q_{23}). \tag{3.26}$$

Put $q = Z^\vee$. Then from (3.26) $(Z^\vee \owedge g)^+_{1213} = 2Z_{1213}$ which implies the first formula. The second one follows from that B is tracefree.

The last formula is similarly seen, since $Z^\vee \in \Lambda^2(\Omega^1)$ satisfies $Z^\vee_{12} = Z^\vee_{34}$, $Z^\vee_{13} = Z^\vee_{42}$ and $Z^\vee_{14} = Z^\vee_{23}$.

The operator $d_L^+\delta_L^+$ operating on $f\Phi \in C^\infty(S^2(\Omega^+))$, $f \in C^\infty(M)$ has the following Weitzenböck formula.

COROLLARY 3.10. *Let g be an arbitrary metric on M. Then*

$$d_L^+\delta_L^+(f\Phi) = \frac{1}{2}\nabla^*\nabla f\Phi - fW^+. \tag{3.27}$$

So, $f\Phi$ in Ker δ_L^+ *must be $c\Phi$, where c is a constant.*

Proof. Since $\Phi^\vee = 3g$, $(g \owedge g)^+ = \Phi$ and $\mathscr{B}^*(\Phi) = -B$, the last three terms of (3.21) reduce to $-\frac{1}{4}\rho(f\Phi)$. Moreover $\mathscr{W}^+(\Phi) = W^+$ and $\mathscr{W}_2^+(\Phi) = W^+$. So the term $\mathscr{W}(\Phi) = -W^+$. Because Φ is parallel, we derive (3.27).

Suppose $f\Phi$ satisfies $\delta_L^+(f\Phi) = 0$. Then by taking the inner product $(d_L^+\delta_L^+(f\Phi), f\Phi)$ we have

$$0 = \frac{1}{2}\int_M \{(\nabla^*\nabla f, f)\,|\,\Phi\,|^2 - (W^+, f\Phi)\}dv_g.$$

Since W and hence W^+ is tracefree, $(W^+, \Phi) = 0$ and then $\int_M |\nabla f|^2 dv_g = 0$. This means that f is constant.

§4. The linearization of the self-dual Weyl conformal tensor

4.1. The aim of this section is to calculate the linearization of the self-dual part of the Weyl conformal tensor W.

The tensor $W = W(g)$ is considered, same as before, as $W \in C^\infty(S_0^2(\Omega^2))$ having the splitting form $W = W^+ + W^- \in C^\infty(S_0^2(\Omega^+) \oplus S_0^2(\Omega^-))$.

For each $h \in C^\infty(S_0^2(\Omega^1))$ the Fréchet differential of W^+ at a metric g to the direction h is given by

$$\delta W_g^+(h) = \frac{d}{dt} W^+(g_t)\,|_{t=0}, \tag{4.1}$$

where g_t, $|t| < \varepsilon$, is a one parameter family of metrics with $g_0 = g$, $\frac{d}{dt} g_t \Big|_{t=0} = h$.

Proposition 4.1. *If g is anti-self-dual, i.e., $W^+ = 0$ for g, then*

$$\delta W_g^+(h) = (\delta W_g(h))^+. \tag{4.2}$$

This means that the linearization of W^+ is exactly the self-dual part of the linearization of W, provided g is anti-self-dual.

Since $W^+(g) = (p_+, p_+) W(g)$, $(\delta W_g^+)(h)$ is given by $(\delta(p_+, p_+))(h) W(g) + (p_+, p_+) \delta W_g(h)$ and the first term vanishes because g is anti-self-dual so that the proposition is obtained (see also [12]).

Proposition 4.2. *The linearization of the self-dual part of the Weyl conformal tensor is written in terms of the left- and right exterior derivatives as*

$$(\delta W_g(h))^+ = -\frac{1}{4}\{(d_R^+ d_L^+ h + d_L^+ d_R^+ h)_0 + (B \circledwedge h)_0^+\} \tag{4.3}$$

for $h \in C^\infty(S_0^2(\Omega^1))$.

Proof. We have in Appendix [12]

$$(\delta W_g(h))^+ = (U(h))_0^+ + (V(h))_0^+, \; h \in C^\infty(S_0^2(\Omega^1)), \tag{4.4}$$

where $U, V : C^\infty(S_0^2(\Omega^1)) \to C^\infty(\Omega^2 \otimes \Omega^2)$ are the operators defined by $U(h) = (U_{ijst})$, $V(h) = (V_{ijst})$, respectively

$$U_{ijst} = \frac{1}{2}\left(\nabla_s \nabla_j h_{it} - \nabla_t \nabla_j h_{is} - \nabla_s \nabla_i h_{jt} + \nabla_t \nabla_i h_{js}\right), \tag{4.5}$$

$$V_{ijst} = \frac{1}{4}\left(R_{sj}\, h_{it} - R_{tj}\, h_{is} - R_{si}\, h_{jt} + R_{ti}\, h_{js}\right). \tag{4.6}$$

By using the left- and right exterior derivatives we can write $U(h)$ as

$$U(h) = -\frac{1}{2}\, d_R d_L\, h,$$

since

$$d_L\, h = (\nabla_i h_{js} - \nabla_j h_{is}) f^i \wedge f^j \otimes f^s$$

and

$$d_R(d_L\, h) = \{\nabla_s(\nabla_i h_{jt} - \nabla_j h_{it}) - \nabla_t(\nabla_i h_{js} - \nabla_j h_{is})\}(f^i \wedge f^j) \otimes (f^s \wedge f^t).$$

The symmetrization of $U(h)$ then becomes

$$\frac{1}{2}\left(-\frac{1}{2}\, d_R d_L\, h - \frac{1}{2}\, d_L d_R\, h\right) = -\frac{1}{4}\left(d_R d_L\, h + d_L d_R\, h\right).$$

Thus the $S^2(\Omega^+)$-component of $U(h)$ is $-\frac{1}{4}\,(d_R^+ d_L^+\, h + d_L^+ d_R^+\, h)$, whose tracefree part $-\frac{1}{4}\,(d_R^+ d_L^+\, h + d_L^+ d_R^+\, h)_0$ gives $U(h)_0^+$.

