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ADDENDUM TO "ON THE BERGMAN KERNEL OF HYPERCONVEX DOMAINS", NAGOYA MATH. J. 129 (1993), 43–52

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0. In [O-1] it was proved that for any bounded hyperconvex domain D in $\mathbf{C}^2$ the Bergman kernel function $K(z, w)$ of D satisfies

$$\lim_{z \to \partial D} K(z, z) = \infty.$$

In case $n = 1$, this is due to a behavior of sublevel sets of the Green function. The general case then follows by the extendability of L^2 holomorphic functions.

1. After the author finished typing the manuscript of [O-1], H. Tanigawa suggested to him an alternative proof of the one variable case. Her argument consisted of an observation that the logarithmic capacity $c_\beta(z)$ of any bounded hyperconvex domain in $\mathbf{C}$ is exhaustive and an assertion that $K(z, z)$ is exhaustive whenever so is $c_\beta(z)$. Unfortunately, her proof of the latter statement was too difficult for the author to follow, and seemingly not to be published anywhere. Therefore, he decided to fix her idea by giving a straightforward proof to the following.

THEOREM. There exists a constant $A \in [\pi, 750\pi]$ such that, for any Riemann surface S and for any local coordinate z on S, $\sqrt{AK(z, z)} \geq c_\beta(z)$ holds.

2. N. Suita [S] conjectured that π can be taken as the above A. In fact he showed that

$$\sqrt{\pi K(z, z)} = c_\beta(z)$$

if $S = \{z \in \mathbf{C} \mid |z| < 1\}$, and that

$$\sqrt{\pi K(z, z)} > c_\beta(z)$$

if $S = \{z \in \mathbf{C} \mid r < |z| < 1\}$ for some $r \in (0,1)$. The author hopes that our method may give a new insight into this subtle question.

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3. Let S be any Riemann surface, and let z be any local coordinate of S defined on a coordinate neighborhood, say U. If S admits the Green function g, the logarithmic capacity $c_\beta(z)$ $(= c_\beta(z(p)))$ is defined by

$$-\log c_\beta(z(p)) = \lim_{q\to p} (g(p, q) + \log |z(p) - z(q)|)$$

for any $p \in U$. Otherwise we set $c_\beta(z) \equiv 0$. The Bergman kernel function $K(z)$ $(= K(z, z))$ is defined by

$$\log K(z) = \sup \log |Q(z)|^2.$$

Here Q runs through the set $\{Q \mid Q(z)$ is holomorphic on U and there exists a holomorphic 1-form f on S of L^2 norm 1 such that $Q(z)\mathrm{d}z = f \mid U\}$.

4. For the proof of theorem we may assume that $c_\beta(z) \not\equiv 0$, since the result is trivial otherwise. For any point $p \in U$, we shall prove that there exists a holomorphic 1-form B_p on S such that

$$B_p \mid p = c_\beta(z)\, dz \mid p$$

and

$$\| B_p \|^2 \le 750\,\pi.$$

Here $\| \;\|$ denotes the L^2 norm.

5. Let $\chi : R \to R$ be any C^∞ function satisfying $\chi(t) = 1$ on $(-\infty, 1]$, $\chi(t) = 0$ on $(2, \infty)$ and $|\chi'(t)| < 2\log 2$ everywhere. For simplicity we put

$$g_p = g(p, \cdot).$$

Then we put

$$f_\varepsilon = \begin{cases} \chi\left(\dfrac{-g_p - \log\varepsilon}{\log 2}\right) c_\beta(z(p)) & \text{on } U \\ 0 & \text{on } S \setminus U. \end{cases}$$

Clearly, for sufficiently small ε, f_ε is a C^∞ function satisfying $f_\varepsilon(p) = c_\beta(z(p))$ and $\| f_\varepsilon\, dz \| \to 0$ as $\varepsilon \to 0$.

6. We assert that there exists a $c_0 > 0$ such that, for any $\varepsilon \in (0, c_0)$ one can find a square integrable (1, 0) form α_ε on S satisfying

(1) $$\bar\partial \alpha_\varepsilon = \bar\partial f_\varepsilon \wedge dz$$

(2) $$\left| \int_U |z|^{-2} \alpha_\varepsilon \wedge \bar\alpha_\varepsilon \right| < \infty$$

and

$$\|\alpha_\varepsilon\|^2 < 750\,\pi. \tag{3}$$

This suffices, since the required 1-form B_p will be obtained by putting $B_p = f_\varepsilon \wedge dz - \alpha_\varepsilon$ for sufficiently small ε.