The term $V(h)$ has the form $V(h) = -\frac{1}{4}\,(\mathrm{Ric} \owedge h)$ by the Nomizu-Kulkarni product (3.20). Since $(g \owedge h)_0^+ = 0$ for a tracefree h from Lemma 3.8, we have $V(h)_0^+ = -\frac{1}{4}\,(B \owedge h)_0^+$, from which (4.3) is obtained.

4.2. The Bach operator. We define an operator

$$D : C^\infty(S_0^2(\Omega^1)) \to C^\infty(S_0^2(\Omega^+)) \tag{4.7}$$

by

$$D(h) = -\,(\delta W_g(h))^+.$$

Moreover we define operators $\mathscr{D}$ and $\mathscr{B} : C^\infty(S_0^2(\Omega^1)) \to C^\infty(S_0^2(\Omega^+))$ as

$$\begin{aligned} \mathscr{D}(h) &= \frac{1}{4}\,(d_R^+ d_L^+\, h + d_L^+ d_R^+\, h)_0, \\ \mathscr{B}(h) &= \frac{1}{4}\,(B \owedge h)_0^+ \end{aligned} \tag{4.8}$$

so that

$$D(h) = \mathscr{D}(h) + \mathscr{B}(h).$$

Notice that in his paper ([19]) C. Taubes uses L_g for the linearization δW_g^+.

PROPOSITION 4.3. *The formal adjoint D^* of D has the form*

$$D^*(Z) = \mathscr{D}^*(Z) + \mathscr{B}^*(Z),\ Z = (Z_{ijst}) \in C^\infty(S_0^2(\Omega^+)), \tag{4.9}$$

where $\mathscr{D}^*$ is given by $\mathscr{D}^*(Z) = (\delta_L^+\delta_R^+ + \delta_R^+\delta_L^+)Z$, in other words

$$(\mathscr{D}^*Z)_{ts} = \nabla_p\nabla_q Z_{ptqs} + \nabla_q\nabla_p Z_{ptqs} \tag{4.10}$$

and the operator $\mathscr{B}^* : S_0^2(\Omega^+) \to S_0^2(\Omega^1)$ was defined in Proposition 3.6, §3.5.

Proof. From (4.4)

$$\int_M (h, D^*Z)dv = \int (Dh, Z)dv = -\int (U(h)_0^+, Z)dv - \int (V(h)_0^+, Z).$$

The inner product $(U(h)_0^+, Z) = (U(h), Z)$ is

$$(U(h)_0^+, Z) = \frac{1}{2}(\nabla_s\nabla_j h_{it} - \nabla_t\nabla_j h_{is} - \nabla_s\nabla_i h_{jt} + \nabla_t\nabla_i h_{js})Z_{ijst}$$

and this reduces to $(\nabla_s\nabla_j h_{it} - \nabla_s\nabla_i h_{jt})Z_{ijst}$ so that one has the integral

$$\begin{aligned}\int_M (U(h)_0^+, Z)dv &= \int (h_{it}\nabla_j\nabla_s Z_{ijst} - h_{jt}\nabla_i\nabla_s Z_{ijst}) \\ &= 2\int h_{is}\nabla_j\nabla_t Z_{ijts}.\end{aligned} \tag{4.11}$$

For the sake of symmetry one may write this as

$$\int_M (U(h)_0^+, Z)dv = -\int h_{is}(\nabla_j\nabla_t + \nabla_t\nabla_j)Z_{jtts}\,dv \tag{4.12}$$

from which one gets (4.10).

Similarly one has

$$-\int_M (V(h)_0^+, Z)dv = \int h_{is}B_{jt}Z_{jits}\,dv, \tag{4.13}$$

which is just the inner product of h and $\mathscr{B}^*(Z)$.

The tracefree symmetric tensor D^*W is called the *Bach tensor* for a Riemannian 4-manifold (M, g).

Consider the functional

$$g \to \int_M | W(g) |^2 \, dv_g. \tag{4.14}$$

This functional is conformally invariant.

PROPOSITION 4.4 (Lemma 1 in [3]). *Let g be a metric on a 4-manifold M. Then g is a critical point of the functional* (4.14) *if and only if the Bach tensor vanishes* (*i.e.,* $D^*W = 0$).

Proof. For $W = W_{ijst}$ we set $W_{ij} = W_{ijst} f^s \wedge f^t$ so that $W = (W_{ij})$ is regarded as an $\mathrm{End}(T_M)$-valued 2-form.

From the Chern-Weil theorem and the property of the Weyl tensor the first Pontrjagin number of M is given, up to a universal constant, by the integral $\int_M \sum_{i,j} W_{ij} \wedge W_{ji}$.

Decompose $W_{ij} = W_{ij}^+ + W_{ij}^-$. Then the integrand gives rise to

$$-\sum_{ij} (W_{ij}^+ \wedge W_{ij}^+ + W_{ij}^- \wedge W_{ij}^-) = -(| W^+ |^2 - | W^- |^2) \, dv_g.$$

Since $\int | W |^2 dv_g = \int (| W^+ |^2 + | W^- |^2) \, dv_g$ and from the argument just above, a metric g is critical to the functional (4.14) if and only if it is critical to the functional $\int_M | W^+ |^2 dv_g$.

Let g_t be a one parameter family of metrics with $g_0 = g$ and $\frac{d}{dt} g_t |_{t=0} = h$ which is in $C^\infty(S_0^2(\Omega^1))$.

Consider the first variation $\frac{d}{dt} \int_M (W^+(g_t), W^+(g_t))_{g_t} dv_{g_t} |_{t=0}$. Because the integral is conformally invariant the volume form dv_{g_t} of g_t is assumed for all t to coincide with dv_g.

So,

$$\frac{d}{dt} \int_M | W^+(g_t) |_{g_t}^2 \, dv_g |_{t=0}$$

$$= 2\int ((\delta W_g(h))^+, W^+(g))_g\, dv_g + 2\int (\delta(p_+, p_+)_g(h)(W^+(g)), W^+(g))_g\, dv_g$$

$$+ \int H(W^+(g), W^+(g))\, dv_g$$

where $H(W^+(g), W^+(g)) = \frac{d}{dt}(W^+(g), W^+(g))_{g_t}|_{t=0}$ is the t-derivative at $t = 0$ of the g_t-inner product of the g-self-dual part of the Weyl conformal tensor W of the metric g.

We show first that the second term vanishes. Let $(p_+)_t$ be the projection $\frac{1}{2}(\mathrm{id} + *_{g_t}) : \Omega^2 \to \Omega^+_{g_t}$. Since the star operators are involutive, $*_g \circ (\delta *_g)(h) + (\delta *_g)(h) \circ *_g = 0$, where $(\delta *_g)(h)$ denotes the derivation $\frac{d}{dt} *_{g_t}|_{t=0}$ of the star operators $*_{g_t}$ in the direction h. This implies that $(\delta *_g)(h)$ and hence $\delta(p_+)_g(h)$ maps Ω^+_g into Ω^-_g. We have then

$$(\delta(p_+, p_+)_g)(h)(W^+(g)) = ((\delta p_+)_g(h), p_+)(W^+(g)) + (p_+, (\delta p_+)_g(h))(W^+(g)).$$

Thus, the inner product $(\delta(p_+, p_+)_g(h)(W^+(g)), W^+(g))$ vanishes, because $W^+ \in \Omega^+ \otimes \Omega^+$.

We compute the third integrand $H(W^+, W^+)$.

For every point q in M we choose a local coordinate $\{x_i\}$ around q such that $g_{ij}(q) = \delta_{ij}$ and $h_{ij}(q)$ is diagonal, that is, $h_{ij} = h_i \delta_{ij}$ at the point. Then

$$H(W^+, W^+) = \frac{d}{dt} \sum g_t^{ik} g_t^{jl} g_t^{ru} g_t^{sv} W^+_{ijrs} W^+_{kluv}|_{t=0}$$

reduces to $-4\sum_{i<j,r<s}(h_i + h_j + h_r + h_s)(W^+(g)_{ijrs})^2$.

It follows from (2.4) that $H(W^+, W^+)$ is zero. Therefore we get the formula

$$\frac{d}{dt}\int_M |W^+|^2\, dv_g|_{t=0} = 2\int ((\delta W_g(h))^+, W^+(g))_g\, dv_g$$

which turns out from simple computations to be

$$\int (D(h), W^+)dv_g = \int (h, D^*(W^+))dv_g$$

so that g is critical to the functional (4.14) if and only if g satisfies the equation $D^*(W^+) = 0$.

On the other hand, we consider the functional $g \to \int_M |W^-(g)|^2 dv_g$. Then it follows that a critical point of this functional is just critical to (4.14) by a way similar to the self-dual Weyl conformal tensor case. So, the equation $D^*W^+ = 0$ is equivalent to the equation $D^*W^- = 0$ and then to $D^*W = D^*(W^+ + W^-) = 0$.

§5. The Weitzenböck formula for Einstein metrics

5.1. We assume throughout this section that a metric g of M is anti-self-dual and moreover Einstein.

From the formulae (4.8), (4.9) in §4 the linearization operator D and its formal adjoint D^* then become as $D = \mathscr{D}$ and $D^* = \mathscr{D}^*$, respectively.

Since $\mathscr{D}^*$ is of second order, we associate the forth order operator $\mathscr{D}\mathscr{D}^*$ to the two-fold of the rough Laplacian $\nabla^*\nabla$.

We state the following formula whose proof is one of the main subject of this section and will be given later.

PROPOSITION 5.1. (Weitzenböck formula). *Let g be an anti-self-dual, Einstein metric. Then the Weitzenböck formula reads as*

$$D\,D^*Z = \frac{1}{24}(3\nabla^*\nabla + 2\rho)(2\nabla^*\nabla + \rho)Z \tag{5.1}$$

for $Z \in C^\infty(S_0^2(\Omega^+))$ (ρ is the scalar curvature).

As an immediate consequence of this formula we have

THEOREM 5.2 (Theorem 1, §1). *Let (M, g) be a compact oriented 4-manifold with an Einstein anti-self-dual metric.*

(i) *If the scalar curvature $\rho > 0$, then* $\operatorname{Ker} D^* = \{0\}$.