7. For that, givin ε we look for a positive number δ, a C^∞ function $\rho: S \to (0, \infty)$ and a conformal metric ds^2 on S satisfying the following conditions (i) through (iii).

(i) $$i\delta \int_U |z|^{-2}\, \alpha \wedge \bar{\alpha} + i \int_S \alpha \wedge \bar{\alpha} \leq 5\, i \int_S \rho\alpha \wedge \bar{\alpha}$$

for any square integrable (1, 0) form α on S.

(ii) For any C^∞ (1,1) form β on $S \backslash \{p\}$ with supp $\beta \subset \{\log 2 < g_p + \log \varepsilon < 2 \log 2\}$, there exists a solution to $\bar{\partial}\alpha = \beta$ satisfying

$$i \int_S \rho\, \alpha \wedge \bar{\alpha} \leq \int_S e^{2g_p} |\beta|^2\, dvol.$$

Here $|\beta|$ denotes the pointwise norm of β and $dvol$ denotes the volume form, both with respect to ds^2.

(iii) $$\int_S e^{2g_p} |\bar{\partial} f_\varepsilon \wedge dz|^2\, dvol < 150\,\pi.$$

8. Obviously, we are through if there exist δ, ρ and ds^2 as above.

9. It is easy to see that (iii) is satisfied if we put

$$ds^2 = 4e^{-2g_p}\varepsilon^2 (e^{-2g_p} + \varepsilon^2)^{-2} \partial g_p\, \bar{\partial} g_p.$$

On the other hand, a general nonsense of elementary functional analysis tells us that (ii) is satisfied provided that there exists a C^∞ positive function η on S such that,

$$-\, i(\partial\bar{\partial}\eta + \eta^{-2} \partial\eta \wedge \bar{\partial}\eta) \geq 4i\varepsilon^2 e^{-2g_p} (e^{-2g_p} + \varepsilon^2)^{-2} \partial g_p \wedge \bar{\partial} g_p \tag{4}$$

and

$$\rho \leq e^{2g_p} (\eta + \eta^2)^{-1} \tag{5}$$

(cf. [O-2] Theorem 1.7).

10. Therefore our problem was reduced to finding ρ and η satisfying (4), (5) and (i) for some $\delta > 0$.

11. For that, we put

$$\eta = -\log(e^{-2(g_p+1)} + \varepsilon^2) + \log(-\log(e^{-2(g_p+1)} + \varepsilon^2))$$

for $\varepsilon \in (0, e^{-1} - e^{-2})$. To simplify the computation, let

$$\phi = \log(e^{-2(g_p+1)} + \varepsilon^2).$$

Then

$$-\partial\bar{\partial}\eta = \partial\bar{\partial}\phi - \phi^{-1}\partial\bar{\partial}\phi + \phi^{-2}\partial\phi \wedge \bar{\partial}\phi$$

and

$$\eta^{-2}\partial\eta \wedge \bar{\partial}\eta = (\phi + \log(-\phi))^{-2}(1 - \phi^{-1})^2\partial\phi \wedge \bar{\partial}\phi.$$

Hence

$$-i(\partial\bar{\partial}\eta + \eta^{-2}\partial\eta \wedge \bar{\partial}\eta) \geq i\partial\bar{\partial}\phi.$$

But a straightforward computation shows that

$$i\,\partial\bar{\partial}\phi \geq 4i\varepsilon^2(e^{-2g_p} + \varepsilon^2)^{-2}e^{-2g_p}\,\partial g_p\bar{\partial}g_p.$$

Thus (4) is satisfied by the above η. As ρ, we have only to put

$$\rho = e^{2g_p}(\eta + \eta^2)^{-1}.$$

In fact, since

$$\begin{aligned}
&\sup_{t>0} e^{-2t}(-\log(e^{-2(t+1)} + \varepsilon^2) + \log(-\log(e^{-2(t+1)} + \varepsilon^2)))\\
&\leq \sup_{t>0} e^{2-2T}(2T + \log T + \log 2)\\
&\leq \sup_{t>0} e^{2-2T}(3T - 1 + \log 2)\\
&< \frac{3}{2}e < 5,
\end{aligned}$$

one has (i) for sufficiently small δ, in view of the behavior of ρ near p.

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