(ii) *If $\rho = 0$, $D^*Z = 0$ if and only if $\nabla Z = 0$, i.e., Z is covariantly constant.*

(iii) *For $\rho < 0$ $\operatorname{Ker} D^*$ is the linear span of the eigenspaces $E_{-2/3\cdot\rho}$ and $E_{-1/2\cdot\rho}$, where $E_\lambda = \{Z\,;\,\nabla^*\nabla Z = \lambda Z\}$.*

5.2. The orbifold case. For 4 dimensional orbifolds we can apply the Weitzenböck formula, same as in the smooth case.

THEOREM 5.3 (Theorem 2, §1). *Let (X, g) be a compact connected oriented 4 dimensional orbifold with an anti-self-dual positive Ricci Einstein metric. Then the second cohomology group $\mathbf{H}^2 = \{0\}$.*

Remark that weighted complex projective planes $\mathbf{C}P^2_{p,q,r}$ with the orientation reversed, for suitable integers p, q, r, admit an Einstein, self-dual orbifold metric of positive scalar curvature ([8]).

5.3. The Ricci flat case. Anti-self-dual Ricci flat 4-manifolds (M, g) are completely classified in [10] as that those manifolds are covered either by a Ricci flat Kähler K3 surface or by a flat Kähler complex 2-torus. More precisely, such a 4-manifold which is covered by a K3 surface is one of the following; i-1) a Ricci flat Kähler K3 surface, i-2) a Ricci flat Kähler Enriques surface (the quotient of a Ricci flat Kähler K3 surface by a free $\mathbf{Z}_2$-actionn), i-3) a $\mathbf{Z}_2 \times \mathbf{Z}_2$-quotient of a Ricci flat Kähler K3 surface. A flat Kähler complex 2-torus, and a flat Kähler hyperelliptic surface, i.e., a finite holomorphic quotient of a flat Kähler complex 2-torus are other examples of anti-self-dual flat 4-manifold.

THEOREM 5.4 (Theohem 3, §1). *Let (M, g) be a compact anti-self-dual Ricci flat 4-manifold. Then the dimension* $\dim \mathbf{H}^2$ *is given in the following way.*
(i) $\dim \mathbf{H}^2 = 5$ *when (M, g) is a Ricci flat Kähler $K3$ complex surface or a flat Kähler complex 2-torus,*
(ii) $\dim \mathbf{H}^2 = 3$ *when (M, g) is a Ricci flat Kähler Enriques surface or a flat Kähler hyperelliptic surface satisfying $\mathcal{O}(K^2_M) = \mathcal{O}$,*
(iii) $\dim \mathbf{H}^2 = 2$ *when (M, g) is a $\mathbf{Z}_2 \times \mathbf{Z}_2$-quotient of a Ricci flat Kähler $K3$ surface and*
(iv) $\dim \mathbf{H}^2 = 1$ *for a hyperelliptic surface with $\mathcal{O}(K^k_M) = \mathcal{O}$ and $\mathcal{O}(K^i_M) \neq \mathcal{O}$, $i < k$,* $k = 3,4,6$.

Proof. It is sufficient from ii) of Theorem 5.2 to compute the dimension of the space of covariantly constant sections Z of $S^2_0(\Omega^+)$.

4-manifolds (M, g) we are now considering are all Kähler except a $\mathbf{Z}_2 \times \mathbf{Z}_2$-quotient of a K3 surface. So, first we may assume

$$S^2_0(\Omega^+) \cong \mathbf{R}\Phi \oplus K_M \oplus K^2_M, \tag{5.2}$$

where Φ denotes the covariantly constant section of $S^2_0(\Omega^+)$ derived from the Kähler structure ([13]). This real bundle isomorphism is invariant with respect to the covariant differentiation.

On the other hand, it is observed in [13] that on a compact Kähler complex surface with zero scalar curvature a section ψ of K^m_M, $m > 0$ is holomorphic if and only if ψ is covariantly constant.

By making use of these facts we see that for a 4-manifold (M, g) having

$\mathcal{O}(K_M) = \mathcal{O}$, namely for a 4-manifold of i-1) or of ii-1) listed above, $S_0^2(\Omega^+)$ has five linearly independent, covariant constant sections. So $\mathbf{H}^2 \cong \mathbf{R}^5$.

Similarly for (M, g) of $\mathcal{O}(K_M^2) = \mathcal{O}$ but $\mathcal{O}(K_M) \neq \mathcal{O}$, i.e., of i-2) or of ii-2), the space of covariantly constant sections of $S_0^2(\Omega^+)$ has exactly dimension three.

For hyperelliptic surface for which K_M is a torsion bundle of order 3, 4 or 6 it is seen from (5.2) that covariantly constant sections of $S_0^2(\Omega^+)$ are of form $c\Phi$, $c \in \mathbf{R}$. This shows (iv).

Finally consider a $\mathbf{Z}_2 \times \mathbf{Z}_2$-quotient (M, g) of a Ricci flat Kähler K3 surface.

We can also regard this manifold as a quotient of a Ricci flat Kähler Enriques surface $(\bar{M}, \bar{g})$ by a free $\mathbf{Z}_2$-isometric action.

Let $s: \bar{M} \to \bar{M}$ be a deck transformation yielding the $\mathbf{Z}_2$-action. So s is an involutive, orientation preserving isometry which freely acts. It is shown in [10] that s is not holomorphic.

It suffices to show that the space of s-invariant, covariantly constant sections of $S_0^2(\Omega^+)$ on $(\bar{M}, \bar{g})$ has dimension two.

Since $b^+(\bar{M}) = 1$ and the pull back 2-form $s^*\theta$ of the Kähler form θ is covariantly constant, $s^*\theta = -\theta$ and hence s must be an anti-holomorphic diffeomorphism of $\bar{M}$.

Consider the section ϕ of $K_{\bar{M}}$ given by $\phi = dz^1 \wedge dz^2$ for a $\bar{g}$-unitary frame of (1,0)-forms $\{dz^1, dz^2\}$. Then we can write the pull back as $s^*\phi = c\bar{\phi}$.

Since s is involutive, $|c| = 1$.

Operate s^* now on the section $\Phi = \frac{1}{4}\theta^2 - \frac{1}{2}\phi \cdot \bar{\phi}$. Then from the above arguments $s^*(\Phi) = \Phi$, in other words, Φ is s-invariant.

Since $\mathcal{O}(K^2) = \mathcal{O}$ for an Enriques surface, K^2 admits a globally defined holomorphic (and hence covariantly constant) section. We may identify this section with ϕ^2. So, the real part Ψ_1 and the imaginary part Ψ_2 of ϕ^2 give exactly covariantly constant sections other than Φ.

The pull back sections $s^*\Psi_1$, $s^*\Psi_2$ are also covariantly constant so that they are written by linear combinations of Ψ_i, $i = 1,2$.

Since $s^*\phi = c\bar{\phi}$, the 2×2 matrix consisting of coefficients of $s^*\Psi_i$ with respect to Ψ_i has trace zero and determinant -1. This matrix then must have eigenvalues $+1, -1$. Therefore $\dim \mathbf{H}^2 = 2$ which completes the proof.

5.4. Now we will show Proposition 5.1. From (4.8) and Proposition 4.3 we have for the operator $DD^*Z = \mathfrak{D}\mathfrak{D}^*Z$, $Z \in C^\infty(S_0^2(\Omega^+))$

$$\mathscr{D}\mathscr{D}^*Z = \frac{1}{4}((d_L^+d_R^+ + d_R^+d_L^+)(\delta_L^+\delta_R^+ + \delta_R^+\delta_L^+)Z)_0 \tag{5.3}$$
$$= \frac{1}{4}\{(d_L^+d_R^+\delta_L^+\delta_R^+Z)_0 + (d_L^+d_R^+\delta_R^+\delta_L^+Z)_0$$
$$+ (d_R^+d_L^+\delta_L^+\delta_R^+Z)_0 + (d_R^+d_L^+\delta_R^+\delta_L^+Z)_0\},$$

where X_0 denotes the trace free component of $X \in S^2(\Omega^+)$.

From the left-right symmetry it suffices for proving the proposition to write down the last two terms with respect to the rough Laplacian.

For this we introduce the operator $\mathscr{F}$ by

$$\mathscr{F}(\phi) = d_L^+(\delta_R^+\phi) - \delta_R^+(d_L^+\phi),\ \phi \in C^\infty(\Omega^1 \otimes \Omega^+). \tag{5.4}$$

As we shall see in Proposition 5.5. $\mathscr{F}$ turns out to be a bundle homomorphism from $\Omega^1 \otimes \Omega^+$ to $\Omega^+ \otimes \Omega^1$.

So, the last two terms become

$$(d_R^+d_L^+\delta_L^+\delta_R^+Z)_0 + (d_R^+d_L^+\delta_R^+\delta_L^+Z)_0$$
$$= ((d_R^+(d_L^+\delta_L^+)\delta_R^+Z)_0 + ((d_R^+\delta_R^+)(d_L^+\delta_L^+)Z)_0 \tag{5.5}$$
$$+ (d_R^+(\mathscr{F}(\delta_L^+Z)))_0.$$

We apply the Weitzenböck formula (3.23) to Z and make use of the fact that Z is tracefree. Then

$$d_L^+\delta_L^+ Z = \frac{1}{2}\nabla^*\nabla Z + \frac{1}{4}\rho Z. \tag{5.6}$$

Noticing $d_L^+\delta_L^+ Z$ is tracefree and (5.6) holds also for the right-exterior-derivative, we get

$$(d_R^+\delta_R^+)(d_L^+\delta_L^+)Z = \left(\frac{1}{2}\nabla^*\nabla + \frac{1}{4}\rho\right)^2 Z \tag{5.7}$$

which gives the second term of the right hand side of (5.5), because (5.7) is tracefree.

That $\delta_R^+Z \in C^\infty(\Omega^+ \otimes \Omega^1)$ is tracefree, namely $(\delta_R^+Z)_{iji} = 0$, together with (3.19) yields

$$d_L^+\delta_L^+(\delta_R^+Z) = \frac{1}{2}\nabla^*\nabla(\delta_R^+Z) + \frac{5}{24}\rho\delta_R^+Z \tag{5.8}$$

(here we applied that $B = 0$) in such a way that the first term is calculated as

$$d_R^+ d_L^+ \delta_L^+ \delta_R^+ Z = \frac{1}{2} d_R^+ (\nabla^* \nabla \delta_R^+ Z) + \frac{5}{24} \rho\Big(\frac{1}{2} \nabla^* \nabla Z + \frac{1}{4} \rho Z\Big). \tag{5.9}$$

What to do next is to derive an explicit formula for the commutator operator $\mathscr{F}$. Set $\psi = \delta_L^+ Z$. $\psi = (\psi_{ist})$ is in $C^\infty(\Omega^1 \otimes \Omega^+)$.
By the definition $\mathscr{F}(\psi)$ is the left self-dual part of $d_L \delta_R^+ \psi - \delta_R^+ d_L \psi$.
One has then

$$(\delta_R^+ \psi)_{is} = - \nabla_a \psi_{ias}$$

and

$$(d_L \delta_R^+ \psi)_{ijs} = - \nabla_i \nabla_a \psi_{jas} + \nabla_j \nabla_a \psi_{ias}.$$

On the other hand,

$$(\delta_R^+ d_L \psi)_{ijs} = - \nabla_a (\nabla_i \psi_{jas} - \nabla_j \psi_{ias}).$$

So,

$$(d_L \delta_R^+ \psi - \delta_R^+ d_L \psi)_{ijs} = [\nabla_a, \nabla_i] \psi_{jas} + [\nabla_j, \nabla_a] \psi_{ias} \tag{5.10}$$

which reduces to

$$\frac{\rho}{4} \psi_{sij} + B_{ti} \psi_{jss} - B_{tj} \psi_{its} + R_{taij} \psi_{tas} - R_{tsai} \psi_{jat} - R_{tsja} \psi_{iat}. \tag{5.11}$$

Here we used the identity, $\psi_{jis} - \psi_{ijs} = \psi_{sij}$, coming from the first Bianchi identity of Z.

Since $B = 0$, the left self-dual part $\mathscr{F}(\psi)$ of $d_L \delta_R^+ \psi - \delta_R^+ d_L \psi$ has the following form, for simplicity for $(i, j) = (1,2)$;

$$\mathscr{F}(\psi)_{12s} = \frac{1}{2} \Big\{ \frac{\rho}{4} (\psi_{s12} + \psi_{s34}) + (R_{ta12} + R_{ta34}) \psi_{tas} - (R_{tsa1} \psi_{2at} + R_{ts2a} \psi_{1at} + R_{tsa3} \psi_{4at} + R_{ts4a} \psi_{3at}) \Big\}. \tag{5.12}$$

We apply (2.1) into the curvature terms and make use of (2.4). Then $\mathscr{F}(\psi)_{12s} = \frac{1}{12} \rho \psi_{s12}$ which gives without loss of generality

Proposition 5.5. *For $\psi \in C^\infty(\Omega^1 \otimes \Omega^+)$ satisfying $\psi_{jis} + \psi_{isj} + \psi_{sji} = 0$*

$$\mathscr{F}(\phi)_{ijs} = \frac{1}{12}\rho\phi_{stj}. \tag{5.13}$$

Since $(\delta_L^+ Z)_{sij} = (\delta_R^+ Z)_{ijs}$, we have then

$$\mathscr{F}(\delta_L^+ Z) = \frac{\rho}{12}\delta_R^+ Z.$$

So,

$$d_R^+(\mathscr{F}(\delta_L^+ Z)) = \frac{\rho}{12} d_R^+\delta_R^+ Z. \tag{5.14}$$

Together with formulas (5.7) and (5.8) this implies

PROPOSITION 5.6. *For* $Z \in C^\infty(S_0^2(\Omega^+))$

$$\begin{aligned}(d_R^+ d_L^+ \delta_L^+ \delta_R^+ Z + d_R^+ d_L^+ \delta_R^+ \delta_L^+ Z) &= \left(\frac{1}{2}\nabla^*\nabla + \frac{1}{4}\rho\right)^2 Z \\ &+ \frac{1}{2} d_R^+(\nabla^*\nabla\delta_R^+ Z) + \frac{7}{24}\rho\left(\frac{1}{2}\nabla^*\nabla + \frac{1}{4}\rho\right)Z.\end{aligned} \tag{5.15}$$

5.5. Only $d_R^+(\nabla^*\nabla\delta_R^+ Z)$ is the term which yet remains to be calculated in the formula (5.15).

We will associate this term to the two-fold rough Laplacian term $(\nabla^*\nabla)^2 Z$. Set $\phi = \delta_R^+ Z \in C^\infty(\Omega^+ \otimes \Omega^1)$. Then, from the Ricci identity,

$$-\nabla_s\nabla_a(\nabla_a\phi_{ijt}) = -\nabla_a\nabla_s(\nabla_a\phi_{ijt}) + S_{ijst} + R_{as}(\nabla_a\phi_{ijt})$$

where

$$S_{ijst} = R_{bisa}\nabla_a\phi_{bjt} + R_{bjsa}\nabla_a\phi_{ibt} + R_{btsa}\nabla_a\phi_{ijb}.$$

We apply again the Ricci identity on $\nabla_s\nabla_a\phi_{ijt}$. So

$$\begin{aligned}-\nabla_s\nabla_a\nabla_a\phi_{ijt} &= -\nabla_a\nabla_a\nabla_s\phi_{ijt} \\ &+ \nabla_a(R_{bisa}\phi_{bjt} + R_{bjsa}\phi_{ibt} + R_{btsa}\phi_{ijb}) \\ &+ S_{ijst} + R_{as}\nabla_a\phi_{ijt},\end{aligned}$$

namely,

$$-\nabla_s\nabla_a\nabla_a\phi_{ijt} = -\nabla_a\nabla_a\nabla_s\phi_{ijt} + R_{as}\nabla_a\phi_{ijt} + 2S_{ijst} + T_{ijst} \tag{5.16}$$

where

(5.17) $$T_{ijst} = \{(\nabla_a R_{bisa})\phi_{bjt} + (\nabla_a R_{bjsa})\phi_{ibt} + (\nabla_a R_{btsa})\phi_{ijb}\}.$$

Therefore, from the first Bianchi identity

$$(d_R \nabla^* \nabla \phi)_{ijst} = -(\nabla_s \nabla_a \nabla_a \phi_{ijt} - \nabla_t \nabla_a \nabla_a \phi_{ijs})$$

is given as

(5.18) $$(d_R \nabla^* \nabla \phi)_{ijst} = (\nabla^* \nabla d_R \phi)_{ijst} + \frac{\rho}{4}(d_R \phi)_{ijst} + B_{as} \nabla_a \phi_{ijt} - B_{at} \nabla_a \phi_{ijs} + 2(S_{ijst} - S_{ijts}) + (T_{ijst} - T_{ijts}).$$

For this we set further

(5.19) $$S_{ijst} - S_{ijts} = N^{(1)}_{ijst} + P^{(1)}_{ijst},$$

where

$$N^{(1)}_{ijst} = -R_{abst} \nabla_a \phi_{ijb}$$

and

$$P^{(1)}_{ijst} = R_{bisa} \nabla_a \phi_{bjt} + R_{bjsa} \nabla_a \phi_{ibt} - R_{bita} \nabla_a \phi_{bjs} - R_{bjta} \nabla_a \phi_{ibs}$$

and

(5.20) $$T_{ijst} - T_{ijts} = N^{(2)}_{ijst} + P^{(2)}_{ijst}$$

where

$$N^{(2)}_{ijst} = -(\nabla_a R_{abst}) \phi_{ijb}$$

and

$$P^{(2)}_{ijst} = \nabla_a R_{bisa} \cdot \phi_{bjt} + \nabla_a R_{bjsa} \cdot \phi_{ibt} - \nabla_a R_{bita} \cdot \phi_{bjs} - \nabla_a R_{bjta} \cdot \phi_{ibs}.$$

Since $B_{ts} = 0$, we have

(5.21) $$(d_R^+ \nabla^* \nabla \phi)_{ij12} = (\nabla^* \nabla d_R^+ \phi)_{ij12} + \frac{\rho}{4}(d_R^+ \phi)_{ij12} + (N^{(1)}_{ij12} + N^{(1)}_{ij34}) + (P^{(1)}_{ij12} + P^{(1)}_{ij34}) + \frac{1}{2}(N^{(2)}_{ij12} + N^{(2)}_{ij34}) + \frac{1}{2}(P^{(2)}_{ij12} + P^{(2)}_{ij34}).$$

Now we calculate the last four terms.

As same as before, we substitute (2.1) into these terms and make use of the fact that $W^+ = 0$ and $B = 0$. Then

ASSERTION 1. (i) $N^{(1)}_{ij12} + N^{(1)}_{ij34} = -\frac{1}{6}\rho(d^+_R\phi)_{ij12}$ *and* (ii) $N^{(2)}_{ij12} + N^{(2)}_{ij34} = 0$

On the other hand, we can write $(P^{(1)}_{ij12} + P^{(1)}_{ij34})$ as the parts $W(\phi)_{ij12}$, $K(\phi)_{ij12}$ and $G(\phi)_{ij12}$ each of which contains W_{ijst}, K_{ijst} and G_{ijst}, respectively;

$$P^{(1)}_{ij12} + P^{(1)}_{ij34} = W(\phi)_{ij12} + K(\phi)_{ij12} + G(\phi)_{ij12}$$

where we see $K(\phi) = 0$ from $B = 0$.

ASSERTION 2.

(i) $W(\phi)_{ij12} = 0$ *and* (ii) $G(\phi)_{ij12} = \frac{\rho}{6}(d^+_R\phi)_{ij12}$

That $W(\phi) = 0$ follows from $W^+ = 0$ by a simple computation.
For $G(\phi)$ we have

$$\begin{aligned} G(\phi)_{ij12} &= G_{311a}\nabla_a\phi_{322} + G_{411a}\nabla_a\phi_{422} - G_{322a}\nabla_a\phi_{131} - G_{422a}\nabla_a\phi_{141} \\ &+ G_{313a}\nabla_a\phi_{324} + G_{323a}\nabla_a\phi_{134} - G_{414a}\nabla_a\phi_{423} - G_{424a}\nabla_a\phi_{143}. \end{aligned}$$

It suffices to show (ii) for $(ij) = (12)$ and (13). From the first Bianchi identity of Z it follows that

$$G(\phi)_{1212} = \frac{\rho}{6}(d^+_R\phi)_{1212}.$$

Similarly, $G(\phi)_{1312}$ is computed

$$G(\phi)_{1312} = \frac{\rho}{12}\{2(d^+_R\phi)_{1312} + 2(\delta_R\phi)_{14}\},$$

and then

$$G(\phi)_{1312} = \frac{1}{6}\rho(d^+_R\phi)_{1312},$$

since $\delta_R\phi = \delta_R\delta^+_R Z = 0$ and this is because from Proposition 3.2 $d^+_R d_R\phi$, $\phi \in C^\infty(M\,;\Omega^+)$ has no $S^2(\Omega^+)$-component. This shows Assertion 2.

Next we calculate the last term $(P^{(2)}_{ij12} + P^{(2)}_{ij34})$.

ASSERTION 3.

$$P^{(2)}_{ij12} + P^{(2)}_{ij34} = 0$$

When $(ij) = (12)$, this term becomes

$$\begin{aligned} P^{(2)}_{1212} + P^{(2)}_{1234} = & \nabla_a R_{b11a} \cdot \psi_{b22} + \nabla_a R_{b21a} \cdot \psi_{1b2} \\ & - \nabla_a R_{b12a} \cdot \psi_{b21} - \nabla_a R_{b22a} \cdot \psi_{1b1} \\ & + \nabla_a R_{b13a} \cdot \psi_{b24} + \nabla_a R_{b23a} \cdot \psi_{1b4} \\ & - \nabla_a R_{b14a} \cdot \psi_{b23} - \nabla_a R_{b24a} \cdot \psi_{1b3}, \end{aligned}$$

which turns out to be zero, because $W^+ = 0$ and ρ is constant.

Similarly, $P^{(2)}_{1312} + P^{(2)}_{1334} = 0$ is easily shown so that Assertion 3 is proven.

Therefore the last four terms in (5.21) reduce completely to zero so that we have

PROPOSITION 5.6.

$$(d_R^+ \nabla^* \nabla \delta_R^+ Z)_0 = (\nabla^* \nabla d_R^+ \delta_R^+ Z) + \frac{1}{4} \rho (d_R^+ \delta_R^+ Z) \tag{5.22}$$

The following proposition gives then the final form of the Weitzenböck formula for the Bach operator in (5.1).

PROPOSITION 5.7.

$$\begin{aligned} & (d_R^+ d_L^+ \delta_L^+ \delta_R^+ Z + d_R^+ d_L^+ \delta_R^+ \delta_L^+ Z)_0 = \\ & \frac{1}{12} (3 \nabla^* \nabla + 2\rho)(2 \nabla^* \nabla + \rho) Z. \end{aligned} \tag{5.23}$$

5.6. The ALE hyperkähler case. Let (M, g) be an ALE hyperkähler 4-manifold. We assert $\mathrm{Ker} D D^* = \{0\}$ for (M, g).

Let $C_o^\infty(M ; S_0^2(\Omega^+))$ be the space of smooth sections of $S_0^2(\Omega^+)$ having a compact support. We denote by W_k^2 the completion of $C_o^\infty(M ; S_0^2(\Omega^+))$ in terms of the L_k^2-norm with respect to the metric g. Here k is a certain integer ≥ 2.

Let $Z \in W_k^2$. From the completion we may assume $Z \in C_o^\infty(M ; S_0^2(\Omega^+))$. Since g is Ricci flat, we have from Proposition 5.1 that

$$D D^* Z = \frac{1}{4} (\nabla^* \nabla)^2 Z$$

so that if $Z \in \mathrm{Ker} D D^*$, then by integration

$$\int_M ((\nabla^*\nabla)^2 Z, Z) dv_g = 0. \tag{5.24}$$

Since Z has a compact support, the partial integral gives us

$$\int_M |\nabla^*\nabla Z|^2 dv_g = 0$$

which implies $\nabla^*\nabla Z = 0$ and hence

$$\int_M (\nabla^*\nabla Z, Z) dv_g = \int_M |\nabla Z|^2 dv_g = 0 \tag{5.25}$$

from which Z must be covariantly constant, whereas the support of Z is compact. Hence we get Theorem 4 in §1.

REFERENCES

[1] W. L. Baily, The decomposition theorem for V-manifolds, Amer. J. Math., **78** (1956), 862–888.
[2] J. P. Bourguignon, Les variétés de dimension 4 à signature non nulle dont la courbure est harmonique sont d'Einstein, Invent. Math., **63** (1981), 263–286.
[3] A. Derdzinski, Self-dual Kähler manifolds and Einstein manifolds of dimension four, Comp. Math., **49** (1983), 405–433.
[4] S. K. Donaldson and R. Friedman, Connected sums of self-dual manifolds and deformations of singular spaces, Nonlinearity, **2** (1989), 197–239.
[5] A. Floer, Self-dual conformal structures on lCP^2, J. Differential Geom., **33** (1991), 551–573.
[6] T. Friendrich and H. Kurke, Compact four-dimensional self-dual Einstein manifolds with positive scalar curvature, Math. Nachr., **106** (1982), 271–299.
[7] M. Furuta, On self-dual pseudo-connections on some orbifolds, Math. Zeit., **209** (1992), 319–337.
[8] K. Galicki and H. B. Lawson, Quaternionic reduction and quaternionic orbifolds, Math. Ann., **282** (1988), 1–21.
[9] K. Galicki and T. Nitta, Non-zero scalar curvature generalizations of the ALE hyperkähler metrics (1991), preprint.
[10] N. Hitchin, On compact four-dimensional Einstein manifolds, J. Differential Geom., **9** (1974), 435–442.
[11] N. Hitchin, Kählerian twistor spaces, Proc. London Math. Soc., **43** (1981), 133–150.
[12] M. Itoh, Moduli of half conformally flat structures, Math. Ann., **296** (1993), 687–708.
[13] M. Itoh, Half conformally flat structures and the deformation obstruction space, Tsukuba J. Math., **17** (1993), 143–158.
[14] M. Itoh, Half conformally flat metrics and connected sums of orbifolds, preparation.

[15] A. D. King and D. Kotshick, The deformation theory of anti-self-dual conformal structures, Math. Ann., **294** (1992), 591–609.
[16] P. B. Kronheimer, A Torelli-type theorem for gravitational instantons, J. Differential Geom., **29** (1989), 685–697.
[17] Y. S. Poon, Compact self-dual manifolds of positive scalar curvature, J. Differential Geom., **24** (1986), 97–132.
[18] I. Satake, On a generalization of the notion of manifolds, Proc. Natl. Acad. Sci. Amer., **42** (1956), 359–363.
[19] C. H. Taubes, The existence of anti-dual conformal structures, J. Differential Geom., **36** (1992), 163–253.

Institute of Mathematics
University of Tsukuba
305 Japan